Journal of Algebraic Combinatorics (2026) 64:23
https://doi.org/10.1007/s10801-026-01559-y

Faithfulness in maniplexes

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Received: 5 December 2025 / Accepted: 3 June 2026

Abstract

Maniplexes generalize the flag graphs of maps and polytopes, and they have enabled the use of new techniques for constructing and analyzing polytopes. Each maniplex gives rise to a natural ranked poset of the faces and their incidence, but it is not always possible to recreate the maniplex from the poset. The first possible problem is that the mapping from flags of the maniplex to maximal chains of the poset may not be injective; in this case we say that the maniplex is *unfaithful*. The second is that the resulting poset may not be *thin* (meaning that some sections of rank 1 may have more than two proper faces), in which case the flag graph of the poset is not a maniplex. In this paper we describe a theory of faithfulness in maniplexes, laying the groundwork for future study in this area. We also classify the unfaithful maps (3-maniplexes) with at most two flag orbits, and we prove that the mix of two polytopes is a faithful maniplex that is thin if the polytopes are orientable.

Keywords Maniplex · Polytope · Faithfullness

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Published online: 01 August 2026

1 Introduction

Abstract polytopes (originally called incidence-polytopes) are ranked posets that were introduced by Danzer and Schulte in [7] as a generalization of the incidence relation between faces of a convex polytope. At the beginning, the great majority of the work on (abstract) polytopes focused on the most symmetric case, where everything can be handled group-theoretically. This led to the publication of "Abstract Regular Polytopes", which laid the groundwork for further research in this area [18].

As the polytope community has moved on to studying less-symmetric polytopes, the combinatorial structure that is most often useful is not the poset $\mathcal{P}$, but rather the *flag graph* $\mathrm{Fl}(\mathcal{P})$ of $\mathcal{P}$. This is a graph whose nodes correspond to the maximal chains in $\mathcal{P}$, and where two nodes are connected by an edge of label i whenever the corresponding chains differ only in their element of rank i. Starting with an n-polytope $\mathcal{P}$, its flag graph is an n-regular n-edge-colored simple graph such that for every pair of edge labels (i, j) with $|j - i| > 1$, the subgraph induced by edges of those colors is a disjoint union of 4-cycles. In fact, these graphs are interesting in their own right, and any graph that satisfies these properties (whether it is the flag graph of a polytope or not) is called a *maniplex* [22]; see Sect. 2.1 for a formal definition. Many operations on polytopes are not guaranteed to produce a polytope, but do produce a maniplex, and so the study of maniplexes has proven important in understanding polytopes. This is evidenced by a series of recent papers where problems pertaining to abstract polytopes have been solved using the theory of maniplexes (see for instance [2, 5, 9, 12, 15]).

Of course, if we want to study polytopes in terms of maniplexes, we need a way of recovering the poset from the flag graph. Indeed, there is a natural way of building a *flagged poset* $\mathrm{Pos}(\mathcal{M})$ from a maniplex $\mathcal{M}$; this is a ranked poset with a maximal and minimal element and such that each maximal chain contains an element in every rank.

If $\mathcal{M} = \mathrm{Fl}(\mathcal{P})$ for some polytope $\mathcal{P}$, then $\mathrm{Pos}(\mathcal{M}) \cong \mathcal{P}$ and thus $\mathrm{Fl}(\mathrm{Pos}(\mathcal{M})) \cong \mathcal{M}$ [8]. Otherwise, if $\mathcal{M}$ is not the flag graph of a polytope, it is not always true that $\mathrm{Fl}(\mathrm{Pos}(\mathcal{M})) \cong \mathcal{M}$. There are two possible ways for this isomorphism to fail. First of all, in order for $\mathrm{Fl}(\mathrm{Pos}(\mathcal{M}))$ to be a maniplex at all, $\mathrm{Pos}(\mathcal{M})$ must be *thin*, meaning that given incident faces $F < G$ of $\mathrm{Pos}(\mathcal{M})$ where $\mathrm{rank}(G) - \mathrm{rank}(F) = 2$, there must be precisely two faces H_1 and H_2 such that $F < H_i < G$ for $i \in \{1, 2\}$. Second, even if $\mathrm{Pos}(\mathcal{M})$ is thin, we also need for the natural map from $\mathcal{M}$ to $\mathrm{Pos}(\mathcal{M})$ that sends each flag to its corresponding maximal chain to be a bijection. When this map is indeed bijective, we say that $\mathcal{M}$ is *faithful*. Thus, $\mathrm{Fl}(\mathrm{Pos}(\mathcal{M})) \cong \mathcal{M}$ if and only if $\mathcal{M}$ is faithful and $\mathrm{Pos}(\mathcal{M})$ is thin. This includes many maniplexes that are not the flag graphs of polytopes; see [14].

The importance of faithfulness in maniplexes has only recently become apparent. A few papers ([8, 12, 14]) have worked with faithfulness in maniplexes, but there has not yet been a systematic study of this concept. In this paper, we aim to build the foundation for further study of faithfulness and thinness. Our approach is to study what could go wrong, namely, to investigate the structural properties of *unfaithful* maniplexes.

One particularly important application of this work concerns the *mix* of two polytopes, which is their minimal common cover. This is a basic ingredient for many constructions of polytopes. The mix of two polytopes is not always a polytope, but it is always a maniplex. In Corollary 6.3, we prove that the mix is always faithful, ruling out some of the most degenerate behavior. In particular, this implies that the smallest regular cover of every finite polytope is faithful; see Corollary 6.4.

We start with background in Sect. 2. Section 3 establishes basic properties of unfaithful maniplexes. Section 4 explores the relationship between unfaithfulness and symmetries, with Theorems 4.2 and 4.6 establishing conditions under which two flags will be chain-equivalent (represented by the same chain in the poset). We expand on this relationship in Sect. 4.2, culminating in Theorems 4.15 and Corollary 4.16, where the faithfulness of a maniplex is determined by properties of the automorphism group. In Sect. 5, we classify all unfaithful two-orbit 3-maniplexes. Section 6 considers faithfulness of the mix of maniplexes, leading to Corollary 6.3 (the mix of two polytopes is always faithful) and Corollary 6.4 (the minimal regular cover of a polytope with finitely many flag orbits is faithful). We then briefly consider thinness in Sect. 7 and refine Corollary 6.3 to Theorem 7.3 (the mix of faithful, thin orientable maniplexes has the same properties). Finally, we end with a few open problems for future research in Sect. 8.

2 Background

2.1 Maniplexes

The definition of a maniplex was introduced in [22] as a generalization of abstract polytopes (see [18]). Here we describe the basic definitions.

An *n-maniplex*[1] is an n-regular connected simple graph (finite or infinite) whose vertices we call *flags*, and such that:

(i) The edges are colored with elements of $\{0, \ldots, n-1\}$;
(ii) every vertex is incident to exactly one edge of each color; and
(iii) for every pair of colors i and j satisfying $|j-i| > 1$, the subgraph induced by edges colored i and j is a disjoint union of 4-cycles.

If Φ is a flag of a maniplex $\mathcal{M}$, then Φ^i denotes the flag that is connected to Φ by an edge of color i. We abbreviate $(\Phi^i)^j$ as $\Phi^{i,j}$ or occasionally as Φ^{ij} and extend this notation naturally to longer sequences. More generally, if Φ and Ψ are flags that are connected by a path using only colors in the interval $[i, j]$ (for $i \leqslant j$), then we write $\Phi \overset{[i,j]}{\text{---}} \Psi$.

A 0-maniplex must consist of a single flag, while a 1-maniplex consists of two flags connected by an edge of color 0. A 2-maniplex either consists of $2k$ flags, connected in a cycle with alternating edge colors, or is an infinite path in both directions.

Rank 3 maniplexes are closely related to maps (cellular embeddings of graphs on surfaces). From a 3-maniplex we can build a simplicial complex whose maximal faces (of dimension 2) correspond to the flags of the maniplex. If the maniplex is finite the

[1] In [22], what we are describing would have been called an $(n-1)$-maniplex, but we are using the new community consensus which calls this an n-maniplex.

resulting simplicial complex is a compact surface and the maniplex naturally defines an embedding of the graph defined by the 0- and 1-faces of the maniplex into the surface (see [22, Sect. 5]). This topological construction can be generalized to n-maniplexes (see [18, Sect. 2C]). Conversely, every map admits a triangulation, which often coincides with its barycentric subdivision, whose incidence (edge-colored) graph is a maniplex, unless the map has leaves or faces defined by a loop, in which case the flag-graph has parallel edges.

For each $i \in \{0, \ldots, n-1\}$, the edges of color i induce an involutory permutation r_i mapping a flag Φ of $\mathcal{M}$ to its i-neighbor Φ^i. The group $\langle r_0, \ldots, r_{n-1}\rangle$ is called the *connection group* of $\mathcal{M}$ (also called the *monodromy group*), denoted $\mathrm{Conn}(\mathcal{M})$. The elements of $\mathrm{Conn}(\mathcal{M})$ will be called *connections*. If $w \in \mathrm{Conn}(\mathcal{M})$, then we write Φ^w for the image of Φ under w. Note that:

(i) Each r_i is fixed-point-free.
(ii) If $i \neq j$, then $r_i r_j$ is fixed-point-free.
(iii) If $|j-i| > 1$, then $(r_i r_j)^2 = 1$ (and in fact, the order of $r_i r_j$ is precisely 2, since otherwise the maniplex would have parallel edges).

Indeed, any transitive permutation group G generated by involutions that satisfies the above (with respect to its distinguished generating set) can be used to build a maniplex $\mathcal{M}$ such that $\mathrm{Conn}(\mathcal{M}) = G$.

For $i \in \{0, \ldots, n-1\}$, the *i-faces* of an n-maniplex $\mathcal{M}$ are the connected components that remain when we remove all the edges of color i. We call *facets*, *vertices* and *edges* the $(n-1)$-, 0- and 1-faces, respectively. Observe that the facets are $(n-1)$-maniplexes themselves; while we can think (and we will) of the vertices as $(n-1)$-maniplexes after reindexing the colors by subtracting 1.

The *dual* of a maniplex $\mathcal{M}$ is the maniplex $\mathcal{M}^*$ obtained from $\mathcal{M}$ by changing every color i to $n-i-1$. It often happens that a result about a maniplex $\mathcal{M}$ has an obvious dual version that follows by applying the result to $\mathcal{M}^*$, and in such a case we say that the dual result follows by duality.

2.2 Automorphisms and regularity

An *automorphism* of a maniplex $\mathcal{M}$ is a graph automorphism that preserves the colors of the edges. Equivalently, it is a permutation of the flags that commutes with every r_i. The automorphism group of $\mathcal{M}$ is denoted $\mathrm{Aut}(\mathcal{M})$. We will write $\Phi\gamma$ for the image of the flag Φ under the automorphism γ.

The automorphism group $\mathrm{Aut}(\mathcal{M})$ acts semiregularly on the flags of $\mathcal{M}$. If the action is transitive, and thus regular, then the maniplex is called *regular* (or also *reflexible*).

Suppose that $\mathcal{M}$ is regular. Let us fix a *base flag* Φ_0. Then since the action of $\mathrm{Aut}(\mathcal{M})$ is transitive, for each $0 \leqslant i \leqslant n-1$ there is an automorphism ρ_i such that $\Phi_0\rho_i = (\Phi_0)^i$. Furthermore, these automorphisms generate the group:

$$\mathrm{Aut}(\mathcal{M}) = \langle \rho_0, \ldots, \rho_{n-1}\rangle.$$

Each ρ_i has order 2, and whenever $|j-i|>1$, then $(\rho_i\rho_j)^2=1$, which implies that ρ_i commutes with ρ_j. Note that

$$\Phi_0\rho_{i_1}\cdots\rho_{i_k}=\Phi_0^{r_{i_1}}\rho_{i_2}\cdots\rho_{i_k}=(\Phi_0\rho_{i_2}\cdots\rho_{i_k})^{r_{i_1}}=\cdots=\Phi_0^{r_{i_k}\cdots r_{i_1}}.$$

If $\Gamma=\langle x_0,\ldots,x_{n-1}\rangle$ is a group where each x_i has order 2, and x_ix_j has order 2 whenever $|j-i|>1$, then we call Γ a *string group generated by involutions*, or *sggi* for short. Given an sggi Γ, its Cayley graph (with respect to the generators $x_0,\ldots,x_{n-1}$) is a regular maniplex $\mathcal{M}$ with $\mathrm{Aut}(\mathcal{M})\cong\Gamma$. Thus, we often identify sggis with their corresponding regular maniplexes.

Notice that every sggi $\Gamma=\langle x_0,\ldots,x_{n-1}\rangle$, where the order of $x_{i-1}x_i$, $(1\leqslant i\leqslant n-1)$ is k_i, is a smooth quotient of the *string Coxeter group* $[k_1,k_2,\ldots,k_{n-1}]=\langle\rho_0,\ldots,\rho_{n-1}\rangle$ defined by the relations

$$\begin{aligned}\rho_i^2&=1\text{ for every }0\leqslant i\leqslant n-1,\\(\rho_{i-1}\rho_i)^{k_i}&=1\text{ for every }1\leqslant i\leqslant n-1,\\(\rho_i\rho_j)^2&=1\text{ if }|j-i|>1.\end{aligned}$$

Therefore, to specify the automorphism group of a regular maniplex, we will sometimes use notation such as

$$[4,4,4]/((\rho_0\rho_1\rho_2)^4,(\rho_1\rho_2\rho_3)^6),$$

meaning that the automorphism group is a quotient of the string Coxeter group [4, 4, 4] by the additional relators $(\rho_0\rho_1\rho_2)^4$ and $(\rho_1\rho_2\rho_3)^6$.

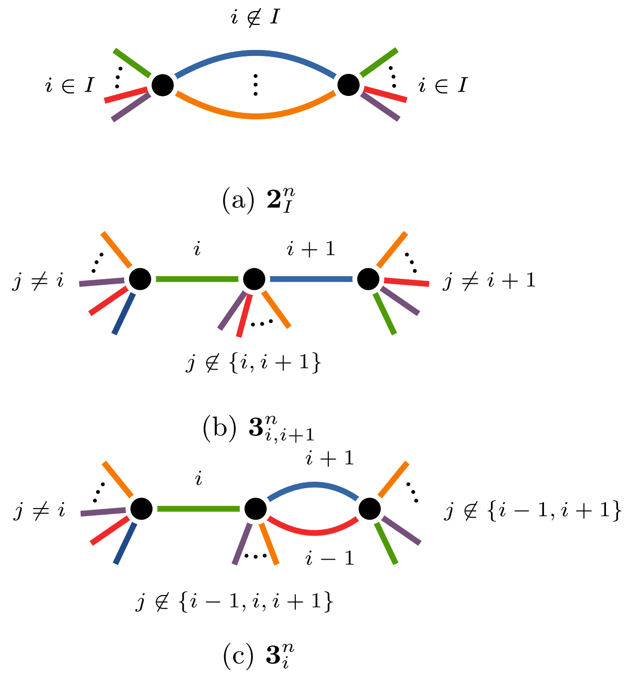
2.3 Quotients, covers, symmetry type graphs and premaniplexes

We often study quotients of maniplexes under automorphisms, and the corresponding induced covers. This naturally gives rise to the notion of *premaniplexes*. An n-premaniplex is an n-regular connected[2] graph that is properly edge colored with colors $\{0,1,\ldots,n-1\}$, such that whenever $|j-i|>1$, $\Phi^{i,j,i,j}=\Phi$. The vertices of a premaniplex are called flags, as well. Note that a premaniplex does not need to be a simple graph: there may be multiple edges, and there may be *semi-edges*, each of which is attached to just one flag.

For each $n\geqslant 0$, there is a single n-premaniplex with one flag, denoted $\mathbf{1}^n$; it has semi-edges of each color. There are 2^n-1 n-premaniplexes with two flags: for each proper subset I of $\{0,\ldots,n-1\}$, we can put semi-edges of each color $i\in I$ at both flags, while connecting them with an edge of color j for each $j\notin I$. We will denote this premaniplex by $\mathbf{2}_I^n$ (see Fig. 1a). We often omit the curly braces in practice, so that instead of $\mathbf{2}^3_{\{1,2\}}$ we would write $\mathbf{2}^3_{1,2}$.

There are $2n-3$ n-premaniplexes with three flags: for each $0\leqslant i\leqslant n-2$, the premaniplex $\mathbf{3}^n_{i,i+1}$ consists of a path with two edges, one labeled i and one labeled

[2] Though it is sometimes convenient to allow disconnected graphs, as in [10].



(a) $\mathbf{2}_I^n$

(b) $\mathbf{3}_{i,i+1}^n$

(c) $\mathbf{3}_i^n$

Fig. 1 The n-premaniplexes with 2 and 3 flags

$i+1$, and where every remaining edge is a semi-edge (Fig. 1b). For $1 \leqslant i \leqslant n-2$, the premaniplex $\mathbf{3}_i^n$ has a flag Φ connected by an i-edge to a flag Ψ, which is then connected by parallel edges of colors $i-1$ and $i+1$ to a flag Λ (see Fig. 1c).

Now let $\mathcal{M}$ be a maniplex and $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$. We define the quotient $\mathcal{M}/\Gamma$ to be the premaniplex whose flags are the orbits of flags in $\mathcal{M}$ under Γ, and where $\Phi\Gamma$ is connected by an i-edge to $\Phi^i\Gamma$. Since $(\Phi^i)\alpha = (\Phi\alpha)^i$ for every automorphism $\alpha \in \Gamma$ and every $\Phi \in \mathcal{M}$, it follows that each orbit $\Phi\Gamma$ is incident to a single edge of color i, which will be a semi-edge if Φ and Φ^i are in the same orbit.

Whenever $\mathcal{T} = \mathcal{M}/\Gamma$, we say that $\mathcal{T}$ is the *symmetry type graph of $\mathcal{M}$ with respect to* Γ. If $\Gamma = \mathrm{Aut}(\mathcal{M})$, then we simply call $\mathcal{T}$ *the symmetry type graph of $\mathcal{M}$* (see [1]).

Let $\mathcal{X}$ and $\mathcal{Y}$ be premaniplexes of rank n. A function from the flags of $\mathcal{X}$ to the flags of $\mathcal{Y}$ that preserves i-adjacency for every i is called a *homomorphism* or *covering projection*. It is easy to see that all coverings are surjective. A bijective homomorphism is an *isomorphism*. If there is a covering projection from $\mathcal{X}$ to $\mathcal{Y}$ we say that $\mathcal{X}$ *covers* $\mathcal{Y}$. The natural function from $\mathcal{M}$ to $\mathcal{M}/\Gamma$ is called a *regular covering projection*, and if $\mathcal{Y}$ is isomorphic to $\mathcal{M}/\Gamma$ we say that $\mathcal{M}$ *regularly covers* $\mathcal{Y}$.

A maniplex is said to be *in class* $\mathbf{2}_I^n$ if its symmetry type graph is $\mathbf{2}_I^n$; if the rank is clear from the context, we denote this simply by $\mathbf{2}_I$. In general, a *k-orbit maniplex* is a maniplex whose automorphism group has k orbits on the flags, or equivalently, a maniplex whose symmetry type graph has k flags.

Note that we can apply duality to premaniplexes as well as maniplexes, and if $\mathcal{M}$ has symmetry type graph $\mathcal{T}$, then $\mathcal{M}^*$ has symmetry type graph $\mathcal{T}^*$.

2.4 Orientability and chirality

A maniplex is *orientable* if it is bipartite; otherwise it is *non-orientable*. A regular maniplex $\mathcal{M}$ is orientable if and only if none of the defining relators of its automorphism group have odd length (as words in the generators ρ_i).

Suppose $\mathcal{M}$ is orientable, so that we can color the flags black and white, with adjacent flags having different colors. If $\mathcal{M}$ is not regular, but $\mathrm{Aut}(\mathcal{M})$ acts transitively on the flags of each color, then we say that $\mathcal{M}$ is *chiral*. Equivalently, a chiral maniplex is in class $\mathbf{2}^n_{\emptyset}$. A maniplex is *rotary* if it is either regular or chiral. See [19] for a comprehensive overview of chiral polytopes.

If $\mathcal{M}$ is a regular maniplex with $\mathrm{Aut}(\mathcal{M}) = \langle \rho_0, \dots, \rho_{n-1} \rangle$, the *rotation group* of $\mathcal{M}$ is

$$\mathrm{Aut}^+(\mathcal{M}) = \langle \sigma_1, \dots, \sigma_{n-1} \rangle,$$

where $\sigma_i = \rho_{i-1}\rho_i$. If $\mathrm{Aut}(\mathcal{M})$ has no relators of odd length, then $\mathrm{Aut}^+(\mathcal{M})$ has index 2 in $\mathrm{Aut}(\mathcal{M})$; otherwise $\mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$. Note that conjugation by ρ_0 inverts σ_1 and sends σ_2 to $\sigma_1^2\sigma_2$, while fixing every other σ_i with $i > 2$.

Now suppose that $\mathcal{M}$ is chiral, with base flag Φ. There are no longer automorphisms ρ_i such that $\Phi\rho_i = \Phi^i$. However, there are automorphisms σ_i such that $\Phi\sigma_i = (\Phi^i)^{i-1}$, and these generate $\mathrm{Aut}(\mathcal{M})$. The *rotation group* of $\mathcal{M}$ is defined as $\mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$. The generators of $\mathrm{Aut}(\mathcal{M})$ satisfy $(\sigma_i \cdots \sigma_j)^2 = 1$ whenever $i < j$. Unlike with regular maniplexes, there is no automorphism that inverts σ_1 and sends σ_2 to $\sigma_1^2\sigma_2$ while fixing every other σ_i.

2.5 Abstract polytopes and posets

Since maniplexes were defined to generalize abstract polytopes, let us discuss the connection briefly. A *flagged poset of rank n* is a ranked poset $\mathcal{P}$, whose elements (called *faces*) have ranks in $\{-1, 0, \dots, n\}$, such that

(i) there is a unique minimal face of rank -1 and a unique maximal face of rank n; and
(ii) every maximal chain of $\mathcal{P}$ consists of $n+2$ faces, one face in every rank.

We call a face of rank i an *i-face*. An *abstract n-polytope* is a flagged poset of rank n that satisfies two additional properties:

(i) (Thinness, also called the *diamond property*) If F is an $(i-1)$-face and G is an $(i+1)$-face with $F < G$, then there are exactly two i-faces H such that $F < H < G$, and
(ii) The poset is *strongly connected*, meaning that if $F < G$ and if F and G differ in rank by at least 3, then the Hasse diagram of the interval $(F, G) := \{H \mid F < H < G\}$ is connected.

The maximal chains of a flagged poset are called *flags*. If two flags differ only in their i-face, we say that they are *i-adjacent*. In an abstract n-polytope, if $0 \leqslant i \leqslant n-1$, then the diamond property ensures that every flag Φ has a unique i-adjacent flag Φ^i. Furthermore, note that if $|j - i| > 1$, then $(\Phi^i)^j = (\Phi^j)^i$. The *flag graph* of an abstract polytope $\mathcal{P}$ is the graph $\mathrm{fl}(\mathcal{P})$ whose vertices are the flags of $\mathcal{P}$, and where for each $0 \leqslant i \leqslant n-1$, each flag Φ is connected to Φ^i by an edge of color i. By the preceding discussion, this will necessarily be an n-maniplex. More generally, we can form the flag graph of any flagged poset; however, some flags may not have any flags that are i-adjacent, or some flags may have multiple i-adjacent flags. Thus, the flag graph of a flagged poset need not be a maniplex.

To reverse this operation, start with an n-maniplex $\mathcal{M}$. We define the i-faces to be the connected components that remain after removing all edges of color i. Then if F is an i-face and G is a j-face with $i < j$, we say $F < G$ if $F \cap G \neq \emptyset$. This defines a ranked poset $\mathrm{Pos}(\mathcal{M})$, to which we adjoin a minimal element of rank -1 and a maximal element of rank n, and the result is a flagged poset. Furthermore, if $\mathcal{M}$ is the flag graph of a polytope $\mathcal{P}$, then $\mathrm{Pos}(\mathcal{M}) \cong \mathcal{P}$. Note that we can similarly construct a flagged poset for a premaniplex.

2.6 Faithful and thin maniplexes

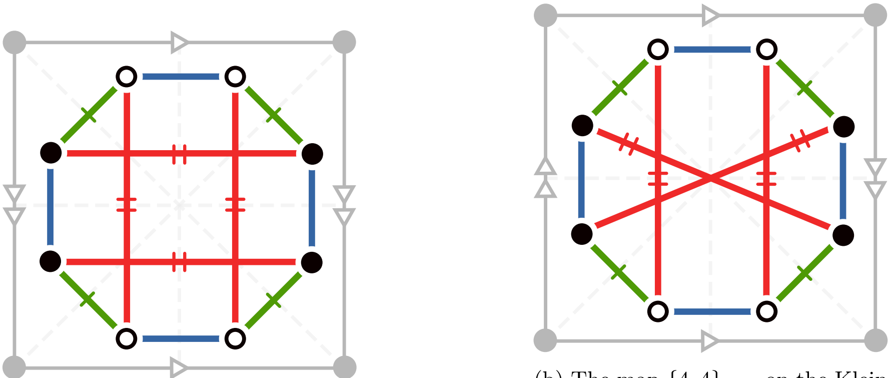
If we start with any (pre)maniplex $\mathcal{M}$, we may form the flagged poset $\mathcal{P}$ as described in the previous section. We say that $\mathcal{M}$ is *faithful* if the function that sends each flag of $\mathcal{M}$ to its corresponding maximal chain in $\mathcal{P}$ is a bijection. Otherwise the maniplex is *unfaithful*.

What goes wrong with unfaithful maniplexes is that multiple flags end up represented by the same maximal chain in the poset. In other words, there are two flags that share their i-face for every i. Furthermore, different maniplexes may give rise to the same poset. For example, take a single square and glue edges together to get the map $\{4,4\}_{(1,0)}$ on the torus (see Fig. 2a). Now, we can also glue edges of a square together as in Fig. 2b to obtain the map $\{4,4\}_{|1,1|}$ on the Klein bottle (see [21]). Then both maps $\{4,4\}_{(1,0)}$ and $\{4,4\}_{|1,1|}$ have one vertex, two edges, and one face, and thus give rise to the same poset. Both maniplexes are unfaithful, each of them with two chain equivalence classes: one consisting of the white flags and the other consisting of the black flags. For a more elaborate example in which a polytope and an unfaithful maniplex are represented by the same poset, see [12, Section 4].

As we will see, even faithful maniplexes may be represented by the same poset. A maniplex $\mathcal{M}$ is called *thin* if, for every flag Φ and every $0 \leqslant i \leqslant n-1$, the intersection of the $(i-1)$-face containing Φ with the $(i+1)$-face containing Φ intersects exactly two i-faces (here, we consider the (-1)-face containing Φ and the n-face containing Φ to be the whole maniplex $\mathcal{M}$). If $\mathcal{M}$ is a faithful maniplex that is not thin, then $\mathrm{Pos}(\mathcal{M})$ will not be thin (that is, will not satisfy the diamond property), and so we cannot recover $\mathcal{M}$ from the poset. In fact, a maniplex can be recovered from its poset (that is, $\mathcal{M}$ is naturally isomorphic to $\mathrm{fl}(\mathrm{Pos}(\mathcal{M}))$) if and only if $\mathcal{M}$ is both faithful and thin. See [14] for constructions of non-polytopal maniplexes that are both faithful and thin.

Example 2.1 Consider the regular 3-maniplex $\mathcal{M}_1$ with automorphism group

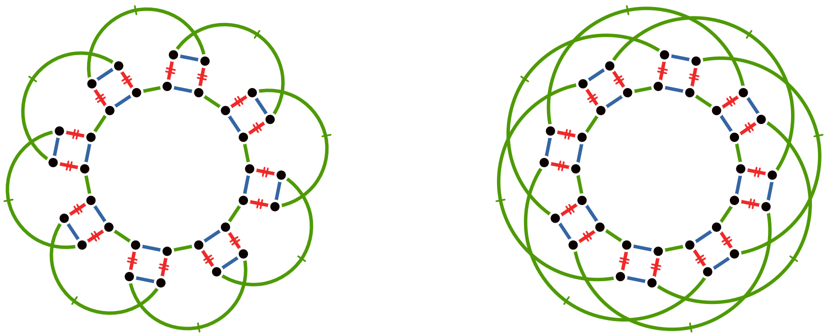
$$[8,8]/(\rho_0\rho_1\rho_2)^2,$$



(a) The map $\{4,4\}_{(1,0)}$ on the torus.

(b) The map $\{4,4\}_{|1,1|}$ on the Klein bottle.

Fig. 2 Two non-isomorphic unfaithful 3-maniplexes with the same associated poset



(a) $\mathcal{M}_1 = [8,8]/(\rho_0\rho_1\rho_2)^2$

(b) $\mathcal{M}_2 = [8,8]/(\rho_0\rho_1)^2(\rho_2\rho_1)^2$.

Fig. 3 Two non-isomorphic *faithful* but not thin 3-maniplexes with the same associated poset

and the regular 3-maniplex $\mathcal{M}_2$ with automorphism group

$$[8,8]/(\rho_0\rho_1)^2(\rho_2\rho_1)^2.$$

Then both $\mathcal{M}_1$ and $\mathcal{M}_2$ are faithful but not thin. In each one, every edge (a red-blue cycle) is incident to both vertices (the green-red cycles) and both facets (the green-blue cycles). Thus, they are represented by the same poset. See Fig. 3.

We note that a similar problem can occur if we start with a poset. That is, if $\mathcal{P}$ is a thin flagged poset, then it need not be the case that $\mathrm{Pos}(\mathrm{fl}(\mathcal{P}))$ is naturally isomorphic to $\mathcal{P}$.

Example 2.2 Consider a cube that has had two opposite vertices identified (without identifying any edges or faces), and let $\mathcal{P}$ be the flagged poset given by the incidence relation between vertices, edges, and faces. Notice that identifying these vertices does not identify any pair of flags. Then the flags in $\mathrm{fl}(\mathcal{P})$ that contain the identified vertex induce a disconnected subgraph of $\mathrm{fl}(\mathcal{P})$, and since the vertices of $\mathrm{Pos}(\mathrm{fl}(\mathcal{P}))$ correspond to connected components of $\mathrm{fl}(\mathcal{P})$ after removing the edges of color 0, we have $\mathrm{Pos}(\mathrm{fl}(\mathcal{P})) \ncong \mathcal{P}$. In fact, $\mathrm{Pos}(\mathrm{fl}(\mathcal{P})) \cong \{4, 3\}$, the ordinary cube.

Unfaithfulness in maniplexes is a counter-intuitive phenomenon; it almost looks as if it is difficult to happen in a natural way. Furthermore, we will see that some important operations on maniplexes preserve faithfulness in Sect. 6. Thus, especially for researchers interested in abstract polytopes, it is useful to distinguish faithful maniplexes from unfaithful ones.

3 Unfaithfulness in maniplexes

Consider a walk in a maniplex $\mathcal{M}$, that is, an alternating sequence of flags and edges. We will call such a walk *i-avoiding* if it does not use any edges of color i. In particular, there is an i-avoiding walk between Φ and Ψ if and only if Φ and Ψ agree in their i-face. In this case, we will write $\Phi \overset{i}{\sim} \Psi$.

Let us define an equivalence relation on the set of flags: we will say that Φ and Ψ are *chain-equivalent* if, for every i, they lie in the same i-face. We denote this by $\Phi \approx \Psi$. In other words, $\Phi \approx \Psi$ if and only if, for every i, there is an i-avoiding walk between Φ and Ψ. Thus, $\mathcal{M}$ is unfaithful if and only if there are flags $\Phi \neq \Psi$ such that $\Phi \approx \Psi$. We will write $[\Phi]$ for the chain-equivalence class of Φ.

Let us first examine one of the simplest cases of unfaithfulness.

Lemma 3.1 $\Phi \approx \Phi^i$ *if and only if* Φ *and* Φ^i *are in the same i-face.*

Proof For every $j \neq i$, the i-edge from Φ to Φ^i constitutes a j-avoiding path, so Φ and Φ^i share their j-face for each $j \neq i$. Thus, if they are in the same i-face, then $\Phi \approx \Phi^i$. □

Proposition 3.2 *Suppose $\mathcal{M}$ is a maniplex. $\mathcal{M}$ has a single i-face if and only if $\Phi^i \approx \Phi$ for every flag Φ, and in this case $\mathcal{M}$ is unfaithful.*

Proof If $\mathcal{M}$ has a single i-face, that implies that every flag $\Phi \overset{i}{\sim} \Phi^i$, and so Lemma 3.1 implies that $\Phi \approx \Phi^i$.

Conversely, suppose that $\Phi^i \approx \Phi$ for every flag Φ. If we fix a flag Φ_0 and consider its i-face, then r_i matches each flag in that i-face with another flag in that i-face. Then by connectivity, there must be exactly one i-face.

Finally, since $\Phi^i \neq \Phi$ for a maniplex, the last claim follows. □

We see that, in some sense, the most straightforward way of constructing an unfaithful $(n+1)$-maniplex $\mathcal{M}$ is to start with an n-maniplex $\mathcal{K}$ and then add a perfect matching of edges labeled n such that the induced permutation r_n commutes with $\langle r_0, \ldots, r_{n-2} \rangle$. This will produce a maniplex with a single facet, which by Proposition 3.2 is therefore unfaithful.

Let us generalize Proposition 3.2 to highlight an important difference between maniplexes and premaniplexes.

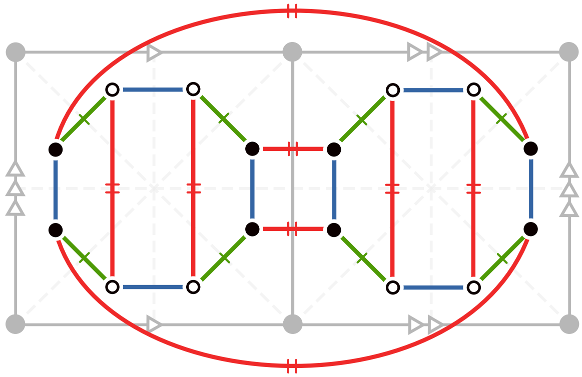


Fig. 4 The maniplex $\{4,4\}_{(2,0),(0,1)}$. Each flag Φ along the top and bottom (white flags) is chain-equivalent to Φ^2, while each of the other flags is chain-equivalent to Φ^0

Proposition 3.3 *Suppose $\mathcal{M}$ is a premaniplex. $\mathcal{M}$ has a single i-face if and only if $\Phi^i \approx \Phi$ for every flag Φ, and in this case $\mathcal{M}$ is unfaithful unless $\Phi^i = \Phi$ for every flag Φ.*

Proof The proof of the first part is the same as the proof of Proposition 3.2, and if there is any Φ such that $\Phi^i \neq \Phi$, then $\{\Phi, \Phi^i\}$ is a witness that $\mathcal{M}$ is unfaithful. □

For example, the premaniplex $\mathbf{1}^n$ has a single i-face for each i, but is nevertheless faithful.

In rank 3, there are not many ways to be unfaithful:

Proposition 3.4 ([12, Lemma 3.1]) *A 3-maniplex $\mathcal{M}$ is unfaithful if and only if there is a flag Φ such that either $\Phi \approx \Phi^0$ or $\Phi \approx \Phi^2$.*

Proposition 3.4 does not necessarily imply that $\mathcal{M}$ has a single face or vertex; for example, consider the maniplex $\{4,4\}_{(2,0),(0,1)}$ on the torus, made of a 1×2 grid of squares (see Fig. 4). In this maniplex, some flags Φ satisfy $\Phi \approx \Phi^0$ while other flags Ψ satisfy $\Psi \approx \Psi^2$. In other words, some edges are loops, and some edges separate a face from itself.

Having analyzed the most straightforward way that two flags can be chain-equivalent (where $\Phi \approx \Phi^i$), let us consider what happens when $\Phi \approx \Phi^w$ for other 'simple' connections $w \in \mathrm{Conn}(\mathcal{M})$.

Proposition 3.5 *Let I and J be disjoint subsets of $\{0, \ldots, n-1\}$. Suppose that $\Phi \approx \Phi^{uv}$ with $u \in \langle r_i \mid i \in I\rangle$ and $v \in \langle r_j \mid j \in J\rangle$. Then $\Phi \approx \Phi^u$.*

Proof Let $k \in \{0, \ldots, n-1\}$. If $k \notin J$, then $\Phi^u \overset{k}{\sim} \Phi^{uv}$, which implies (by hypothesis) that $\Phi^u \overset{k}{\sim} \Phi$. If $k \notin I$, then $\Phi^u \overset{k}{\sim} \Phi$. The result follows from the fact that I and J are disjoint. □

Corollary 3.6 *If $\Phi \approx \Phi^{i,j}$ for some $i \neq j$, then $\Phi \approx \Phi^i$.*

Proposition 3.5 lets us generalize the argument of [12, Lemma 3.1] to higher ranks.

Proposition 3.7 *Suppose $\mathcal{M}$ is an unfaithful n-maniplex. Then for each $1 \leqslant i \leqslant n-2$, there are flags $\Phi \approx \Psi$ such that either $\Phi \overset{[0, i-1]}{\text{———}} \Psi$ or $\Phi \overset{[i+1, n-1]}{\text{———}} \Psi$.*

Proof If $\mathcal{M}$ is unfaithful, then there are flags Φ and $\Phi' \neq \Phi$ such that $\Phi \approx \Phi'$. In particular, Φ and Φ' are in the same i-face, so there is an i-avoiding walk between them. Since $\langle r_0, \dots, r_{i-1} \rangle$ commutes with $\langle r_{i+1}, \dots, r_{n-1} \rangle$, that means that $\Phi' = \Phi^{uv} = \Phi^{vu}$, where $u \in \langle r_0, \dots, r_{i-1} \rangle$ and $v \in \langle r_{i+1}, \dots, r_{n-1} \rangle$. Then by Proposition 3.5, $\Phi \approx \Phi^u$ and $\Phi \approx \Phi^v$. It could happen that $\Phi^u = \Phi$ or $\Phi^v = \Phi$, but since $\Phi^{uv} \neq \Phi$, at least one of the flags in $\{\Phi^u, \Phi^v\}$ must be distinct from Φ, and this flag is our desired Ψ. □

Proposition 3.8 *Let $\mathcal{M}$ be a 3-maniplex and Φ be a flag in $\mathcal{M}$ such that $\Phi \approx \Phi^1$. Then $\Phi^1 = \Phi^{0,2}$, and $[\Phi] = \{\Phi, \Phi^0, \Phi^2, \Phi^{0,2}\}$.*

Proof If $\Phi \approx \Phi^1$, then $\Phi^1 \overset{1}{\sim} \Phi$, meaning that $\Phi^1 \in \{\Phi, \Phi^0, \Phi^2, \Phi^{0,2}\}$. Since a maniplex has no semi-edges or multiple edges, we must have $\Phi^1 = \Phi^{0,2}$. So $\Phi \approx \Phi^{0,2} = \Phi^{2,0}$, and then Corollary 3.6 proves that $\Phi \approx \Phi^0 \approx \Phi^2$. □

Note that the chain-equivalence class in Proposition 3.8 is as large as possible, since every flag that is chain-equivalent to Φ must agree with Φ in its edge, which is precisely $\{\Phi, \Phi^0, \Phi^2, \Phi^{0,2}\}$.

Now let us see how chain-equivalence of some pairs of flags induces chain-equivalence of some closely-related pairs.

Proposition 3.9 *If Φ and Ψ are flags of an n-maniplex with $\Phi \approx \Psi$, then $\Phi^0 \approx \Psi^0$ and $\Phi^{n-1} \approx \Psi^{n-1}$.*

Proof Clearly $\Phi^0 \overset{i}{\sim} \Psi^0$ for $i \neq 0$. In particular, $\Phi^0 \overset{1}{\sim} \Psi^0$, so there is a 1-avoiding path from Φ^0 to Ψ^0. Now, given that the action of r_0 commutes with the action of $\langle r_2, \dots, r_{n-1} \rangle$, it follows that $\Psi^0 = (\Phi^0)^{r_0^{\varepsilon} w}$, with $\varepsilon \in \{0, 1\}$ and $w \in \langle r_2, \dots, r_{n-1} \rangle$. If $\varepsilon = 0$, then w yields a 0-avoiding path from Φ^0 to Ψ^0, and so $\Phi^0 \approx \Psi^0$. Otherwise, w yields a 0-avoiding path from Φ to Ψ^0, so that $\Phi \overset{0}{\sim} \Psi^0$. Since $\Phi \approx \Psi$, it follows that $\Psi^0 \overset{0}{\sim} \Psi$, so $\Psi^0 \approx \Psi$. By a similar argument but interchanging the roles of Φ and Ψ, we see that $\Phi \approx \Phi^0$. It follows that $\Phi^0 \approx \Psi^0$. The fact that $\Phi^{n-1} \approx \Psi^{n-1}$ follows by duality. □

Example 3.10 It is not true in general that if $\Phi \approx \Psi$ then $\Phi^i \approx \Psi^i$. For example, consider the regular 3-maniplex with group

$$[8, 8]/((\rho_0\rho_1)^4\rho_0\rho_2).$$

In this maniplex, the chain-equivalence class of each flag Φ is as large as possible, consisting of $\{\Phi, \Phi^0, \Phi^2, \Phi^{0,2}\}$. However, $\Phi^1 \not\approx \Phi^{0,1}$. Indeed, if it were true that $\Phi \approx \Psi$ implied $\Phi^1 \approx \Psi^1$, then $\{\Phi^1, \Phi^{0,1}, \Phi^{2,1}, \Phi^{0,2,1}\}$ would need to be a chain-equivalence class, implying that these flags are the $\langle r_0, r_2 \rangle$-orbit of Φ^1. Then by connectivity, such a maniplex can only have at most 8 flags, whereas the maniplex above has 16.

Let us briefly explore unfaithfulness in premaniplexes in more detail.

Proposition 3.11 *If Φ and Ψ are connected by multiple edges in a premaniplex, then $\Phi \approx \Psi$.*

Proof If, say, $\Psi = \Phi^i = \Phi^j$ with $i \neq j$, then Ψ is in the same i-face as Φ, and Lemma 3.1 shows that $\Phi \approx \Psi$. □

Proposition 3.12 *If $\mathcal{M}$ is a premaniplex that is a caterpillar (a path with possible semi-edges), then $\mathcal{M}$ is faithful.*

Proof Given any two distinct flags Φ and Ψ in a caterpillar, there is exactly one path between them that does not use any semi-edges, and that path cannot be i-avoiding for every i. □

Together, Proposition 3.11 and Proposition 3.12 imply the following.

Proposition 3.13 *Let $n \geqslant 3$. The following statements hold:*

(i) For $I \subseteq \{0, \dots, n-1\}$, the premaniplex $\mathbf{2}_I^n$ is faithful if and only if $|I| = n-1$.
(ii) For $0 \leqslant i \leqslant n-1$, the premaniplex $\mathbf{3}_{i,i+1}^n$ is faithful.
(iii) For $1 \leqslant i \leqslant n-2$, the premaniplex $\mathbf{3}_i^n$ is unfaithful.

The relationship between chain-equivalence and covers is simple:

Proposition 3.14 *If $\pi : \mathcal{M} \to \mathcal{M}'$ is a covering of premaniplexes, and $\Phi \approx \Psi$ in $\mathcal{M}$, then $\pi(\Phi) \approx \pi(\Psi)$.*

Proof This follows directly from the fact that an i-avoiding path in $\mathcal{M}$ induces an i-avoiding path in $\mathcal{M}'$. □

Of course, it may happen that $\Phi \approx \Psi \neq \Phi$ but $\pi(\Phi) = \pi(\Psi)$. Indeed, sometimes this is forced to happen:

Corollary 3.15 *If $\Phi \approx \Psi$ and the symmetry type graph of $\mathcal{M}$ with respect to Γ is faithful, then Φ and Ψ are in the same Γ-orbit.*

Now, let us study the relationship between unfaithfulness of $\mathcal{M}$ and the unfaithfulness of its facets.

Proposition 3.16 *Suppose that $\Phi \approx \Psi$ in an n-maniplex $\mathcal{K}$. If $\mathcal{M}$ is an $(n+1)$-maniplex with a facet isomorphic to $\mathcal{K}$, then $\mathcal{M}$ has induced flags Φ' and Ψ' with $\Phi' \approx \Psi'$. In particular, if $\mathcal{K}$ is unfaithful, then so is $\mathcal{M}$.*

Proof The induced flags are connected by an i-avoiding path for each $0 \leqslant i \leqslant n-1$, and since these paths are induced by paths in $\mathcal{K}$, they are also n-avoiding. □

Corollary 3.17 *Each chain-equivalence class of a maniplex is the union of chain-equivalence classes of its facets.*

Naturally, $\mathcal{M}$ may have larger chain-equivalence classes than its facets. For example, if we build a 1-facet extension of an n-maniplex by picking a matching r_n, then we will have $\Phi \approx \Phi^n$ for each flag. Thus, if we match Φ to a flag that it was not already equivalent to, then we have enlarged the chain-equivalence class. Furthermore, this matching may have additional consequences, so it need not even be true that the chain-equivalence classes only double in size.

Example 3.18 Let F be the faithful regular maniplex (in fact, polytope) $\{2, 2\}$. Then if we add a matching r_3 such that for each flag, $\Phi^3 = \Phi^{0,2}$, the result is an unfaithful maniplex where each chain-equivalence class has size 4.

Finally, we briefly explore maniplexes with small chain-equivalence classes.

Definition 3.19 A maniplex $\mathcal{M}$ is *almost faithful* if it is unfaithful and if every chain-equivalence class has size 1 or 2.

The following result follows immediately from Corollary 3.17

Proposition 3.20 *If $\mathcal{M}$ is an almost faithful maniplex, then its facets are either faithful or almost faithful.*

Proposition 3.21 *Suppose that $\mathcal{M}$ is an almost faithful n-maniplex with one facet F and that F is almost faithful. If $\Phi \overset{[0, n-2]}{\text{———}} \Psi \neq \Phi$ (that is, Ψ and Φ induce flag equivalent flags of F), then $\Psi = \Phi^{n-1}$ in $\mathcal{M}$.*

Proof On the one hand, we have $\Phi \overset{[0, n-2]}{\text{———}} \Psi$ from F, and on the other, $\Phi \approx \Phi^{n-1}$ by Proposition 3.2. Since the chain-equivalence classes have size at most 2 and $\Psi \neq \Phi$, the result follows. □

Proposition 3.22 *If $\mathcal{M}$ is a maniplex with a unique i-face and a unique j-face for $i \neq j$, then $\mathcal{M}$ is not almost faithful.*

Proof By Proposition 3.2, having one i-face and one j-face means that $\Phi \approx \Phi^i \approx \Phi^j$, and these flags are distinct in a maniplex, giving a chain-equivalence class of size at least three. □

4 Unfaithfulness and symmetries

4.1 Chain-equivalence classes

We now explore the interactions between unfaithfulness and symmetry. The first result is an immediate consequence of the fact that automorphisms commute with connections.

Proposition 4.1 *If $\Phi \approx \Psi$ and $\alpha \in \mathrm{Aut}(\mathcal{M})$, then $\Phi\alpha \approx \Psi\alpha$.*

Proposition 4.1 implies that the chain-equivalence classes of flags form a partition that is invariant under $\mathrm{Aut}(\mathcal{M})$.

The automorphisms α such that $\Phi \approx \Phi\alpha$ can be easily characterized:

Theorem 4.2 *Let $\mathcal{M}$ be an n-maniplex and let $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$. Fix a flag Φ, and for each i, let Γ_i be the stabilizer in Γ of the i-face of Φ. If $\alpha \in \Gamma$, then $\Phi \approx \Phi\alpha$ if and only if $\alpha \in \Gamma_0 \cap \cdots \cap \Gamma_{n-1}$.*

Proof If $\alpha \in \Gamma_0 \cap \cdots \cap \Gamma_{n-1}$ so that α fixes the i-face of Φ for every i, then clearly $\Phi\alpha \approx \Phi$. Conversely, if $\Phi \approx \Phi\alpha$, then $\Phi\alpha \overset{i}{\sim} \Phi$ for every i, and so $\alpha \in \Gamma_i$ for every i. $\square$

Corollary 4.3 *Every chain-equivalence class of a regular n-maniplex $\mathcal{M}$ has size equal to $|\Gamma_0 \cap \cdots \cap \Gamma_{n-1}|$, where Γ_i is the stabilizer of the base i-face. In particular, $\mathcal{M}$ is unfaithful if and only if $\Gamma_0 \cap \cdots \cap \Gamma_{n-1}$ is nontrivial.*

The group $\Gamma_0 \cap \cdots \cap \Gamma_{n-1}$ plays an important role, so let us give it a name.

Definition 4.4 The *chain-stabilizer of Φ in Γ*, denoted $\Gamma_{[\Phi]}$ is the group $\Gamma_0 \cap \cdots \cap \Gamma_{n-1}$, where Γ_i is the stabilizer of the i-face of Φ in Γ.

We also state the following definitions to make it easier to distinguish between chain-equivalent flags that are in the same Γ-orbit from those that are not.

Definition 4.5 A maniplex $\mathcal{M}$ is *monochromatically unfaithful (with respect to Γ)* if there are a flag Φ and a non-trivial automorphism $\alpha \in \mathrm{Aut}(\mathcal{M})$ (resp. in Γ) such that $\Phi \approx \Phi\alpha$.

We will say that $\mathcal{M}$ is *polychromatically unfaithful (with respect to Γ)* if there are flags Φ and Ψ in different $\mathrm{Aut}(\mathcal{M})$-orbits (resp. different Γ-orbits) such that $\Phi \approx \Psi$.

If we color the flags of a maniplex according to their orbits, a maniplex is monochromatically unfaithful if there are two chain-equivalent flags of the same color, and polychromatically unfaithful if there are two chain-equivalent flags of different colors. Notice that a maniplex may be both monochromatically and polychromatically unfaithful at the same time.

By Theorem 4.2, $[\Phi]$ contains $\Phi\Gamma_{[\Phi]}$. Let us now explore those flags in $[\Phi]$ that are not in the same $\Gamma_{[\Phi]}$-orbit.

Theorem 4.6 *Let $\mathcal{M}$ be an n-maniplex and let $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$. Fix a flag Φ, suppose $\Psi \approx \Phi$, and let $\varphi \in \Gamma$. Then $\Psi\varphi \approx \Phi$ if and only if $\varphi \in \Gamma_{[\Phi]}$.*

Proof Since $\Psi \approx \Phi$, Proposition 4.1 implies that $\Psi\varphi \approx \Phi\varphi$. Since $\Phi\varphi \approx \Phi$ if and only if $\varphi \in \Gamma_{[\Phi]}$, the result follows. $\square$

Theorem 4.6 implies that the chain-equivalence classes are unions of $\Gamma_{[\Phi]}$-orbits, with at most one $\Gamma_{[\Phi]}$-orbit per Γ-orbit. Then Theorem 4.6 directly implies the following.

Corollary 4.7 *Let $\mathcal{M}$ be an n-maniplex, and let $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$. Fix a flag Φ and let Ω be a maximal set of flags with at most one flag per Γ-orbit and such that every flag in Ω is chain-equivalent to Φ. Then $[\Phi] = \Omega\Gamma_{[\Phi]}$.*

Corollary 4.8 *If $\mathcal{M}$ is an unfaithful k-orbit maniplex that is not monochromatically unfaithful, then the chain-equivalence classes consist of at most k flags.*

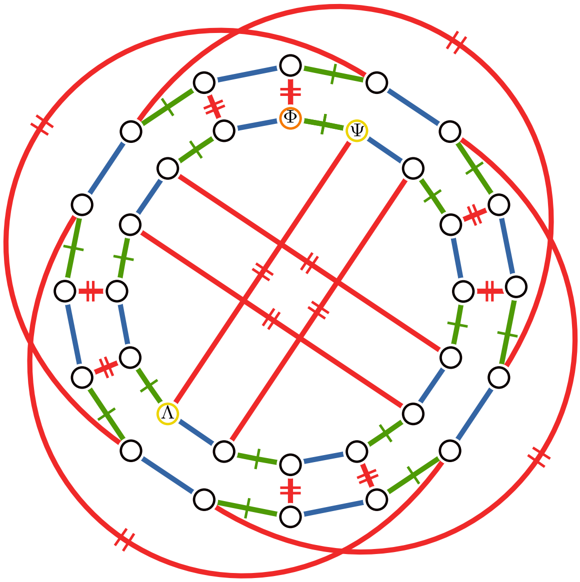


Fig. 5 A 3-maniplex in class $\mathbf{2}_{0,2}$ with chain-equivalence classes of size 1 and 2. See Example 4.10

Note that the subgroup $\Gamma_{[\Phi]}$ may have a different size with respect to different choices of Φ. Similarly, the size of Ω can be different for different choices of Φ. Let us explore the possibilities for two-orbit maniplexes.

Proposition 4.9 *Suppose $\mathcal{M}$ is a two-orbit maniplex in class $\mathbf{2}_I^n$ with $0 \notin I$ or $n-1 \notin I$. Then the chain-equivalence classes of $\mathcal{M}$ have the same size.*

Proof Proposition 4.1 already implies that flags in the same orbit have the same size of chain-equivalence class. If $0 \notin I$, then Φ^0 and Φ are in different orbits, and Proposition 3.9 implies that the base flag Φ and Φ^0 have the same size chain-equivalence classes; the argument is analogous when $n-1 \notin I$. □

Example 4.10 It is not true in general that the chain-equivalence classes have the same size, even for other highly-symmetric maniplexes. Among two-orbit 3-maniplexes, the sizes can only be different for class $\mathbf{2}_{0,2}^3$. In Fig. 5 we show an example of a 3-maniplex in class $\mathbf{2}_{0,2}$ The flag Φ has a chain-equivalence class of size 1, while flag Ψ is chain-equivalent to flag Λ.

Intuitively, this maniplex is obtained by taking two octagons, gluing alternating edges from one octagon to the other, and then gluing the remaining edges to their opposites within each octagon. In this way, half of the flags are 2-adjacent to a flag in the same face. We will generalize this example in Sect. 5.

4.2 Unfaithfulness in *k*-orbit maniplexes

Let us describe in more detail those flags that are chain-equivalent to Φ but in a different Γ-orbit. We start with maniplexes in class 2_I^n. Note that by Proposition 3.13 and Corollary 3.15, this is only possible if $|I| < n-1$.

Proposition 4.11 *Suppose $\mathcal{M}$ is a two-orbit maniplex in class $\mathbf{2}_I$ with $i, j \notin I$ and $i \neq j$. Let $\Gamma = \mathrm{Aut}(\mathcal{M})$ and let the base flag be Φ. Then $\Phi \approx \Phi^i\alpha$ if and only if*

$$\alpha \in \Gamma_0 \cap \cdots \cap \Gamma_{i-1} \cap \tau_{ji}\Gamma_i \cap \Gamma_{i+1} \cap \cdots \cap \Gamma_{n-1},$$

where τ_{ji} is the automorphism such that $\Phi\tau_{ji} = \Phi^{j,i}$.

Proof Set $\Psi = \Phi^i\alpha = (\Phi\alpha)^i$ and suppose $\Phi \approx \Psi$. Now, $\Psi^i \overset{k}{\sim} \Psi$ (and thus $\Psi^i \overset{k}{\sim} \Phi$) for every $k \neq i$, so

$$\alpha \in \Gamma_0 \cap \cdots \cap \Gamma_{i-1} \cap \Gamma_{i+1} \cap \cdots \cap \Gamma_{n-1}.$$

Now consider Ψ^j, and let τ_{ij} be the automorphism such that $\Phi\tau_{ij} = \Phi^{i,j}$. Then

$$\Psi^j = (\Psi^i)^{i,j} = (\Phi\alpha)^{i,j} = (\Phi^{i,j})\alpha = \Phi\tau_{ij}\alpha.$$

Now, since $j \neq i$, $\Psi^j \overset{i}{\sim} \Psi \overset{i}{\sim} \Phi$, so $\tau_{ij}\alpha \in \Gamma_i$. This means that $\alpha \in \tau_{ij}^{-1}\Gamma_i = \tau_{ji}\Gamma_i$. Putting everything together shows that α lies in the desired intersection. Conversely, for every α in the intersection and $k \neq i$,

$$\Phi \overset{k}{\sim} \Phi\alpha \overset{k}{\sim} (\Phi\alpha)^i = \Phi^i\alpha,$$

while

$$\Phi \overset{i}{\sim} \Phi\tau_{ij}\alpha = \Phi^{i,j}\alpha = (\Phi^i\alpha)^j \overset{i}{\sim} \Phi^i\alpha.$$

Therefore $(\Phi\alpha)^i \approx \Phi$. □

Corollary 4.12 *Let $\mathcal{M}$ be an orientable rotary n-maniplex with base flag Φ and with $\Gamma = \mathrm{Aut}^+(\mathcal{M})$. Then $\mathcal{M}$ is faithful if and only if $\Gamma_{[\Phi]}$ is trivial and*

$$\sigma_1\Gamma_0 \cap \Gamma_1 \cap \cdots \cap \Gamma_{n-1} = \emptyset.$$

Proof Recall that σ_1 was defined for rotary maniplexes in Sect. 2.4. If $\mathcal{M}$ is chiral, then the result follows directly from Theorem 4.2 and Proposition 4.11. Otherwise, if $\mathcal{M}$ is regular and orientable, then the argument of Proposition 4.11 works but taking $\Gamma = \mathrm{Aut}^+(\mathcal{M})$ instead. □

Note that Corollary 4.12 uses $i = 0$ and $j = 1$ in Proposition 4.11; other choices are also possible.

Corollary 4.13 *Suppose that $\mathcal{M}$ is an orientable rotary n-maniplex with rotation group $\Gamma = \langle \sigma_1, \ldots, \sigma_{n-1} \rangle$. If $\langle \sigma_i \rangle \cap \langle \sigma_j \rangle$ is nontrivial for some i and j with $|j - i| \geqslant 2$, then $\mathcal{M}$ is unfaithful.*

Proof Suppose that the intersection has nontrivial element $\sigma_i^a = \sigma_j^b$. Note that in general, σ_i^a is contained in every Γ_k with $k \notin \{i-1, i\}$. Indeed, the only generators not present in Γ_{i-1} are σ_{i-1} and σ_i, and the only generators not present in Γ_i are σ_i and σ_{i+1}. Since $|j-i| \geqslant 2$, it follows that $\sigma_i^a = \sigma_j^b \in \Gamma_{i-1} \cap \Gamma_i$, and so $\sigma_i^a \in \Gamma_{[\Phi]}$ (where Φ is the base flag) and thus $\mathcal{M}$ is unfaithful by Corollary 4.12. $\square$

Proposition 4.14 *Let $\mathcal{M}$ be a rotary n-maniplex, with $\Gamma = \mathrm{Aut}^+(\mathcal{M})$, and let $S = \sigma_1\Gamma_0 \cap \Gamma_1 \cap \cdots \cap \Gamma_{n-1}$. Then $\mathcal{M}$ has one facet if and only if $\sigma_1 \cdots \sigma_{n-1} \in S$, and $\mathcal{M}$ has one vertex if and only if $1 \in S$.*

Proof If $\sigma_1 \cdots \sigma_{n-1} \in S$, then $\sigma_1 \cdots \sigma_{n-1} \in \Gamma_{n-1} = \langle \sigma_1, \ldots, \sigma_{n-2} \rangle$, and it follows that $\sigma_{n-1} \in \Gamma_{n-1}$ and so $\mathcal{M}$ has one facet. Similarly, if $1 \in S$, then $1 \in \sigma_1\Gamma_0$ which implies that $\sigma_1 \in \Gamma_0 = \langle \sigma_2, \ldots, \sigma_{n-1} \rangle$, and so $\mathcal{M}$ has one vertex.

Conversely, suppose that $\mathcal{M}$ has base flag Φ and has one vertex. Then $\Phi \approx \Phi^0$ by Proposition 3.2. Then taking $\alpha = 1$ (that is, the identity), and using $i = 0$ and $j = 1$ in Proposition 4.11, we get that $1 \in S$. If instead, $\mathcal{M}$ has one facet, then $\Phi \approx \Phi^{n-1}$, again by Proposition 3.2. Note that $\Phi^{n-1} = (\Phi^0)\sigma_1 \cdots \sigma_{n-1}$, and so again applying Proposition 4.11 with $i = 0$, $j = 1$, and $\alpha = \sigma_1 \cdots \sigma_{n-1}$, we see that $\sigma_1 \cdots \sigma_{n-1} \in S$. $\square$

Proposition 4.11 can be generalized for arbitrary symmetry type graphs as follows:

Theorem 4.15 *Let $\mathcal{M}$ be a maniplex, $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$ and let $\mathcal{X} = \mathcal{M}/\Gamma$. Let x and y be flag-orbits of $\mathcal{M}$ (flags of $\mathcal{X}$) and let Φ_x and Φ_y be flags of $\mathcal{M}$ in the fibers of x and y, respectively. For every $i \in \{0, 1, \ldots, n-1\}$, let $\tau_i \in \Gamma$ be such that $\Phi_y\tau_i \overset{i}{\sim} \Phi_x$. Then, $\Phi_x \approx \Phi_y\alpha$ if and only if $\alpha \in \bigcap_{i=0}^{n-1} \tau_i\Gamma_i$ where Γ_i is the stabilizer of the i-face of Φ_x under the action of Γ.*

Proof Let us assume that $\Phi_x \approx \Phi_y\alpha$. By assumption, for every i we have $\Phi_x \overset{i}{\sim} \Phi_y\tau_i$. Applying $\tau_i^{-1}\alpha$ we get that $\Phi_x\tau_i^{-1}\alpha \overset{i}{\sim} \Phi_y\alpha$, which itself satisfies $\Phi_y\alpha \overset{i}{\sim} \Phi_x$. Thus, $\tau_i^{-1}\alpha$ fixes the i-face of Φ_x, which is to say that $\alpha \in \tau_i\Gamma_i$. Since this is true for every i, $\alpha \in \bigcap_{i=0}^{n-1} \tau_i\Gamma_i$.

On the other hand, if $\alpha \in \bigcap_{i=0}^{n-1} \tau_i\Gamma_i$ we get that $\tau_i^{-1}\alpha$ fixes the i-face of Φ_x, and since $\Phi_y\tau_i$ has that i-face, so does $\Phi_y\alpha = \Phi_y\tau_i(\tau_i^{-1}\alpha)$. $\square$

If $x \approx y$ in $\mathcal{X}$, there is an i-avoiding path W_i from x to y in $\mathcal{X}$. The lift of this path (the path that follows the same colors) starting at Φ_x must end in a flag in the same orbit as Φ_y and with the same i-face as Φ_x. Therefore we can find the τ_i in the hypothesis of Theorem 4.15. On the other hand, if $x \not\approx y$, because of Proposition 3.14 it is not possible that $\Phi_x \approx \Phi_y\alpha$ for any $\alpha \in \Gamma$.

Corollary 4.16 *Let $\mathcal{M}$ be a maniplex, $\Gamma \leqslant \mathrm{Aut}(\mathcal{M})$ and let $\mathcal{X} = \mathcal{M}/\Gamma$. For every flag x of $\mathcal{X}$ let Φ_x be a base flag in that orbit and let Γ_i^x be the stabilizer of the i-face of Φ_x. For every pair of distinct flags x and y in $\mathcal{X}$ such that $x \approx y$ in $\mathcal{X}$, let $\tau_i^{x,y}$ be such that $\Phi_y\tau_i^{x,y} \overset{i}{\sim} \Phi_x$. Then $\mathcal{M}$ is faithful if and only if $\bigcap_{i=0}^{n-1} \Gamma_i^x = \{1\}$ for all x and $\bigcap_{i=0}^{n-1} \tau_i^{x,y}\Gamma_i^x = \emptyset$ when $x \approx y$ but $x \neq y$.*

5 Classification of unfaithful two-orbit 3-maniplexes

Before considering unfaithful two-orbit 3-maniplexes, let us classify the unfaithful regular 3-maniplexes. Up to duality, such a maniplex must have one facet, by Proposition 3.4 (since if $\Phi \approx \Phi^2$ for one flag of a regular maniplex, then $\Psi \approx \Psi^2$ for every flag Ψ). Thus, $\rho_2 \in \langle \rho_0, \rho_1 \rangle$. Since ρ_2 commutes with ρ_0, the only possibilities are

(i) $\rho_2 = \rho_0$ (which does not yield a maniplex, since there would be multiple edges),
(ii) $\rho_2 = (\rho_0\rho_1)^k$ if the order of $\rho_0\rho_1$ is $2k$, and
(iii) $\rho_2 = (\rho_0\rho_1)^k\rho_0$ if the order of $\rho_0\rho_1$ is $2k$.

These are precisely the one-facet maps described in [20]. We will use $\mathbb{P}\{k\}$ to denote the regular map (called δ_k in [20]) with group

$$[2k, 2]/(\rho_0\rho_1)^k\rho_2,$$

and $\mathbb{T}\{k\}$ to denote the regular map (called M_k in [20]) with group

$$[2k, k]/(\rho_0\rho_1)^k\rho_0\rho_2$$

if k is odd and

$$[2k, 2k]/(\rho_0\rho_1)^k\rho_0\rho_2$$

if k is even. Thus, we have proven:

Proposition 5.1 *An unfaithful regular 3-maniplex is $\mathbb{T}\{k\}$, $\mathbb{P}\{k\}$, or the dual of one of these.*

Note that $\mathbb{P}\{k\}$ is the embedding of a $2k$-gon in the projective plane, and $\mathbb{T}\{k\}$ is the embedding of a $2k$-gon in an orientable surface of genus $\lfloor k/2 \rfloor$.

We now move on to two-orbit 3-maniplexes. Let us start by showing that for most classes $\mathbf{2}_I$, the unfaithful 3-maniplexes in those classes also have one facet or one vertex.

Proposition 5.2 *If $\mathcal{M}$ is an unfaithful two-orbit 3-maniplex in class $\mathbf{2}_I$ and $I \neq \{0, 2\}$, then $\mathcal{M}$ has one facet or one vertex.*

Proof Up to duality, we may assume that there is a flag Φ such that $\Phi \approx \Phi^0$. It then suffices to show that there is a flag Ψ in the orbit opposite to Φ such that $\Psi \approx \Psi^0$; then Proposition 4.1 will imply that every flag Λ satisfies $\Lambda \approx \Lambda^0$, and then Proposition 3.2 will show that there is a single vertex. If $0 \notin I$, then Φ^0 is in the opposite orbit of Φ, and $\Phi^0 \approx (\Phi^0)^0 = \Phi$. If $2 \notin I$, then Φ^2 is in the opposite orbit of Φ, and then Proposition 3.9 implies that $\Phi^2 \approx \Phi^{0,2} = \Phi^{2,0}$. Thus, $\mathcal{M}$ has one vertex (or one facet) unless $\{0, 2\} \subseteq I$ which implies $I = \{0, 2\}$. □

Note that the example $\{4, 4\}_{(2,0),(0,1)}$ from before (see Fig. 4) is in fact in class $\mathbf{2}_{0,2}$ and has two facets and two vertices.

Before we continue, let us define another common operation on 3-maniplexes. The *Petrie dual* of a 3-maniplex $\mathcal{M}$, denoted $\mathcal{M}^\pi$, is the 3-premaniplex whose flags are the flags of $\mathcal{M}$ and where i-adjacency is as follows:

(i) if $i \in \{1, 2\}$, then two flags are i-adjacent in $\mathcal{M}^\pi$ if and only if they are i-adjacent in $\mathcal{M}$.
(ii) if $i = 0$ then two flags are i-adjacent in $\mathcal{M}^\pi$ if and only if they are joined by a 02-path in $\mathcal{M}$.

That is, $\mathcal{M}^\pi$ is obtained from $\mathcal{M}$ by deleting all the edges of color 0, and drawing a new edge of color 0 between each Φ and $\Phi^{0,2}$. This operation is involutory (that is, $(\mathcal{M}^\pi)^\pi = \mathcal{M}$), and it produces a bona fide maniplex unless there is some flag Φ such that $\Phi^{0,2} = \Phi^1$. We define the *opposite* of $\mathcal{M}$ to be $((\mathcal{M}^*)^\pi)^*$. See [22, Section 8] for more information about the Petrie dual.

Just as with regular duality, we can apply Petrie duality to premaniplexes, and if $\mathcal{M}$ has symmetry type graph $\mathcal{T}$, then $\mathcal{M}^\pi$ has symmetry type graph $\mathcal{T}^\pi$ (see [10, Thm. 6.1]).

Now let us show that several classes of two-orbit 3-maniplexes do not admit any unfaithful examples.

Proposition 5.3 *There are no unfaithful chiral* 3*-maniplexes.*

Proof Up to duality, a rotary unfaithful 3-maniplex has one facet (by Proposition 5.2), so $\sigma_2 \in \langle \sigma_1 \rangle$. Thus $\langle \sigma_1, \sigma_2 \rangle = \langle \sigma_1 \rangle$ is cyclic, with a group automorphism that maps σ_1 to σ_1^{-1} and thus σ_2 to σ_2^{-1}, proving that it is regular. □

Proposition 5.4 *There are no unfaithful* 3*-maniplexes in classes* $\mathbf{2}_0$ *or* $\mathbf{2}_2$.

Proof Up to duality, it suffices to prove the claim for class $\mathbf{2}_0$. The automorphism group Γ of a maniplex $\mathcal{M}$ in class $\mathbf{2}_0$ is generated by an involution ρ_0 and an element σ_2 satisfying that $\Phi\rho_0 = \Phi^0$ and $\Phi\sigma_2 = \Phi^{21}$ for some base flag Φ (see [13, Page 951]). If $\mathcal{M}$ is unfaithful, then by Proposition 5.2, there is either one single vertex or one single facet. If there is one single vertex, then the vertex-stabilizer Γ_0 is the whole group Γ. Then $\rho_0 \in \Gamma_0 = \langle \sigma_2 \rangle$, and there is an automorphism that inverts σ_2 and thus inverts (therefore fixes) ρ_0, showing that $\mathcal{M}$ is regular. Since $\mathcal{M}$ is not regular, there must be one facet. Then the facet-stabilizer Γ_2 is the whole group Γ, so that $\sigma_2 \in \Gamma_2 = \langle \rho_0, \sigma_2^{-1}\rho_0\sigma_2 \rangle$. The latter group is dihedral. Clearly, if σ_2 has order 2, then $\mathcal{M}$ is regular since the identity automorphism fixes ρ_0 and inverts σ_2. Otherwise, $\sigma_2 = (xy)^a$ for some a where $x = \rho_0$ and $y = \sigma_2^{-1}\rho_0\sigma_2$. Then conjugation by x fixes ρ_0 while inverting σ_2, proving again that $\mathcal{M}$ is regular. Thus $\mathcal{M}$ must be faithful. □

Remark 5.5 We will see that the remaining classes of two-orbit 3-maniplexes all admit unfaithful examples. Again, the map $\{4, 4\}_{(2,0),(0,1)}$ encountered earlier is an unfaithful maniplex in class $\mathbf{2}_{0,2}$. Let us describe an unfaithful *pre*maniplex in class $\mathbf{2}_{1,2}$, and then we will generalize the construction.

Consider a $2k$-gon and color the vertices alternatively white and black. Then *zip* every pair of edges that share a black vertex, that is, make $\Psi^2 = \Psi^1$ if Ψ is black, and $\Psi^2 = \Psi^{0,1,0}$ if Ψ is white (see Fig. 6). The result $\mathcal{M}$ is the flag graph of a k-star embedded on the sphere. This is unfaithful (it has one facet) and is in class $\mathbf{2}_{1,2}$. The dual $\mathcal{M}^*$ can be seen as a bouquet (non-nested loops sharing one vertex) embedded on a sphere, yielding an unfaithful premaniplex in class $\mathbf{2}_{0,1}$. Finally, the Petrie dual of $\mathcal{M}^*$ will also have one vertex (making it unfaithful) and has symmetry type $\mathbf{2}_1$.

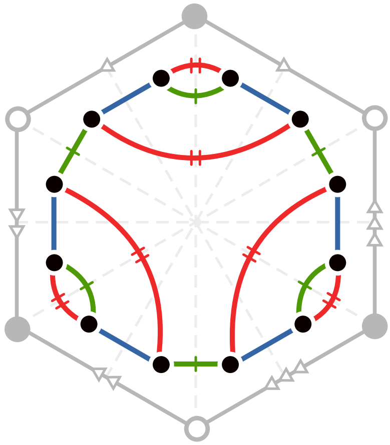


Fig. 6 A construction to build an unfaithful 3-premaniplex in class $\mathbf{2}_{1,2}$

The premaniplex of class $\mathbf{2}_{1,2}$ in Remark 5.5 suggests the following construction. Let F be a $2k$-gon (k might be ∞) with properly colored vertices (black and white); notice that this induces a coloring of the flags: a flag is white (resp. black) if its vertex is white (resp. black). Let w be an involutory connection of F that preserves the colors of the vertices (i.e. it maps flags in a black vertex to flags in a black vertex). Define $\Psi^{r_2} = \Psi^w$ if Ψ is black and $\Psi^{r_2} = \Psi^{r_0 w r_0}$ if Ψ is white. We define $\mathbf{2}_{1,2}(k, w)$ as the (pre)maniplex obtained in this way.

Proposition 5.6 *A 3-maniplex $\mathcal{M}$ with symmetry type $\mathbf{2}_{1,2}$ is unfaithful if and only if $\mathcal{M} \cong \mathbf{2}_{1,2}(k, w)$ for some $k \in [2, \infty]$ and some color-preserving involutory connection $w \notin \{id, r_1, r_0 r_1 r_0, (r_1 r_0)^k, r_0(r_1 r_0)^k\}$.*

Proof First we prove that $\mathcal{M} = \mathbf{2}_{1,2}(k, w)$ is an unfaithful maniplex with symmetry type $\mathbf{2}_{1,2}$. By definition and the fact that 0-adjacent flags have different color, r_2 commutes with r_0. Since $w \neq \text{id}$, r_2 does not have any fixed points. We know that r_2 does not coincide with r_0 at any flag, because r_2 preserves the coloring. On the other hand, since $w \neq r_1$, r_2 acts differently than r_1 on every black flag, and since $w \neq r_0 r_1 r_0$, r_2 acts differently than r_1 on every white flag. Therefore, $\mathcal{M}$ is a maniplex.

By construction, $\mathcal{M}$ has only one facet, and it is therefore unfaithful. We only need to prove that it has symmetry type $\mathbf{2}_{1,2}$. Note that the automorphism group of $\mathcal{M}$ is contained in $\langle \rho_0, \rho_1 \rangle$, the automorphism group of the $2k$-gon, where $\Phi\rho_i = \Phi^i$. Now, if α is an automorphism of the $2k$-gon that preserves the color of the vertices, then clearly it commutes with r_0 and r_1. If Ψ is black, then $(\Psi^{r_2})\alpha = (\Psi^w)\alpha = (\Psi\alpha)^w$, and since $\Psi\alpha$ is black, this is $(\Psi\alpha)^{r_2}$. An analogous argument works when Ψ is white. So α also commutes with r_2. Thus, $\text{Aut}(\mathcal{M})$ contains at least these automorphisms; this shows that $\mathcal{M}$ is a regular cover of $\mathbf{2}_{1,2}$.

It remains to show that $\mathcal{M}$ is not regular, for which it suffices to show that $\rho_0 \notin \text{Aut}(\mathcal{M})$. Indeed, if $\rho_0 \in \text{Aut}(\mathcal{M})$, then the flag

$$(\Phi^{r_2})\rho_0 = (\Phi^w)\rho_0 = (\Phi\rho_0)^w = \Phi^{r_0 w}$$

must be equal to

$$(\Phi\rho_0)^{r_2} = (\Phi^0)^{r_2} = \Phi^{wr_0},$$

and since we are assuming that $\mathcal{M}$ is regular, this implies that $r_0 w = wr_0$. The centralizer of r_0 in the dihedral group $\mathrm{Conn}(\mathcal{M})$ is $\langle r_0, (r_1r_0)^k\rangle$, but since w preserves color, $w \neq r_0$ and the other possibilities are ruled out by the fact that

$$w \notin \{\mathrm{id}, r_1, r_0r_1r_0, (r_1r_0)^k, r_0(r_1r_0)^k\}.$$

Therefore, $\mathcal{M}$ is in class $\mathbf{2}_{1,2}$.

Now let us assume that $\mathcal{M}$ is unfaithful with symmetry type $\mathbf{2}_{1,2}$. By Proposition 5.2, $\mathcal{M}$ must have either one facet or one vertex, but since it has two orbits on vertices, it must have one facet and that facet must be a $2k$-gon for some $k \in [1, \infty]$ (as 0-adjacent flags belong to different orbits). Let Φ be a base black flag. Let $w \in \langle r_0, r_1\rangle$ be such that $\Phi^2 = \Phi^w$. If Ψ is a black flag, then $\Psi = \Phi\alpha$ for some $\alpha \in \Gamma$, and we get

$$\Psi^2 = (\Phi\alpha)^2 = (\Phi^2)\alpha = (\Phi^w)\alpha = (\Phi\alpha)^w = \Psi^w.$$

If Ψ is white, then $\Psi = (\Phi^0)\alpha = (\Phi\alpha)^0$ for some $\alpha \in \Gamma$ and we get

$$\Psi^2 = (\Phi\alpha)^{0,2} = (\Phi\alpha)^{2,0} = (\Phi\alpha)^{wr_0} = ((\Phi\alpha)^0)^{r_0wr_0} = \Psi^{r_0wr_0}.$$

This shows that $\mathcal{M} = \mathbf{2}_{1,2}(k, w)$. In addition, w cannot be in

$$\{\mathrm{id}, r_1, r_0r_1r_0, (r_1r_0)^k, r_0(r_1r_0)^k\},$$

because otherwise $\mathcal{M}$ would not be a maniplex or it would be regular. Finally, the fact that $w \notin \{\mathrm{id}, r_1, r_0r_1r_0, (r_1r_0)^k, r_0(r_1r_0)^k\}$ implies that $k \geqslant 2$. □

Remark 5.7 The reason for excluding r_1 and $r_0r_1r_0$ from the values of w are different than those for excluding $(r_1r_0)^k$ and $r_0(r_1r_0)^k$. The first two are excluded to make sure that $\mathbf{2}_{1,2}(k, w)$ is a maniplex, while the latter two are excluded to make sure that $\mathbf{2}_{1,2}(k, w)$ has type $\mathbf{2}_{1,2}$. Our proof also shows that $\mathbf{2}_{1,2}(k, r_0)$ and $\mathbf{2}_{1,2}(k, r_0r_1r_0)$ are premaniplexes of type $\mathbf{2}_{1,2}$, while $\mathbf{2}_{1,2}(k, (r_1r_0)^k)$ and $\mathbf{2}_{1,2}(k, r_0(r_1r_0)^k)$ are regular maniplexes. The identity is excluded for both reasons, as $\mathbf{2}_{1,2}(k, \mathrm{id})$ is regular and not a maniplex.

Applying duality and Petrie duality then gives us the following result.

Theorem 5.8 *Let $\mathcal{M}$ be an unfaithful 3-maniplex.*

(i) $\mathcal{M}$ has symmetry type $\mathbf{2}_{1,2}$ if and only if it is $\mathbf{2}_{1,2}(k, w)$ for some $k \in [2, \infty]$ and some color-preserving involutory connection w of the $2k$-gon, with $w \notin \{id, r_1, r_0r_1r_0, (r_1r_0)^k, r_0(r_1r_0)^k\}$.

(ii) $\mathcal{M}$ has symmetry type $\mathbf{2}_{0,1}$ if and only if it is the dual of $\mathbf{2}_{1,2}(k, w)$ for some $k \in [2, \infty]$ and some color-preserving involutory connection w of the $2k$-gon, with $w \notin \{id, r_1, r_0r_1r_0, (r_1r_0)^k, r_0(r_1r_0)^k\}$.

(iii) *$\mathcal{M}$ has symmetry type $\mathbf{2}_1$ and has a single vertex if and only if it is the Petrie dual of the dual of $\mathbf{2}_{1,2}(k, w)$ for some $k \in [2, \infty]$ and some color-preserving involutory connection w of the $2k$-gon, with $w \notin \{id, (r_1 r_0)^k, r_0(r_1 r_0)^k\}$.*

(iv) *$\mathcal{M}$ has symmetry type $\mathbf{2}_1$ and has a single facet if and only if it is the opposite (dual of the Petrie dual of the dual) of $\mathbf{2}_{1,2}(k, w)$ for some $k \in [2, \infty]$ and some color-preserving involutory connection w of the $2k$-gon with $w \notin \{id, (r_1 r_0)^k, r_0(r_1 r_0)^k\}$.*

Proof (i) This is just Proposition 5.6.

(ii) If $\mathcal{M}$ is an unfaithful 3-maniplex of type $\mathbf{2}_{0,1}$, its dual is also unfaithful and has type $\mathbf{2}_{1,2}$. Then we use Proposition 5.6.

(iii) If $\mathcal{M}$ is a 3-maniplex of type 2_1 with a single vertex, then its Petrie dual has type $2_{0,1}$ and has the same vertex-set, so it is unfaithful (although it might not be a maniplex). By (ii) and Remark 5.7 we get that $\mathcal{M}$ must be the Petrie dual of the dual of $2_{1,2}(k, w)$ where $w \notin \{\mathrm{id}, (r_1 r_0)^k, r_0(r_1 r_0)^k\}$ (so that $2_{1,2}(k, w)$ is not regular). Note that to get from $\mathbf{2}_{1,2}(k, w)$ to the Petrie dual of its dual we must replace the 2-edges with 0-edges, remove the old 0-edges and draw new ones connecting each flag Φ to $\Phi^{r_0 r_2}$. Recall that r_2 acts like w on the black flags of $\mathbf{2}_{1,2}(k, w)$ and like $r_0 w r_0$ on the white ones. This means that in $(\mathbf{2}_{1,2}(k, w)^*)^\pi$, the flag 0-adjacent to a black flag Φ is $\Phi^{r_0 w}$, and the flag 0-adjacent to a white flag Ψ is $\Psi^{w r_0}$. Since w is color-preserving and r_0 switches colors, $\Phi^{r_0 w}$ must be white and therefore different from Φ^1 which is black. For a similar reason $\Psi^{w r_0} \neq \Psi^1$.

(iv) It follows in a similar fashion as (iii) by noting that the opposite of a (pre)maniplex has the same set of facets and that the opposite of a maniplex of type $\mathbf{2}_1$ has type $\mathbf{2}_{1,2}$.

□

Notice that for (iii) and (iv) we do not demand $w \notin \{r_1, r_0 r_1 r_0\}$, since for these values of w it turns out that $(\mathbf{2}_{1,2}(k, w)^*)^\pi$ and $((\mathbf{2}_{1,2}(k, w)^*)^\pi)^*$ are maniplexes.

In general, the Petrie dual of a 3-maniplex with one facet may not have a single vertex or facet, so in principle $\mathbf{2}_{1,2}(k, w)^\pi$ and $(\mathbf{2}_{1,2}(k, w)^*)^\pi$ could be faithful. In fact, the Petrie dual of $\mathbf{2}_{1,2}(6, r_0(r_1 r_0)^3)$ has 3 square facets, 2 black vertices of valency 3 and one white vertex of valency 6. Since it is still in class $\mathbf{2}_{1,2}$, $\mathbf{2}_{1,2}(6, r_0(r_1 r_0)^3)^\pi$ must be faithful by Proposition 5.2.

Finally, we want to characterize 3-maniplexes with symmetry type $\mathbf{2}_{0,2}$, which is the only symmetry type of 2-orbit maniplexes where it is possible to have more than one vertex and more than one facet. Example 4.10 can be constructed from a quadrangular dihedron by truncating the vertices of each facet (turning squares into octagons) and then gluing each new edge to its opposite in that facet. This construction can be generalized for any 3-maniplex with finite facets of even length. Let us be more precise. Given a 3-maniplex $\mathcal{M}$ with finite facets of even length (possibly different lengths for different facets), we will define two new maniplexes $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ and $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ as follows. The set of flags for both two new maniplexes is the set of pairs (Φ, i) with Φ a flag of $\mathcal{M}$ and $i \in \{0, 1\}$. For both maniplexes we define

$$(\Phi, 0)^{r_0} := (\Phi^{r_0}, 0),$$

$$
\begin{aligned}
(\Phi, 0)^{r_1} &:= (\Phi, 1),\\
(\Phi, 0)^{r_2} &:= (\Phi^{r_2}, 0),\\
(\Phi, 1)^{r_0} &:= (\Phi^{r_1}, 1),\\
(\Phi, 1)^{r_1} &:= (\Phi, 0).
\end{aligned}
$$

For the action of r_2 on $(\Phi, 1)$ let $p(\Phi)$ be the least positive number k such that $\Phi^{(r_1 r_0)^k} = \Phi$. Then in $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ we define

$$
(\Phi, 1)^{r_2} := (\Phi^{(r_0 r_1)^{p(\Phi)/2} r_0}, 1),
$$

while in $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ we define

$$
(\Phi, 1)^{r_2} := (\Phi^{((r_0 r_1)^{p(\Phi)/2}}, 1).
$$

One can verify that $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ and $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ are maniplexes. The difference between them is that in $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$, opposite edges are glued without twisting (as when forming a torus from a square), while in $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ they are glued by twisting (as when forming a projective plane from a square). Note that there is no restriction that $\mathcal{M}$ itself be finite.

We can see that Example 4.10 is just $\mathrm{tr}_{\mathbb{T}}\{4, 2\}$.

When stating Proposition 5.6 we ask for w to not be $(r_1 r_0)^k$ or $r_0(r_1 r_0)^k$, since for those values of w the maniplexes $\mathbf{2}_{1,2}(k, w)$ turn out to be regular (instead of type $\mathbf{2}_{1,2}$). Note that $\mathbf{2}_{1,2}(k, (r_1 r_0)^k)$ is just $\mathbb{P}\{k\}$ and $\mathbf{2}_{1,2}(k, r_0(r_1 r_0)^k)$ is $\mathbb{T}\{k\}$ (as defined at the beginning of this section). Since r_0 commutes with both $(r_1 r_0)^k$ and $r_0(r_1 r_0)^k\}$, we do not need to make a distinction between the white and black flags in $\mathbb{T}\{k\}$ and $\mathbb{P}\{k\}$ when defining r_2.

Proposition 5.9 *If $\mathcal{M}$ is a regular 3-maniplex with finite facets of even length $2k$, then $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ and $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ are unfaithful 3-maniplexes. Furthermore, they are in class $\mathbf{2}_{0,2}$ unless $\mathcal{M}$ is $\mathbb{T}\{k\}$ or $\mathbb{P}\{k\}$ (resp.), in which case $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ or $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ (resp.) are regular.*

Proof If $\tau \in \mathrm{Aut}(\mathcal{M})$ we define $(\Phi, i)\tau = (\Phi\tau, i)$. We can verify that in this way τ acts as an automorphism of both $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ and $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$. Clearly, $\mathrm{Aut}(\mathcal{M})$ acts with two orbits and $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})/\mathrm{Aut}(\mathcal{M}) = \mathrm{tr}_{\mathbb{P}}(\mathcal{M})/\mathrm{Aut}(\mathcal{M}) = \mathbf{2}_{1,2}$. Let us assume that $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ is regular. Then, there is some $\tau \in \mathrm{Aut}(\mathrm{tr}_{\mathbb{P}}(\mathcal{M}))$ mapping a flag $(\Phi, 0)$ to its 1-adjacent flag $(\Phi, 1)$. Note that $(\Phi, 0)^{(r_0 r_1)^{2k}} = (\Phi^{(r_0 r_1)^k}, 0)$ and $(\Phi, 1)^{(r_0 r_1)^{2k}} = (\Phi^{(r_1 r_0)^k}, 1) = (\Phi^{(r_0 r_1)^k}, 1)$. It follows that

$$
(\Phi^{(r_0 r_1)^k}, 0)\tau = (\Phi, 0)^{(r_0 r_1)^{2k}}\tau = ((\Phi, 0)\tau)^{(r_0 r_1)^{2k}} = (\Phi, 1)^{(r_0 r_1)^{2k}} = (\Phi^{(r_0 r_1)^k}, 1).
$$

On the other hand,

$$
(\Phi^2, 0)\tau = (\Phi, 0)^{r_2}\tau = ((\Phi, 0)\tau)^{r_2} = (\Phi, 1)^{r_2} = (\Phi^{(r_0 r_1)^k}, 1).
$$

Since automorphisms act semiregularly on flags, this means that $\Phi^2 = \Phi^{(r_0r_1)^k}$, in other words $\mathcal{M}$ is $\mathbb{P}\{k\}$. Analogously, if there is an automorphism of $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ mapping $(\Phi, 0)$ to $(\Phi, 1)$, that implies that $\Phi^2 = \Phi^{(r_0r_1)^k r_0}$, and therefore $\mathcal{M}$ is $\mathbb{T}\{k\}$.

Finally, notice that $(\Phi, 1)$ and $(\Phi, 1)^2$ share a 2-face, and therefore are chain-equivalent. □

Proposition 5.10 *Let $\mathcal{M}$ be an unfaithful 3-maniplex in class $\mathbf{2}_{0,2}$. Then there exists some regular 3-maniplex $\mathcal{K}$ such that at least one of the following statements is true:*

- $\mathcal{M} \cong \mathrm{tr}_{\mathbb{T}}(\mathcal{K})$
- $\mathcal{M} \cong \mathrm{tr}_{\mathbb{P}}(\mathcal{K})$
- $\mathcal{M}^* \cong \mathrm{tr}_{\mathbb{T}}(\mathcal{K})$
- $\mathcal{M}^* \cong \mathrm{tr}_{\mathbb{P}}(\mathcal{K})$

Proof Let $\mathcal{M}$ be unfaithful in class $\mathbf{2}_{0,2}$. Let Φ be the base black flag and Φ^1 the base white flag. The automorphism group $\Gamma = \mathrm{Aut}(\mathcal{M})$ is generated by $\alpha_0, \alpha_2, \beta_0$ and β_2 where $\Phi\alpha_i = \Phi^i$ and $\Phi^1\beta_i = \Phi^{1i}$ (see [13, Proposition 7]). Let Γ_i be the stabilizer of the i-face of Φ. Up to duality, there is some flag that is chain-equivalent to its 2-adjacent flag (by Proposition 3.4). Without loss of generality, we may assume that $\Phi^{1,2} \approx \Phi^1$. Since $\Phi^1\beta_2 = \Phi^{1,2}$ and this flag is in the same facet as Φ and Φ^1, we have that $\beta_2 \in \Gamma_2 = \langle \alpha_0, \beta_0 \rangle$, which is a dihedral group. Now, since $\mathcal{M}$ is a maniplex, $\beta_2 \notin \{\mathrm{id}, \beta_0\}$. Then the fact that β_2 commutes with β_0 implies that k must be even and $\beta_2 \in \{(\beta_0\alpha_0)^{k/2}, (\beta_0\alpha_0)^{k/2}(\beta_0)\}$ where k is the order of $\beta_0\alpha_0$.

Now notice that by making $\rho_0 = \alpha_0$, $\rho_1 = \beta_0$ and $\rho_2 = \alpha_2$ we get a regular maniplex $\mathcal{K}$ with automorphism group Γ (since $\beta_2 \in \langle \alpha_0, \beta_0 \rangle$). It is straightforward to see that if $\beta_2 = (\beta_0\alpha_0)^{k/2}$ then $\mathcal{M} \cong \mathrm{tr}_{\mathbb{P}}(\mathcal{K})$, and if $\beta_2 = (\beta_0\alpha_0)^{k/2}\beta_0$ then $\mathcal{M} \cong \mathrm{tr}_{\mathbb{T}}(\mathcal{K})$. □

More visually, we can obtain $\mathcal{K}$ from $\mathcal{M}$ by ignoring the white flags and making each black flag Φ 1-adjacent to $\Phi^{1,0,1}$. The operation $\mathcal{M} \mapsto \mathcal{K}$ is a voltage operation (as defined in [10, 11]) that does not preserve connectivity. In contrast, the operations $\mathcal{K} \mapsto \mathrm{tr}_{\mathbb{T}}(\mathcal{K})$ and $\mathcal{K} \mapsto \mathrm{tr}_{\mathbb{P}}(\mathcal{K})$ are only voltage operations when restricted to maniplexes with facets whose number of sides is a divisor of some fixed number k. This is so that $(r_0r_1)^{k/2}$ acts on such maniplexes as an involution, so it can be assigned as the voltage of a semi-edge.

Note that if $\mathcal{M}$ is faithful the chain-equivalence classes of $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ consist of single black flags and pairs of 2-adjacent white flags. If $\mathcal{M}$ is unfaithful, the chain-equivalence classes in $\mathrm{tr}_{\mathbb{T}}(\mathcal{M})$ and $\mathrm{tr}_{\mathbb{P}}(\mathcal{M})$ are unions of these sets running over the chain-equivalence classes of $\mathcal{M}$.

6 Faithful maniplexes and covers

We return to the relationship between covers and faithfulness, expanding on Proposition 3.14. Let us start by proving an analogue of the *quotient criterion* for polytopality (see [18, Thm. 2E17]).

Proposition 6.1 *Suppose $\mathcal{M}_1$ is a maniplex that covers a faithful maniplex $\mathcal{M}_2$. If this cover is one-to-one on each facet (i.e., if the facets of $\mathcal{M}_2$ are isomorphic to those of $\mathcal{M}_1$), then $\mathcal{M}_1$ is faithful.*

Proof If $\Phi \approx \Psi$ in $\mathcal{M}_1$, every i-avoiding walk between them induces an i-avoiding walk in $\mathcal{M}_2$, and so this will induce an unfaithful pair in $\mathcal{M}_2$ unless Φ is identified with Ψ. However, since Φ and Ψ lie in the same facet and the covering is one-to-one on each facet, they will not be identified, and so $\mathcal{M}_1$ must be faithful. □

The remainder of our results concern the following construction. If $\mathcal{M}_1$ and $\mathcal{M}_2$ are n-maniplexes, with base flags (roots) Φ_1 and Φ_2, then their *mix*, denoted $\mathcal{M}_1 \diamond \mathcal{M}_2$ is the maniplex obtained in the following way. Start with the product of the flag sets, and then connect each (Λ, Δ) with an i-edge to (Λ^i, Δ^i). Finally, set the root to be (Φ_1, Φ_2), and take the connected component containing this flag. The maniplex $\mathcal{M}_1 \diamond \mathcal{M}_2$ is the minimal common cover of $\mathcal{M}_1$ and $\mathcal{M}_2$. See [3, Section 3.2] for more information on the mix of maniplexes. Our first result follows directly from Proposition 3.14.

Proposition 6.2 *If $(\Phi_1, \Phi_2) \approx (\Psi_1, \Psi_2)$ in $\mathcal{M}_1 \diamond \mathcal{M}_2$, then $\Phi_1 \approx \Phi_2$ and $\Psi_1 \approx \Psi_2$. In particular, $[(\Phi_1, \Phi_2)] \subseteq [\Phi_1] \times [\Phi_2]$.*

Corollary 6.3 *If $\mathcal{M}_1$ and $\mathcal{M}_2$ are faithful, then $\mathcal{M}_1 \diamond \mathcal{M}_2$ is faithful. In particular, the mix of two polytopes is faithful.*

Thus, while the mix of two polytopes might not be a polytope, it cannot be "too degenerate".

Note that Corollary 6.3 really does require that both $\mathcal{M}_1$ and $\mathcal{M}_2$ are faithful. Certainly the mix can be unfaithful if only $\mathcal{M}_1$ is unfaithful; for example, if $\mathcal{M}_1$ is the regular 4-maniplex with group $[6, 3, 6]/((\rho_0\rho_1)^3(\rho_2\rho_3)^3)$, and $\mathcal{M}_2$ is the 4-simplex, then $\mathcal{M}_1$ is an unfaithful cover of the faithful maniplex $\mathcal{M}_2$, and thus $\mathcal{M}_1 \diamond \mathcal{M}_2 \cong \mathcal{M}_1$.

Every maniplex has a *smallest regular cover*. If $\mathcal{M}$ is a k-orbit maniplex, then this can be obtained by picking one representative from each flag orbit and mixing the k rooted maniplexes (see [3, Example 3.4]). Thus, we obtain the following corollary.

Corollary 6.4 *The smallest regular cover of a polytope with finitely many flag orbits is a faithful maniplex.*

It is worth noting that prescribed conditions on a polytope could imply stronger conditions on its smallest regular cover. The smallest regular cover of a 3-polytope is always a polytope (see [17, Proposition 6.1]). If $\mathcal{M}$ is a polytope whose facets are isomorphic regular polytopes, its smallest regular cover is a polytope (see [17, Corollary 6.5]). In particular, the smallest regular cover of a chiral polytope with regular facets is a polytope. More generally, [6, Theorem 3.1] characterizes which chiral polytopes have a smallest regular cover that is a polytope (see also [19, Proposition 4.3.12]), and [2, Corollary 6.15] characterizes when the smallest regular cover of any two-orbit polytope is a polytope.

In full generality, Corollary 6.4 is almost best possible: in [16] the authors construct a 4-polytope whose minimal regular cover is not a polytope.

7 Thinness

Having thoroughly investigated faithfulness in maniplexes, let us turn to thinness. In the following two results, we describe an alternative characterization of thinness for faithful maniplexes.

Proposition 7.1 *Suppose $\mathcal{M}$ is faithful and thin, and fix a flag Φ and an integer $0 \leqslant i \leqslant n-1$. Let F_j be the j-face that contains Φ. Then $\bigcap_{j \neq i} F_j = \{\Phi, \Phi^i\}$.*

Proof It is clear that $\{\Phi, \Phi^i\}$ is contained in $\bigcap_{j \neq i} F_j$. Now, let $\Psi \in \bigcap_{j \neq i} F_j$. Note that, since $\mathcal{M}$ is faithful, Φ and Φ^i are in separate i-faces, by Proposition 3.2. Let F_i' be the i-face of Φ^i. Then since $\mathcal{M}$ is thin and $\Psi \in F_{i-1} \cap F_{i+1}$, that implies that Ψ either lies in F_i or F_i'. If $\Psi \in F_i$, then $\Psi \approx \Phi$ (since already we had $\Psi \in F_j$ for every $j \neq i$), and thus $\Psi = \Phi$ since $\mathcal{M}$ is faithful. Otherwise, $\Psi \in F_i'$. Note that $\Phi^i \overset{j}{\sim} \Phi$ for every $j \neq i$, and thus the same j-face as Ψ for every $j \neq i$. Then since $\Psi \in F_i'$, it follows that $\Psi \approx \Phi^i$ and thus $\Psi = \Phi^i$, by faithfulness. □

Proposition 7.2 *Suppose $\mathcal{M}$ is faithful. Suppose that for each flag Φ and each integer $0 \leqslant i \leqslant n-1$, we have $\bigcap_{j \neq i} F_j = \{\Phi, \Phi^i\}$, where F_j is the j-face that contains Φ. Then $\mathcal{M}$ is thin.*

Proof Let $\Psi \in F_{i-1} \cap F_{i+1}$. Since $\Psi \in F_{i+1}$, then $\Psi = \Phi^{uv}$ with $u \in \langle r_0, \ldots, r_i \rangle$ and $v \in \langle r_{i+2}, \ldots, r_{n-1} \rangle$. Then $\Psi^{v^{-1}}$ is also in $F_{i-1} \cap F_{i+1}$, and is equal to Φ^u. Let us write $\Psi' = \Psi^{v^{-1}}$, and note that Ψ' is contained in F_k for each $k \geqslant i+1$. Now, we similarly have that $\Psi' = \Phi^{yx}$ with $x \in \langle r_0, \ldots, r_{i-2} \rangle$ and $y \in \langle r_i, \ldots, r_{n-1} \rangle$. Then $(\Psi')^{x^{-1}}$ will still be contained in $F_{i-1} \cap F_{i+1} \cap F_{i+2} \cap \cdots \cap F_{n-1}$ and will be equal to Φ^y. Let $\Psi'' = (\Psi')^{x^{-1}}$. Then it follows that Ψ'' is also contained in F_k for each $k \leqslant i-1$. So we have $\Psi'' \in \bigcap_{j \neq i} F_j$, and by assumption, either $\Psi'' = \Phi$ or $\Psi'' = \Phi^i$. Thus, either $\Psi = \Phi^{xv}$ or $\Psi = \Phi^{ixv}$. In the first case, xv yields an i-avoiding path from Φ to Ψ, and so $\Psi \in F_i$. Otherwise, xv yields an i-avoiding path from Φ^i to Ψ, and so Ψ is in the i-face of Φ^i. In either case, we found that Ψ had to be in the i-face of either Φ or Φ^i, and these i-faces are distinct since $\mathcal{M}$ is faithful. □

Theorem 7.3 *The mix of two faithful, thin, orientable maniplexes is faithful, thin, and orientable.*

Proof That the mix is faithful is proven in Corollary 6.3, and that the mix is orientable is well-known. It remains to show that the mix is thin. Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be faithful, thin, orientable n-maniplexes. To prove that $\mathcal{M}_1 \diamond \mathcal{M}_2$ is thin, Proposition 7.2 says that it suffices to show that for each flag (Φ, Ψ) of $\mathcal{M}_1 \diamond \mathcal{M}_2$ and for each $0 \leqslant i \leqslant n-1$, the intersection $\bigcap_{j \neq i} F_j(\Phi, \Psi)$ is equal to $\{(\Phi, \Psi), (\Phi^i, \Psi^i)\}$, where $F_j(\Phi, \Psi)$ is the j-face of (Φ, Ψ). Suppose (Λ, Δ) is in that intersection. Then for each $j \neq i$, there is a j-avoiding path from (Φ, Ψ) to (Λ, Δ), and this induces a j-avoiding path from Φ to Λ and from Ψ to Δ. Then, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are thin, $\Lambda \in \{\Phi, \Phi^i\}$ and $\Delta \in \{\Psi, \Psi^i\}$. Now, since $\mathcal{M}_1$ and $\mathcal{M}_2$ are orientable and $\mathcal{M}_1 \diamond \mathcal{M}_2$ contains (Φ, Ψ), the flags of $\mathcal{M}_1 \diamond \mathcal{M}_2$ all must have the same color flag in each coordinate. Thus,

(Φ, Ψ^i) and (Φ^i, Ψ) are not flags of $\mathcal{M}_1 \diamond \mathcal{M}_2$, and so (Λ, Δ) must be either (Φ, Ψ) or (Φ^i, Ψ^i), which implies that $\mathcal{M}_1 \diamond \mathcal{M}_2$ is thin. □

Corollary 7.4 *The mix of two orientable polytopes is a faithful, thin, orientable maniplex.*

Example 7.5 The orientability condition in Theorem 7.3 is essential (even if we do not require orientability of the output). Indeed, we can find an example of its necessity in [4, Example 2.40]; there, faithfulness (of regular polytopes) was referred to as the "empty intersection property" and thinness as the "weak intersection property". The example is this: let $\mathcal{P}_1$ be the regular 5-polytope with group [2, 3, 3, 2], and let $\mathcal{P}_2$ be the regular 5-polytope with group

$$[3, 4, 4, 3]/((\rho_0\rho_1\rho_2)^3, (\rho_4\rho_3\rho_2)^3).$$

Let $\Gamma = \mathrm{Aut}(\mathcal{P}_1 \diamond \mathcal{P}_2)$. Then $|\bigcap_{j \neq 2} \Gamma_j| = 4$. In other words, the sections consisting of a 3-face and an incident 1-face consist of four flags, and so the result is not thin.

Corollary 7.4 tells us that, though the mix of two orientable polytopes need not be a polytope, the result will still be a 'nice' maniplex that can be reconstructed from its poset.

8 Open problems

We conclude with a few open problems for further research. First, considering that a one-facet maniplex must be unfaithful, we wonder how restricted a two-facet maniplex must be in order to be faithful.

Problem 1 Develop a theory of faithfulness for maniplexes with two facets.

It was shown in [12] that there are unfaithful maniplexes $\mathcal{M}$ such that $\mathrm{Pos}(\mathcal{M})$ is a polytope $\mathcal{P}$, but $\mathcal{M}$ is not the flag graph of $\mathcal{P}$ (or of any polytope). This strange phenomenon is worth further study.

Problem 2 Characterize the abstract polytopes $\mathcal{P}$ such that $\mathcal{P} \cong \mathrm{Pos}(\mathcal{M})$ for an unfaithful maniplex $\mathcal{M}$ (as in [12]).

We saw in Example 2.2 that for a thin flagged poset $\mathcal{P}$, it need not be true that $\mathrm{Pos}(\mathrm{fl}(\mathcal{P}))$ is naturally isomorphic to $\mathcal{P}$.

Problem 3 Determine the connectivity conditions required for a thin flagged poset $\mathcal{P}$ to satisfy $\mathrm{Pos}(\mathrm{fl}(\mathcal{P})) \cong \mathcal{P}$.

Finally, the authors are working on a classification of unfaithful rotary 4-maniplexes, to appear in a sequel.

Acknowledgements The authors thank the anonymous referees for many helpful comments which helped improve the paper.

Funding This research project was supported by the Slovenian Research and Innovation Agency (ARIS), research projects P1-0294, J1-4351, and the bilateral activity BI-US/24-26-025. The second author had a postdoctoral position at the Department of Mathematics of Northeastern University while working on this paper.

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