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Fuzzy Bunch Graphs and Properties

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Abstract

This paper presents an approach to graph theory that combines the well-established concept of fuzzy graph theory with recent developments in bunch graphs. Fuzzy graph theory, which is widely applied across various fields, graphically represents significant objects and provides a comprehensive overview of complex areas. By contrast, bunch graphs consolidate nodes with identical activities into a single node, thereby reducing the number of active nodes. However, the application scope of bunch graphs, being crisp graphs, is limited. We propose these concepts in a fuzzy bunch graph, expanding the application area and simplifying the graphical representation. We establish the membership value of a fuzzy bunch node adhering to the rules of the aforementioned theorems, which ranges between zero and unity. This approach not only reduces the number of nodes but also broadens the application scope. However, through an example, we observed that the values of certain properties fall between two specific numbers. This innovative approach enhances the versatility and efficiency of graph theory, making fuzzy bunch graphs a more effective tool.

Keywords: Fuzzy graph, Bunch graph, Matrix

1. Introduction

Graph theory provides a structural mathematical model in which nodes (or vertices) represent entities and edges denote the relationships between them. Directed graphs illustrate the directional relationships from one node to another. However, constructing bunch graphs from large-scale networks presents challenges—especially when selecting the appropriate nodes for bunching and determining the strength or cohesion of these bunches.

To address these challenges, this paper introduces the concept of fuzzy bunch graphs (FBGs), which combines fuzzy logic with the principle of homophily—the tendency of nodes to associate with others that share similar attributes. In social networks, homophily is a key factor influencing node connectivity. Here, it is employed as a criterion for bunch formation under the hypothesis that high-homophily nodes are more likely to belong to the same bunch, owing to shared characteristics and increased interaction.

For example, consider a wholesaler that distributes groceries to multiple cities within a district. Each city can be modeled as a node, with edges representing distribution links. Numerous grocery shops interact in each city to form a co-opetition graph. As the number of

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shops increases, the graphs become more complex. To address this, Samanta et al. (2022) proposed grouping several similar nodes into a bunch node to simplify the graph structure.

This study also draws on the foundational work on set theory by George Cantor (1874), later extended to fuzzy set theory by Zadeh (1965), which allows the partial membership of elements in a set with degrees ranging from zero to unity. Rosenfeld (1975) extended the classical graph theory to fuzzy graph theory, recognizing the inherent uncertainties in modern networks. Subsequently, many researchers have generalized the core concepts of graph theory using fuzzy sets to better model real-world systems.

We propose that FBGs offer a powerful tool for analyzing large and complex networks. By integrating homophily and fuzziness, FBGs simplify the network representation, uncover hidden patterns, quantify connection strengths, and support multi-dimensional comparisons.

1.1 Literature Review

Clustering is a technique that divides a set of objects into groups or clusters based on their similarities or dissimilarities. In the context of graph theory, clustering can be used to analyze the structure and properties of networks, where nodes symbolize objects and edges denote relationships. Clustering can reveal the underlying patterns and communities in networks and simplify them by reducing their complexity and dimensionality. Various methods and algorithms exist for clustering graphs, including spectral, modularity-based, hierarchical, and k-means clustering.

Bhadoria et al. [43] introduced the concept of bunch graphs, in which nodes are grouped into clusters based on their similarity. They also employed autoencoders, a type of neural network, to reduce the data dimensionality and noise for character recognition. They demonstrated the effectiveness of their proposed method using the MNIST dataset, which contains handwritten digits.

Das et al. [44] expanded the concept of competition graphs, which depicts the competition between nodes over a common resource, into a picture fuzzy environment, which is a generalization of the intuitionistic fuzzy environment. They introduced several picture fuzzy competition graph variations and provided applications across various domains, such as education, ecology, business, and job markets.

Sahoo and Pal [45] introduced the concept of intuitionistic fuzzy competition graphs, which generalize competition graphs within an intuitionistic fuzzy framework. They proposed multiple generalizations, explored their properties, and demonstrated their applications in ecological systems.

Fuzzy algebra plays a crucial role in modeling uncertainty in mathematical and engineering problems. Saqib et al. [12] proposed a numerical solution approach for bipolar fuzzy initial value problems and demonstrated significant improvements in terms of modeling imprecision. Chehlabi and Allahviranloo [13] provided concrete solutions for fuzzy linear fractional differential equations using generalized fuzzy techniques. Similarly, Akram et al. [14] studied LR-type fully Pythagorean fuzzy linear programming problems with equality constraints and showcased their applications in complex decision-making systems.

Armand et al. [15] obtained several fundamental results for fuzzy calculus and formed a theoretical backbone for more advanced fuzzy systems. Pirmuhammadi et al. [16] extended the scope of fuzzy algebra by presenting a parametric form of Z-numbers and applying it to fuzzy value problems. These studies collectively contribute to expanding the applicability of fuzzy set theory and its integration with numerical and optimization techniques.

In 2024, P. Giri et al. [49] recognized and evaluated the barriers hindering the adoption of renewable energy technologies and prioritized various renewable energy sources in India using matrix permanent in a fuzzy environment. In a 2023 study, Islam and Pal [50] analyzed railway crimes in India using fuzzy methods and compared their findings with three other topological indices. In 2024, Fujita [51] explored various uncertain models for bunch graphs, including fuzzy and neutrosophic graphs. In 2023, Bera et al. [52] examined the vertex connectivity in intuitionistic fuzzy graphs, along with related properties and applications. In 2024, Khushal et al. [53] proposed a novel fuzzy data transformation technique for hormonally imbalanced datasets. This was extended in 2025 by Khushal et al. [54], who integrated fuzzy logic with quantum machine learning to analyze a medical dataset related to chronic diseases. In 2025, Fatima et al. [55] introduced a dynamic community detection method using fuzzy weighting techniques, termed fuzzy-variant community groups, to track evolving network structures over time.

1.2 Research Gap and Contribution

Although previous studies on fuzzy graph theory and bunch graph modeling have made significant strides, a clear gap exists in terms of integrating the uncertainty handling power of fuzzy sets with the structural simplification provided by bunch graphs. Bhadoria et al. [43] introduced bunch graphs to reduce network complexity; however, their model operated in a crisp domain and lacked the flexibility to represent uncertainty. Similarly, fuzzy graph theory—pioneered by Rosenfeld [56] and extended by others—focuses on handling ambiguity in node and edge relationships; however, it does not directly address the issue of node consolidation for large-scale networks.

The proposed FBG framework bridges this gap by

- Integrating fuzzy logic with bunching mechanisms to reduce the number of nodes while preserving uncertainty.
- Introducing a membership function for bunch nodes based on homophily and fuzzy association strength.
- Demonstrating real-world application through a case study on terrorist activity data, highlighting the proposed method's relevance to complex sociopolitical networks.

To the best of our knowledge, this is the first attempt to formally and mathematically define and analyze FBGs. Our model provides a novel structure that supports scalable and uncertainty-aware network analysis across disciplines.

The remainder of this paper is organized as follows: Section 2 describes Key definitions related to fuzzy and bunch graphs; Section 3, FBG definition and method to determine the membership value (MV) of bunch nodes; Section 4, definition and discussion of connected fuzzy bunch graphs (CFBGs); Section 5, graph coloring in the context of FBGs; Section 6, matrix representation of bunch graphs; Section 7, matrix representation of FBGs; Section 8, energy computation of FBGs; Section 9, application involving data on terrorist attacks in India over the last ten years; and Section 10, conclusion and future research directions.

All abbreviations and basic notations are listed in Tables 1 and 2, respectively.

Table 1. List of abbreviations

Abbreviation	Meaning
CpG	Crisp graph
FG	Fuzzy graph
FL	Fuzzy logic
CFG	Complete fuzzy graph
BG	Bunch graph
FBG	Fuzzy bunch graph
MV	Membership value
SBG	Super bunch graph
CBG	Complete bunch graph
CFBG	Complete fuzzy bunch graph
CpBG	Crisp bunch graph
AM	Adjacency matrix
BN	Bunch node
LM	Laplacian matrix
DM	Degree matrix
EFBG	Energy of fuzzy bunch graph
EV	Eigen value

Table 2. Basic notations

Notation	Meaning
G	Crisp graph
F_{BG}	Fuzzy bunch graph
CF_{BG}	Complete FBG
(σ_B)	MV of BG
$A(B_G)$	AM of BG
$D(B_G)$	DM of BG
$L(B_G)$	LM of BG
$A(F_{BG})$	AM of FBG
$E(F_{BG})$	Energy of FBG

2. Preliminaries

Definition 1 (Graph) [1]**.** If an arbitrary set of objects V and E is denoted by $G = (V, E)$, where V represents the vertices and E represents the edges by joining any two members of V, and G is called a graph.

Definition 2 (Vertex degree)**.** The number of edges contained in any vertex in a graph is called the vertex degree. This value is denoted as $d(V)$, and varies for different vertices.

Definition 3 (Complete graph [CG])**.** Weisstein et al. [32] introduced the CG concept. In CG, every vertex is joined by all other vertices. If the number of vertices in a CG is n, then the degree of all vertices must be $(n-1)$. Therefore, every CG is also a regular graph; however, the reverse is not always true.

Definition 4 (Sub-graph)**.** For any two graphs $G = (V, E)$ and $G^* = (V^*, E^*)$ be such that $V^* \subseteq V$ and $E^* \subseteq E$, then G^* is called a sub–graph of G. Therefore, if a graph G has n vertices, then the number of sub-graphs of G is $\binom{n}{1} + \binom{n}{2} +$

$\binom{n}{3} + \cdots + \binom{n}{n} = 2^n$.

Definition 5 (Fuzzy graph [FG]). Pal et al. [33] and Mathew et al. [34] introduced the FG concept. If $G = (V, E)$ is a crisp graph (CpG), then $G^* = (V^*, \sigma, \mu)$ is also a graph, where V^* denotes the vertex along with the two functions $\sigma : V^* \to [0, 1]$ and $\mu : V^* \times V^* \to [0, 1]$, such that $\mu(a, b) \leq min\{\sigma(a), \sigma(b)\}$ $\forall a, b \in V^*$ is called the FG, where σ and μ represent the MVs of the vertices and edges, respectively.

Definition 6 (Bunch graph [BG]). Sammanta et al.[22] introduced the BG. Let X be any non-empty set and V_X be a set of vertices where some elements of X form a new vertex for the same activity, such that $\phi \notin V_X$, $V_X - X \neq \phi$, that is, $V_X - X \neq \phi$ and $V_X \subset P(X)$. If we denote a set of edges as E, where $E \subset V_X \times V_X$, then $G = (V_X, E)$ is called the BG.

Definition 7 (Super bunch graph [SBG]). In a BG, if any bunch contains other bunches, then this type of BG is called a SBG.

Definition 8 (Degree of maximal and non-maximal nodes). In a BG, $G = (V_X, E)$, where X is any non-empty set, the degree of a maximal node v is denoted by $d(v)$ and defined as $d(v) =$ number of edges connected to $V+$ number of simple nodes inside the maximal node; the degree of a non-maximal node u is denoted by $d(u)$ and defined as $d(u) =$ number of edges connected to $u+$ number of bunches containing u.

Definition 9 (Complete bunch graph [CBG]). In a BG, if an edge exists between any two maximal nodes and all non-maximal nodes within a bunch node (BN) are connected, then it is called a CBG. Therefore, a CBG is always bunch-connected.

3. Fuzzy Bunch Graph

FBG is a mathematical concept used in fuzzy logic (FL) and graph theory. In FBG, each node represents a pixel or region in an image, and the edges between the nodes represent the degree of similarity or connectivity between them. Here, the term 'fuzzy' refers to the notion of FL, where instead of crisp distinctions, there are degrees of truth or membership. Similarly, the term 'Bunch' implies a collection or group of things. Putting these together, i.e., FBG, might be a graph where the relationships between nodes or edges are not precisely defined but are instead represented in terms of degrees of truth or membership in fuzzy sets (FSs).

Definition 10. Let X be a non–empty set and $V_X \subset P(X)$, where $P(X)$ is the power set of X, such that $\phi \notin V_X$, $X \subset V_X$, i.e., $V_X - X \neq \phi$. Now, a non-empty set V_X together with a pair of functions $\sigma_X : V_X \to [0, 1]$ and $\mu_X : V_X \times V_X \to [0, 1]$, such that for all $A, B \in V_X$, $\mu_X(A, B) \leq min\{\sigma_X(A), \sigma_X(B)\}$, then graph $G = (V_X, \sigma_X, \mu_X)$ is called a FBG.

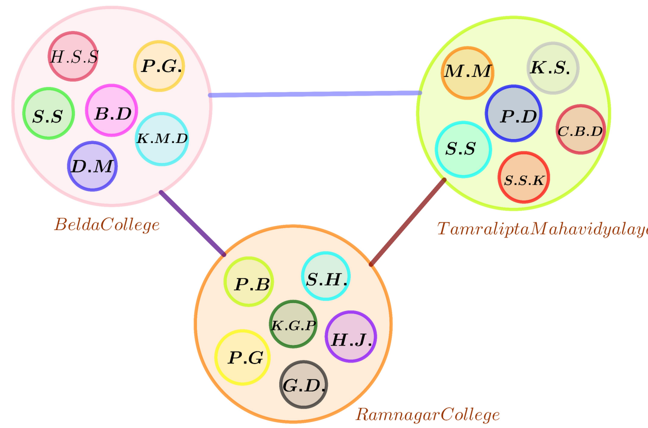

Figure 1. Representation of departments in three different colleges.

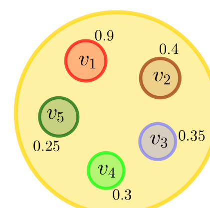

Figure 2. Example of bunch nodes.

3.1 Determination of the MVs of Bunch Nodes

Let us consider a bunch B having p nodes in P choosing MVs greater than 0.5, and q nodes in Q having MVs less than 0.5. Then, the MV of bunch (σ_B) can be obtained as follows:

$$\sigma_B = w \times \frac{1}{q}\Sigma_{U \subset Q}\sigma(U) + (1 - w) \times \frac{1}{p}\Sigma_{U \subset P}\sigma(U),$$

where $P = \{U \subset B \subset V_X / \sigma_P(U) \geq 0.5\}$, $|P| = p$, $Q = \{U \subset B \subset V_X / \sigma_Q(U) < 0.5\}$, and $|Q| = q$, where w is a parameter such that $0 < w < 1$.

Example 1. Let a bunch contain the nodes v_1, v_2, v_3, v_4, and v_5 having MVs of 0.9, 0.4, 0.35, 0.3, and 0.25, respectively. The MV of the bunch is calculated as follows:

$$\begin{aligned}\sigma_B &= w \times v_1 + (1 - w)\frac{1}{4}\Sigma_{i=2}^{5} v_i \\ &= w \times (0.9) + (1 - w) \times \frac{1}{4}(0.4 + 0.35 + 0.3 + 0.25),\end{aligned}$$

when $w = 0.5$, $\sigma_B = (0.5 \times 0.9) + (0.5 \times 0.325) = 0.612$.

Theorem 1. If a bunch B consists n nodes whose p nodes in P having MVs less than 0.5 and q nodes in Q having MVs greater than or equal to 0.5, then the MV of the bunch node (σ_B) holds the inequality

$$\frac{1}{q}w\sqrt{\Sigma_{U\subset Q}\sigma_U^2} < \sigma_B < \frac{1}{p(1-w)\sqrt{\sum_{U\subset P}\sigma_U^2}},$$

where $0 < w < 1$ and $n = p + q$, $\mid P \mid= p$, $\mid Q \mid= q$.

Proof. As σ_B is the MV of the BN that consists of n nodes with MVs σ_1, σ_2, σ_3, ..., σ_n. Therefore, we may assume σ_B is representative of σ_1, σ_2, σ_3, ..., σ_n out of n nodes if the MV of the p nodes is ≥ 0.5 in P, and that of the q nodes is < 0.5 in Q. Then, $\Sigma_{U\subset Q}\sigma_U < q\sigma_B$; or

$$\begin{aligned}
&\frac{1}{q}\Sigma_{U\subset Q}\sigma_U < \sigma_B\\
&\Rightarrow \frac{1}{q}.w\Sigma_{U\subset Q}\sigma_U < \sigma_B[as 0 < w < 1]\\
&\Rightarrow .\frac{1}{q}.w\sqrt{\Sigma{\sigma_U}^2} < \sigma_B\\
&\Rightarrow \frac{1}{q}.w\sqrt{\sigma_1^2+\sigma_2^2+\cdots+\sigma_q^2+2(\sigma_i\times\sigma_j)} < \sigma_B\\
&\Rightarrow \frac{1}{q}.w\sqrt{\sigma_1^2+\sigma_2^2+\cdots+\sigma_q^2} < \sigma_B\\
&\Rightarrow \frac{1}{q}.w\sqrt{\Sigma_{U\subset Q}\sigma_U^2} < \sigma_B. \qquad (1)
\end{aligned}$$

Additionally,

$$\begin{aligned}
&p.\sigma_B < \Sigma_{U\subset P}\sigma_U\\
&\Rightarrow \sigma_B < \frac{1}{p}\Sigma_{U\subset Q}\sigma_U\\
&\Rightarrow \sigma_B < \frac{1}{p(1-w)}\Sigma_{U\subset P}\sigma_U\\
&\Rightarrow \sigma_B < \frac{1}{p(1-w)}(\Sigma_{U\subset P}\sigma_U)^2, \quad [\text{as } (1-w)<1]\\
&\Rightarrow \sigma_B < \frac{1}{p(1-w)}\frac{1}{(\Sigma_{U\subset P}\sigma_U)^2} \quad [\text{as } \Sigma\sigma_U > 1]\\
&\Rightarrow \sigma_B < \frac{1}{p(1-w)}\frac{1}{\Sigma_{U\subset P}{\sigma_U}^2},\\
&\qquad\qquad [\text{as } (\sigma_1+\sigma_2)^2 > \sigma_1^2+\sigma_2^2]\\
&\Rightarrow \sigma_B < \frac{1}{p(1-w)\sqrt{\Sigma_{U\subset P}{\sigma_U}^2}}. \qquad (2)
\end{aligned}$$

From (1) and (2), we obtain

$$\frac{1}{q}w\sqrt{\Sigma_{U\subset Q}\sigma_U^2} < \sigma_B < \frac{1}{p(1-w)\sqrt{\sum_{U\subset P}\sigma_U^2}}.$$

□

Example 2. Let the bunch consist of four nodes with MVs of 0.2, 0.3, 0.7, and 0.8, where the MVs of two nodes $(0.2, 0.3)$ are less than 0.5 and those of the other two nodes $(0.7, 0.8)$ are greater than 0.5, that is, $p = 2$ and $q = 2$. Then, we obtain

$$\frac{1}{q}.w\sqrt{\Sigma_{U\subset Q}\sigma_U^2} = \frac{1}{2}.w\sqrt{0.4+0.9} = 0.09,$$

(taking $w = 0.5$).

Additionally,

$$\frac{1}{p(1-w)\sqrt{\Sigma_{U\subset P}{\sigma_U}^2}} = \frac{1}{2(1-w)\sqrt{0.49+0.64}} = 0.94.$$

Then,

$$\begin{aligned}
\sigma_B &= w\times\frac{1}{q}\Sigma_{U\subset Q}\sigma(U) + (1-w)\times\frac{1}{p}\Sigma_{U\subset P}\sigma(U)\\
&= w.\frac{1}{2}.(0.2+0.3) + (1-w).\frac{1}{2}.(0.7).(0.8)\\
&= 0.5.
\end{aligned}$$

As $0.09 < 0.5 < 0.94$, the value of σ_B satisfies the above inequality.

Theorem 2. If a BN consists of n nodes with MV σ_V ($V = 1$, 2, 3,..., n) and the MV of the bunch σ_B occurs by the inequality

$$\frac{1}{n}[1-\frac{1}{\prod(1+\sigma_V)}] < \sigma_B < \frac{1}{n}[\prod(1+\sigma_V)-1],$$

where $\Sigma\sigma_V < 1$.

Proof. Assuming n.σ_B instead of S_n, we obtain, from the 'Weierstrass' inequality $[1+S_n < \prod(1+v_i) < \frac{1}{1-S_n}]$, $1 + n.\sigma_B < \prod(1+\sigma_V) < \frac{1}{1-n.\sigma_B}$. As

$$\begin{aligned}
1+n.\sigma_B < \prod(1+\sigma_V) &\Rightarrow n.\sigma_B < \prod(1+\sigma_V)-1\\
&\Rightarrow \sigma_B < \frac{1}{n}[\prod(1+\sigma_V)-1]. \quad (3)
\end{aligned}$$

Again,

$$\begin{aligned}
\prod(1+\sigma_V) < \frac{1}{1-n.\sigma_B} &\Rightarrow \frac{1}{\prod(1+\sigma_V)} > 1-n.\sigma_B\\
&\Rightarrow n.\sigma_B > 1-\frac{1}{\prod(1+\sigma_V)}\\
&\Rightarrow \sigma_B > \frac{1}{n}\left[1-\frac{1}{\prod(1+\sigma_V)}\right]. \quad (4)
\end{aligned}$$

From (3) and (4), we obtain $\frac{1}{n}[1-\frac{1}{\prod(1+\sigma_V)}] < \sigma_B < \frac{1}{n}[\prod(1+\sigma_V)-1]$. $\square$

Example 3. Let the values of σ_V of a bunch be 0.2, 0.3, 0.1, 0.2, where

$$\begin{aligned}
&\sum \sigma_V < 1,\\
&\frac{1}{n}\left[1-\frac{1}{\prod(1+\sigma_V)}\right] = \frac{1}{4}\left[1-\frac{1}{1.2\times 1.3\times 1.1\times 1.2}\right]\\
&\qquad = 0.1292,\\
&\frac{1}{n}\left[\prod(1+\sigma_V)-1\right] = \frac{1}{4}[1.2\times 1.3\times 1.1\times 1.2-1]\\
&\qquad = 0.2648,\\
&\sigma_B = \frac{1}{3}\times w(0.1+0.1+0.2)+\frac{1}{1}(1-w)\times(0.3)\\
&\qquad [q \text{ bunches for} \leq 0.2 \text{ and } p \text{ bunches} > 0.2,\\
&\qquad \text{assuming that } w = 0.5]\\
&= \frac{1}{3}\times(0.5)\times(0.5)+(1-0.5)(0.3) = 0.23.
\end{aligned}$$

Therefore, $0.1292 < \sigma_B < 0.2648$.

Definition 11. Let X be a non-empty set and V_X be a subset of $P(X)$ such that $\phi \notin V_X$ and $X \subset V_X$. Now, let $E \subset V_X \times V_X$ be a set such that $\phi \notin E$, where $V_X = B_X \cup S_X$ (B_X is the set of all bunches and S_X is the set of all simple nodes), and there are functions $\sigma : S_X \to [0,1]$, $\rho : B_X \to [0,1]$ and $\mu : E \to [0,1]$ such that

$$\mu(u,v) \leq \begin{cases} min[\sigma(u),\sigma(v)] & \text{for all } u,\ v \in S_X,\\ min[\rho(u),\rho(v)] & \text{for all } u,\ v \in B_X,\\ min[\sigma(u),\rho(v)] & \text{forall } u \in S_X,\ v \in B_X, \end{cases}$$

for all the edges $(u,v) \in E$.

Then, $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ is said to be an FBG where σ denotes the degree of membership of S_X, ρ denotes the degree of membership of B_X (where ρ depends on the membership of simple nodes within a bunch), and μ denotes the degree of membership of E (where μ depends on the membership of the nodes of any edge).

Example 4. Let $X = a, b, c, d, e, f, g$, $V_X = (a,0.5)$, $(b,0.6)$, $(c,0.7)$, $(abc,0.6)$, $(g,0.8)$, $(d,0.)$, $(e,0.4)$, $(de,0.5)$, $(f,0.6)$, $S_X = (a,0.5)$, $(b,0.6)$, $(c,0.7)$, $(g,0.8)$, $(d,0.6)$, $(e,0.4)$, $(f,0.7)$, $B_X = (abc,0.6)$, $(de,0.5)$, where $\rho(abc) = average[\sigma(a),\sigma(b),\sigma(c)]$, $\rho(de) = average[\sigma(d),\sigma(e)]$, $E = [(abc,de),\ 0.5]$, $[(abc,a),\ 0.5]$, $[(abc,b),\ 0.6]$, $[(abc,c),\ 0.6]$,

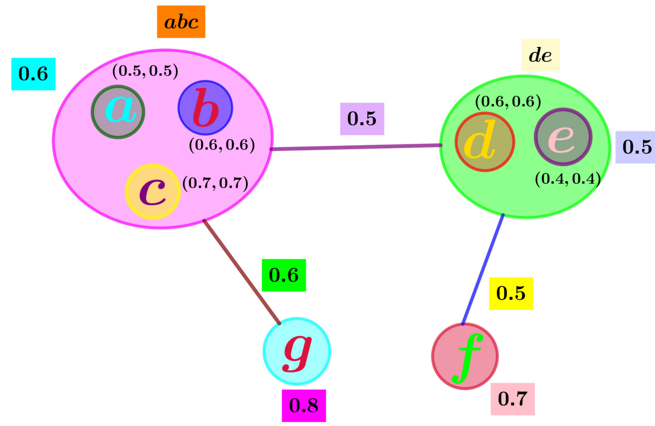

Figure 3. Example of a fuzzy bunch graph.

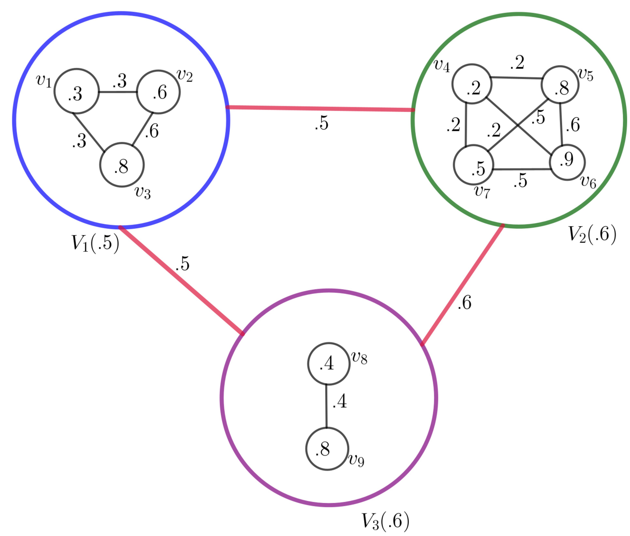

Figure 4. Example of a complete fuzzy bunch graph.

$[(abc,g),0.6]$, $[(de,f),0.5]$, $[(de,d),0.5]$, $[(de,e),0.4]$, where

$$\mu(u,v) = \begin{cases} min[\sigma(u),\sigma(v)] & \text{for all } u,\ v \in S_X,\\ min[\rho(u),\rho(v)] & \text{for all } u,\ v \in B_X,\\ min[\sigma(u),\rho(v)] & \text{for all } u \in S_X,\ v \in B_X, \end{cases}$$

Then, $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ is the FBG (Figure 3).

4. Complete Fuzzy Bunch Graph

An FBG is called a CFBG if any two maximal nodes or bunches are liked by an edge and all the non-maximal nodes within a maximal bunch are connected, as shown in Figure 4.

Definition 12. Let X be a non-empty set and $V_X \subset P(X)$, where $P(X)$ is the power set of X, such that $\phi \notin V_X$, $X \subset V_X$, i.e., $V_X - X \neq \phi$. Now, a non-empty set V_X together with a pair of functions $\sigma_X : V_X \to [0,1]$ and $\mu_X : V_X \times V_X \to [0,1]$, such that for all $A, B \in V_X$, $\mu_X(A,B) = min[\sigma_X(A), \sigma_X(B)]$, then the graph $G = (V_X, \sigma_X, \mu_X)$ is called a CFBG.

Note 1. (1) If the number of n non-maximal nodes with MVs v_i $(i = 1, 2, ..., n)$ make the number of m maximal BNs with MVs V_j $(j = 1, 2, ..., m)$ in a CFBG, then $\frac{1}{m}\sum_{j=1}^{m} V_j < \frac{1}{n}\sum_{i=1}^{n} v_i$.

(2) If $E(CFBG) \geq \frac{n+m}{2}$, G is called a strong CFBG; otherwise, it is called a weak CFBG.

(3) An FBG can be defined as either strong or weak based on the maximum number of strong or weak edges. If $\frac{\mu(a,b)}{\sigma_a \bigwedge \sigma_b} \geq 0.5$,then the edge is strong; if $\frac{\mu(a,b)}{\sigma_a \bigwedge \sigma_b} < 0.5$, then the edge is weak.

(4) As in an CFBG $\mu(a,b) = \sigma_a \bigwedge \sigma_b$, then the ratio $\frac{\mu(a,b)}{\sigma_a \bigwedge \sigma_b} = 1$.

Therefore, the CFBG is always strong.

5. Coloring in a FBG

Definition 13. Let $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ be an FBG. The coloring of F_{BG} is a function $c : V_X \to [0,1]$ such that for every edge $(u,v) \in E$, we have $|c(u) - c(v)| \geq \delta$,where $(\delta > 0)$ is a fixed color distance.

In this definition, $c(u)$ can be interpreted as the "color" of node (u). The condition $|c(u) - c(v)| \geq \delta$ ensures that the colors of adjacent nodes are not too close to each other.

Example 5. Consider the FBG shown in Figure 3. We define the color (c) of the FBG by setting $(c(u) = \sigma(u))$ for all $(u \in S_X)$ and $(c(u) = \rho(u))$ for all $(u \in B_X)$. For instance, we have

(i) $c(a) = \sigma(a) = 0.5$,

(ii) $c(b) = \sigma(b) = 0.6$,

(iii) $c(abc) = \rho(abc) = 0.6$.

For every edge $[(u,v) \in E]$, the colors of the end nodes satisfy $(|c(u) - c(v)| \geq \delta)$. For example, for edge (abc, a), we have $(|c(abc) - c(a)| = |0.6 - 0.5| = 0.1)$, which is greater than or equal to (δ) for a suitable choice of (δ), such as $(\delta = 0.05)$. Therefore, (c) is a valid coloring of F_{BG}.

5.1 Chromatic Number of a FBG

The chromatic number of a FBG $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ is the smallest number (k) such that there exists a coloring $(c : V_X \to [0,1])$ of F_{BG} with $(|c(u) - c(v)| \geq \delta)$ for every edge $((u,v) \in E)$, and the range of (c) is a set of (k) distinct values.

In other words, the chromatic number is the minimum number of different "colors" needed to color the graph nodes so that no two adjacent nodes have colors that are too close to each other. The "closeness" of colors is determined by the color distance (δ).

Several factors influence the bounds of the chromatic number in an FBG:

- Independence and clique numbers: The chromatic number can be bounded in terms of the independence and the clique numbers: $\chi(F_{BG}) \geq \omega(F_{BG})$ and $\chi(F_{BG}) \leq \left[\frac{|V|}{\alpha F_{BG}}\right]$.
- Maximum degree: The chromatic number can also be bounded in terms of the maximum degree: $\chi F_{BG} \leq \Delta F_{BG} + 1$.
- Given the definition of the chromatic number of an FBG, we can derive some properties and bounds:
  1. Lower bound: The chromatic number of FBG $F_{BG} = (V_X, E, \sigma, \rho, \mu))$ is at least 1, i.e.,$(k \geq 1)$. This is because at least one color is needed to color a graph.
  2. Upper bound: The chromatic number of FBG $F_{BG} = (V_X, E, \sigma, \rho, \mu))$ is at most equal to the graph's order, i.e., $(k \leq |V_X|)$. This is because in the worst-case scenario, each vertex may require a distinct color.
- Color distance constraint: For any two adjacent vertices (u) and (v) in (E), the absolute difference between their colors must be greater than or equal to the color distance (δ), i.e., $(|c(u) - c(v)| \geq \delta)$. This ensures that no two adjacent vertices have colors that are too close to each other.
- Color range: The range of coloring function $(c : V_X \to [0,1])$ is a set of (k) distinct values. This means that the colors used to color the graph are distinct and lie within the interval $[0,1]$.

Theorem 3. If $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ is an FBG and $(\delta > 0)$, the chromatic number of F_{BG} is unbounded.

Proof. SupposeF_{BG} is a complete FBG with (n) vertices. Subsequently, each vertex must be colored such that the color difference between any two adjacent vertices is at least (δ). Because F_{BG} is complete, each vertex pair is adjacent. Therefore, we require at least $(n\delta)$ distinct colors to color F_{BG}. As (n) can be arbitrarily large, the chromatic number of F_{BG} is unbounded. □

This theorem shows that the chromatic number of an FBG can be very large depending on its structure and (δ). This highlights the complexity of the coloring problem in FBGs.

Based on the information provided, we can derive the following lemmas for an FBG.

5.2 Lemma

1. Membership boundaries: The MVs σ, ρ, and μ are mapped to the interval $[0, 1]$. This implies that the membership degree of any node or edge in a graph is bounded by 0 and 1.

Proof. By definition, $\sigma : S_X \to [0, 1], \rho : B_X \to [0, 1] and \mu : E \to [0, 1]$. Therefore, the range of these functions is $[0, 1]$. □

2. Edge membership: The membership of edge (u, v) in E is less than or equal to the minimum membership of nodes u and v.

Proof. By definition, for all edges $(u, v) \in E$, $\mu(u, v) \le min\{\sigma(u), \sigma(v)\}$ for all $u, v \in S_X$, $\mu(u, v) \le min\{\rho(u), \rho(v)\}$ for all $u, v \in B_X$, and $\mu(u, v) \le min\{\sigma(u), \rho(v)\}$ for all $u \in S_X, v \in B_X$. □

3. Bunch membership: The membership of a bunch in B_X is dependent on the membership of the simple nodes within the bunch.

Proof. Per the given example, $\rho(abc) = average\{\sigma(a), \sigma(b), \sigma(c)\}$, $\rho(de) = average\{\sigma(d)$, and $\sigma(e)\}$. This indicates that the membership of a bunch is calculated as the average membership of simple nodes within the bunch. □

4. Graph formation: FBG (F_{BG}) is formed by the node set V_X, set of edges E, and membership functions σ, ρ, and μ.

Proof. By definition, $F_{BG} = (V_X, E, \sigma, \rho, \mu)$ is an FBG. □

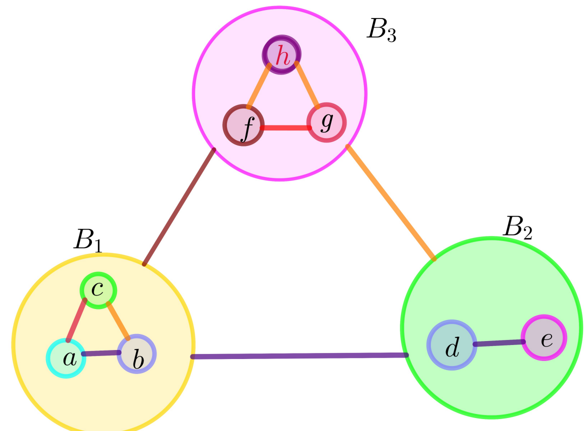

Figure 5. Bunch graph.

Table 3. Adjacency matrix of an FBG

	a	b	c	d	e	f	g	h	B_1	B_2	B_3
a	0	1	1	0	0	0	0	0	1	0	0
b	1	0	1	0	0	0	0	0	1	0	0
c	1	1	0	0	0	0	0	0	1	0	0
d	0	0	0	0	1	0	0	0	0	1	0
e	0	0	0	1	0	0	0	0	0	1	0
f	0	0	0	0	0	0	1	1	0	0	1
g	0	0	0	0	0	1	0	1	0	0	1
h	0	0	0	0	1	1	0	0	0	0	1
B_1	1	1	1	0	0	0	0	0	0	1	1
B_2	0	0	0	1	1	0	0	0	1	0	1
B_3	0	0	0	0	0	1	1	1	1	1	0

6. Matrix Representation of a BG

A crisp bunch graph (CpBG) is represented by an adjacency matrix (AM), which is always square (Table 3). The matrix order must be equal to the total number of vertices. The elements of this graph are either 0 or 1 depending on the connectivity between any two vertices. If an edge exists between any two vertices, then the entry is 1; otherwise, it is 0.

The AM of the above CBG can be written as

Note 2. 1. The AM of the BG is always a square matrix of order $(n + r)$, where n is the total number of nodes and r is the number of bunches.

2. This type of matrix is symmetrical about the principal diagonal.

Table 4. Adjacency degree matrix of a BG

	a	b	c	d	e	f	g	h	B_1	B_2	B_3
a	2	0	0	0	0	0	0	0	0	0	0
b	0	2	0	0	0	0	0	0	0	0	0
c	0	0	2	0	0	0	0	0	0	0	0
d	0	0	0	1	0	0	0	0	0	0	0
e	0	0	0	0	1	0	0	0	0	0	0
f	0	0	0	0	0	2	0	0	0	0	0
g	0	0	0	0	0	0	2	0	0	0	0
h	0	0	0	0	0	0	0	2	0	0	0
B_1	0	0	0	0	0	0	0	0	2	0	0
B_2	0	0	0	0	0	0	0	0	0	2	0
B_3	0	0	0	0	0	0	0	0	0	0	2

Table 5. Laplacian matrix of a BG

	a	b	c	d	e	f	g	h	B_1	B_2	B_3
a	2	-1	-1	0	0	0	0	0	-1	0	0
b	-1	2	-1	0	0	0	0	0	-1	0	0
c	-1	-1	2	0	0	0	0	0	-1	0	0
d	0	0	0	1	-1	0	0	0	0	-1	0
e	0	0	0	-1	1	0	0	0	0	-1	0
f	0	0	0	0	0	2	-1	-1	0	0	-1
g	0	0	0	0	0	-1	2	-1	0	0	-1
h	0	0	0	0	0	-1	-1	2	0	0	-1
B_1	-1	-1	-1	0	0	0	0	0	2	-1	-1
B_2	0	0	0	-1	-1	0	0	0	-1	2	-1
B_3	0	0	0	0	0	-1	-1	-1	-1	-1	2

3. It is always a Boolean matrix, that is, of the $(0, 1)$ type.
4. For a simple BG all the diagonal entries are '0'.
5. The elements of any node v_i are '1' for an existing bunch and '0' for any other bunch.
6. Eigenvalues of $A(B_G)$ can be obtained by solving $\{A(B_G) -x \times I_{n+r}\}$, where I_{n+r} is an identity matrix of order $(n + r)$.
7. Because the AM of a BG is a real equable matrix, all eigenvalues are also real.
8. The degree of any BN is equal to the sum of the bunch entries.

6.1 Degree Matrix of a BG

The degree matrix (DM) of a BG (B_G) is a diagonal matrix $D(B_G)$ of order $(n+r)$, where n is the number of simple nodes and r is the number of BNs (Table 4). Then, $D(B_G)$ is defined as $D(B_G) = (d_{ij})_{(n+r)\times(n+r)}$, where $d_{ij} = d(v_i)$ if $(i = j)$; otherwise, $= 0$. $D(B_G)$ is obtained from Figure 5.

6.2 Laplacian Matrix of a BG

The Laplacian matrix (LM) of a BG (B_G) of the same order as $A(B_G)$ and $D(B_G)$ is defined as $L(B_G) = D(B_G) - A(B_G)$, where $D(B_G)$ and $A(B_G)$ are the DM and AM of BG (B_G), respectively (Table 5). Then, the elements of LM $L(B_G)$ = $(l_{ij})_{(n+r)\times(n+r)}$ are given by $l_{ij} = d(v_i)$ for $i = j$, -1 if v_i is adjacent to v_j in G_B, $i \neq j$, 0 otherwise.

$L(B_G)$ is obtained from Figure 5.

7. Matrix Representation of an FBG

An FBG can be represented by a matrix; however, its entries are not only 0 or 1 as in a CpBG. If m simple nodes create n nodes, then the order of the AM of an FBG is $(m + n)$. The entry is zero, as well as for other bunches, and for any two nodes within a bunch depends on the MV of their connecting edge.

Therefore, the FBG is a complex structure in FG theory. To represent this using matrices, we must extend the concepts of adjacency and incidence matrices from CpG to the fuzzy context.

Definition 14. The AM of an FBG is represented by $A(F_{BG})$ and is defined by $A(F_{BG}) = \mu_{ij}$, $i \neq j$, 0 for $i = j$ in the same bunch, $i \neq j$ in otherwise. $\forall$ simple nodes, and $A(F_{BG}) = (\mu_b)_{ij}$, for $i \neq j$, 0 for $i = j$ $\forall$ BNs.

Example 6. Let us consider FBG (F_{BG}) with bunches $B_1(v_1, v_2, v_3)$ and $B_2(v_4, v_5)$.

Then,

$$\sigma_{B_1} = w \times (0.5) + (1 - w) \times \frac{1}{2}\sum(0.7 + 0.8), \quad \text{[where } w = 0.4]$$

$$= 0.4 \times 0.5 + 0.6 \times 0.75 = 0.65,$$

$$\sigma_{B_2} = \frac{(0.8 + 0.7)}{2} = 0.75.$$

Now, $A(F_{BG}) =$

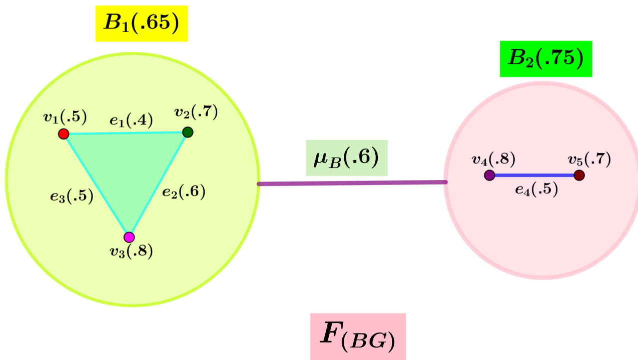

Figure 6. Example of an FBG.

	$v_1(0.5)$	$v_2(0.7)$	$v_3(0.8)$	$v_4(0.8)$	$v_5(0.7)$	$B_1(0.65)$	$B_2(0.75)$
$v_1(0.5)$	0	0.4	0.5	0	0	0	0
$v_2(0.7)$	0.4	0	0.6	0	0	0	0
$v_3(0.8)$	0.5	0.6	0	0	0	0	0
$v_4(0.8)$	0	0	0	0	0.5	0	0
$v_5(0.7)$	0	0	0	0.5	0	0	0
$B_1(0.65$	0	0	0	0	0	0	0.6
$B_2(0.75)$	0	0	0	0	0	0.6	0

which can also be represented as $\begin{pmatrix} A_{F_{BG_1}} & 0 & 0 \\ 0 & A_{F_{BG_2}} & 0 \\ 0 & 0 & A_{F_{BG}} \end{pmatrix}$, where $A_{F_{BG_1}} = \begin{pmatrix} 1 & 0.4 & 0.5 \\ 0.4 & 1 & 0.6 \\ 0.5 & 0.6 & 1 \end{pmatrix}$, $A_{F_{BG_2}} = \begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}$, $A_{F_{BG}} = \begin{pmatrix} 0 & 0.6 \\ 0.6 & 0 \end{pmatrix}$, where $A_{F_{BG_1}}$, $A_{F_{BG_2}}$, and $A_{F_{BG}}$ are the AMs of the graphs containing vertices (v_1, v_2, v_3), (v_4, v_5) and bunches (B_1, B_2).

Note.

1. The AM of the FBG is also a square matrix.
2. It is always a diagonal matrix and symmetric about the principal diagonal.
3. All entries in this matrix are $[0, 1]$.
4. If a simply connected FBG contains m bunches, then the number of entries in the separated $A(F_{BG})$ is $(m+1)^2$.
5. Except for the principal diagonal, the rest are null matrices in separate $A(F_{BG})$.
6. All eigenvalues of $A(F_{BG})$ can be determined separately from the diagonal matrices in the separated $A(F_{BG})$.

8. Energy of Fuzzy Bunch Graph

Traditionally, the concept of energy of a graph is derived from the eigenvalues of its AM. Extending this idea to FBGs involves considering the membership degrees of the vertices and edges in the AM. Energy of fuzzy bunch graph (EFBG) is defined similarly to CpG, but considers the fuzzy nature of the graph. In Figure 6, EFGB can also be determined by the sum of the absolute value of its eigenvalue (EV).

$$\left|A(F_{BG_1})\right| = 0 \Rightarrow \begin{vmatrix} -\lambda & 0.4 & 0.5 \\ 0.4 & -\lambda & 0.6 \\ 0.5 & 0.6 & -\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda = 0.9,\ -0.6,\ -0.3.$$

The EVs of $A(F_{BG_1})$ are $\lambda = 0.9, -0.6, -0.3$. Additionally,

$$\left|A(F_{BG_2})\right| = 0\ \Rightarrow \begin{vmatrix} -\lambda & 0.5 \\ 0.5 & -\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda = 0.5,\ -0.5.$$

The EVs of $A(F_{BG_2})$ are $\lambda = 0.5, --.5$.

Additionally,

$$\left|A(F_{BG})\right| = 0 \Rightarrow \begin{vmatrix} -\lambda & 0.6 \\ 0.6 & -\lambda \end{vmatrix} = 0$$
$$\Rightarrow \lambda = 0.6,\ -0.6.$$

The EVs of $A(F_{BG})$ are $\lambda = 0.6, -0.6$.

The sum of the numerical values of all EVs of the AM is zero, which satisfies the FG energy theory. Thus, the EFBG defined by $E(F_{BG})$ can be determined as

$$E(F_{BG}) = \Sigma \left|\lambda_i\right|$$
$$\Rightarrow= 0.9 + 0.6 + 0.3 + 0.5 + 0.5 + 0.6 + 0.6 \Rightarrow= 4.0.$$

Note 3. The bounds for the EFBG and FBG are:

(1) $E(F_{BG}) \leq 2\Sigma\mu_{e_i}$ can be extended to $E(F_{BG}) \leq 2\Sigma\mu_{e_i} \leq 2\sigma_{v_i}$.

(2) $E(F_{BG}) \leq (n-1)\sum_{i=1}^{n}\sigma_{v_i}$ can also be extended for the energy of only BNs in $E(B_i) \leq (n-1)^2\sigma_{B_i}$, as the number of n nodes can make the maximum number of

$(n-1)$ BNs.

Clearly, the above bounds are satisfied by the results shown in Figure 6.

8.1 Computational Complexity and Scalability

The computational complexity of the FBG construction involves three key steps: homophily-based node grouping, fuzzy membership computation, and matrix representation. For a graph with n nodes and e edges

- Pairwise homophily calculation requires $O(n^2)$ in the worst case.
- Membership value assignment depends on fuzzy rule evaluation, which scales with $O(k)$ for k bunches.
- Matrix construction for FBG takes $O(n^2)$ for adjacency representation.

Therefore, the overall complexity lies in the range of $O(n^2)$–$O(n^3)$ depending on the density of the network.

To handle large-scale networks, future work will explore greedy algorithms, sampling strategies, and parallel computation to reduce the overhead.

8.2 Comparative Performance Framework

To benchmark the performance of FBGs against that of FGs, BGs, and simplified bunch graphs, we propose a comparative evaluation based on the following metrics:

- **Accuracy:** How closely the structure preserves the original community relationships.
- **Reduction rate:** Percentage decrease in active nodes.
- **Computation time:** Processing time for graph transformation.
- **Memory usage:** Size of matrix representations.

In future work, we plan to implement and test these comparisons using synthetic and real datasets to highlight the efficiency and interpretability of FBGs.

8.3 Real-World Utility of the FBG Framework

The FBG model is applied as follows:

- **Social networks:** Detecting community boundaries based on fuzzy similarity, and identifying central bunch nodes representing groups of similar users.
- **Biological networks:** Simplifying protein–protein interaction networks by bundling functionally similar proteins, while preserving uncertainty in interaction strength.
- **Transportation systems:** Representing clusters of routes or stations with fuzzy demand and connectivity.

By computing the fuzzy membership values and edge strengths, FBGs provide a tool for identifying the overlapping communities and uncertainty zones that are often present in real-world systems.

9. Application Area

We work every day, everywhere, and in any field. If we succeed many times, we enjoy it; if we fail a maximum number of times, we try to avoid it differently. If we graphically represent our activities using a fuzzy graph, the membership values of different nodes and edges are clearly shown regarding success and failure. By comparing these values, we can determine the areas in which our activities are not satisfactory.

For example, a country's cybercrime department protects its citizens from terrorism. If we represent the cybercrime network in different areas of a country using a fuzzy graph or FBG in the case of more than one node in a specific area, the number of communications among terrorists yields the membership values of the nodes and edges in such a graph. When these values lie between 0.5 and 1, the cybercrime department may alert the government to increase its defensive system to protect the country.

Table 6 lists the terrorist attacks in India over the last 10 years in different states, with dates, locations, and attackers.

Table 6 is shown graphically in Figure 7.

As can be seen, the maximum number of attacks occurred at JK. Here, every attack location is treated as a node, and all

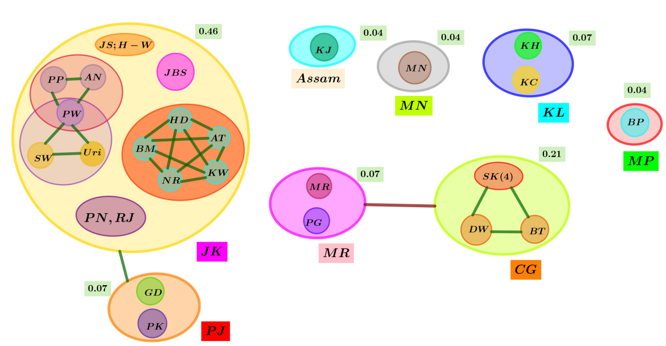

Figure 7. Terrorist attack graph.

Table 6. List of terrorist attacks

Serial no.	Date	Location	State	Attacker
1	20.03.2015	Jammu Busstand (JBS)	Jammu Kashmir (JK)	Fidayeen
2	09.06.2015	Monipur (MN)	Monipur (MN)	NSCN-K
3	27.07.2015	Gurdaspur (GD)	Panjab (PUN)	Millitantes
4	02.01.2016	Pattanket (PK)	PUN	U.J.C
5	25.06.2016	Pampora (PP)	JK	Lashkar-e-Taiba
6	05.08.2016	Kokrajhar (KJ)	Assam	N.D.F.B
7	18.09.2016	Uri	JK	Jaish-e-Mahammed
8	03.10.2016	Baramulla (BM)	JK	Millitants
9	06.10.2016	Handwara (HD)	JK	Millitants
10	29.11.2016	Nagrota (NR)	JK	Millitants
11	07.03.2016	Bhopal (BP)	Madhyapradesh (MP)	Not found
12	24.04.2017	Sukhma (SK)	Chhattishgarh (CG)	Maoist
13	11.07.2017	Anantanag (AN)	JK	Lashkar-e-Taiba
14	10.02.2018	Sunjuwar (SW)	CG	Jaish-e-Mahammed
15	13.03.2018	SK	CG	Maoist
16	14.02.2019	Pulwama (PW)	JK	Lashkar-e-Taiba
17	07.03.2019	J-Srinagar HW (JSHW)	JK	Not found
18	09.04.2019	Dantewada (DW)	CG	Maoist
19	09.04.2019	Kishtwar (KW)	JK	Millitant
20	01.05.2019	Maharastra (MR)	MR	Maoist
21	12.06.2019	Awantipora (AT)	JK	Millitant
22	21.03.2020	SK	CG	Maoist
23	03.04.2023	SK	CG	Maoist
24	08.04.2023	Kozhikoda (KH)	Kerala	Terroist
25	20.04.2023	Poonch (PN), Rajouri (RJ)	JK	Terroist
26	26.04.2023	Bastar (BT)	CG	Maoist
27	31.07.2023	Palghar (PG)	MR	RPF-Constable
28	29.10.2023	Kochi (KC)	Kerala	Jehovah's Chri. rel. gp

nodes within a state form a bunch. This graph also shows that some of the nodes within a bunch or some of the bunches are linked by an edge, which indicates that there is a relationship among the attack locations by the same terrorist organization. The MV of each bunch is represented by the ratio of the number of attacks in a bunch or state and the total number of attacks. For JK, this ratio is $\frac{13}{28} = 0.46$. This figure also shows that some separate bunches are present in the bunch for JK. Therefore, this bunch may be considered a superbunch.

In this way, this graphical representation can be used in travelling sales, marketing distribution, transportation networks, mobile banking in deferent branches, number of followers for a popular singer, film stars, or political leader on social media, and many other real-life activities.

10. Conclusion and Future Work

10.1 Limitations and Scope

This study focused on the integration of fuzzy set theory with bunch graphs to develop the FBG concept. Although the proposed model introduces a novel structure for uncertainty-aware network simplification, certain limitations remain.

- We did not explore dynamic or time-evolving FBGs, which could be relevant in modeling continuously changing networks.
- The theoretical bounds and computational complexity of the FBG model were not fully addressed and will be addressed in future work.
- The model currently supports membership-based fuzziness; however, it does not consider advanced extensions such as intuitionistic fuzzy, type-2 fuzzy, or neutrosophic frameworks.
- The application was demonstrated on a single domain (terrorist networks); generalizability to other domains, such as biological, communication, and transportation networks, needs empirical testing.

Despite these limitations, this study established a foundational framework for FBGs and set the stage for future research on their analytical properties, algorithmic processing, and domain-specific customizations.

10.2 Generalization and Future Validation

Although the FBG was validated through a real-world case involving terrorist networks in India, additional experimentation on standard datasets, such as MNIST (for character graphs) and CIFAR (for image-based networks), is planned. These datasets will allow us to evaluate the performance of the FBG in high-dimensional and structured domains, further proving its robustness and flexibility across domains.

10.3 Conclusion

In this paper, we introduce the concept of a FBG by integrating the key principles of both crisp and fuzzy bunch graphs. This generalized framework allows for the inclusion of a large number of nodes within a single graph structure while retaining the fuzzy nature of membership, where each bunch node has a membership value in the range $[0, 1]$, similar to traditional fuzzy graphs.

We demonstrated a method for computing the membership values of bunch nodes that provides meaningful insights into the strength and significance of node associations. Such values can be used in real-world applications, where partial associations and uncertainties play a crucial role. Additionally, we defined the concept of CFBG and presented matrix representations for various fuzzy and bunch graph structures.

We believe that this work offers a natural and practical approach to modeling complex, uncertain networks and can aid in solving real-life problems more effectively.

The future scope of FBG research is both deep and wide, offering exciting prospects in theoretical mathematics, artificial intelligence, data science, healthcare, and engineering. As uncertainty becomes increasingly central to computational models, the role of FBGs is likely to increase significantly. Future studies may explore extensions into intuitionistic, interval-valued, Pythagorean, and neutrosophic FBGs as well as investigations into algebraic properties such as homomorphism, isomorphism, and automorphism.

Conflict of Interest

No potential conflict of interest relevant to this article was reported.

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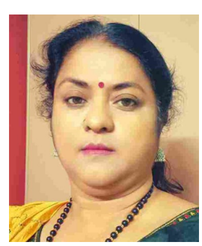

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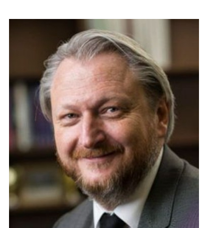

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