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On a continuation of quaternionic and octonionic logarithm along curves and the winding number ☆

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ABSTRACT

This paper deals with the problem of finding a continuous extension of the hypercomplex (quaternionic or octonionic) logarithm along (quaternionic or octonionic) paths which avoid the origin. The main difficulty depends upon this fact: while a branch of the complex logarithm can be defined in a small open neighbourhood of a strictly negative real point, no continuous branch of the hypercomplex logarithm can be defined in any open set which contains a strictly negative real point. To overcome this difficulty, we use the *logarithmic manifold* introduced in [4]: in general, the existence of a lift of a path to this manifold is not guaranteed and, indeed, the problem of lifting a path to the logarithmic manifold is completely equivalent to the problem of finding a continuation of the hypercomplex logarithm along this path.

The second part of the paper scrutinizes the existence of a notion of winding number (with respect to the origin) for hypercomplex loops that avoid the origin, even though it is known that the definition of winding number for such loops is not natural in $\mathbb{R}^n$ when n is greater than 2. The surprise is that, in the hypercomplex setting, the new definition of winding number introduced in this paper can be given and has full meaning for a large class of hypercomplex loops (*untwisted loops with companion* that avoid the origin).

Finally an original but rather natural notion of homotopy for these hypercomplex loops (the *c-homotopy*) is presented and it is proved to be suitable to comply with the intrinsic geometrical meaning of the winding number for this class of loops, namely, two such hypercomplex loops are c-homotopic if, and only if, they have the same winding number.

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1. Introduction

This paper focuses on the problem of finding a continuous extension of the hypercomplex logarithm along a path. As pointed out in [4], while a branch of the complex logarithm can be defined in a small open neighbourhood of a strictly negative real point, no continuous branch of the hypercomplex logarithm can be defined in any open set $A \subset \mathbb{K} \setminus \{0\}$ which contains a strictly negative real point x_0 (here $\mathbb{K}$ represents the algebra of quaternions or of octonions).

To overcome these difficulties, in [4] we introduced the *logarithmic manifold* $\mathscr{E}^+_{\mathbb{K}}$ and then showed that, if $q \in \mathbb{K},\ q = x + Iy$ then $E(x + Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$ is an immersion and a diffeomorphism between $\mathbb{K}$ and $\mathscr{E}^+_{\mathbb{K}}$.

In this paper, we consider lifts of paths in $\mathbb{K} \setminus \{0\}$ to the logarithmic manifold $\mathscr{E}^+_{\mathbb{K}}$; even though $\mathbb{K} \setminus \{0\}$ is simply connected, in general, given a path in $\mathbb{K} \setminus \{0\}$, the existence of a lift of this path to $\mathscr{E}^+_{\mathbb{K}}$ is not guaranteed. There is an obvious equivalence between the problem of lifting a path in $\mathbb{K} \setminus \{0\}$ and the one of finding a continuation of the hypercomplex logarithm $\log_{\mathbb{K}}$ along this path.

We want to recall that the slice regular logarithm $\log_*(f)$ of a slice regular function f (see [2,5]) over the quaternions or octonions, introduced as the slice regular inverse of the slice regular exponential $\exp_*(f)$ of a slice regular function f (see [1]), is not defined in general via the lift of f to $\mathscr{E}^+_{\mathbb{K}}$. In particular it turns out that, in general, $\log_*(f)(q) \neq \log_{\mathbb{K}}(f(q))$.

The paper is organized as follows: in Sections 2 and 3, after recalling the basic notions on slice regular exponential and logarithmic functions, we provide explicit examples of paths intersecting the real axis and show how a branch of the hypercomplex logarithm can be defined along certain curves even when they encounter the real axis at negative points, providing a so called *continuation of the logarithm along a continuous curve*.

Furthermore, we introduce the notion of path and of loop *with a companion* (see Subsection 4.1) and then give a definition of *winding number with respect to* 0 that has a full meaning for a class of loops in $\mathbb{K} \setminus \{0\} \simeq \mathbb{R}^{2^s} \setminus \{0\}$ $(s = 2, 3)$ with companion; this fact is quite novel and original since it is well known that a definition of winding number for a loop (with respect to a point) is not in general possible in $\mathbb{R}^n$ when n is greater than 2. Moreover this notion of winding number is invariant for the class of c-homotopic loops with companion.

Finally, in the last Section 5, we extend the continuation of the hypercomplex logarithm to the case of curves with an infinite number of intersections with the real axis. These represent the set of obstructions for such an extension. When these obstructions are “mild” and “reasonable”, then we also present an effective way to calculate the winding numbers using the so-called notion of *signature*.

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2. Preliminary results

We denote by $\mathbb{K}$ either the algebra of quaternions or octonions. Let $\mathbb{S}$ be the sphere of imaginary units, i.e. the set of $I \in \mathbb{K}$ such that $I^2 = -1$. Given any $z \in \mathbb{K} \setminus \mathbb{R}$, there exist (and are uniquely determined) an imaginary unit I, and two real numbers x, y (with $y > 0$) such that $z = x + Iy$. With this notation, the conjugate of z will be $\bar{z} := x - Iy$ and $|z|^2 = z\bar{z} = \bar{z}z = x^2 + y^2$. Each imaginary unit I generates (as a real algebra) a copy of a complex plane denoted by $\mathbb{C}_I$. We call such a complex plane a *slice*. The upper half-plane in $\mathbb{C}_I$, namely $\{x + yI \ : \ y > 0\}$ will be denoted by $\mathbb{C}^+_I$. Similarly, the lower half-plane in $\mathbb{C}_I$ $\{x + yI \ : \ y < 0\}$ will be denoted by $\mathbb{C}^-_I$; each of these two half–planes will be called a *leaf* of $\mathbb{C}_I$. On any leaf $\mathbb{C}^+_I$ we define the function $\arg_I : \mathbb{C}^+_I \to (0, \pi)$ as $z = x + Iy \in \mathbb{C}^+_I \mapsto \cot^{-1}(x/y) := \arg_I(z)$.

The function $\arg_I$ can be continuously extended as a function $\arg_I : \mathbb{C}^+_I \cup \mathbb{R}^+ \cup \mathbb{R}^- \to [0, \pi]$.

It is also useful to define the imaginary unit function on $\mathbb{K} \setminus \mathbb{R}$ in the following way: if $z \in \mathbb{C}_I^+$, i.e. if $z = x + Iy$, with $x, y \in \mathbb{R}$ and $y > 0$, then $\mathcal{I}(z) = I$; if $z \in \mathbb{C}_I^-$, i.e. if $z = x - Iy$, with $x, y \in \mathbb{R}$ and $y > 0$, then $\mathcal{I}(z) = -I$.

Remark 2.1. It is worthwhile noticing that the function $\mathcal{I}$ cannot be extended as a continuous function to any single point of the real axis $\mathbb{R}$ of $\mathbb{K}$. At the same time, if we set $\mathbb{S}(-\pi, \pi) = \{Iy : I \in \mathbb{S},\ y \in (-\pi, \pi)\}$, then the function

$$\mathrm{Arg} : \mathbb{K} \setminus (-\infty, 0] \to \mathbb{S}(-\pi, \pi)$$

defined as the product

$$\mathrm{Arg}(q) := \mathcal{I}(q) \arg_{\mathcal{I}(q)}(q)$$

can be extended (as the zero function) to the positive real axis $\mathbb{R}^+$ of $\mathbb{K}$.

3. The hypercomplex exponential and logarithm

Let us recall that the exponential map on $\mathbb{K}$

$$\exp : \mathbb{K} \to \mathbb{K} \setminus \{0\}$$

defined as

$$\exp(q) = \sum_{k \geq 0} \frac{q^k}{k!}$$

is a *slice regular* and *slice preserving* entire function on $\mathbb{K}$ ([1,6]). Let $\mathscr{E}_{\mathbb{K}}^+ = T(\mathbb{K}^+)$ denote the *logarithm manifold*, i.e., the image $T(\mathbb{K}^+)$ of $\mathbb{K}^+ = \{q \in \mathbb{K} : \mathrm{Re}\, q > 0\}$ of the map $T : \mathbb{K} \to \mathbb{K} \times \mathrm{Im}(\mathbb{K})$ defined by

$$T(x + Iy) = (\sinh x \cos y + I \sinh x \sin y, Iy)$$

for $I \in \mathbb{S}$, $x, y \in \mathbb{R}$. The $\mathscr{E}_{\mathbb{K}}^+$-exponential map

$$E : \mathbb{K} \to \mathscr{E}_{\mathbb{K}}^+ \subset \mathbb{K} \times \mathrm{Im}(\mathbb{K})$$

defined by:

$$E(x + Iy) = (\exp(x + Iy), Iy) = (\exp x \cos y + I \exp x \sin y, Iy)$$

is an immersion and a diffeomorphism between $\mathbb{K}$ and $\mathscr{E}_{\mathbb{K}}^+$ (see [4]). In the case of quaternions, it endows $\mathscr{E}_{\mathbb{H}}^+$ with a structure of slice quaternionic manifold (see, e.g., [3]), which is different from the structure of hypercomplex Riemann manifold defined in Proposition 4.3. [4])

The next definition and result appear in [4] (Proposition 5.3).

Definition 3.1. Let $\mathscr{E}_{\mathbb{K}}^+$ be the logarithm manifold. The $\mathscr{E}_{\mathbb{K}}^+$-logarithm

$$L : \mathscr{E}_{\mathbb{K}}^+ \subset \mathbb{K} \times \mathrm{Im}(\mathbb{K}) \to \mathbb{K}$$

is defined as follows, in terms of the real logarithm log:

$$L(q,p) = \log|q| + p$$

Indeed, if $(q,p) \in \mathscr{E}_{\mathbb{K}}^{+}$, then $q = r\exp p$ for $r = |q|$ and our definition can be rewritten as:

$$L(r\exp p, p) = \log r + p$$

The hypercomplex manifold $\mathscr{E}_{\mathbb{K}}^{+}$ plays the role of an "adapted" blow-up of $\mathbb{K}$ at points of the form $x + 2Ik\pi$, for $k \in \mathbb{Z}$ and $k \neq 0$.

Proposition 3.2. *The map*

$$L : \mathscr{E}_{\mathbb{K}}^{+} \to \mathbb{K}$$

is the inverse of the $\mathscr{E}_{\mathbb{K}}^{+}$-exponential E, and a diffeomorphism from the logarithm manifold $\mathscr{E}_{\mathbb{K}}^{+}$ to $\mathbb{K}$.

Note that if $\mathrm{pr}_1 : \mathbb{K} \times \mathrm{Im}(\mathbb{K}) \to \mathbb{K}$ denotes the projection on the first factor, then by definition the following equality holds

$$\mathrm{pr}_1 \circ E(q) = \exp(q)$$

for all $q \in \mathbb{K}$. Indeed, the map L is a slice regular map from $\mathscr{E}^{+}$ to $\mathbb{K}$, with respect to the structure of slice regular manifold induced by E on $\mathscr{E}_{\mathbb{K}}^{+}$ (see, e.g., [3]). This map allows the definition of the hypercomplex logarithm (see [4,5]):

Definition 3.3. Let $\mathrm{pr}_1 : \mathscr{E}_{\mathbb{K}}^{+} \subset \mathbb{K} \times \mathrm{Im}(\mathbb{K}) \to \mathbb{K} \setminus \{0\}$ denote the natural projection

$$(q,p) \mapsto q$$

and let $\Omega \subset \mathscr{E}_{\mathbb{K}}^{+}$ be a path connected subset such that ${\mathrm{pr}_1}_{|\Omega}$ is injective. Then, the map

$$\log_{\mathbb{K}} : \mathrm{pr}_1(\Omega) \to \mathbb{K}$$

defined by

$$\log_{\mathbb{K}} q = L({\mathrm{pr}_1}_{|\Omega}^{-1}(q))$$

is called a *branch* or *a determination of the hypercomplex logarithm* on $\mathrm{pr}_1(\Omega)$.

As one can expect, it holds

$$\exp(\log_{\mathbb{K}} q) = \mathrm{pr}_1(E(L({\mathrm{pr}_1}_{|\Omega}^{-1}(q)))) = \mathrm{pr}_1({\mathrm{pr}_1}_{|\Omega}^{-1}(q)) = q$$

for all q in $\mathrm{pr}_1(\Omega)$. It is worthwhile noticing that, if we consider the open, path-connected subset

$$\Omega = \{(q, \mathrm{Arg}(q)) : q \in \mathbb{K} \setminus (-\infty, \pi]\} \subset \mathscr{E}_{\mathbb{K}}^{+},$$

then the projection on the first factor

$$\mathrm{pr}_1 : \Omega \to \mathbb{K} \setminus (-\infty, 0]$$

is injective. Therefore, in this way, one defines the *principal branch* of the logarithm (see [7]) in $\mathrm{pr}_1(\Omega) = \mathbb{K} \setminus (-\infty, \pi]$ (see [5,2]). The principal branch of the hypercomplex logarithm

$$\log_0 : \mathbb{K} \setminus (-\infty, 0] \to \mathbb{R} \times [0, \pi)\mathbb{S} \subset \mathbb{K}$$
$$q \mapsto \log|q| + \mathrm{Arg}(q)$$

is well defined and, for all $I \in \mathbb{S}$, coincides with the principal branch of the complex logarithm in the slice $\mathbb{C}_I \setminus (-\infty, 0]$. As a consequence, $\log_0$ is a slice regular function in the symmetric slice domain $\mathbb{K} \setminus (-\infty, 0]$.

As already observed in the Introduction, despite the analogy with the complex holomorphic case, in general no continuous branch of the hypercomplex logarithm can be defined in any open set $A \subset \mathbb{K}\setminus\{0\}$ which contains a strictly negative real point x_0. Nevertheless, we will now see how a branch of the hypercomplex logarithm can be defined along certain curves even when they encounter the real axis at negative points, providing a so called *continuation of the logarithm along a continuous curve*.

Throughout the paper, a continuous curve will be called a *path*, and a closed path will be called a *loop*.

Definition 3.4. Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a path. Then a path $\widetilde{\gamma} : [a, b] \to \mathbb{K}$ is called *a continuation of the logarithm along* γ if

$$\exp \circ \widetilde{\gamma} = \gamma,$$

i.e., if the following diagram commutes:

$$\begin{tikzcd} & \mathbb{K} \arrow[d, "\exp"] \\ {[a,b]} \arrow[ur, "\widetilde{\gamma}"] \arrow[r, "\gamma"] & \mathbb{K} \setminus \{0\} \end{tikzcd}$$

The point $\widetilde{\gamma}(a) \in \mathbb{K}$ will be called the initial point of the continuation $\widetilde{\gamma}$.

To study the possible continuations of the logarithm along a path, we need to specifically define the various branches of the hypercomplex argument of an element from $\mathbb{K} \setminus \{0\}$.

Definition 3.5. If $k \in \mathbb{Z}$, for all $q \in \mathbb{K} \setminus \mathbb{R}$, $q = x + \mathcal{I}y$ with $y > 0$, let us define for

$$\begin{aligned} k = 2l : \quad & \mathcal{I}_{2l}(q) = \mathcal{I}(q),\ \arg_{2l}(q) = \arg_{\mathcal{I}}(q) + 2l\pi, \\ & \text{where } \arg_{2l}(q) \in (2l\pi, (2l+1)\pi), \\ k = 2l+1 : \quad & \mathcal{I}_{2l+1}(q) = -\mathcal{I}(q),\ \arg_{2l+1}(q) = (2\pi - \arg_{\mathcal{I}}(q)) + 2l\pi, \\ & \text{where } \arg_{2l+1}(q) \in ((2l+1)\pi, (2l+2)\pi). \end{aligned}$$

The k-th branch of the hypercomplex argument

$$\mathrm{Arg}_k : \mathbb{K} \setminus \mathbb{R} \to \mathbb{S}(k\pi, (k+1)\pi)$$

is defined by setting

$$\mathrm{Arg}_k(q) := \mathcal{I}_k(q) \arg_k(q).$$

As a consequence,

$$\begin{aligned}\mathrm{Arg}_{2l+1}(q) &= -\mathcal{I}(q)(2\pi - \arg_{\mathcal{I}}(q) + 2l\pi) \\ &= \mathcal{I}(q)(\arg_{\mathcal{I}}(q) - 2(l+1)\pi) \\ &= \mathcal{I}(q)\arg_{-2(l+1)}(q) \\ &= \mathrm{Arg}_{-(2l+2)}(q)\end{aligned} \tag{3.1}$$

Therefore, the only different branches of the hypercomplex argument of a quaternion $q \in \mathbb{K} \setminus \mathbb{R}$, $q = x + \mathcal{I}y$ with $y > 0$, can be listed for $k \in \mathbb{Z}$ as

$$\mathrm{Arg}_{2k}(q) := \mathcal{I}(q)\arg_{2k}(q).$$

It is worthwhile noticing that for any fixed $q \in \mathbb{K} \setminus \mathbb{R}$, we have that for all $k \in \mathbb{Z}$

$$\exp(\mathrm{Arg}_{2k}(q)) = \exp(\mathrm{Arg}(q));$$

indeed

$$\begin{aligned}\exp(\mathrm{Arg}_{2l}(q)) &= \exp[\mathcal{I}(q)(\arg_{\mathcal{I}}(q) + 2l\pi)] = \exp[\mathcal{I}(q)\arg_{\mathcal{I}}(q)] \\ &= \exp(\mathrm{Arg}(q)).\end{aligned}$$

Consequently, we define

$$\log_k(q) := \log|q| + \mathrm{Arg}_k(q),$$

where log is the real logarithm.

4. Continuation of hypercomplex logarithms along paths

The construction of a continuation of the logarithm along a path naturally involves the notion of a lift of a path.

Definition 4.1. Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path. Then a path $\Gamma : [a,b] \to \mathcal{E}_{\mathbb{K}}^{+}$ is *a lift of* γ *(to* $\mathcal{E}_{\mathbb{K}}^{+}$*)* if $\mathrm{pr}_1 \circ \Gamma = \gamma$, i.e., if the following diagram commutes:

$$\begin{tikzcd} & \mathcal{E}_{\mathbb{K}}^{+} \arrow[d, "\mathrm{pr}_1"] \\ {[a,b]} \arrow[ur, "\Gamma"] \arrow[r, "\gamma"] & \mathbb{K} \setminus \{0\} \end{tikzcd}$$

For $(q,p) \in \mathcal{E}_{\mathbb{K}}^{+}$, a lift Γ of γ such that $\Gamma(a) = (q,p)$ will be said to have *initial point* (q,p).

The existence of a lift of a path γ is equivalent to the existence of a continuation of the hypercomplex logarithm along it.

Proposition 4.2. *Let* $\gamma : [a,b] \subset \mathbb{R} \to \mathbb{K} \setminus \{0\}$ *be a path. Then, there exists a lift of* γ *to* $\mathcal{E}_{\mathbb{K}}^{+}$ *if, and only if, there exists a continuation of the logarithm along* γ*.*

Proof. Suppose that there exists a continuation of the logarithm $\widetilde{\gamma}$ along γ. Then the path Γ defined by $\Gamma(t) = ((\exp \circ \widetilde{\gamma})(t), \operatorname{Im}(\widetilde{\gamma}(t))$ is obviously a lift of γ to $\mathcal{E}^+_{\mathbb{K}}$.

$$\begin{tikzcd}
 & \mathcal{E}^+_{\mathbb{K}} \arrow[r, "L"] \arrow[d, "\mathrm{pr}_1"] & \mathbb{K} \arrow[dl, "\exp"] \\
{[a,b]} \arrow[ur, "\Gamma"] \arrow[r, "\gamma"'] & \mathbb{K} \setminus \{0\} &
\end{tikzcd}$$

Conversely, if a lift Γ of the path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ to $\mathcal{E}^+_{\mathbb{K}}$ exists, then a continuation of the hypercomplex logarithm along γ can be defined by $\widetilde{\gamma}(t) := L(\Gamma(t))$. $\square$

Thanks to the result just stated, we are left to find conditions under which a path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ can be lifted to $\mathcal{E}^+_{\mathbb{K}}$. Since the map $\mathrm{pr}_1 : \mathcal{E}^+_{\mathbb{K}} \to \mathbb{K} \setminus \{0\}$ is not a covering, we have to specifically study the existence of lifts of γ.

Let us first consider the easy cases: it is not difficult to see that if we restrict the map $\mathrm{pr}_1 : \mathcal{E}^+_{\mathbb{K}} \to \mathbb{K} \setminus \{0\}$ to the preimage of $\mathbb{K} \setminus \mathbb{R}$, then the restriction $\mathrm{pr}_{1|{\mathrm{pr}_1}^{-1}(\mathbb{K}\setminus\mathbb{R})}$ becomes a covering. Indeed it becomes a trivial covering, since ${\mathrm{pr}_1}^{-1}(\mathbb{K} \setminus \mathbb{R})$ is homeomorphic (namely diffeomorphic) through the diffeomorphism

$$E : \mathbb{K} \to \mathscr{E}^+_{\mathbb{K}}$$

to the countable collection of open simply connected domains given by

$$\mathbb{K} \setminus \left\{ \bigcup_{k\in\mathbb{Z}} \mathbb{R} \times \mathbb{S}k\pi \right\} = \bigcup_{k\in\mathbb{Z}} \mathbb{R} \times \mathbb{S}(k\pi, (k + 1)\pi).$$

Let us now set, for any $k \in \mathbb{Z}$,

$$D_k = \mathbb{R} \times \mathbb{S}(k\pi, (k + 1)\pi), \qquad E(D_k) = E(\mathbb{R} \times \mathbb{S}(k\pi, (k + 1)\pi)) \subset \mathscr{E}^+_{\mathbb{K}}.$$

Notice that $\mathbb{S}(k\pi, (k + 1)\pi) = \mathbb{S}(-(k + 1)\pi, -k\pi))$ and hence $D_{2k} = D_{-(2k+1)}$, so that, among all D_k's, it suffices to consider only those with even k's. With this in mind, we can now state the following proposition.

Proposition 4.3. *Assume the path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ is such that $\gamma([a, b]) \cap \mathbb{R} = \varnothing$, and let*

$$\gamma(t) = x(t) + \mathcal{I}(t)y(t)$$

with $y(t) > 0$ for all $t \in [a, b]$. Then, for any $k \in \mathbb{Z}$, there exists one, and only one, lift Γ_k of γ to $E(D_{2k}) \subset \mathscr{E}^+_{\mathbb{K}}$ with initial point

$$\Gamma_k(a) = (\gamma(a), \mathrm{Arg}_{2k}(\gamma(a))).$$

Namely, for all $t \in [a, b]$, we have

$$\Gamma_k(t) := (\gamma(t), \mathrm{Arg}_{2k}(\gamma(t))) \in E(D_{2k}).$$

Finally, for all $k \in \mathbb{Z}$, the map defined on the interval $[a, b]$ by

$$(L \circ \Gamma_k)(t) = \log|\gamma(t)| + \mathrm{Arg}_{2k}(\gamma(t)) = (\log_{2k} \circ \gamma)(t) \tag{4.2}$$

is the unique continuation of the hypercomplex logarithm along γ *with initial point* $\log|\gamma(a)| + \operatorname{Arg}_{2k}(\gamma(a))$*, and is called* the k-th branch of the hypercomplex logarithm along γ with initial point $\log|\gamma(a)| + \operatorname{Arg}_{2k}(\gamma(a))$.

Proof. For each $k \in \mathbb{Z}$, the proof of the existence and uniqueness of Γ_k as in the statement is a straightforward consequence of what already established. To prove the last part of the statement, let us consider the graph Ω_k of the lift Γ_k of γ to $E(D_{2k}) \subset \mathscr{E}^+_{\mathbb{K}}$ with initial point $\Gamma_k(a) = (\gamma(a), \operatorname{Arg}_{2k}(\gamma(a)))$, i.e.,

$$\Omega_k := \{(q, \operatorname{Arg}_{2k}(q)) : q \in \gamma([a,b])\} \subset E(D_{2k}) \subset \mathcal{E}^+_{\mathbb{K}}.$$

Since the projection $\pi : \mathcal{E}^+_{\mathbb{K}} \to \mathbb{K} \setminus \{0\}$ restricted to Ω_k is injective, following Definition 3.3, we obtain (4.2). □

Under the hypotheses of the preceding proposition, loops lift to loops, hence:

Corollary 4.4. *Assume the loop* $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ *is such that* $\gamma([a,b]) \cap \mathbb{R} = \varnothing$*. Then for each* $k \in \mathbb{Z}$*, the lift* Γ_k *found in Proposition 4.3 is a loop. As a consequence, for each* $k \in \mathbb{Z}$*,*

$$\log_{2k}(\gamma(a)) = \log_{2k}(\gamma(b))$$

Among the initial cases, there is the one corresponding to what is stated in Remark 2.1.

Proposition 4.5. *Assume the path* $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ *is such that* $\gamma([a,b]) \cap \mathbb{R}^- = \varnothing$*. Then there exists a lift* Γ *of* γ *to* $\mathcal{E}^+_{\mathbb{K}}$*.*

Proof. The proof is a consequence of what observed in Remark 2.1. Indeed, taking into account that $\mathbb{S}(-\pi,\pi) = \mathbb{S}[0,\pi) = \mathbb{S}(-\pi,0]$, the mentioned remark implies that $\mathrm{pr}_1 : E(\mathbb{R} \times \mathbb{S}(-\pi,\pi)) \to \mathbb{K} \setminus (-\infty, 0]$ is a homeomorphism. □

As pointed out in the Introduction, even though $\mathbb{K} \setminus \{0\}$ is simply connected, in general, given a path in $\mathbb{K} \setminus \{0\}$, the existence of a lift of this path to $\mathcal{E}^+_{\mathbb{K}}$ is not guaranteed. Indeed, consider the following examples.

Example 4.6.

(a) Let $\sigma : [\pi/2, 3\pi/2] \to \mathbb{K} \setminus \{0\}$ be the path (depicted in Fig. 1) defined by

$$\sigma(t) = \cos(t) + I(t)\sin(t)$$

where $I : [\pi/2, 3\pi/2] \to \mathbb{S}$ is defined as

$$I(t) = i \quad \text{for} \quad \pi/2 \le t < \pi \quad \text{and} \quad I(t) = -j \quad \text{for} \quad \pi \le t \le 3\pi/2.$$

The curve σ is continuous, but the function

$$\operatorname{Arg}(\sigma(t)) = I(t)\arg_I(\sigma(t))$$

is not continuous at π (the left and right limits are different). Therefore σ cannot be lifted to $\mathcal{E}^+_{\mathbb{K}}$.

(b) Consider now the loop $\gamma : [0,1] \to \mathbb{K} \setminus \{0\}$ defined by

$$\gamma(t) = \cos(\pi - 2\pi t) + t(1-t)(i\cos(2\pi/t) + j\sin(2\pi/t)),$$

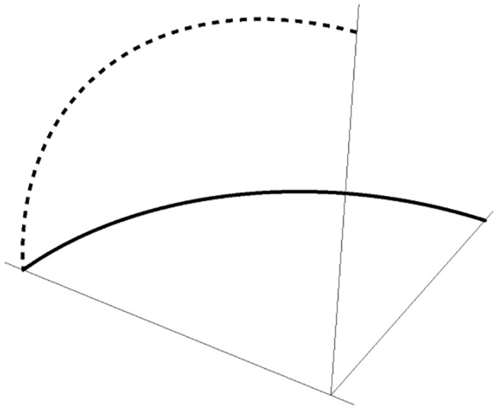


Fig. 1. The arc σ.

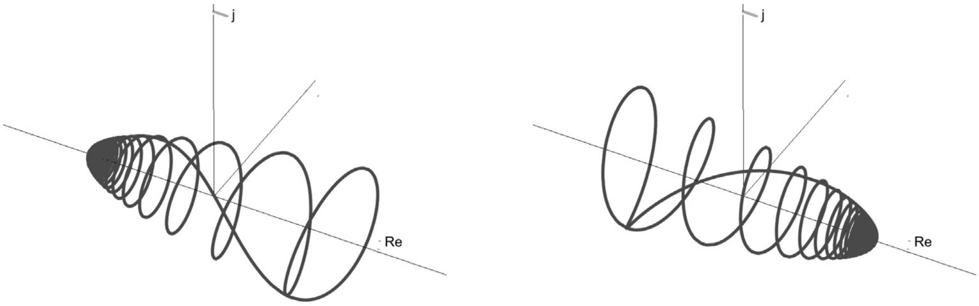


Fig. 2. The path γ (negative rocket) of the Example 4.6 (b) is drawn on the left: it cannot be lifted to $\mathcal{E}^+_{\mathbb{K}}$. Its reflection on the right (positive rocket) can be lifted to $\mathcal{E}^+_{\mathbb{K}}$.

where i, j are the usual orthogonal imaginary units (see Fig. 2). Notice that the imaginary part of γ is continuous at all points of the interval $[0, 1]$ (including 0). Nevertheless, for t near to 0, we have that the function

$$\begin{aligned}&\mathrm{Arg}(\gamma(t)) = \\ &= (i\cos(2\pi/t) + j\sin(2\pi/t))\arccos\left(\frac{\cos(\pi - 2\pi t)}{\sqrt{\cos^2(\pi - 2\pi t) + t^2(1-t)^2}}\right)\end{aligned}$$

has no limit for t approaching 0^+. Therefore γ cannot be lifted to $\mathcal{E}^+_{\mathbb{K}}$.

(c) Notice that in both the preceding cases, the paths $\hat{\sigma} := -\overline{\sigma}$ and $\hat{\gamma} := -\overline{\gamma}$ can be lifted to $\mathcal{E}^+_{\mathbb{K}}$, since their images are included in $\mathbb{K} \setminus (-\infty, 0]$ (see Proposition 4.5).

It is useful to point out that the existence of a lift Γ of a path γ to $\mathcal{E}^+_{\mathbb{K}}$ is equivalent to the existence of a continuous function $\mathrm{Arg}^\gamma : [a, b] \to \mathrm{Im}(\mathbb{H})$, such that

$$\Gamma(t) = (\gamma(t), \mathrm{Arg}^\gamma(t)) \in \mathcal{E}^+_{\mathbb{K}}.$$

As noticed in Remark 2.1, the function Arg^γ will be decomposed, where possible, with obvious notation, as

$$\mathrm{Arg}^\gamma := \mathcal{I}^\gamma \arg^\gamma$$

where $\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$ and $\arg^\gamma : [a,b] \to \mathbb{R}$. The existence of $\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$ implies that we can assign to each $t \in [a,b]$ a complex plane $\mathbb{C}_{\mathcal{I}^\gamma}$ which contains the point $\gamma(t)$ and hence determines the argument up to a multiple of 2π.

Complex slices $\{\mathbb{C}_J\}_{J\in\mathbb{S}}$ are naturally parameterized by the elements of $\mathbb{S}/\{\pm \mathrm{Id}\}$, the real projective space $\mathbb{RP}^{\dim_{\mathbb{R}}\mathbb{K}-2}$ of dimension $\dim \mathbb{K} - 2$. The projection $[\ ] : \mathbb{S} \to \mathbb{S}/\{\pm \mathrm{Id}\} = \mathbb{RP}^{\dim_{\mathbb{R}}\mathbb{K}-2}$ is the classical $2:1$ universal covering map and, as customary, for $J \in \mathbb{S}$, the symbol $[J]$ denotes the equivalence class whose representatives are the opposite (conjugate) imaginary units $J, -J \in \mathbb{S}$. Each element $[J] \in \mathbb{S}/\{\pm \mathrm{Id}\}$ uniquely defines the complex slice $\mathbb{C}_{[J]} = \mathbb{C}_J = \mathbb{C}_{-J}$. A continuous imaginary unit function $\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$ naturally defines a continuous function $\mathfrak{I}^\gamma : [a,b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ when we set $\mathfrak{I}^\gamma(t) = [\mathcal{I}^\gamma(t)]$.

Definition 4.7. Let $[a,b] \subset \mathbb{R}$ and let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path.

A path $\mathfrak{I}^\gamma : [a,b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ such that $\gamma(t) \in \mathbb{C}_{\mathfrak{I}^\gamma(t)}$ for every $t \in [a,b]$ is called a *companion* of the path γ.

If a companion $\mathfrak{I}^\gamma$ of the path γ exists, then γ is called a *path with a companion* and the pair $(\gamma, \mathfrak{I}^\gamma)$ is called a *path with companion* $\mathfrak{I}^\gamma$.

If the path γ has a unique companion $\mathfrak{I}^\gamma$, then both γ and the pair $(\gamma, \mathfrak{I}^\gamma)$ are called a *tame* path.

Proposition 4.8. *Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path with companion $\mathfrak{I}^\gamma$. If $\mathcal{I}^\gamma, -\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$ are the two lifts of $\mathfrak{I}^\gamma$, then there exist continuous functions $x, y : [a,b] \to \mathbb{R}$ such that, for all $t \in [a,b]$,*

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t) = x(t) + (-\mathcal{I}^\gamma(t))(-y(t)).$$

These last expressions are called canonical forms of $(\gamma, \mathfrak{I}^\gamma)$.

Proof. Since $\gamma(t)$ and $\mathcal{I}^\gamma(t)$ are both continuous, then $\mathrm{Re}(\gamma(t)) = x(t)$ and $-\mathcal{I}^\gamma(t)\mathrm{Im}(\gamma(t)) = y(t)$ are continuous as well on $[a,b]$. $\square$

It is easy to see that all paths lying entirely in a complex slice have a companion. Notice as well that a path $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ may have more than one companion: this happens for example when the path γ is such that $\gamma([a,b]) \subset (0,\infty)$; in this case, for an arbitrary path $\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$, the induced path $\mathfrak{I}^\gamma : [a,b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ is a companion of γ; consequently γ is not tame. For a similar reason, a path $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ which maps a closed sub-interval of $[a,b]$ to a real number has more than one companion, and hence is not tame.

Remark 4.9. There exist paths in $\mathbb{K} \setminus \{0\}$ which can be lifted to $\mathcal{E}^+_{\mathbb{K}}$, but have no companion. Indeed, set

$$\hat{\sigma} = -\overline{\sigma} : [\pi/2, 3\pi/2] \to \mathbb{K} \setminus \{0\}$$

where σ is the path defined in Example 4.6 (a). The path $\hat{\sigma}$ is the symmetric image of the path σ with respect to the plane of purely imaginary quaternions (see Fig. 1) and, as pointed out in Example 4.6 (c), it can be lifted to $\mathcal{E}^+_{\mathbb{K}}$. Obviously $\hat{\sigma}$ has no companion: the continuity of a companion cannot hold at $t = \pi$.

The following definition will play a central role in the sequel.

Definition 4.10. Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path with a companion $\mathfrak{I}^\gamma : [a,b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$, let $\mathcal{I}^\gamma, -\mathcal{I}^\gamma : [a,b] \to \mathbb{S}$ be the two (continuous) lifts of $\mathfrak{I}^\gamma$ to $\mathbb{S}$ and let

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t) = x(t) + (-\mathcal{I}^\gamma)(t)(-y(t))$$

be the canonical forms of $(\gamma, \mathfrak{I}^\gamma)$. The paths $\gamma_{\mathcal{I}^\gamma}, \gamma_{-\mathcal{I}^\gamma} : [a,b] \to \mathbb{C} \setminus \{0\}$ defined by

$$\gamma_{\mathcal{I}^\gamma} = x(t) + iy(t), \qquad \gamma_{-\mathcal{I}^\gamma} = x(t) - iy(t)$$

are called *the (two conjugated) shadows associated with the pair* $(\gamma, \mathfrak{I}^\gamma)$. If the path γ is tame, then the paths $\gamma_{\mathcal{I}^\gamma}$ and $\gamma_{-\mathcal{I}^\gamma}$ are simply called *the (two) shadows associated with the path* γ.

Remark 4.11. The two shadows associated with the pair $(\gamma, \mathfrak{I}^\gamma)$ are conjugate paths.

Paths with a companion are of interest because they can all be lifted to $\mathcal{E}^+_{\mathbb{K}}$.

Proposition 4.12. *Let* $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ *be a path with companion* $\mathfrak{I}^\gamma : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$*. Then there exist*

- *a path* $\mathcal{I}^\gamma : [a, b] \to \mathbb{S}$ *with* $[\mathcal{I}^\gamma(t)] = \mathfrak{I}^\gamma(t)$*, for all* $t \in [a, b]$*,*
- *a path* $\arg^\gamma : [a, b] \to \mathbb{R}$*,*

such that, after setting $\mathrm{Arg}^\gamma = \mathcal{I}^\gamma \arg^\gamma : [a, b] \to \mathrm{Im}(\mathbb{H})$*, the path*

$$\Gamma(t) = (\gamma(t), \mathrm{Arg}^\gamma(t))$$

is a lift of γ *to* $\mathcal{E}^+_{\mathbb{K}}$ *with* $\arg^\gamma(a) \in [0, \pi]$*, called a* $\mathfrak{I}^\gamma$-lift of γ*.*

If, as in Definition 3.5, for every $k \in \mathbb{Z}$ *we set* $\arg^\gamma_{2k} := \arg^\gamma + 2k\pi$ *and* $\mathrm{Arg}^\gamma_{2k} := \mathcal{I}^\gamma \arg^\gamma_{2k}$*, then the path*

$$\Gamma_k(t) = (\gamma(t), \mathrm{Arg}^\gamma_{2k}(t))$$

is a $\mathfrak{I}^\gamma$*-lift of* γ *to* $\mathcal{E}^+_{\mathbb{K}}$ *with* $\arg^\gamma_{2k}(a) \in [2k\pi, (2k + 1)\pi]$*.*

Proof. There exist exactly two continuous lifts $\mathcal{I}^\gamma, -\mathcal{I}^\gamma$ of $\mathfrak{I}^\gamma$ to the universal covering $\mathbb{S}$ of $\mathbb{S}/\{\pm \mathrm{Id}\}$. Correspondingly, there exist two shadows $\gamma_{\mathcal{I}^\gamma}, \gamma_{-\mathcal{I}^\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$ associated with $\mathfrak{I}^\gamma$. Exchange $\mathcal{I}^\gamma$ and $-\mathcal{I}^\gamma$ if necessary, so that $\mathcal{I}^\gamma$ is such that $\arg(\gamma_{\mathcal{I}^\gamma}(a)) \in [0, \pi]$. As a complex path, $\gamma_{\mathcal{I}^\gamma}$ has a well defined argument $\arg^{\gamma_{\mathcal{I}^\gamma}} : [a, b] \to \mathbb{R}$ such that $\arg^{\gamma_{\mathcal{I}^\gamma}}(a) \in [0, \pi]$. Set $\arg^\gamma := \arg^{\gamma_{\mathcal{I}^\gamma}}$. Then the chosen paths $\mathcal{I}^\gamma$ and $\arg^\gamma$ have the properties required in the statement. The rest of the proof is straightforward. □

The lifts Γ and Γ_k (for $k \in \mathbb{Z}$) appearing in the last Proposition are not unique, when $\Gamma(a)$ and $\Gamma_k(a)$ are real.

At this point, Proposition 4.2 implies directly the existence of all branches of the logarithm, along all paths in $\mathbb{K} \setminus \{0\}$ having a companion.

Corollary 4.13. *Let* $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ *be a path with companion* $\mathfrak{I}^\gamma : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$*. For every* $k \in \mathbb{Z}$*, let*

$$\Gamma_k = (\gamma, \mathrm{Arg}^\gamma_{2k})$$

be a $\mathfrak{I}^\gamma$*-lift of* γ *to* $\mathcal{E}^+_{\mathbb{K}}$ *with* $\arg^\gamma_{2k}(a) \in [2k\pi, (2k + 1)\pi]$*. Then, the map defined on the interval* $[a, b]$ *by*

$$(L \circ \Gamma_k)(t) = \log |\gamma(t)| + \mathrm{Arg}_{2k}(\gamma(t)) = (\log_{2k} \circ \gamma)(t)$$

is a continuation of the hypercomplex logarithm along γ *with initial point* $\log |\gamma(a)| + \mathrm{Arg}_{2k}(\gamma(a))$*. This map is called* a k-th branch of the hypercomplex logarithm along γ with initial point $\log |\gamma(a)| + \mathrm{Arg}_{2k}(\gamma(a))$.

Proof. The proof is a straightforward consequence of Proposition 4.12 and Proposition 4.2. □

Observe that Proposition 4.3 is a special case of Corollary 4.13, since any path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ such that $\gamma([a, b]) \cap \mathbb{R} = \varnothing$ has only one companion. We will now turn our attention to the case of loops of $\mathbb{K} \setminus \{0\}$.

Definition 4.14. Let $[a, b] \subset \mathbb{R}$ and let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a path with a companion $\mathfrak{J}^\gamma : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$.

If both γ and $\mathfrak{J}^\gamma$ are closed, then the path γ is called *a loop with companion* $\mathfrak{J}^\gamma$, and the pair $(\gamma, \mathfrak{J}^\gamma)$ is called *a loop with companion*.

The loop with companion $(\gamma, \mathfrak{J}^\gamma)$ is called *untwisted* if $\mathfrak{J}^\gamma$ is homotopic to a constant in $\mathbb{S}/\{\pm \mathrm{Id}\}$; if instead $\mathfrak{J}^\gamma$ is not homotopic to a constant, then $(\gamma, \mathfrak{J}^\gamma)$ is said to be *twisted*.

In the most relevant case in which γ is tame, we can specialize the definition as follows.

Definition 4.15. Let $[a, b] \subset \mathbb{R}$ and let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a tame path with companion $\mathfrak{J}^\gamma : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$.

If both γ and $\mathfrak{J}^\gamma$ are closed, then γ is called *a tame loop (with companion* $\mathfrak{J}^\gamma$*)*, and the pair $(\gamma, \mathfrak{J}^\gamma)$ is called *a tame loop*.

The tame loop $(\gamma, \mathfrak{J}^\gamma)$ is called *untwisted* if $\mathfrak{J}^\gamma$ is homotopic to a constant in $\mathbb{S}/\{\pm \mathrm{Id}\}$; if instead $\mathfrak{J}^\gamma$ is not homotopic to a constant, then $(\gamma, \mathfrak{J}^\gamma)$ is said to be *twisted*.

Remark 4.16. For any fixed $I \in \mathbb{S}$, let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a path lying in the complex slice $\mathbb{C}_I$. The path γ has always a particularly simple companion, namely $\mathfrak{J}^\gamma : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ constantly equal to $[I]$. Moreover, the two different lifts of $\mathfrak{J}^\gamma$ to $\mathbb{S}$ are both constantly equal to I or $-I$, respectively. As a consequence, if the given path γ is closed and tame, it is a tame loop and is untwisted.

A twisted loop necessarily intersects the real axis. Indeed the following result holds.

Proposition 4.17. *Let* $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ *be a loop which misses the real axis. Then* γ *is a tame loop and is untwisted.*

Proof. By Proposition 4.5, the loop γ can be lifted to a path $\Gamma : [a, b] \to \mathcal{E}^+_{\mathbb{K}}$ with $\Gamma = (\gamma, \mathrm{Arg}^\gamma)$. Let us consider the map $\mathrm{Arg}^\gamma = \mathcal{I}^\gamma \arg^\gamma : [a, b] \to \mathrm{Im}(\mathbb{K})$. By the hypothesis, there exists $k \in \mathbb{Z}$ such that the map $\arg^\gamma = \arg^\gamma_{2k} : [a, b] \to (2k\pi, (2k + 1)\pi)$ is never vanishing and hence has constant sign. Now, since γ is closed, we have that

$$\mathcal{I}^\gamma(a) \arg^\gamma_{2k}(a) = \mathcal{I}^\gamma(b) \arg^\gamma_{2k}(b).$$

Since $\arg^\gamma_{2k}(a)$ and $\arg^\gamma_{2k}(b)$ have the same sign and both belong to the interval $(2k\pi, (2k + 1)\pi)$, we obtain

$$\arg^\gamma_{2k}(a) = \arg^\gamma_{2k}(b)$$

and hence

$$\mathcal{I}^\gamma(a) = \mathcal{I}^\gamma(b).$$

Therefore the path $\mathcal{I}^\gamma : [a, b] \to \mathbb{S}$ is a loop, and hence the unique companion $[\mathcal{I}^\gamma] : [a, b] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ is a loop, homotopic to a constant. As a consequence the path γ is an untwisted, tame loop. $\square$

4.1. Winding number for untwisted loops with companion in $\mathbb{K} \setminus \{0\}$

It is well known that the definition of winding number for a loop (with respect to a point) is not natural in $\mathbb{R}^n$ when n is greater than 2. Nevertheless, in our setting, we can start by giving a definition of winding number that has full meaning for loops with companion that are untwisted and lie in $\mathbb{K} \setminus \{0\}$.

The following result opens a way to this definition of winding number.

Proposition 4.18. *A loop $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$, $\gamma([a, b]) \not\subset \mathbb{R}$, with companion $\mathfrak{I}^\gamma$ is untwisted if, and only if, for any chosen non real initial point of γ, both shadows associated with $\mathfrak{I}^\gamma$ are loops.*

Proof. Let $\gamma_{\mathcal{I}^\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$, $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$, be one of the shadows associated with $\mathfrak{I}^\gamma$.

If the loop γ is untwisted, then any lift $\mathcal{I}^\gamma$ of the companion $\mathfrak{I}^\gamma$ of γ is a loop, and hence it has coinciding endpoints. Therefore, the path

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t)$$

being a loop, the continuous function $y : [a, b] \to \mathbb{R}$ is such that $y(a) = y(b)$. Hence the associated shadow $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$ is closed.

On the other hand, suppose the associated shadow $\gamma_{\mathcal{I}^\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$, $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$, is a loop and assume that $y(a) = y(b) \neq 0$. Since the path

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t)$$

is a loop by assumption, we obtain $\mathcal{I}^\gamma(a) = \mathcal{I}^\gamma(b)$ and so the lift $\mathcal{I}^\gamma$ of $\mathfrak{I}^\gamma$ is a loop. In conclusion, γ is untwisted. □

We are now ready to use the well established definition of winding number for complex loops in $\mathbb{C} \setminus \{0\}$ to define the winding number in the case of untwisted loops with companion in $\mathbb{K} \setminus \{0\}$.

Definition 4.19. Let the loop $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ with companion $\mathfrak{I}^\gamma$ be untwisted. The *winding number* (with respect to zero) of the loop $(\gamma, \mathfrak{I}^\gamma)$, denoted $wind(\gamma, \mathfrak{I}^\gamma)$, is defined as the absolute value of the winding number (with respect to zero), $wind(\gamma_{\mathcal{I}^\gamma})$, of a shadow $\gamma_{\mathcal{I}^\gamma}$ associated with $\mathfrak{I}^\gamma$:

$$wind(\gamma, \mathfrak{I}^\gamma) = |wind(\gamma_{\mathcal{I}^\gamma})|.$$

In the case in which the loop $(\gamma, \mathfrak{I}^\gamma)$ is tame, there is one and only one companion of γ, and hence we can simply denote the winding number of γ by $wind(\gamma)$.

Of course, we need to show that the given definition of winding number of an untwisted loop with companion $(\gamma, \mathfrak{I}^\gamma)$ does not depend on the choice of the shadow associated with $\mathfrak{I}^\gamma$. Indeed, the two shadows associated with $\mathfrak{I}^\gamma$ are conjugate loops: as a consequence, their winding numbers are opposite. Therefore, Definition 4.19 is consistent.

One of the important features of the classical winding number (with respect to zero) of loops of $\mathbb{C} \setminus \{0\}$ is its invariance with respect to homotopy between such loops. The winding number of an untwisted loop with companion (in $\mathbb{K} \setminus \{0\}$) just defined cannot be invariant with respect to standard homotopy in $\mathbb{K} \setminus \{0\}$: all such loops are homotopic to a constant loop since $\mathbb{K} \setminus \{0\}$ is simply connected, and a constant loop has vanishing winding number.

A special notion of homotopy comes into the scenery in our setting. The next definition is useful to define such a notion.

Definition 4.20. Let $[a,b] \times [c,d] \subset \mathbb{R}^2$ and let $F : [a,b] \times [c,d] \to \mathbb{K} \setminus \{0\}$ be a continuous map.

A continuous map $\mathfrak{J}^F : [a,b] \times [c,d] \to \mathbb{S}/\{\pm \mathrm{Id}\}$ such that $F(t,s) \in \mathbb{C}_{\mathfrak{J}^F(t,s)}$ for every $(t,s) \in [a,b] \times [c,d]$ is called a *companion* of the map F.

If a companion $\mathfrak{J}^F$ of the map F exists, then F is called a *continuous map with companion* $\mathfrak{J}^F$, and $(F, \mathfrak{J}^F)$ is called a *continuous map with companion.*

If the map F has a unique companion $\mathfrak{J}^F$, then it is called a *tame* map.

Proposition 4.21. *Let $F : [a,b] \times [c,d] \to \mathbb{K} \setminus \{0\}$ be a continuous map with companion $\mathfrak{J}^F$. If $\mathcal{I}^F, -\mathcal{I}^F : [a,b] \times [c,d] \to \mathbb{S}$ are the two lifts of $\mathfrak{J}^F$, then there exist continuous functions $x, y : [a,b] \times [c,d] \to \mathbb{R}$ such that, for all $(t,s) \in [a,b] \times [c,d]$,*

$$F(t) = x(t,s) + \mathcal{I}^F(t,s)y(t,s) = x(t,s) + (-\mathcal{I}^F(t,s))(-y(t,s)).$$

These last expressions are called canonical forms of $(F, \mathfrak{J}^F)$.

Proof. See the proof of Proposition 4.8. □

As announced, the idea is now to define a special type of homotopy between paths, each having a companion and sharing the same endpoints. As customary, also in this paper homotopy between paths will always be meant with fixed endpoints.

Definition 4.22. Let $\gamma_1, \gamma_2 : [a,b] \to \mathbb{K} \setminus \{0\}$ be two paths with the same endpoints $\gamma_1(a) = \gamma_2(a) = p$ and $\gamma_1(b) = \gamma_2(b) = q$. Let $\mathfrak{J}^{\gamma_1}$ and $\mathfrak{J}^{\gamma_2}$ be companions of γ_1 and γ_2 respectively. If there exists a continuous map $F : [a,b] \times [0,1] \to \mathbb{K} \setminus \{0\}$ with companion $\mathfrak{J}^F$ such that:

(1) $\mathfrak{J}^F(t,0) = \mathfrak{J}^{\gamma_1}(t)$ and $\mathfrak{J}^F(t,1) = \mathfrak{J}^{\gamma_2}(t)$, for all $t \in [a,b]$;
(2) $F(t,0) = \gamma_1(t)$ and $F(t,1) = \gamma_2(t)$, for all $t \in [a,b]$;
(3) $F(0,s) = p$ and $F(1,s) = q$, for all $s \in [0,1]$;

then we will say that $(\gamma_1, \mathfrak{J}^{\gamma_1}), (\gamma_2, \mathfrak{J}^{\gamma_2})$ are *companion homotopic (or c-homotopic)* and that $(F, \mathfrak{J}^F)$ is a *c-homotopy* between $(\gamma_1, \mathfrak{J}^{\gamma_1})$ and $(\gamma_2, \mathfrak{J}^{\gamma_2})$.

Let $\gamma_1, \gamma_2 : [a,b] \to \mathbb{K} \setminus \{0\}$ be two paths with the same endpoints. If there exist a companion $\mathfrak{J}^{\gamma_1}$ of γ_1 and a companion $\mathfrak{J}^{\gamma_2}$ of γ_2 such that $(\gamma_1, \mathfrak{J}^{\gamma_1}), (\gamma_2, \mathfrak{J}^{\gamma_2})$ are c-homotopic, then we say that γ_1 and γ_2 are *weakly c-homotopic.*

The following simple result will be helpful in the sequel.

Proposition 4.23. *Let the continuous map $F : [a,b] \times [c,d] \to \mathbb{K} \setminus \{0\}$ with companion $\mathfrak{J}^F$ be a c-homotopy between $(\gamma_1, \mathfrak{J}^{\gamma_1})$ and $(\gamma_2, \mathfrak{J}^{\gamma_2})$. Then:*

(i) *the map $\mathfrak{J}^F$ is a homotopy between $\mathfrak{J}^{\gamma_1}$ and $\mathfrak{J}^{\gamma_2}$;*
(ii) *the homotopy $\mathfrak{J}^F$ can be lifted to a homotopy $\mathcal{I}^F$ between a lift $\mathcal{I}^{\gamma_1}$ of $\mathfrak{J}^{\gamma_1}$ and a lift $\mathcal{I}^{\gamma_2}$ of $\mathfrak{J}^{\gamma_2}$ in such a way that the canonical form of F*

$$F(t,s) = x(t,s) + \mathcal{I}^F(t,s)y(t,s) \tag{4.3}$$

is a homotopy between the canonical forms

$$\gamma_1(t) = x_1(t) + \mathcal{I}^{\gamma_1}(t)y_1(t)$$

and

$$\gamma_2(t) = x_2(t) + \mathcal{I}^{\gamma_2}(t) y_2(t)$$

of γ_1 *and* γ_2*, respectively;*

(iii) *the shadows*

$$x_1(t) + i y_1(t), \qquad x_2 + i y_2(t)$$

of $(\gamma_1, \mathfrak{I}^{\gamma_1})$ *and* $(\gamma_2, \mathfrak{I}^{\gamma_2})$*, respectively, are homotopic in* $\mathbb{C} \setminus \{0\}$*.*

Proof. The proofs of *(i)* and *(ii)* are a straightforward consequence of Definition 4.22 and of what is stated in Propositions 4.8 and 4.21. Let us prove *(iii)*. To this aim, consider the canonical form of F that appears in (4.3) and the following continuous maps, for $(t, s) \in [a, b] \times [0, 1]$:

$$\begin{aligned}
&\mathcal{L}_1(t, s) = \operatorname{Re}(F(t, s)), \\
&\mathcal{L}_2(t, s) = -\mathcal{I}^F(t, s) \operatorname{Im}(F(t, s))
\end{aligned}$$

We will prove that $\mathcal{L} = (\mathcal{L}_1, \mathcal{L}_2) : [a, b] \times [0, 1] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ is a homotopy between the two given shadows of γ_1 and γ_2. Indeed, using directly formula (4.3) for the canonical form of F, it is easy to check that on $[a, b] \times [0, 1]$,

$$\begin{aligned}
&\mathcal{L}(t, 0) = (\mathcal{L}_1(t, 0), \mathcal{L}_2(t, 0)) = (x_1(t), y_1(t)), \\
&\mathcal{L}(t, 1) = (\mathcal{L}_1(t, 1), \mathcal{L}_2(t, 1)) = (x_2(t), y_2(t)), \\
&\mathcal{L}(a, s) = (\mathcal{L}_1(a, s), \mathcal{L}_2(a, s)) = (x_1(a), y_1(a)) = (x_2(a), y_2(a)), \\
&\mathcal{L}(b, s) = (\mathcal{L}_1(b, s), \mathcal{L}_2(b, s)) = (x_1(b), y_1(b) = (x_2(b), y_2(b)).
\end{aligned}$$

The proof is now complete. □

Example 4.24. To better illustrate the major difference between complex and quaternionic cases, consider the curve, defined by

$$\gamma(t) = 3e^{it},\ t \in [0, \pi], \qquad \gamma(t) = -4 + \frac{t}{\pi},\ t \in [\pi, 3\pi],$$

$$\gamma(t) = e^{-it},\ t \in [3\pi, 4\pi], \quad \gamma(t) = -3 + \frac{t}{\pi},\ t \in [4\pi, 6\pi].$$

If we associate γ with the (constant) companion $\mathcal{I}^\gamma(t) \equiv i$, then the curve γ, regarded as a complex curve, has winding number 0 around the origin. Furthermore γ and its shadow with respect to the companion $\mathcal{I}^\gamma(t) \equiv i$ coincide.

At the same time, we can associate the same curve γ with other companions, for instance

$$\begin{aligned}
&\mathcal{I}^\gamma(t) = i,\ t \in [0, \pi], && \mathcal{I}^\gamma(t) = \mathcal{J}(t),\ t \in [\pi, 3\pi], \\
&\mathcal{I}^\gamma(t) = -i,\ t \in [3\pi, 4\pi], && \mathcal{I}^\gamma(t) = -\mathcal{J}(t),\ t \in [4\pi, 6\pi];
\end{aligned}$$

and $\mathfrak{I}^\gamma(t) = [\mathcal{I}^\gamma(t)]$, where $\mathcal{J} : [\pi, 3\pi] \to \mathbb{S}$ is an arbitrary continuous curve with $\mathcal{J}(\pi) = i$ and $\mathcal{J}(3\pi) = -i$. Correspondingly, the shadow of $(\gamma, \mathfrak{I}^\gamma)$ is

$$\gamma_{\mathfrak{I}^\gamma}(t) = 3e^{it},\ t \in [0, \pi], \qquad \gamma_{\mathfrak{I}^\gamma}(t) = -4 + \frac{t}{\pi},\ t \in [\pi, 3\pi],$$

$$\gamma_{\mathfrak{I}^\gamma}(t) = e^{it},\ t \in [3\pi, 4\pi], \quad \gamma_{\mathfrak{I}^\gamma}(t) = -3 + \frac{t}{\pi},\ t \in [4\pi, 6\pi]$$

and so the winding number of $(\gamma, \mathfrak{I}^\gamma)$ around the origin is 1. The pairs (γ, i) and $(\gamma, \mathfrak{I}^\gamma)$ are not c-homotopic.

The notion of c-homotopy is particularly useful in this setting, because of the following result.

Proposition 4.25. *Let $(\gamma_1, \mathfrak{I}^{\gamma_1}), (\gamma_2, \mathfrak{I}^{\gamma_2}) : [a, b] \to \mathbb{K} \setminus \{0\}$ be two paths with companions and with the same endpoints $\gamma_1(a) = \gamma_2(a) = p$ and $\gamma_1(b) = \gamma_2(b) = q$. Then the following statements are equivalent:*

(1) *$(\gamma_1, \mathfrak{I}^{\gamma_1}), (\gamma_2, \mathfrak{I}^{\gamma_2})$ are c-homotopic;*
(2) *$\mathfrak{I}^{\gamma_1}$ and $\mathfrak{I}^{\gamma_2}$ are homotopic in $\mathbb{S}/\{\pm Id\}$, and, in addition, for each of the shadows of $(\gamma_1, \mathfrak{I}^{\gamma_1})$ there is a shadow of $(\gamma_2, \mathfrak{I}^{\gamma_2})$ so that these two shadows are homotopic in $\mathbb{C} \setminus \{0\}$.*

Proof. Suppose first that *(2)* holds. Then there exist:

- a homotopy $\mathbf{G} : [a, b] \times [0, 1] \to \mathbb{S}/\{\pm Id\}$ between $\mathfrak{I}^{\gamma_1}$ and $\mathfrak{I}^{\gamma_2}$;
- a lift of $\mathbf{G}$, i.e. a homotopy $G : [a, b] \times [0, 1] \to \mathbb{S}$ between a lift $\mathcal{I}^{\gamma_1}$ of $\mathfrak{I}^{\gamma_1}$ and a lift $\mathcal{I}^{\gamma_2}$ of $\mathfrak{I}^{\gamma_2}$;
- a homotopy $L = (L_1, L_2) : [a, b] \times [0, 1] \to \mathbb{R}^2 \setminus \{(0, 0)\}$ between a shadow of γ_1 and a shadow of γ_2 (its "conjugate" being a homotopy between the corresponding conjugate shadows).

In this situation, the map $F : [a, b] \times [0, 1] \to \mathbb{K} \setminus \{0\}$ defined by

$$F(t, s) = L_1(t, s) + G(t, s)L_2(t, s)$$

is a homotopy between γ_1 and γ_2. Indeed, F is obviously continuous, and such that, for all $t \in [a, b]$ and all $s \in [0, 1]$,

$$\begin{aligned}
&F(t, 0) = L_1(t, 0) + G(t, 0)L_2(t, 0) = x_1(t) + \mathcal{I}^{\gamma_1}(t)y_1(t) = \gamma_1(t);\\
&F(t, 1) = L_1(t, 1) + G(t, 1)L_2(t, 1) = x_2(t) + \mathcal{I}^{\gamma_2}(t)y_2(t) = \gamma_2(t);\\
&F(a, s) = L_1(a, s) + G(a, s)L_2(a, s) = x_1(a) + \mathcal{I}^{\gamma_1}(a)y_1(a) = \gamma_1(a) = \gamma_2(a);\\
&F(b, s) = L_1(b, s) + G(b, s)L_2(b, s) = x_1(b) + \mathcal{I}^{\gamma_1}(b)y_1(b) = \gamma_1(b) = \gamma_2(b).
\end{aligned}$$

Moreover, the continuous map $G : [a, b] \times [0, 1] \to \mathbb{S}$ defines, by construction, a companion of F given by

$$\mathfrak{I}^F(t, s) = [G(t, s)] = \mathbf{G}(t, s)$$

for all $(t, s) \in [a, b] \times [0, 1]$. As a consequence, $(F, \mathfrak{I}^F)$ is a c-homotopy between $(\gamma_1, \mathfrak{I}^{\gamma_1}), (\gamma_2, \mathfrak{I}^{\gamma_2})$.

Let us now suppose that *(1)* holds, i.e. that $(\gamma_1, \mathfrak{I}^{\gamma_1})$ and $(\gamma_2, \mathfrak{I}^{\gamma_2})$ are c-homotopic. In this case $\mathfrak{I}^{\gamma_1}$ and $\mathfrak{I}^{\gamma_2}$ are homotopic by definition, and the rest of the assertion follows from Proposition 4.23. □

Proposition 4.26. *Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a loop with companion $\mathfrak{I}^\gamma$. Then $(\gamma, \mathfrak{I}^\gamma)$ is untwisted if, and only if, it is c-homotopic to one of its (closed and conjugate) shadows in $\mathbb{C}_{\mathfrak{I}^\gamma(a)}$.*

Proof. If $(\gamma, \mathfrak{I}^\gamma)$ is untwisted, then any lift $\mathcal{I}^\gamma$ of the companion $\mathfrak{I}^\gamma$ with initial point $\mathcal{I}^\gamma(a)$ is homotopic in $\mathbb{S}$ to the constant loop $\mathcal{I}^\gamma(a)$, and therefore the loop γ is c-homotopic to its (closed) shadow in $\mathbb{C}_{\mathcal{I}^\gamma(a)}$ (see Proposition 4.18). On the other hand, if the loop with companion $(\gamma, \mathfrak{I}^\gamma)$ is c-homotopic to its shadow, then the lift of its companion $\mathfrak{I}^\gamma$ with initial point $\mathcal{I}^\gamma(a)$ has to be homotopic in $\mathbb{S}$ to the constant loop $\mathcal{I}^\gamma(a)$. As a consequence the loop $(\gamma, \mathfrak{I}^\gamma)$ is untwisted by definition. □

The notion of c-homotopy is suitable to comply with the meaning of the winding number of loops in the setting of $\mathbb{K} \setminus \{0\}$. In this panorama, all untwisted tame loops play a special role: any such a loop has an "intrinsically defined" winding number that depends only on its geometric properties. Indeed, we can state the following result.

Theorem 4.27. *Let $\gamma, \delta : [a,b] \to \mathbb{K} \setminus \{0\}$ be two untwisted, tame loops. Then γ and δ are c-homotopic if, and only if,* $\mathrm{wind}(\gamma) = \mathrm{wind}(\delta)$.

Proof. By Proposition 4.25, γ and δ are c-homotopic if, and only if, the unique companions $\mathfrak{J}^\gamma$ and $\mathfrak{J}^\delta$ are homotopic and a shadow of $(\gamma, \mathfrak{J}^\gamma)$ is homotopic to a shadow of $(\delta, \mathfrak{J}^\delta)$, in $\mathbb{C} \setminus \{0\}$. According to Definition 4.19, the winding number of $(\gamma, \mathfrak{J}^\gamma)$ (or $(\delta, \mathfrak{J}^\delta)$) is defined as the absolute value of the winding number of one of the two (closed) shadows of $(\gamma, \mathfrak{J}^\gamma)$ (or $(\delta, \mathfrak{J}^\delta)$). Therefore the proof is a straightforward consequence of the properties of the fundamental group $\Pi_1(\mathbb{C} \setminus \{0\}) \equiv \mathbb{Z}$, where the class of each loop is determined by its winding number (with respect to zero). □

The given definition of winding number, which has particularly transparent geometrical meanings, cannot be adopted as it is in the twisted case, due to the two following results.

Proposition 4.28. *Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a loop with a companion $\mathfrak{J}^\gamma$. Then $(\gamma, \mathfrak{J}^\gamma)$ is twisted if, and only if, for any chosen non real initial point of γ, any shadow associated with $(\gamma, \mathfrak{J}^\gamma)$ has conjugate endpoints.*

Proof. Let $\gamma_{\mathcal{I}^\gamma} : [a,b] \to \mathbb{C} \setminus \{0\}$, $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$, be one of the shadows associated with $\mathfrak{J}^\gamma$.

If the loop $(\gamma, \mathfrak{J}^\gamma)$ is twisted, then the lift $\mathcal{I}^\gamma$ is not closed, and hence it has opposite endpoints. Therefore, the path

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t)$$

being closed, the continuous function $y : [a,b] \to \mathbb{R}$ is such that $y(a) = -y(b)$. Hence the associated shadow $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$ has conjugate endpoints.

On the other hand, suppose the associated shadow $\gamma_{\mathcal{I}^\gamma} : [a,b] \to \mathbb{C} \setminus \{0\}$, $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$, has conjugate nonreal endpoints. Then $y(a) = -y(b) \neq 0$ and, the path

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t)$$

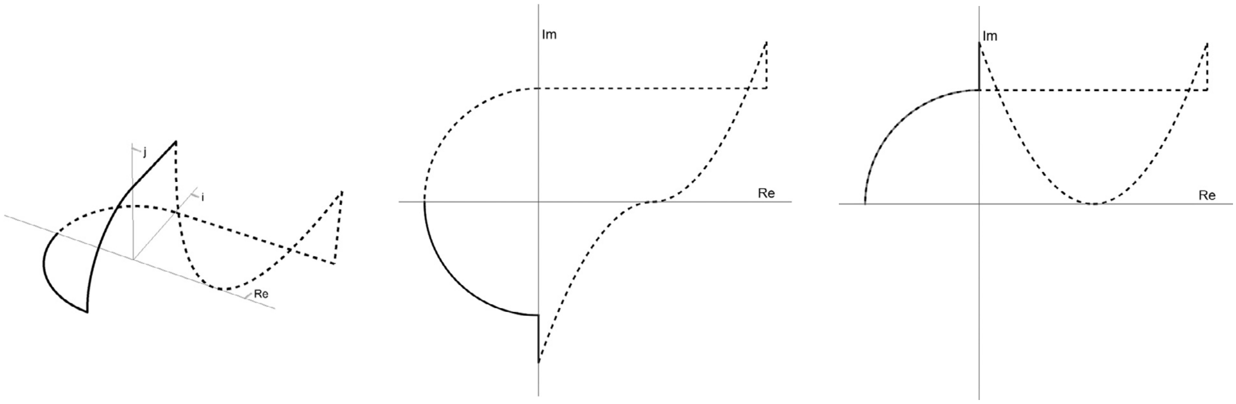
being closed by assumption, we obtain $\mathcal{I}^\gamma(a) = -\mathcal{I}^\gamma(b)$ and so the lift $\mathcal{I}^\gamma$ of $\mathfrak{J}^\gamma$ is not closed. In conclusion, $(\gamma, \mathfrak{J}^\gamma)$ is twisted. □

Corollary 4.29. *Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a loop with companion $\mathfrak{J}^\gamma$. Then the two shadows associated with $\mathfrak{J}^\gamma$ are closed if, and only if, the endpoints of γ are real.*

Proof. The proof follows immediately from Proposition 4.28. □

We might be encouraged to think that, in the case of a loop with companion which is twisted, we should first parameterise the loop in such a way that it has real endpoints (see Proposition 4.17), and then use Definition 4.19. Indeed, this approach gives a weird result, if tested, for instance, in the case of the twisted, tame loop λ presented in the next example.

Example 4.30. Consider the loop λ in the hyperplane of $\mathbb{H}$ generated by the orthogonal units $\{1, i, j\}$. The path consists of several arcs: the arc of parabola $t + 1 + t^2(i + j), t \in [-1, 1]$, the segments from $(2, 1, 0)$ to $(2, 1, 1)$, from $(2, 1, 0)$ to $(0, 1, 0)$ and from $(0, 0, 1)$ to $(0, 1, 1)$, the halfcircle $\cos t + i \sin t, t \in [\pi/2, 3\pi/2]$ and the quarter of circle $i \cos t + j \sin t, t \in [\pi/2, \pi]$. Let the orientation be such that it coincides with the



(a) (b) (c)

Fig. 3. From left to right: (a) the path γ and two of its shadows (b), (c).

positive orientation of the halfcircle part in the plane containing $1, i$. The path intersects the real axis at points $z = 1$ and $z = -1$.

In the previous example, the proposed winding number of the twisted, tame loop λ would be 1 if the loop is parameterised with real endpoints equal to $1 \in \mathbb{R}$ (see Fig. 3 (b)). On the other hand, the same loop λ parameterised with endpoints equal to $-1 \in \mathbb{R}$ would have winding number 0 (see Fig. 3 (c)). What we just illustrated clarifies that a notion of winding number for twisted, tame loops in $\mathbb{K} \setminus \{0\}$ (if it exists) has to be given by following a different approach.

In the spirit of the above example and Proposition 4.28 the definition of the winding number for a closed tame twisted loop $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ cannot be given by considering the change of the argument since this depends on the choice of the initial point.

Assume that γ is a twisted loop in $\mathbb{K} \setminus \{0\}$ which intersects both the positive and the negative real axis; let $\gamma_{\mathcal{I}^\gamma} : [a, b] \to \mathbb{C} \setminus \{0\}$, $\gamma_{\mathcal{I}^\gamma}(t) = x(t) + iy(t)$, be one of the shadows associated with γ. Let $\arg^\gamma(t), t \in [a, b]$, be the corresponding argument and choose the initial argument so that $\arg^\gamma(a) \in [0, \pi]$. The set $\Delta = \{\arg^\gamma(a)$ for all possible initial points$\}$ is an interval contained in $[0, \pi]$. Because the loop γ is twisted, the argument at b is $\arg^\gamma(b) = 2n\pi - \arg^\gamma(a)$ and hence

$$\arg^\gamma(b) - \arg^\gamma(a) = 2n\pi - 2\arg^\gamma(a)$$

and is not an integer multiple of 2π unless $\arg^\gamma(a) = 0, \pi$. Even if we set the initial point to be real, so that the change of argument is $2n\pi$, the number n can have more than one value as shown in the following example.

Example 4.31. Let γ_1 be the positively oriented unit circle with initial point -1 and companion i and define $\gamma_2(t) := \cos(t) + i\sin(t) + j(\cos(t) + 1), t \in [-\pi, \pi]$. Choose the lift of the companion $\mathcal{I}_2$ of γ_2 so that $\mathcal{I}_2(-\pi) = i$ and $\mathcal{I}_2(\pi) = -i$. Let $\gamma = \gamma_1^m \cdot \gamma_2$ denote the loop composed first of m copies of γ_1 followed by a copy of γ_2. If the initial point is assumed to be the point -1 on the first copy of γ_1, then the change of the argument is $2\pi m$. If the initial point is the point -1 on the second copy of γ_1, then the $m - 1$ copies of γ_1 before γ_2 give the winding number $m - 1$, but then the curve γ_2 reverses the orientation so the last copy of γ_1 has negative orientation with respect to the unit $-i$, hence the winding number is $m - 2$. Starting at -1 on the third copy of γ_1, would therefore result in the winding number $m - 4$ and so forth.

A few words seem now appropriate, to present a suggestive geometrical explanation of the reason why the notion of winding number as given in the case of untwisted loops does not work for the case of twisted loops. Indeed, consider an untwisted loop $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$

$$\gamma(t) = x(t) + \mathcal{I}^\gamma(t)y(t)$$

If we regard all points $\{x(t)\}_{t\in[a,b]}$ as distinct points except the endpoints, such a γ has values in the surface $S_\gamma = \{x(t) + \mathcal{I}^\gamma(t)s : t \in [a, b], s \in \mathbb{R}\}$; since γ is untwisted, then $\mathcal{I}^\gamma : [a, b] \to \mathbb{S}$ is a loop, and hence it is homotopic to the constant loop $\mathcal{I}^\gamma(0) = \mathcal{I}^\gamma(1)$. As a consequence, the surface S_γ is homeomorphic to a twodimensional cylinder. Therefore there is a notion of $\gamma(t)$ being a point of this surface lying on one side or the other of the "real axis" formed by the points $\{x(t)\}_{t\in[a,b]}$, and hence a notion of winding number with respect to the origin becomes possible: the situation reduces, naively speaking, to a planar one. If instead $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ is twisted, then the path $\mathcal{I}^\gamma : [a, b] \to \mathbb{S}$ has antipodal endpoints, and the surface S_γ turns out to be homeomorphic to a Moebius strip. In this last situation, the lack of orientability seems to exclude the possibility of defining coherently a winding number for the loop γ.

5. Obstructions to the existence of lifts of a path

In this section we present sufficient conditions for a path to have a lift, a companion and to be tame.

As already mentioned, if the path $\gamma : [a, b] \to \mathbb{K}\setminus\{0\}$ misses the real axis, then the lift to $\mathscr{E}^+_{\mathbb{K}}$ always exists. On the other hand, if $\gamma([a, b]) \subset \mathbb{R}$, then, necessarily either $\gamma([a, b]) \subset \mathbb{R}^-$ and we have the lifts of the form $\Gamma(t) = \log|\gamma(t)| + I(2k+1)\pi$, or $\gamma([a, b]) \subset \mathbb{R}^+$ and then we have the lifts of the form $\Gamma(t) = \log|\gamma(t)| + I2k\pi$ for any $I \in \mathbb{S}$. From now on assume that $\gamma([a, b])$ is not entirely contained in the real axis but it intersects it.

Definition 5.1. For a path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ we define the set $T := \gamma^{-1}(\mathbb{R})$ to be the *obstruction set* (for the lift of γ) and its points as *obstruction parameters*.

It is clear that the necessary assumption for a lift of γ to $\mathcal{E}^+_{\mathbb{K}}$ to exist is the requirement that γ has a lift on a neighbourhood of every parameter t, in particular, for each $t \in T$. It turns out that the existence of local lifts does not necessarily imply the existence of a global lift; recall that complex curves avoiding 0 always have local and global lifts.

In what follows, we start establishing the conditions on the behaviour of γ locally near its obstruction parameters in order to guarantee the existence first of local lifts and local companions and then of a global lift and a global companion.

As these conditions depend on the structure of the obstruction set, we start by considering paths with a finite obstruction set.

Definition 5.2. Let the path $\gamma(t) = x(t) + Y(t) : [a, b] \to \mathbb{H} \setminus \{0\}$, with $x(t) = \mathrm{Re}(\gamma(t))$, be such that $T = \gamma^{-1}(\mathbb{R}) = \{a \le t_1 < \ldots < t_p \le b\}$. Consider the limits

$$\lim_{t\to t_s^\pm} \frac{Y(t)}{|Y(t)|}. \tag{5.4}$$

Let $t_s \in (a, b)$. Then

1) γ is *tame* at t_s if both limits are either equal or opposite. In particular, if these limits are opposite, then the parameter t_s is called a *flip*, whereas if they are the same it is called *a bounce*;
2) γ is *semi-tame* at t_s if it is not tame at t_s but both limits in (5.4) exist;

3) γ is *not tame* at t_s if at least one of the limits in (5.4) does not exist.

If $t_s = a$ (resp. $t_s = b$) then the path is tame at t_s from the right (left) if the right (left) limit in (5.4) exists and not tame in all other cases.

If, in addition, the path γ is closed, we adapt the definition of tameness at the endpoints in the natural way. In particular, γ is semi-tame at $a \simeq b$ if it is tame at a from the right and at b from the left. If the limits are the same, then $a \simeq b$ is called a *bounce* and if they are opposite it is called a *flip*. In all other cases γ is not tame at $a \simeq b$.

Remark 5.3. The path γ in the Example 4.6 (b) does not have the limit (5.4) at $t = 0$.

Remark 5.4. The definition of tameness of γ at parameters $\{t_1, \dots, t_p\}$ means precisely that the projectivized imaginary unit function $[\mathcal{J}(\gamma(t))]$ defined on $[a, b] \setminus \{t_1, \dots, t_p\}$ has a continuous extension to $\{t_1, \dots, t_p\}$.

The proposition below gives a motivation for the previous definitions.

Proposition 5.5. *Let $\gamma(t) : [a, b] \to \mathbb{K} \setminus \{0\}$ be a path with finite obstruction set $\gamma^{-1}(\mathbb{R}) = \{a \leq t_1 < \dots < t_p \leq b\}$. Then γ is tame if and only if it is tame at each $t_s, s = 1, \dots, p$.*

If γ is a loop, then it is a tame loop if and only if it is tame at each $t_s, s = 1, \dots, p$.

Proof. By assumption, the projectivized imaginary unit function $[\mathcal{J}(\gamma(t))]$ defined on $[a, b] \setminus \{t_1, \dots, t_p\}$ has a continuous extension to $[a, b]$. $\square$

If γ is a tame loop, the lift to $\mathcal{E}^+_{\mathbb{K}}$ exists by Proposition 4.12. However, we want to present also a constructive proof, because we will use the same techniques to obtain lifts of non tame paths and to explain the definition of winding number through local data on the obstruction set.

Without loss of generality we assume that $a \neq t_1, b \neq t_p$. Consider the intervals $I_0 = [a =: t_0, t_1], I_1 = [t_1, t_2], \dots, I_p = [t_p, t_{p+1} =: b]$ and denote the restrictions of γ on I_s by $\gamma^s := \gamma|_{I_s}$. The existence of limits (5.4) provides, for any $s = 0, \dots, p$, continuous extensions of all functions $\arg_k(\gamma^s)(t), \mathcal{I}_k(\gamma^s)(t)$ to the endpoints of I_s.

Choose an arbitrary $k_0 \in \mathbb{Z}$. Setting $\arg_{k_0}(\gamma^0(t)) =: \arg^\gamma(t)$ and $\mathcal{I}_{k_0}(\gamma^0(t)) =: \mathcal{I}^\gamma(t)$ we define continuous functions $\arg^\gamma$ and $\mathcal{I}^\gamma$ on $[a, t_1]$. We set $\operatorname{Arg}^\gamma(t) := \arg^\gamma(t)\mathcal{I}^\gamma(t)$ and define the lift $\Gamma^0 := (\gamma^0, \operatorname{Arg}^{\gamma^0})$. Consider the endpoint $\gamma(t_1)$. If it is a flip, then we define

$$k_1 := \begin{cases} k_0 + 1, & \text{if } \arg_{k_0}(\gamma^0(t_1)) = (k_0 + 1)\pi, \\ k_0 - 1, & \text{if } \arg_{k_0}(\gamma^0(t_1)) = k_0\pi. \end{cases}$$

If it is a bounce then we set $k_1 := k_0$. By setting $\arg^\gamma(t) := \arg_{k_1}(\gamma^1(t))$ and $\mathcal{I}^\gamma(t) := \mathcal{I}_{k_1}(\gamma^1(t))$ we extend the functions $\arg^\gamma$ and $\mathcal{I}^\gamma$ continuously to $[a, t_2]$. We extend the above functions to $[a, b]$ by repeating this process.

Proposition 5.6. *A tame loop γ with $\gamma^{-1}(\mathbb{R}) = \{a \leq t_1 < \dots < t_p \leq b\}$ has an even number of flips if and only if γ is untwisted.*

Proof. Assume that the loop does not have flips. Then $\mathcal{I}^\gamma$ equals $\mathcal{I}_{k_0} \circ \gamma$ for some $k \in \mathbb{Z}$ and so it is obviously a loop.

For the case of a loop with flips, we assume, without loss of generality, that $k_0 = 0$, so we have started with the principal branch and moreover, we also assume that the parameterisation $\gamma : [a, b] \to \mathbb{H}$ is such that $\gamma(a) \in \mathbb{R}$.

As in the previous proof, all the functions $\arg_0(\gamma^s)(t), \mathcal{I}_0(\gamma^s)(t)$ have continuous extensions to the endpoints of I_s.

If a is a bounce, then the imaginary unit at $\gamma(b)$ is the same as the one at $\gamma(a)$, i.e. $\mathcal{I}_0(\gamma^0)(a) = \mathcal{I}_0(\gamma^p)(b)$. The even number of flips ensures that the sign of the imaginary unit at the endpoint remains the same with respect to the one at the principal branch.

If the initial point is a flip, then the imaginary unit function at endpoint has the opposite sign with respect to the one at the initial point, $\mathcal{I}_0(\gamma^0)(a) = -\mathcal{I}_0(\gamma^p)(b)$, and to end up with the same sign there must be an odd number of additional flips following the first one to ensure that the sign of the unit at the endpoint remains the same with respect to the one at the principal branch. □

The proofs of Propositions 5.5 and 5.6 show that once the lift near the initial point is chosen, only the flips are relevant for the determination of the lift near the endpoint; bounces can be discarded. This enables us to calculate the change of argument and the winding number out of local data at the intersections of γ with the real axis. To determine the change of the argument we introduce a notion of *signature*.

Definition 5.7. Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a given path with $\gamma^{-1}(\mathbb{R}) = \{a \leq t_1 < \ldots < t_p \leq b\}$, with points of $\gamma^{-1}(\mathbb{R}) \cap (a, b)$ all tame. Let $a < \xi_1 < \ldots < \xi_m < b$ be those parameters in $\gamma^{-1}(\mathbb{R})$ which are flips. The signature $\sigma(\gamma)$ is defined by

$$\sigma(\gamma) := \sum_{l=1}^{m} \operatorname{sign}(\gamma(\xi_l))(-1)^l.$$

If there are no flips, then we define $\sigma(\gamma) := 0$.

The connection between the signature and the change of argument is described in the following

Proposition 5.8. *Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a tame path with $\gamma^{-1}(\mathbb{R}) = \{a_1 \leq t_1 < \ldots < t_p \leq b\}$ with all the parameters $\gamma^{-1}(\mathbb{R}) \cap (a, b)$ tame. Assume that a lift Γ of $\gamma|_{[a,t_1]}$ in $\mathcal{E}^+_{\mathbb{K}}$ exists and equals $\Gamma(t) = (\gamma(t), \operatorname{Arg}^\gamma_{k_0}(t)) \in \mathcal{E}^+_{\mathbb{K}}, t \in [a, t_1]$ for some $k_0 \in \mathbb{Z}$. Then the lift of $\gamma|_{[t_p,b)}$ is given by $(\gamma(t), \operatorname{Arg}^\gamma_{k_0+(-1)^{k_0}\sigma(\gamma)}(t)), t \in [t_p, b)$.*

Remark 5.9. If $\gamma(b) \in \mathbb{R}$, then a lift of γ on $[a, b)$ can be extended continuously to b if and only if b is tame from the left.

According to Definition 3.5, if $(-1)^{k_0}\sigma(\gamma)$ is even, then $\operatorname{Arg}_{k_0+(-1)^{k_0}\sigma(\gamma)}(b) = \mathcal{I}_b \arg^\gamma_{k_0}(b)$ and $\operatorname{Arg}_{k_0}(a) = \mathcal{I}_a \arg^\gamma_{k_0}(a)$. Therefore it follows that

Corollary 5.10. *Let γ be as in Proposition 5.8 and let $\gamma(a) = \gamma(b) \notin \mathbb{R}$. Then γ is a tame loop. Moreover $\sigma(\gamma)$ is even if and only if γ is a tame and untwisted loop. If this is the case, then*

$$\omega(\gamma) = |\sigma(\gamma)|/2.$$

Proof of Proposition 5.8. For the sake of simplicity, assume first that $k_0 = 0$, and so $\arg_0(\gamma(a)) \in (0, \pi)$ and let the sequence of signs of flips be alternating and starting with -1, i.e. $-1, 1, -1, \ldots$. Then $\arg^\gamma(\gamma(t))$ increases when the path γ crosses the real axis, and, to be more precise, in a small neighbourhood of each flip (corresponding to the parameter t_s), it turns out that if $\arg^\gamma(\gamma(t)) \in (k\pi, (k + 1)\pi)$ for $t < t_s$ (and $|t - t_s|$ small enough), necessarily $\arg^\gamma(\gamma(t)) \in ((k + 1)\pi, (k + 2)\pi)$ for $t > t_s$ (and $|t - t_s|$ small enough). Altogether, this occurs $\sigma(\gamma) = \sum_{l=1}^m \operatorname{sign}(\gamma(\xi_l))(-1)^l = \sum_{l=1}^m (-1)^l(-1)^l$ times. Geometrically, this means that the corresponding shadow associated with γ winds around the origin in the positive direction.

If the sequence of signs of flips is still alternating but starts with 1, then $\arg^{\gamma}(\gamma(t))$ decreases when the path γ crosses the positive real axis and this results in the translation of the interval $[0, \pi]$ by $\pi \sum_{l=1}^{m}(-1)^{l-1}(-1)^{l} = \pi\sigma(\gamma)$.

To prove the general assertion it suffices to investigate what happens if the sequence of signs of flips is not alternating at one position.

Assume that we insert in the alternating sequence $-1, 1 - 1 \dots$ the integer 1 in the second position, so the sequence is no longer alternating: $-1, 1, 1, -1, \dots$. This means that we have started from the upper half-plane, crossed the negative real axis, then the positive real axis with the arguments in $[2\pi, 3\pi]$. Then we have crossed the positive real axis again, hence the choice of argument at this intersection must be 2π. Because the point is a flip, this means that the argument decreases and keeps decreasing till the end. This is faithfully reflected in the sequence $s_l = (-1)^l \operatorname{sign}(\gamma(\xi_l))$, because it equals $1, 1, -1, -1, \dots$ and so the sum $\sum_{l=1}^{m} \operatorname{sign}(\gamma(\xi_l))(-1)^l = \sigma(\gamma)$ multiplied by π corresponds with the total translation of the initial interval for the Arg.

Similarly, if we insert -1 on the second position, this means that we have crossed the negative real axis and we have the argument in $[\pi, 2\pi]$ but then we have returned to the negative real axis and in order to have the argument continuous, at the second crossing the argument π must be chosen and because we have a flip, the argument $\arg^{\gamma}(\gamma(t))$ decreases and keeps decreasing till the end. The corresponding sequence $s_1, s_2, \dots$ now equals $1, -1, -1, -1, \dots$ and $\sum_{l=1}^{m} s_l = \sigma(\gamma)$.

The proof for k_0 even is the same. If k_0 is odd, this coincides with considering the conjugate shadow and hence reversed orientation compared, to k_0 even, so the signature has to be multiplied by $-\pi$ to get the total translation of the initial interval for the Arg. $\square$

In practice this means that once the sequence of ± 1-s is given, we start by cancelling the pairs of the same numbers until we end up with an alternating sequence. The number of elements multiplied by minus the first element is the signature.

If the path $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ is closed, i.e. $\gamma(a) = \gamma(b)$, then we identify points a and b of $[a, b]$ and write $b \simeq a$; hence, we consider the parameterisation as $\gamma : S^1 \to \mathbb{K} \setminus \{0\}$, so there is no distinguished initial point. Therefore, in this case, we require that for each $s \in S^1$ there exists a neighbourhood of U_s of s in S^1 such that the lift of γ exists on U_s.

Definition 5.11. Let $\gamma : S^1 \to \mathbb{K} \setminus \{0\}$ be a continuous loop. Then a continuous function $\Gamma : i\mathbb{R} \to \mathcal{E}^+$ is *a lift of* γ if the following diagram commutes:

$$\begin{tikzcd} i\mathbb{R} \arrow[r, "\Gamma"] \arrow[d, "\exp"] & \Gamma(i\mathbb{R}) \subset \mathcal{E}^+_{\mathbb{K}} \arrow[d, "\mathrm{pr}_1"] \\ S^1 \arrow[r, "\gamma"] & \gamma(S^1) \subset \mathbb{K} \setminus \{0\} \end{tikzcd}$$

Remark 5.12. A loop with companion always has a (not necessarily closed) lift in the sense of Definition 5.11. The loop presented in Fig. 4 does not have a lift in the sense of Definition 5.11. The curve is defined by $\gamma(t) = \cos(t) + i(\sin(t) - t/10)$ for $t \in [0, \pi]$ and $\gamma(t) = \cos(2\pi - t) - i(2\pi - t)/10 + j\sin(2\pi - t)$ for $t \in [\pi, 2\pi]$. However, starting from any point with $\gamma(t) \in \mathbb{R} \times i\mathbb{R}^+$ and using the principal branch one can obtain the local lift of γ and prolong it to the interval $[0, 2\pi]$.

Corollary 5.13. *Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a continuous loop with $\gamma^{-1}(\mathbb{R})$ nonempty and assume it is not tame at least at one of the obstruction parameters. Let $a = \xi_1 < \dots < \xi_m = b \simeq a$ be all the obstruction parameters where γ is not tame and assume, moreover, that $\gamma(\xi_k) > 0$. Then a lift of γ in $\mathcal{E}^+_{\mathbb{K}}$ exists if and only if $\sigma(\gamma|_{[\xi_l, \xi_{l+1}]}) \in \{0, -1\}$ for each $l = 1, \dots, m-1$. If it exists, the lift is a loop.*

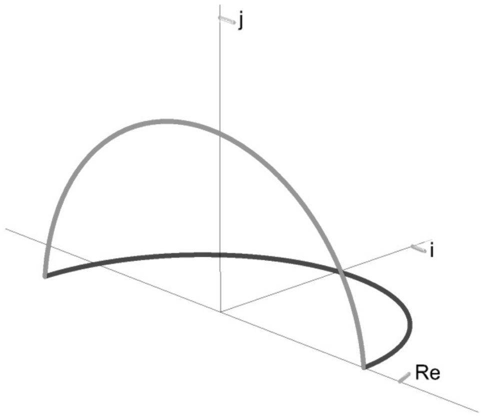


Fig. 4. A loop without a lift in the sense of Definition 5.11.

Proof. Because γ is not tame at ξ_l we can only choose either $k_0 = 0$ or $k_0 = -1$ and lift the curve in a neighbourhood of the point $\gamma(\xi_l)$ to $\mathcal{E}^+_{\mathbb{K}}$ using the principal branch of the logarithm. Assume that we have chosen $k_0 = 0$. Then on $[\xi_{l+1} - \delta, \xi_{l+1})$ for some small $\delta > 0$ we may only have $k = 0, -1$, hence the signature can be either 0 or -1 in order to be able to extend the lift to ξ_{l+1}. If we have $k_0 = -1$, then, since we have to end up with $k = 0, -1$ near ξ_{i+1}, the condition is $(-1)^{k_0}\sigma(\gamma|_{[\xi_l,\xi_{l+1}]}) \in \{0, 1\}$ hence $\sigma(\gamma|_{[\xi_l,\xi_{l+1}]}) \in \{0, -1\}$. $\square$

When we do not have additional information about the set $\gamma^{-1}(\mathbb{R})$, we must assume that the continuous lift of γ exists on a neighbourhood U of $\gamma^{-1}(\mathbb{R}) \cap (-\infty, 0)$. This means that the path $\gamma|_U$ has a companion, since the restriction of the function $\arg^\gamma$ to U is not vanishing. Recall that, on a neighbourhood of $\gamma^{-1}(\mathbb{R}) \cap (0, \infty)$, the principal branch of the logarithm is well-defined and hence a lift of γ always exists. This does not imply that a global lift exists.

We now proceed with the detailed description of the possible situations when γ has a companion on a neighbourhood of real points and omit the (trivial) case $\gamma([a, b]) \subset \mathbb{R}$.

Since on the complement of the obstruction set the companion of a path γ exists and is unique, one immediately obtains the following

Proposition 5.14. *Let $\gamma : [a, b] \to \mathbb{K} \setminus \{0\}$ be a path. Then γ has a companion if and only if it has a companion on a neighbourhood of the obstruction set. The same holds for a loop γ with $\gamma(a) \notin \mathbb{R}$.*

In the sequel we explain how to extend the notion of signature to paths with infinite obstruction set. Since $\gamma([a, b])$ is compact, there are only finitely many connected components of $\gamma([a, b]) \setminus \mathbb{R}$ with endpoints of opposite sign.

Definition 5.15. Let $L_1, \dots, L_m$ be all the connected components of $\gamma([a, b]) \setminus \mathbb{R}$, $L_l(t) = \gamma(t), t \in (s_l, e_l) \subset [a, b]$ satisfying $\gamma(s_l)\gamma(e_l) < 0$ and $a \le s_l < e_l \le s_{l+1} < e_m \le b$, $l = 1, \dots, m$. We call the components *the big arcs* and the subdivision $a \le s_l < e_l \le s_{l+1} < e_m \le b$, $l = 1, \dots, m$ *the induced subdivision*. The intervals $[e_l, s_{l+1}]$ are called *obstruction intervals*. If γ is closed, then we identify a and b, $e_0 := e_m, s_{m+1} := s_1$ and define also $[e_0, s_1]$ as the obstruction interval.

Because $\gamma([e_l, s_{l+1}])$ misses either the positive or the negative real axis, we define the sign of the obstruction interval as follows.

Definition 5.16. If $\gamma([e_l, s_{l+1}]) \cap (-\infty, 0) = \varnothing$, then $\operatorname{sign}([e_l, s_{l+1}]) = 1$; otherwise, if $\gamma([e_l, s_{l+1}]) \cap (0, \infty) = \varnothing$, then $\operatorname{sign}([e_l, s_{l+1}]) = -1$.

Extend the domains of definition of each L_l to its endpoints and let

$$\mathcal{I}^l(t) := \mathcal{J}(\gamma(t)), t \in (s_l, e_l) \text{ and } I_l^s := \lim_{t \to s_l^-} \mathcal{J}^l(t), I_l^e := \lim_{t \to e_l^+} \mathcal{J}^l(t)$$

be the imaginary units of L_l at its endpoints, if the limits exist.

Definition 5.17. Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path with companion $\mathfrak{I}$ with lifts $\pm\mathcal{I}$. Let $a \le s_1 < e_1 \le s_2 < e_2 < \ldots \le s_m < e_m \le b$ be the induced subdivision and L_l the big arcs with limits I_l^e and $I_l^s, l = 1, \ldots, m$.

The interval $[e_l, s_{l+1}], 1 \le l \le m-1$, is *a bounce with respect to* $\mathfrak{I}$ if $\mathcal{I}$ (or $-\mathcal{I}$) satisfies $\mathcal{I}(e_l) = \pm I_l^e, \mathcal{I}(s_{l+1}) = \pm I_{l+1}^s$ and *a flip with respect to* $\mathfrak{I}$ if $\mathcal{I}$ (or $-\mathcal{I}$) satisfies $\mathcal{I}(e_l) = \pm I_l^e, \mathcal{I}(s_{l+1}) = \mp I_{l+1}^s$.

Remark 5.18. If γ has a companion and $\gamma([e_l, s_{l+1}]) \cap \mathbb{R}$ contains an open set then γ always has a companion that makes it a bounce and a companion that makes it a flip. If the interval $[e_l, s_{l+1}]$ reduces to a point, then the definition of tameness for intervals coincides with the definition of tameness for points.

We can now extend the definition of signature also to this general case.

Definition 5.19. Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path with the induced subdivision $a \le s_1 < e_1 \le s_2 < e_2 < \ldots \le s_m < e_m \le b$ and a companion $\mathfrak{I}$. Let $1 \le j_1 < \ldots < j_k \le m$ be the indices for which the intervals $[e_{j_i}, s_{j_i+1}]$ are flips. The signature $\sigma(\gamma, \mathfrak{I})$ with respect to the companion $\mathfrak{I}$ is defined as

$$\sigma(\gamma, \mathfrak{I}) := \sum_{l=1, j_k \neq m}^{k} \operatorname{sign}([e_{j_l}, s_{j_l+1}])(-1)^l.$$

If γ is a loop, then we define the circular signature with respect to $\mathfrak{I}$ to be

$$\sigma^c(\gamma, \mathfrak{I}) := \sum_{l=1}^{k} \operatorname{sign}([e_l, s_{l+1}]))(-1)^l.$$

If there are no flips, then we define $\sigma(\gamma, \mathfrak{I}) := 0, \sigma^c(\gamma, \mathfrak{I}) := 0$.

The following are straightforward generalisations of Proposition 5.8 and Corollary 5.13

Proposition 5.20. *Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a path with the companion $\mathfrak{I}$ and the induced subdivision $a = e_0 \le s_1 < e_1 \le s_2 < e_2 < \ldots \le s_m < e_m \le b = s_{m+1}$. Assume that a lift Γ of γ in $\mathcal{E}_{\mathbb{K}}^+$ is given by $\log_{k_0}$, $k_0 \in \mathbb{Z}$ on $[s_1 - \delta, s_1]$ for some $\delta > 0$. The lift on $[s_m, e_m]$ is given by $k := k_0 + (-1)^{k_0}\sigma(\gamma)$.*

To define the winding number for a closed curve we have to take into account also the last interval I_m and hence consider the closed signature.

Corollary 5.21. *Let γ be a loop and $\sigma^c(\gamma)$ even. Then $\omega(\gamma, \mathfrak{I}) = |\sigma^c(\gamma, \mathfrak{I})|/2$.*

Corollary 5.22. *Let $\gamma : [a,b] \to \mathbb{K} \setminus \{0\}$ be a loop with the induced subdivision $a \le s_1 < e_1 \le s_2 < e_2 < \ldots < s_m < e_m \le b$.*

Let $1 \le j_1 < \ldots < j_k \le m$ be the indices for which γ, restricted to the neighbourhoods of the intervals $J_{j_l} := [e_{j_l}, s_{j_{l+1}}] \subset (0, \infty)$, does not have a companion and assume γ has a companion on a neighbourhood of the closure of $[a,b] \setminus \cup_{l=1}^k J_{j_l}$. Then a lift of γ in $\mathcal{E}_{\mathbb{K}}^+$ exists if and only if $\sigma(\gamma|_{[s_{j_l+1}, e_{j_{l+1}}]}) \in \{0, -1\}$ for each $l = 1, \ldots, m-1$. If it exists, the lift is a loop.

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