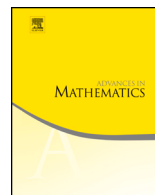




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Advances in Mathematics

journal homepage: www.elsevier.com/locate/aim

Runge and Mergelyan theorems on families of open Riemann surfaces

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ARTICLE INFO

Article history:

Received 2 February 2025

Received in revised form 27 May 2026

Accepted 6 July 2026

Available online xxxx

Communicated by C. Fefferman

MSC:

primary 32Q56, 32H02

secondary 32E10, 53A10

Keywords:

Riemann surface

Runge theorem

Mergelyan theorem

Oka manifold

Stein manifold

Minimal surface

ABSTRACT

Given a smooth open oriented surface X , endowed with a family of complex structures $\{J_b\}_{b \in B}$ of some Hölder class and depending continuously or smoothly on the parameter b in a suitable topological space B , we construct continuous or smooth families $F_b : X \rightarrow Y$, $b \in B$, of J_b -holomorphic maps to any Oka manifold Y , with approximation on a suitable family of compact Runge sets in X . Along the way, we prove Runge and Mergelyan approximation theorems and Weierstrass interpolation theorem for functions on such families. We include applications to the construction of families of directed holomorphic immersions and conformal minimal immersions to Euclidean spaces.

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1. Introduction and main results

In this paper, we construct families of holomorphic maps from families of open Riemann surfaces to Oka manifolds. Our main results are applied to the construction of families of directed holomorphic immersions and of conformal minimal immersions from families of open Riemann surfaces to Euclidean spaces. We expect further applications of the techniques developed in the paper.

A complex manifold Y is said to be an Oka manifold if maps from any Stein manifold X to Y satisfy all forms of the homotopy principle, called the Oka principle in this context. Basically, this means that holomorphic maps $X \rightarrow Y$ satisfy the same approximation and extension properties as holomorphic functions $X \rightarrow \mathbb{C}$ in the absence of topological obstructions; see [38, Theorem 5.4.4] for a summary statement. Oka manifolds appear naturally in many existence results in complex geometry. Every complex homogeneous manifold and, more generally, every Gromov elliptic manifold is an Oka manifold (see Grauert [53] and Gromov [55]). Further examples and properties of Oka manifolds can be found in [38,37,41,42,45,71,72] and in other sources.

It is often desirable to construct families of holomorphic maps depending continuously or smoothly on parameters. For maps from a Stein space X with a fixed complex structure, see [38, Theorem 2.8.4] for the parametric Cartan–Oka–Weil theorem for functions $X \rightarrow \mathbb{C}$ and [38, Theorem 5.4.4] for the parametric Oka principle for maps from X to any Oka manifold. In those results, the maps depend continuously on the parameter in a compact Hausdorff space B , and they remain unchanged for parameter values in a closed subset of B for which they are already holomorphic on all of X .

In the present paper, we consider the more general situation where not only the maps but also the complex structures on the source manifold depend on a parameter in a suitable topological space. In this first work on the subject, we limit ourselves to the case of a smooth open oriented surface X endowed with a family $\mathcal{J} = \{J_b\}_{b \in B}$ of complex structures. Recall that every open Riemann surface is a Stein manifold according to Behnke and Stein [18]. A compact set K in an open Riemann surface (X, J) is called *Runge* if the complement $X \setminus K$ has no relatively compact connected components. Our main result, Theorem 1.6, is an Oka principle saying that for suitable parameter spaces B and regularity conditions on the family of complex structures $\mathcal{J} = \{J_b\}_{b \in B}$ on X , every family of continuous maps $f_b : X \rightarrow Y$ ($b \in B$) to an Oka manifold Y is homotopic

to a family of J_b -holomorphic maps $F_b : X \rightarrow Y$, with approximation on a suitable family of compact Runge subsets $K_b \subset X$ on which the given maps f_b are already holomorphic. Our method also applies to families of products of open Riemann surfaces with a fixed Stein manifold; see Theorem 6.4. Furthermore, we obtain Mergelyan-type theorems for families of maps to complex manifolds; see Theorems 6.5 and 6.7. On the way to the main results, we obtain approximation of functions on families of open Riemann surfaces — the Runge Theorem 1.1 and the Mergelyan Theorem 1.3. Their proofs are simpler since one can use partitions of unity, instead of dealing with homotopies as we must do for nonlinear target manifolds, and in this case B may be an arbitrary paracompact Hausdorff space. In particular, these two results apply to the universal family of complex structures on a given smooth open orientable surface.

Our main results, combined with techniques of Gromov's convex integration theory, enable the construction of families of holomorphic curves with prescribed conformal types having additional properties (immersed, directed by a conical subvariety of $\mathbb{C}^n$, etc.), and of families of immersed minimal surfaces with prescribed conformal types in Euclidean spaces $\mathbb{R}^n$ for $n \geq 3$. A sample of such applications is given in Section 8, where we also indicate several further problems which can likely be treated by these methods.

We now turn to the detailed presentation. Let X be a smooth, connected, orientable surface with a countable base of topology, which will be an open surface in most results. We endow X with a smooth Riemannian metric, which is used to define Hölder spaces of functions or maps from domains in X ; see (3.10). A complex structure on X is given by an endomorphism J of its tangent bundle TX satisfying $J^2 = -\text{Id}$. Thus, J is a section of the smooth vector bundle $T^*X \otimes TX \rightarrow X$ whose fibre over $x \in X$ is the space $\text{Hom}(T_x X, T_x X)$ of linear maps $T_x X \mapsto T_x X$. (Note that this bundle is trivial if X is an open orientable surface.) A differentiable function $f : X \rightarrow \mathbb{C}$ is said to be *J-holomorphic* if the Cauchy–Riemann equation $df_x \circ J_x = i df_x$ holds for every $x \in X$, where $i = \sqrt{-1}$. We say that J is of local Hölder class $\mathcal{C}^{(k, \alpha)}$ for some $k \in \mathbb{Z}_+ = \{0, 1, 2, \dots\}$ and $0 < \alpha < 1$ if for every relatively compact domain $\Omega \subset X$, the restriction $J|_\Omega \in \Gamma^{(k, \alpha)}(\Omega, T^*\Omega \otimes T\Omega)$ is a section of class $\mathcal{C}^{(k, \alpha)}(\Omega)$ of the restricted vector bundle $T^*\Omega \otimes T\Omega \rightarrow \Omega$. For such J , there is an atlas $\{(U_i, \phi_i)\}_i$ of open sets $U_i \subset X$ with $\bigcup_i U_i = X$ and J -holomorphic charts $\phi_i : U_i \rightarrow \phi_i(U_i) \subset \mathbb{C}$ of class $\mathcal{C}^{(k+1, \alpha)}(U_i)$; see Theorem 2.2. Since the transition maps $\phi_i \circ \phi_j^{-1}$ are biholomorphic in the standard structure J_{st} on $\mathbb{C}$, J determines on X the structure of a Riemann surface (X, J) whose underlying smooth structure is $\mathcal{C}^{(k+1, \alpha)}$ compatible with the smooth structure on X . In fact, the inverse of a diffeomorphism of local class $\mathcal{C}^{(k+1, \alpha)}$ is again of the same class; see Norton [85] and Bojarski et al. [21, Theorem 2.1].

Let $l \in \mathbb{Z}_+$ be an integer, and let B be a topological space which is a manifold of class $\mathcal{C}^l$ if $l \in \mathbb{N} = \{1, 2, \dots\}$. A family $\mathcal{J} = \{J_b\}_{b \in B}$ of complex structures on X is said to be of class $\mathcal{C}^{l, (k, \alpha)}$ if for any relatively compact domain $\Omega \Subset X$ the map $B \ni b \mapsto J_b|_\Omega \in \Gamma^{(k, \alpha)}(\Omega, T^*\Omega \otimes T\Omega)$ is of class $\mathcal{C}^l$. Such a family $\mathcal{J}$ can equivalently be given by a family $\{\mu_b\}_{b \in B}$ of maps from X to the unit disc $\mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ of the same smoothness class $\mathcal{C}^{l, (k, \alpha)}$; see Section 2. Following Kodaira and Spencer [69] and

Kirillov [68], the collection $\{(X, J_b)\}_{b \in B}$ is called a *family of Riemann surfaces* of class $\mathcal{C}^{l,(k,\alpha)}$. Consider the projection $\pi : Z = B \times X \rightarrow B$. We endow each fibre $X_b = \pi^{-1}(b)$ with the complex structure J_b . A map $f : B \times X \rightarrow Y$ to a complex manifold Y is said to be $\mathcal{J}$ -holomorphic if the map $f_b = f(b, \cdot) : X_b \rightarrow Y$ is J_b -holomorphic for every $b \in B$. Assuming that the family $\mathcal{J}$ is of class $\mathcal{C}^{l,(k,\alpha)}$, the space $Z = B \times X$ admits fibre preserving $\mathcal{J}$ -holomorphic charts of class $\mathcal{C}^{l,(k+1,\alpha)}$ with values in $B \times \mathbb{C}$ (see Theorem 4.1). If $0 < l \leq k + 1$ then $Z = B \times X$ endowed with such an atlas is a *mixed manifold* of class $\mathcal{C}^l$ in the sense of Douady [30] and Jurchescu [62,65], and a Levi-flat CR manifold of CR-dimension one in the sense of the Cauchy–Riemann geometry; see [81] and [16]. In Jurchescu’s papers, maps which are holomorphic on complex leaves of a mixed manifold are called *morphic*, while in CR geometry they are called CR maps.

Recall that a topological space is said to be paracompact if every open cover has an open locally finite refinement. A Hausdorff space is paracompact if and only if it admits a locally finite partition of unity subordinate to any open cover. Every metrisable space is Hausdorff and paracompact [94,90].

Our first result, which is proved in Section 5, extends the classical Runge–Behnke–Stein approximation theorem on open Riemann surfaces [91,18], combined with the Weierstrass–Florack interpolation theorem [98,35], to families of open Riemann surfaces.

Theorem 1.1. *Assume that $l \in \mathbb{Z}_+$, B is a paracompact Hausdorff space if $l = 0$ and a manifold of class $\mathcal{C}^l$ if $l > 0$, X is a smooth open surface, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}$ ($k \in \mathbb{Z}_+$, $l \leq k + 1$, $0 < \alpha < 1$), K is a compact Runge subset of X , A is a closed discrete subset of X , $U \subset B \times X$ is an open set containing $B \times (K \cup A)$, $f : U \rightarrow \mathbb{C}$ is a function of class $\mathcal{C}^{l,0}$ such that $f_b = f(b, \cdot)$ is J_b -holomorphic on $U_b = \{x \in X : (b, x) \in U\}$ for every $b \in B$, and $r \in \{0, 1, \dots, k + 1\}$. Then, f is of class $\mathcal{C}^{l,(k+1,\alpha)}$ on U and there is a function $F : B \times X \rightarrow \mathbb{C}$ of class $\mathcal{C}^{l,(k+1,\alpha)}$ satisfying the following conditions.*

- (a) *The function $F_b = F(b, \cdot) : X \rightarrow \mathbb{C}$ is J_b -holomorphic for every $b \in B$.*
- (b) *F approximates f as closely as desired in the fine $\mathcal{C}^{l,(k+1,\alpha)}$ topology on $B \times K$.*
- (c) *$F_b - f_b$ vanishes to order r at every point $a \in A$ for every $b \in B$.*

The reason for assuming $l \leq k + 1$ will become evident in the proof of Lemma 5.6. As explained above, the complex structures J_b in the theorem are compatible with one another only to order $k + 1$, which necessitates the assumption $r \leq k + 1$ in condition (c). See also Corollary 5.2 for approximation on variable families of compact J_b -convex subsets $K_b \subset X$, $b \in B$.

Remark 1.2. If B is a manifold of class $\mathcal{C}^l$ then Theorems 1.1 and 4.1 imply that the manifold $B \times X$, endowed with the family of complex structures $\{J_b\}_{b \in B}$ as in the theorem, is a *Cartan manifold* of class $\mathcal{C}^l$ in the sense of Jurchescu [65, Sect. 6]; see

also his papers [62–64,66] as well as [34,96]. Cartan manifolds are analogues of Stein manifolds in the category of mixed manifolds. In the cited papers, Cartan manifolds are assumed to be of class $\mathcal{C}^\infty$ in order to avoid problems which appear under the finite regularity assumptions, such as those in the present paper.

A more sophisticated approximation theorem was proved by Mergelyan [79] in 1951. Its original version says that a continuous function on a compact Runge set $K \subset \mathbb{C}$ which is holomorphic in the interior $\mathring{K}$ of K is a uniform limit on K of holomorphic polynomials. In view of Runge's theorem, the main new point is that every function in the algebra $\mathcal{A}(K)$ of continuous functions on K which are holomorphic in $\mathring{K}$ can be approximated by functions holomorphic on open neighbourhoods of K . A compact set K in a Riemann surface X for which this condition holds is said to have the *Mergelyan property*. For results on this subject we refer to the surveys [36, Sect. 2] and [50]. The following Mergelyan theorem for families of complex structures on a smooth surface is proved in Section 5.

Theorem 1.3. *Assume that X is a smooth oriented surface without boundary, B is a paracompact Hausdorff space, $\mathcal{J} = \{J_b\}_{b \in B}$ is a continuous family of complex structures on X of Hölder class $\mathcal{C}^{(k,\alpha)}$ for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, K is compact set in X such that, for some $c > 0$ and a Riemannian distance function on X , each relatively compact connected component of $X \setminus K$ has diameter at least c , A is a finite subset of $\mathring{K}$, $r \in \{0, 1, \dots, k+1\}$, and $f : B \times K \rightarrow \mathbb{C}$ is a continuous function such that $f_b = f(b, \cdot) : K \rightarrow \mathbb{C}$ is J_b -holomorphic on $\mathring{K}$ for every $b \in B$. Given a continuous function $\epsilon : B \rightarrow (0, +\infty)$, there is a continuous $\mathcal{J}$ -holomorphic function F on a neighbourhood $U \subset B \times X$ of $B \times K$ such that for every $b \in B$ we have $\sup_{x \in K} |F_b(x) - f_b(x)| < \epsilon(b)$ and $F_b - f_b$ vanishes to order r in every point $a \in A$.*

The condition on the set K in the above theorem ensures that K has the Mergelyan property with respect to any complex structure on X ; see Bishop [20]. See also Corollary 5.7 for Mergelyan approximation on certain variable families of holomorphically convex sets $K_b \subset X$, $b \in B$.

Remark 1.4. The $l = 0$ case of Theorems 1.1 and 1.3 applies to the universal or tautological family on X . (I wish to thank the referee for pointing this out.) The universal family has parameter space B_u consisting of pairs (J, f) , where J is a complex structure on X of class $\mathcal{C}^{(k,\alpha)}$ and f is a continuous function on K , J -holomorphic on a neighbourhood of K (in case of Theorem 1.1) or on the interior of K (in case of Theorem 1.3). The fibre of the universal family over $b = (J, f) \in B_u$ is the Riemann surface (X, J) . There is a natural metrizable topology on B_u , so it is Hausdorff and paracompact by a theorem of Stone [94] (see also Rudin [90]). Any family $\{(J_b, f_b)\}_{b \in B}$ as in Theorem 1.1 or 1.3 can be pulled back from this universal family. A solution of the approximation problem for the universal family will therefore pull back to a solution for a general family, at least

if ϵ is constant. For this result, one does not need any assumption on the topological space B .

We now introduce the parameter spaces used in our main result, Theorem 1.6.

Definition 1.5. In the following, all topological spaces are assumed to be metrizable.

- (i) A space B is an *absolute neighbourhood retract* (ANR) if, whenever B is a closed subset of a space B' , then B is a retract of a neighbourhood of B in B' (a neighbourhood retract).
- (ii) A space B is a *Euclidean neighbourhood retract* (ENR) if it admits a closed topological embedding $\iota : B \hookrightarrow \mathbb{R}^n$ for some n whose image $\iota(B) \subset \mathbb{R}^n$ is a neighbourhood retract.
- (iii) A space B is a *local ENR* if every point of B has an ENR neighbourhood.

We refer to Mardešić [76] for the theory of ANRs and ENRs. Clearly, every local ENR is locally compact. The class of ENRs includes many geometrically relevant classes of spaces such as certain CW complexes. Note that CW complexes generalise both manifolds and simplicial complexes, and they have particular significance for algebraic topology; see Hatcher [58] and May [78]. A CW complex is locally compact if and only if its collection of closed cells is locally finite, if and only if it is metrizable (see Fritsch and Piccinini [49, Theorem B]). If a CW complex B can be embedded in a Euclidean space $\mathbb{R}^m$, then B has at most countably many cells, it is locally compact and has dimension at most m [49, Theorem D]. Conversely, every countable locally compact CW-complex B of finite dimension m is an ENR. Indeed, by [49, Theorem A] such B admits a closed embedding in $\mathbb{R}^{2m+1}$. Since every metrizable CW-complex is an ANR [32], the image of the embedding is a neighbourhood retract, so B is an ENR. In particular, every finite CW complex is an ENR [58, Corollary A.10].

A topological space is said to be σ -compact if it is the union of countably many compact subspaces. According to Michael [80], every locally compact and σ -compact Hausdorff space is paracompact.

The main result of this paper is the following Oka principle with approximation for maps from families of open Riemann surfaces to Oka manifolds. It is proved in Section 6.

Theorem 1.6. Assume the following:

- (a) B is a σ -compact Hausdorff local ENR (see Definition 1.5). In particular, B may be a finite CW complex or a countable locally compact CW-complex of finite dimension.
- (b) X is a smooth open surface and $\pi : B \times X \rightarrow B$ is the projection.
- (c) $\mathcal{J} = \{J_b\}_{b \in B}$ is a continuous family of complex structures on X of Hölder class $\mathcal{C}^\alpha$, $0 < \alpha < 1$.

- (d) $K \subset B \times X$ is a closed subset such that the projection $\pi|_K : K \rightarrow B$ is proper, and for every $b \in B$ the fibre $K_b = \{x \in X : (b, x) \in K\}$ is a compact Runge set in X , possibly empty. (In the sequel, such a set K is called a proper Runge set.)
- (e) Y is an Oka manifold endowed with a distance function dist_Y inducing its manifold topology.
- (f) $f : B \times X \rightarrow Y$ is a continuous map, and there is an open set $U \subset B \times X$ containing K such that $f_b = f(b, \cdot) : X \rightarrow Y$ is J_b -holomorphic on $U_b = \{x \in X : (b, x) \in U\}$ for every $b \in B$.

Given a continuous function $\epsilon : B \rightarrow (0, +\infty)$, there are a neighbourhood $U' \subset U$ of K and a homotopy $f_t : B \times X \rightarrow Y$ ($t \in I = [0, 1]$) satisfying the following conditions.

- (i) $f_0 = f$.
- (ii) The map $f_{t,b} = f_t(b, \cdot) : X \rightarrow Y$ is J_b -holomorphic on $U'_b \supset K_b$ for every $b \in B$ and $t \in I$.
- (iii) $\sup_{x \in K_b} \text{dist}_Y(f_t(b, x), f(b, x)) < \epsilon(b)$ for every $b \in B$ and $t \in I$.
- (iv) The map $F = f_1$ is such that $F_b = F(b, \cdot) : X \rightarrow Y$ is J_b -holomorphic for every $b \in B$.
- (v) If Q is a closed subset of B and $U_b = X$ for all $b \in Q$, then the homotopy $f_{t,b}$ ($t \in I$) can be chosen to be fixed for every $b \in Q$, and in particular $F = f$ on $Q \times X$.

If B is a manifold of class $\mathcal{C}^l$ ($l \in \mathbb{N}$), the set Q in (v) is a closed $\mathcal{C}^l$ submanifold of B , the family $\mathcal{J} = \{J_b\}_{b \in B}$ is of class $\mathcal{C}^{l, (k, \alpha)}$ where $l \leq k + 1$, and the map $f : B \times X \rightarrow Y$ is $\mathcal{J}$ -holomorphic on a neighbourhood U of K and $f|_U \in \mathcal{C}^{l, 0}(U, Y)$, then $f|_U \in \mathcal{C}^{l, (k+1, \alpha)}(U, Y)$ and there is a homotopy $f_t : B \times X \rightarrow Y$ ($t \in I$) which is of class $\mathcal{C}^{l, (k+1, \alpha)}$ on a neighbourhood of K , it satisfies conditions (i)–(v), f_t approximates f in the fine $\mathcal{C}^{l, (k+1, \alpha)}$ -topology on K to a desired precision uniformly in $t \in I$, and $F = f_1 : B \times X \rightarrow Y$ is of class $\mathcal{C}^{l, (k+1, \alpha)}(B \times X, Y)$ and $\mathcal{J}$ -holomorphic.

The proof of Theorem 1.6, given in Section 6, also applies to products $X \times Z$, where X is a smooth open surface endowed with a family $\mathcal{J} = \{J_b\}_{b \in B}$ of complex structures as above and Z is a Stein manifold with a fixed complex structure; see Theorem 6.4. Essentially the same proof, combined with Theorem 1.3, yields Mergelyan approximation for maps from families of open Riemann surfaces to an arbitrary complex manifold; see Theorems 6.5 and 6.7.

Remark 1.7. Condition in (v) of Theorem 1.6 can be strengthened as follows.

- (v') If Q is a closed subset of B and f is $\mathcal{J}$ -holomorphic on $U \cup (Q \times X)$ then the homotopy $f_{t,b}$ ($t \in I$) can be chosen to be fixed for every $b \in Q$, and in particular $F = f$ on $Q \times X$.

Thus, we need not assume that the open set $U \subset B \times X$ contains $Q \times X$. The sets K and U may be empty if we are not interested in an interpolation statement. The proof of (v') requires a minor modification in the induction step and is given (in the more general context of tame families of Stein structures on a smooth manifold X of any dimension) in [47, proof of Theorems 6.1 and 6.3]. The same remark applies to Theorem 1.1.

Model Oka manifolds are the complex Euclidean spaces $\mathbb{C}^n$. For $Y = \mathbb{C}^n$, Theorem 1.6 generalises the approximation statement in Theorem 1.1 to any proper Runge subset $K \subset B \times X$. This generalisation holds for any paracompact Hausdorff parameter space B if we replace condition (d) in Theorem 1.6 by the condition that the fibres K_b of K are upper semicontinuous with respect to $b \in B$ and K_b is Runge in (X, J_b) for every $b \in B$; see Definition 5.1 and Corollary 5.2. If B is locally compact then these two conditions are easily seen to be equivalent. We do not know whether the interpolation statement in Theorem 1.1 carries over to the more general setting in Theorem 1.6.

The condition on the parameter space B to be a local ENR enables us to reduce the proof of Theorem 1.6 to the Oka principle in [38, Theorem 5.4.4]. It is likely that Theorem 1.6 holds for more general parameter spaces. Such a generalisation would require a proof from the first principles, developing the gluing techniques of Oka theory (see [38, Chapter 5]) on Levi-flat CR foliations and laminations. We shall not pursue this approach in the present paper.

The Oka theory has recently been developed for maps from open Riemann surfaces to the class of Oka-1 manifolds, which properly contains the class of Oka manifolds; see [8, 46]. However, I do not know whether this bigger class of manifolds can be used in Theorem 1.6 since its proof relies on the Oka principle for maps from higher dimensional Stein manifolds to Oka manifolds.

Theorem 1.6 and the related results in Section 6 apply in particular if B is a Teichmüller space $T(g, k)$ of k -punctured compact Riemann surfaces of genus g and $\mathcal{J} = \{J_b\}_{b \in B}$ is the associated universal family of complex structures on X . The space $T(g, k)$ is finite dimensional and

$$(X, J_b) = \widehat{X}_b \setminus \{p_1(b), \dots, p_k(b)\}, \quad b \in T(g, k),$$

where $\widehat{X}_b$ is a compact Riemann surface with the complex structure J_b determined by $b \in T(g, k)$ and the pairwise disjoint punctures $p_i(b)$ ($i = 1, \dots, k$) are holomorphic sections of the universal family $\pi : \widehat{V}(g, k) \rightarrow T(g, k)$. (See Nag [84, pp. 322–323] or Iwayoshi and Taniguchi [61].) The open subset $V(g, k) = \widehat{V}(g, k) \setminus \bigcup_{i=1}^k p_i(T(g, k))$ of the complex manifold $\widehat{V}(g, k)$ is the universal family of k -punctured compact Riemann surfaces of genus g . If $2g + k \geq 3$ then the Teichmüller family $\pi : V(g, k) \rightarrow T(g, k)$ is the universal object in the category of topologically marked holomorphically varying families of k -punctured compact Riemann surfaces of genus g (see [84, Theorem 5.4.3]). It is shown in [40] that $V(g, k)$ is a Stein manifold. This gives analogues of the main results of this paper for holomorphic maps $V(g, k) \rightarrow Y$ to any Oka manifold Y . Furthermore,

holomorphic functions on $V(g, k)$ which are algebraic on every fibre of the Teichmüller submersion $\pi : V(g, k) \rightarrow T(g, k)$ are dense in the space of all holomorphic functions [40, Theorem 3.4], and they provide an affine embedding of $V(k, n)$ over any relatively compact domain in $T(g, n)$ [40, Theorem 3.1]. However, the setting in Theorem 1.6 is more general than the one in Teichmüller theory since the complex structures in a given family need not be quasiconformally equivalent. For example, a punctured Riemann surface can be a member of a family in which the punctures develop into boundary curves and vice versa. Our results concerning approximation of functions, such as Theorems 1.1 and 1.3, also hold on infinite dimensional Teichmüller spaces, which are known to be metrisable and hence paracompact Hausdorff; see the surveys by Fletcher and Marković [33] and Marković and Šarić [77] for this topic.

The results and methods developed in the paper can be used in constructions of families of holomorphic curves with special properties and of related objects, such as conformal minimal surfaces. To illustrate the point, we give two such applications in Section 8. The first one, in Theorem 8.2, gives families of J_b -holomorphic immersions $X \rightarrow \mathbb{C}^n$ directed by an irreducible conical complex subvariety $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ such that $A = \overline{A} \setminus \{0\}$ is an Oka manifold. By taking $A = \mathbb{C}_*^n$ we obtain families of ordinary immersions $X \rightarrow \mathbb{C}^n$. For $n = 1$ this gives an extension of the Gunning–Narasimhan theorem [56] to families of J_b -holomorphic immersions $X \rightarrow \mathbb{C}$; see Corollary 8.3.

Another major application pertains to the null cone $\mathbf{A} \subset \mathbb{C}^n$ for $n \geq 3$, see (8.3). The real and the imaginary part of holomorphic immersions $X \rightarrow \mathbb{C}^n$ directed by the null cone are conformally immersed minimal surfaces $X \rightarrow \mathbb{R}^n$. We thus obtain continuous or smooth families of conformally immersed minimal surfaces $X \rightarrow \mathbb{R}^n$, $n \geq 3$, for any continuous or smooth family of complex structures J_b on X (see Corollary 8.6). Several other possible applications are indicated in Problem 8.7. A common feature of these examples is that their construction combines Oka theory with methods from Gromov's convex integration theory to ensure the period vanishing conditions of the derivative maps on a basis of the first homology group $H_1(X, \mathbb{Z})$.

The paper is organised as follows. In Section 2 we recall the connection between Riemannian metrics, conformal structures, and the Beltrami equation. Section 3 contains preparatory results on the Cauchy and Beurling transforms on open Riemann surfaces. In Section 4 we obtain results on deformations of complex structures which are used in the proofs. Theorem 4.1 gives a solution of the Beltrami equation on any smoothly bounded relatively compact domain Ω in an open Riemann surface X for Beltrami coefficients $\mu : \Omega \rightarrow \mathbb{D} = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ with small Hölder $\mathcal{C}^{(k, \alpha)}(\Omega)$ norm ($k \in \mathbb{Z}_+$, $0 < \alpha < 1$), with analytic dependence on μ . It is then shown in Theorem 4.3 that any sufficiently small $\mathcal{C}^{(k, \alpha)}$ perturbation of the complex structure on Ω can be realised by a small $\mathcal{C}^{(k+1, \alpha)}$ diffeomorphic perturbation of Ω in X , with analytic dependence of the map on the complex structure. This extends the Ahlfors–Bers theory [3] of quasiconformal maps of the plane.

With these tools in hand, Theorems 1.1 and 1.3 are proved in Section 5.

In Section 6 we prove our main result, Theorem 1.6, and obtain further Runge and Mergelyan type approximation results for families of manifold-valued maps; see Theorems 6.4, 6.5, and 6.7.

In Section 7 we show that for a family of complex structures on a smooth open surface, the family of their holomorphic cotangent (canonical) bundles admits a family of holomorphic trivialisations (see Theorem 7.1), which can be given by a family of holomorphic immersions to $\mathbb{C}$ (see Corollary 8.3).

Finally, in Section 8 we apply our results to the construction of families of directed holomorphic immersions and conformal minimal immersions to Euclidean spaces.

We conclude this introduction by mentioning some recent developments in which the results of an earlier version of the present paper were used in an important way.

A standard application of the Runge–Behnke–Stein approximation theorem on open Riemann surfaces and, more generally, of the Oka–Weil theorem on Stein manifolds, is the global solvability of the $\bar{\partial}$ -equation for $(0,1)$ -forms. Using Theorem 1.1 and the techniques developed in Sections 3 and 4, I obtained in [39] an optimal solution of the $\bar{\partial}$ -equation on families of open Riemann surfaces with the gain one of derivative in the space variable and without any loss of regularity in the parameter;

Note that solvability of the tangential $\bar{\partial}$ -complex in all bidegrees on $\mathcal{C}^\infty$ smooth Cartan manifolds was shown by Jurchescu [67, Sect. 3]. See also his paper [66] for the approximation theorems.

Open Riemann surfaces are Stein manifolds of complex dimension one. The Oka theory for continuous tame families of sufficiently smooth integrable Stein structures $\{J_b\}_{b \in B}$ on smooth manifolds X of dimension ≥ 4 was developed by Sigurðardóttir and the author in [47]. (Every family of complex structures on a surface is tame, but there are nontame families of Stein structures in higher dimensions; see [47, Sect. 4].) The construction relies on the approach developed in this paper and uses in particular Lemmas 5.3 and 6.3. The use of Theorem 4.3 is replaced by a parametric version of Hamilton’s theorem [57] on strongly pseudoconvex domains, whose proof relies on the work by Greene and Krantz [54] on stability of Kohn’s solutions of the $\bar{\partial}$ -equation under small integrable deformations of the Stein structure on a strongly pseudoconvex domain; see [47, Theorem 3.1].

In [31], Drinovec Drnovšek and Kališnik proved that for any smooth open surface X endowed with a family of complex structures $\{J_b\}_{b \in B}$ depending continuously on the parameter b in a metrisable space B , there is a continuous family of proper holomorphic maps $F_b : (X, J_b) \rightarrow \mathbb{C}^2$, $b \in B$. Their result provides a partial affirmative answer to the first part of Problem 8.7 (a). Several further applications of the results and techniques of this paper were obtained by Alarcón and the author in [9] (May 2026); see the discussion following Problem 8.7.

2. Riemannian metrics, complex structures, and the Beltrami equation

In this section, we recall the relevant background on the topics mentioned in the title. The details can be found in standard texts on quasiconformal mappings and Teichmüller

spaces; see e.g. Ahlfors [2], Lehto and Virtanen [73], Nag [84], and Imayoshi and Taniguchi [61].

Let $z = x + iy$ be the complex coordinate on $\mathbb{C}$. Set $\partial_x = \partial/\partial x$, $\partial_y = \partial/\partial y$, $dz = dx + i dy$, $d\bar{z} = dx - i dy$,

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_x + i\partial_y).$$

For a differentiable function f we shall write $f_z = \partial_z f$ and $f_{\bar{z}} = \partial_{\bar{z}} f$. Note that f is holomorphic if and only if $f_{\bar{z}} = 0$. The exterior differential on functions splits in the sum of its $\mathbb{C}$ -linear and $\mathbb{C}$ -antilinear parts: $d = \partial + \bar{\partial} = \partial_z dz + \partial_{\bar{z}} d\bar{z}$.

A Riemannian metric on a smooth surface X is given in any local coordinates (x, y) by

$$g = E dx \otimes dx + F(dx \otimes dy + dy \otimes dx) + G dy \otimes dy = E dx^2 + 2F dx dy + G dy^2, \quad (2.1)$$

where E, F, G are real functions satisfying $EG - F^2 > 0$. The area form determined by the metric g is $\sqrt{EG - F^2} dx \wedge dy$. The Euclidean metric and the area form on $\mathbb{R}^2 \cong \mathbb{C}$ with the coordinate $z = x + iy$ are given by $g_{\text{st}} = dx^2 + dy^2 = |dz|^2$ and $dx \wedge dy = \frac{i}{2} dz \wedge d\bar{z}$. On every tangent space $T_p X$, a Riemannian metric g defines a scalar product having the matrix $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ in the basis ∂_x, ∂_y . Hence, g determines a unique conformal structure on X , and two Riemannian metrics g_1, g_2 determine the same conformal structure if and only if $g_2 = \lambda g_1$ for a positive function λ . A pair of nonzero tangent vectors $\xi, \eta \in T_p X$ is said to be a conformal frame if ξ and η have the same g -length and are g -orthogonal to each other. If X is oriented, there is a unique endomorphism $J : TX \rightarrow TX$ on the tangent bundle of X such that for any tangent vector $0 \neq v \in T_p X$, (v, Jv) is a positively oriented g -conformal frame. Note that $J^2 = -\text{Id}$; an endomorphism of TX satisfying this condition is called an *almost complex structure* on X . We have the following local expression for the matrix of J (in the standard oriented basis ∂_x, ∂_y) in terms of the metric g (2.1):

$$[J] = \frac{1}{\sqrt{EG - F^2}} \begin{pmatrix} -F & -G \\ E & F \end{pmatrix} = \begin{pmatrix} -b & -c \\ (b^2 + 1)/c & b \end{pmatrix} \quad (2.2)$$

where $\delta = EG - F^2 > 0$, $b = F/\sqrt{\delta}$, and $c = G/\sqrt{\delta} > 0$. Every almost complex structure J is of this form for some Riemannian metric g , which is unique up to conformal equivalence. The standard almost complex structure J_{st} on $\mathbb{C}$, defined by the Euclidean metric g_{st} , has the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. In complex notation, J_{st} amounts to multiplication by i . A differentiable function $f : U \rightarrow \mathbb{C}$ on a domain $U \subset X$ is said to be J -holomorphic (more precisely, (J, J_{st}) -holomorphic) if it satisfies the Cauchy–Riemann equation $df_p \circ J_p = J_{\text{st}} \circ df_p$ at all points $p \in U$, where J_p denotes the restriction of J to $T_p X$. At a point where $df_p \neq 0$, such f is an orientation preserving conformal map from the conformal structure on X determined by $J = J_g$ to the standard conformal structure on $\mathbb{C}$.

Assume that the metric g is given in local coordinates (x, y) on an open set $U \subset X$ by (2.1). Taking $z = x + iy$ as a complex coordinate on U , we can write g in the complex form as

$$g = \lambda |dz + \mu d\bar{z}|^2 \quad (2.3)$$

for a positive function $\lambda > 0$ and the complex function

$$\mu = \frac{1 - c + ib}{1 + c + ib} : U \rightarrow \mathbb{D} \quad (2.4)$$

with values in the unit disc, where the numbers b and c are as in (2.2); see [12, p. 51]. A diffeomorphism $f : U \rightarrow f(U) \subset \mathbb{C}$ is conformal from the g -structure on X to the standard conformal structure on $\mathbb{C}$ if and only if $g = h|df|^2$ for a positive function $h > 0$. A chart f with this property is said to be *isothermal* for g . Assume that f is orientation preserving, which amounts to $|f_z| > |f_{\bar{z}}|$. Then

$$|df|^2 = |f_z dz + f_{\bar{z}} d\bar{z}|^2 = |f_z|^2 \cdot \left| dz + \frac{f_{\bar{z}}}{f_z} d\bar{z} \right|^2,$$

and comparison with (2.3) shows that f is isothermal if and only if it satisfies the Beltrami equation

$$f_{\bar{z}} = \mu f_z \quad (2.5)$$

with the Beltrami coefficient μ given by (2.4). We shall say that f is μ -conformal if (2.5) holds. Equivalently, f is a biholomorphic map from (U, J) to $(f(U), J_{\text{st}})$ where J is the complex structure on X determined by g (or by μ).

One can also consider quasiconformal maps $f : X \rightarrow Y$ between a pair of Riemann surfaces. The quantity $\mu_f(z) = f_{\bar{z}}/f_z$, defined in a local holomorphic coordinate z on X , is independent of the choice of the local holomorphic coordinates on Y , and $\mu_f(z)d\bar{z}/dz$ is a section of the bundle $K_X^{-1} \otimes \overline{K}_X \rightarrow X$ where $K_X = T^*X$ is the canonical bundle of X (see [84, p. 46]).

Remark 2.1. The formulas (2.2)–(2.4) show that the conformal class of a Riemannian metric g , the associated complex structure J , and the Beltrami coefficient μ are of the same smoothness class.

The situation is especially simple if we fix a reference complex structure on X , so it is an open Riemann surface. By a theorem of Gunning and Narasimhan [56], such a surface admits a holomorphic immersion $z = u + iv : X \rightarrow \mathbb{C}$. Its differential $dz = du + idv$ is a nowhere vanishing holomorphic 1-form on X trivialising the canonical bundle $T^*X = K_X$, $|dz|^2 = du^2 + dv^2$ is a Riemannian metric on X determining the given complex structure, $\frac{i}{2}dz \wedge d\bar{z} = du \wedge dv$ is the associated area form, and $d\sigma = du \, dv$ is the surface

measure on X . The function z provides a local holomorphic coordinate on X at every point. Given a differentiable function $f : X \rightarrow \mathbb{C}$, its partial derivatives

$$f_z = \partial_z f = \partial f / dz, \quad f_{\bar{z}} = \partial_{\bar{z}} f = \bar{\partial} f / d\bar{z} \quad (2.6)$$

are globally defined functions on X . Any Riemannian metric g on X is globally of the form (2.1) for some real functions E, F, G on X . (However, these coefficients are not functions of z unless z is injective, in which case X is a plane domain.) We can write g in the form (2.3) where the function $\mu : X \rightarrow \mathbb{D}$ is given by (2.4). Conversely, any such function μ determines a Riemannian metric by (2.3), and hence a complex structure J_μ by (2.2). Note that $\mu = 0$ corresponds to the given reference complex structure on X . This global viewpoint will be important in the sequel.

The study of isothermal charts on Riemannian surfaces is a classical subject going back to Lagrange and Gauss. For a Hölder continuous μ , see Korn [70], Lichtenstein [74], and Chern [24]. The existence of global quasiconformal homeomorphisms $\mathbb{C} \rightarrow \mathbb{C}$ follows from the local theorem by use of the uniformization theorem, with direct proofs given by Ahlfors [1] and Vekua [97]. For a measurable function μ satisfying $\|\mu\|_\infty \leq k < 1$ (where $\|\mu\|_\infty$ denotes the essential supremum), see Morrey [82] and Bojarski [22]. In this case, solutions of (2.5) are k -quasiconformal homeomorphisms having distributional derivatives in L^p for some $p \geq 1$. More precise results in $L^p(\mathbb{C})$ spaces, with smooth dependence of solutions of the Beltrami equation (2.5) on the Beltrami coefficient μ , are due to Ahlfors and Bers [3]; see also Ahlfors [2, Chapter V] and Astala et al. [15]. We shall use the following result; see [15, Theorem 5.3.4] for the first part and [21, Theorem 2.1] for the second part.

Theorem 2.2. *An almost complex structure J of Hölder class $\mathcal{C}^{(k,\alpha)}$ ($k \in \mathbb{Z}_+$, $0 < \alpha < 1$) on a smooth surface X admits a J -holomorphic chart of class $\mathcal{C}^{(k+1,\alpha)}$ at any point of X . Hence, the smooth structure on X determined by J is $\mathcal{C}^{(k+1,\alpha)}$ compatible with the given smooth structure.*

The following result shows that the assumptions in our main results (Theorems 1.1, 1.3, 1.6) are independent of the choice of the smooth structure on X in the equivalence class of $\mathcal{C}^{(k+1,\alpha)}$ -equivalent structures. It will be used in the proof of Corollary 4.5.

Proposition 2.3. *Assume that X and Y are smooth surfaces and $\phi : Y \rightarrow X$ is a diffeomorphism of (local) class $\mathcal{C}^{(k+1,\alpha)}$ for some $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$. Let $\{J_b\}_{b \in B}$ be a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}$. Then, the family of complex structures $\{J'_b = \phi^* J_b\}_{b \in B}$ on Y is also of class $\mathcal{C}^{l,(k,\alpha)}$. Furthermore, if $f : B \times X \rightarrow \mathbb{C}$ is of class $\mathcal{C}^{l,s}$ for some $s \leq k + 1 + \alpha$ then the function $\tilde{f} : B \times Y \rightarrow \mathbb{C}$ given by $\tilde{f}(b, y) = f(b, \phi(y))$ is also of class $\mathcal{C}^{l,s}$ on $B \times Y$.*

Proof. For any point $y \in Y$ we have that $(J'_b)_y = (d\phi_y)^{-1} \circ (J_b)_{\phi(y)} \circ d\phi_y$. Let $A(y)$ denote the matrix of the differential $d\phi_y$ in a pair of smooth local trivialisations of the

tangent bundle on X and Y , and let $[J_b(x)]$ denote the matrix of J_b at $x \in X$. The above then says that

$$[J'_b(y)] = A(y)^{-1}[J_b(\phi(y))]A(y)$$

where the operation is the matrix product. By the assumption, the matrix $[J_b(x)]$ is of class $\mathcal{C}^{l,(k,\alpha)}$ on $B \times X$. It is elementary to see that inserting $x = \phi(y)$, with ϕ of class $\mathcal{C}^{(k+1,\alpha)}$, gives a matrix function $[J_b(\phi(y))]$ of the same class $\mathcal{C}^{l,(k,\alpha)}$ on $B \times Y$. Finally, the conjugation by the matrix function $A(y)$ of class $\mathcal{C}^{(k,\alpha)}$ preserves the class $\mathcal{C}^{l,(k,\alpha)}$. The last part of the proposition is a simple exercise. $\square$

3. The Cauchy and Beurling transforms on open Riemann surfaces

In this section, we consider regularity properties of the Cauchy and Beurling transforms on smoothly bounded relatively compact domains in open Riemann surfaces. Theorem 3.2 is an important analytic ingredient for solving the Beltrami equation on such domains; see Theorems 4.1 and 4.3.

Let X be an open Riemann surface. Fix a holomorphic immersion $z = u + iv : X \rightarrow \mathbb{C}$ (see [56]) and let $d\sigma = du dv$ denote the associated area measure on X . Given a differentiable function $f : U \rightarrow \mathbb{C}$ on a domain $U \subset X$, its derivatives f_z and $f_{\bar{z}}$ given by (2.6) are well-defined functions on U . The pullback of the Cauchy kernel $C(\zeta, z) = \frac{dz}{z - \zeta}$ on $\mathbb{C}$ by the immersion $z : X \rightarrow \mathbb{C}$ is a Cauchy-type kernel on X with the correct behaviour near the diagonal $D_X = \{(x, x) : x \in X\}$ (see (3.2)), but with additional poles if z is not injective. Since D_X has a basis of Stein neighbourhoods in $X \times X$ and $X \times X \setminus D_X$ is also Stein, one can remove the extra poles by solving a Cousin problem (see Scheinberg [92, Lemma 2.1]). This gives a meromorphic 1-form on $X \times X$ of the form

$$\omega(q, x) = \xi(q, x)dz(x) \quad \text{for } q, x \in X, \quad (3.1)$$

where $dz(x)$ denotes the restriction of dz to $T_x X$, ξ is a meromorphic function on $X \times X$ which is holomorphic on $X \times X \setminus D_X$, and the 1-form $\omega(q, \cdot)$ has a simple pole at $q \in X$ with residue 1. In a neighbourhood $U \subset X \times X$ of D_X the coefficient ξ of ω is of the form

$$\xi(q, x) = \frac{1}{z(x) - z(q)} + h(q, x), \quad (3.2)$$

where h is a holomorphic function on U . Such Cauchy kernels were constructed by Scheinberg [92] and Gauthier [51], following the work by Behnke and Stein [18, Theorem 3]. (See also Behnke and Sommer [17, p. 584] and [36, Remark 1, p. 141] for additional references.) Given a relatively compact smoothly bounded domain $\Omega \Subset X$, the usual argument using Stokes formula and the residue calculation gives the following Cauchy–Green formula for any $f \in \mathcal{C}^1(\bar{\Omega})$ and $q \in \Omega$:

$$\begin{aligned}
 f(q) &= \frac{1}{2\pi i} \int_{x \in b\Omega} f(x) \omega(q, x) - \frac{1}{2\pi i} \int_{x \in \Omega} \bar{\partial} f(x) \wedge \omega(q, x) \\
 &= \frac{1}{2\pi i} \int_{x \in b\Omega} f(x) \xi(q, x) dz(x) - \frac{1}{\pi} \int_{x \in \Omega} f_{\bar{z}}(x) \xi(q, x) d\sigma(x).
 \end{aligned}$$

If f is holomorphic in Ω , we obtain the Cauchy representation formula

$$f(q) = \frac{1}{2\pi i} \int_{x \in b\Omega} f(x) \omega(q, x), \quad q \in \Omega.$$

On the other hand, for a function $f \in \mathcal{C}_0^1(X)$ with compact support we have

$$f(q) = -\frac{1}{\pi} \int_{x \in X} f_{\bar{z}}(x) \xi(q, x) d\sigma(x), \quad q \in X. \quad (3.3)$$

To the Cauchy kernel ω we associate two transforms, defined for $\phi \in \mathcal{C}_0(X)$ and $q \in X$ by

$$P(\phi)(q) = -\frac{1}{\pi} \int_X \phi(x) \xi(q, x) d\sigma(x), \quad (3.4)$$

$$S(\phi)(q) = \partial_z P(\phi)(q) = -\frac{1}{\pi} \int_X \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x). \quad (3.5)$$

Here, $\partial_{z(q)} \xi(q, x)$ denotes the ∂_z derivative (2.6) of the function $\xi(\cdot, x)$ at the point $q \in X$. The integral defining S is understood as the Cauchy principal value (compare with (3.8)), and the existence of $S(\phi)$ for any $\phi \in \mathcal{C}_0(X)$ follows from Theorem 3.2 (c).

The operator P is called the Cauchy–Green transform associated to the Cauchy kernel (3.1). The integral converges absolutely, and we have that

$$\partial_{\bar{z}} \circ P = \text{Id} = P \circ \partial_{\bar{z}} \quad \text{on } \mathcal{C}_0^1(X).$$

The second identity follows from (3.3). The first identity holds in a more precise form: for every relatively compact domain $\Omega \subset X$ with piecewise $\mathcal{C}^1$ boundary,

$$\partial_{\bar{z}} P(\phi) = \phi \quad \text{holds on } \Omega \text{ for every } \phi \in \mathcal{C}^1(\overline{\Omega}), \quad (3.6)$$

where the integral defining P is applied only over Ω . The equation (3.6) holds in the distributional sense for every integrable ϕ . It is obtained by following the proof in the case when $\xi(q, x) = \frac{1}{x-q}$ on $X = \mathbb{C}$, when P equals the standard Cauchy–Green operator on $\mathbb{C}$:

$$\mathcal{C}(\phi)(z) = -\frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(\zeta)}{\zeta - z} d\sigma(\zeta), \quad z \in \mathbb{C}. \quad (3.7)$$

The operator S (3.5) is an analogue of the Beurling transform $\mathcal{B}$ in the plane (see [2] or [15, p. 94]):

$$\mathcal{B}(\phi)(z) = -\frac{1}{\pi} \lim_{\epsilon \rightarrow 0} \int_{|z-\zeta|>\epsilon} \frac{\phi(\zeta)}{(z-\zeta)^2} d\sigma(\zeta), \quad z \in \mathbb{C}. \quad (3.8)$$

This is a singular convolution operator of Calderón–Zygmund type with nonintegrable kernel $-1/\pi z^2$. It extends to a bounded linear operator $L^p(\mathbb{C}) \rightarrow L^p(\mathbb{C})$ for every $1 < p < \infty$ (see [15, Corollary 4.5.1]). Its main property is that $\mathcal{B} \circ \partial_{\bar{z}} = \partial_z$ on $\mathcal{C}_0^1(\mathbb{C})$, so $\mathcal{B}$ interchanges the operators $\partial_{\bar{z}}$ and ∂_z . Likewise, it follows from (3.3)–(3.5) that

$$S(\phi_{\bar{z}}) = \partial_z P(\phi_{\bar{z}}) = \phi_z \quad \text{for every } \phi \in \mathcal{C}_0^1(\Omega). \quad (3.9)$$

In order to understand the local regularity properties of P and S , we look more closely at their kernel functions $\xi(q, x)$ and $\partial_{z(q)}\xi(q, x)$. We consider the latter one, which is more involved; the analogous analysis applies to the former. Let $U \subset X \times X$ be an open neighbourhood of the diagonal D_X on which (3.2) holds. On U we have

$$\partial_{z(q)}\xi(q, x) = \partial_{z(q)} \frac{1}{z(x) - z(q)} + \partial_{z(q)} h(q, x) = \frac{1}{(z(x) - z(q))^2} + \partial_{z(q)} h(q, x),$$

and $\partial_{z(q)} h(q, x)$ is holomorphic on U . Fix $q_0 \in X$ and choose a neighbourhood $V \subset X$ of q_0 such that $V \times V \subset U$ and the immersion $z : X \rightarrow \mathbb{C}$ is injective on V . Pick a smooth function $\chi : X \rightarrow [0, 1]$ with $\text{supp } \chi \subset V$ such that $\chi = 1$ on a smaller neighbourhood $V' \Subset V$ of q_0 . For $q \in V$ we have

$$\begin{aligned} S(\phi)(q) &= -\frac{1}{\pi} \int_X \chi(x) \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x) + \frac{1}{\pi} \int_X (\chi(x) - 1) \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x) \\ &= S_1(\phi)(q) + S_2(\phi)(q), \end{aligned}$$

where the operators S_1 and S_2 are given by

$$\begin{aligned} S_1(\phi)(q) &= -\frac{1}{\pi} \int_X \frac{\chi(x) \phi(x)}{(z(x) - z(q))^2} d\sigma(x), \\ S_2(\phi)(q) &= -\frac{1}{\pi} \int_X \chi(x) \phi(x) \partial_{z(q)} h(q, x) d\sigma(x) \\ &\quad + \frac{1}{\pi} \int_X (\chi(x) - 1) \phi(x) \partial_{z(q)} \xi(q, x) d\sigma(x). \end{aligned}$$

In the complex coordinate z on V , $S_1(\phi) = \mathcal{B}(\chi\phi)$ is the Beurling operator applied to $\chi\phi$, while S_2 has smooth kernel. The same construction can be carried out with the operator P .

The conclusion is that the operators P and S have the same local regularity properties as their classical models $\mathcal{C}$ (3.7) and $\mathcal{B}$ (3.8), respectively.

Let Ω be a relatively compact smoothly bounded domain in X . One may consider truncated operators P and S defined by integration over Ω . While P has the expected regularity on Hölder spaces, the regularity of S fails at the boundary points of Ω since the effect of averaging in (3.7) is lost. To circumvent this problem, we shall use a bounded linear extension operator, which we now describe.

Let dist denote the distance function on a surface X induced by a smooth Riemannian metric. We recall the basics concerning Hölder spaces; see Gilbarg and Trudinger [52, Sect. 4.1] for more information. Let Ω be a domain in X with piecewise $\mathcal{C}^1$ boundary. For $\alpha \in (0, 1)$, the Hölder $\mathcal{C}^\alpha(\Omega)$ norm of a function $f : \Omega \rightarrow \mathbb{C}$ is given by

$$\|f\|_\alpha = \sup_{x \in \Omega} |f(x)| + \sup\{|f(x) - f(y)|/\text{dist}(x, y)^\alpha : x, y \in \Omega, x \neq y\}, \quad (3.10)$$

and the associated Hölder space is $\mathcal{C}^\alpha(\Omega) = \{f : \Omega \rightarrow \mathbb{C} : \|f\|_\alpha < \infty\}$. Similarly we define the norm $\|f\|_{(k, \alpha)}$ for $k > 0$, and the corresponding Hölder space $\mathcal{C}^{(k, \alpha)}(\Omega)$, by adding to $\|f\|_\alpha$ in (3.10) the $\mathcal{C}^\alpha(\Omega)$ norms of partial derivatives of f of the highest order k . In particular, $\mathcal{C}^\alpha(\Omega) = \mathcal{C}^{(0, \alpha)}(\Omega)$. These spaces are Banach algebras with the pointwise product of functions. Compositions of $\mathcal{C}^{(k, \alpha)}$ maps with $k \geq 1$ are again of the same class (but this fails for $k = 0$). The inverse of a $\mathcal{C}^{(k, \alpha)}$ diffeomorphism with $k \geq 1$ is of the same class (see [21, Theorem 2.1]). Every function in $\mathcal{C}^{(k, \alpha)}(\Omega)$ has a unique extension to a function in $\mathcal{C}^{(k, \alpha)}(\overline{\Omega})$. We shall need the following lemma. (The analogous result holds in a smooth Riemannian manifold X of arbitrary dimension.)

Lemma 3.1. *Given a smoothly bounded relatively compact domain $\Omega \Subset X$ in a smooth open Riemannian surface X and a domain $\Omega' \subset X$ containing $\overline{\Omega}$, there is for every $k \in \mathbb{Z}_+$ and $0 \leq \alpha < 1$ a continuous linear extension operator $E : \mathcal{C}^{(k, \alpha)}(\Omega) \rightarrow \mathcal{C}_0^{(k, \alpha)}(\Omega')$ with range in the space of compactly supported functions in $\mathcal{C}^{(k, \alpha)}(\Omega')$.*

Proof. For domains in Euclidean spaces and $k \geq 1$, this is [52, Lemma 6.37]; it is clear from the construction that one obtains a linear extension operator. We can reduce to this case by noting that every component S of $\partial\Omega$ has a neighbourhood $U \subset \Omega'$ smoothly diffeomorphic to an annulus in $\mathbb{R}^2$, with S corresponding to the unit circle. (See Bellettini [19, Theorem 1.18, p. 14].) Assume now that $k = 0$. Using the above notation, let $\tau : U \rightarrow S$ denote the smooth radial projection of the annulus onto the circle S . Set $U_+ = U \setminus \Omega$, and let $\chi : \overline{\Omega} \cup U \rightarrow [0, 1]$ be a smooth function which equals 1 on $\overline{\Omega}$ and the restriction $\chi|_{U_+}$ has compact support. Given $f \in \mathcal{C}(\overline{\Omega})$, we let $E(f) : \overline{\Omega} \cup U \rightarrow \mathbb{C}$ be defined by $E(f)(x) = f(x)$ for $x \in \overline{\Omega}$ and $E(f)(x) = \chi(x)f(\tau(x))$ for $x \in U_+$. We

perform the same construction on each of the finitely many boundary components of Ω . $\square$

With the notation of Lemma 3.1 we define the operators P_Ω and S_Ω on $\phi \in \mathcal{C}(\overline{\Omega})$ and $q \in \overline{\Omega}$ by

$$P_\Omega(\phi)(q) = -\frac{1}{\pi} \int_{x \in \Omega'} E(\phi)(x) \xi(q, x) d\sigma(x), \quad (3.11)$$

$$S_\Omega(\phi)(q) = -\frac{1}{\pi} \int_{x \in \Omega'} E(\phi)(x) \partial_{z(q)} \xi(q, x) d\sigma(x). \quad (3.12)$$

Theorem 3.2. *Let X be an open Riemann surface with a Cauchy kernel (3.1), (3.2).*

- (a) $P_\Omega : \mathcal{C}^{(k, \alpha)}(\overline{\Omega}) \rightarrow \mathcal{C}^{(k+1, \alpha)}(\overline{\Omega})$ is a bounded linear operator for every $0 < \alpha < 1$ and $k \in \mathbb{Z}_+$, and it satisfies $\partial_{\bar{z}} P_\Omega(\phi) = \phi$ on $\overline{\Omega}$ for every $\phi \in \mathcal{C}^\alpha(\overline{\Omega})$.
- (b) $S_\Omega : \mathcal{C}^{(k, \alpha)}(\overline{\Omega}) \rightarrow \mathcal{C}^{(k, \alpha)}(\overline{\Omega})$ is a bounded linear operator for every $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$, and it satisfies $S_\Omega(\phi) = \partial_z P_\Omega(\phi)$ for every $\phi \in \mathcal{C}^{(k, \alpha)}(\overline{\Omega})$.
- (c) S_Ω extends to a bounded linear operator $L^p(\Omega) \rightarrow L^p(\Omega)$ for every $1 < p < \infty$.

Proof. We have seen above that the operators P and S have the same local regularity properties as their classical models $\mathcal{C}$ (3.7) and $\mathcal{B}$ (3.8), respectively. Part (a) then follows from [15, Theorem 4.7.2] and (3.3), part (b) from [15, Theorem 4.7.1] and (3.12), and part (c) from [15, Corollary 4.5.1]. (Part (c) is only stated to justify the existence of the integral for continuous functions.) The analogous properties hold on Sobolev spaces $W^{k, p}$, but we shall not need them. $\square$

4. Quasiconformal deformations of the identity map

In this section, $z : X \rightarrow \mathbb{C}$ denotes a holomorphic immersion from an open Riemann surface X (see [56]). We shall call the pair (X, z) a *Riemann domain over $\mathbb{C}$* . Given a $\mathcal{C}^1$ function $f : X \rightarrow \mathbb{C}$, the derivatives $f_z = \partial f / dz$ and $f_{\bar{z}} = \bar{\partial} f / d\bar{z}$ (2.6) are well-defined continuous functions on X . We endow X with the smooth structure determined by its Riemann surface structure and define the Hölder norms on domains $\Omega \Subset X$ with respect to a fixed smooth Riemannian metric on X . The following result gives a solution of the Beltrami equation on any smoothly bounded relatively compact domain for Beltrami coefficients with sufficiently small Hölder norm. Recall that $\mathbb{D}$ is the unit disc in $\mathbb{C}$.

Theorem 4.1. *Let Ω be a smoothly bounded relatively compact domain in a Riemann domain (X, z) . For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is a constant $c = c(\Omega, k, \alpha) > 0$ such that for every $\mu \in \mathcal{C}^{(k, \alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{(k, \alpha)} < c$ there is function $f = f(\mu) \in \mathcal{C}^{(k+1, \alpha)}(\Omega)$ solving the Beltrami equation $f_{\bar{z}} = \mu f_z$, with $f(\mu)$ depending analytically on μ and satisfying $f(0) = z|_\Omega$.*

Interpreting $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ as a complex structure J_μ on Ω (see (2.2)–(2.4)), with J_0 coinciding with the initial complex structure, the function $f(\mu) : \Omega \rightarrow \mathbb{C}$ is J_μ -holomorphic, and it is an immersion for μ close to 0 since $f(\mu)$ is then close to $f(0) = z|_\Omega$ in $\mathcal{C}^{(k+1,\alpha)}(\Omega)$. Thus, $(\Omega, J_\mu, f(\mu))$ is a family of Riemann domains over $\mathbb{C}$ depending analytically on μ in a neighbourhood of $\mu = 0$.

Proof of Theorem 4.1. The idea is inspired by the proof of the corresponding result for $\mu \in L^p(\mathbb{C})$ ($p > 2$), due to Ahlfors and Bers [3, Theorem 4].

Recall that the algebra $\text{Lin}(E)$ of all bounded linear operators on a Banach space E , with functional composition as multiplication and the operator norm, is a unital Banach algebra (see Conway [27]). In our case, E will be the Banach space $\mathcal{C}^{(k,\alpha)}(\Omega)$.

We look for a solution of the Beltrami equation $f_{\bar{z}} = \mu f_z$ on Ω in the form

$$f = f(\mu) = z|_\Omega + P(\phi), \quad \phi \in \mathcal{C}^{(k,\alpha)}(\Omega). \quad (4.1)$$

Here, $P = P_\Omega : \mathcal{C}^{(k,\alpha)}(\Omega) \rightarrow \mathcal{C}^{(k+1,\alpha)}(\Omega)$ is the Cauchy–Green operator (3.11). Thus, $\phi = 0$ corresponds to $f = z|_\Omega$. By Theorem 3.2 (a), P is a continuous linear operator. We have that

$$f_{\bar{z}} = \partial_{\bar{z}} P(\phi) = \phi, \quad f_z = 1 + \partial_z P(\phi) = 1 + S(\phi),$$

where $S = S_\Omega \in \text{Lin}(\mathcal{C}^{(k,\alpha)}(\Omega))$ is the Beltrami operator (3.12). The first identity follows from Theorem 3.2 (a), and the second one follows from the definition (3.12) of S . Inserting the above expressions in the Beltrami equation $f_{\bar{z}} = \mu f_z$ gives the following equation for ϕ :

$$\phi = \mu(S(\phi) + 1) = \mu S(\phi) + \mu \iff (I - \mu S)\phi = \mu, \quad (4.2)$$

where I denotes the identity map on $\mathcal{C}^{(k,\alpha)}(\Omega)$. By Theorem 3.2 (b), S is a bounded linear operator on $\mathcal{C}^{(k,\alpha)}(\Omega)$. Hence, for μ small enough we have $\|\mu S\|_{(k,\alpha)} < 1$, so the operator $I - \mu S$ is invertible with

$$\Theta(\mu) = (I - \mu S)^{-1} = \sum_{j=0}^{\infty} (\mu S)^j \in \text{Lin}(\mathcal{C}^{(k,\alpha)}(\Omega)). \quad (4.3)$$

The equation (4.2) then has a unique solution $\phi = \Theta(\mu)\mu = \sum_{j=0}^{\infty} (\mu S)^j \mu$. Inserting into (4.1) gives the following solution to the Beltrami equation $f_{\bar{z}} = \mu f_z$ on Ω :

$$f(\mu) = z|_\Omega + P(\Theta(\mu)\mu) = z|_\Omega + P((I - \mu S)^{-1}\mu) \in \mathcal{C}^{(k+1,\alpha)}(\Omega). \quad (4.4)$$

By standard results (see Mujica [83, 29.3 Theorem]), $\Theta(\mu)$ and hence $f(\mu)$ (4.4) depend analytically on μ , and the other properties of f are obvious from the construction. $\square$

Remark 4.2. The fact that the map $\mu \rightarrow f(\mu)$ is analytic implies that if $\mu(t)$ is a function of class $\mathcal{C}^l$ on an open set $t \in U \subset \mathbb{R}^m$ (or $t \in U \subset \mathbb{C}^n$), with $\|\mu(t)\|_{(k,\alpha)} < c$ for all $t \in U$, then the map $U \ni t \mapsto f(\mu(t)) \in \mathcal{C}^{(k+1,\alpha)}(\Omega)$ is also of class $\mathcal{C}^l(U)$. This holds for any $l \in \{0, 1, \dots, \infty\}$ as well as for real analytic or holomorphic dependence on t . The analogous statement was proved by Ahlfors and Bers [3, Theorem 2] for solutions of the Beltrami equation with $\mu \in L^p(\mathbb{C})$ for $p > 2$.

Given an open Riemann surface (X, J) and a domain $\Omega \subset X$, a family of smooth diffeomorphisms $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ ($b \in B$) induces a family of complex structures $J_b = \Phi_b^* J$ on Ω . The following result shows that the converse holds on any smoothly bounded relatively compact domain $\Omega \Subset X$ for sufficiently small variations of the complex structure.

Theorem 4.3. *Assume that (X, z) is a Riemann domain over $\mathbb{C}$, Ω is a relatively compact smoothly bounded domain in X , and $a_1, \dots, a_m \in \Omega$ are distinct points. For any $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$ there is a constant $c = c(k, \alpha) > 0$ such that for every function $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega, \mathbb{D})$ with $\|\mu\|_{(k,\alpha)} < c$ there is a μ -conformal diffeomorphism $\Phi_\mu : \Omega \rightarrow \Phi_\mu(\Omega) \subset X$ in $\mathcal{C}^{(k+1,\alpha)}(\Omega, X)$, depending analytically on μ , such that $\Phi_0 = \text{Id}_\Omega$ and $\Phi_\mu(a_j) = a_j$ for all such μ and $j = 1, \dots, m$.*

Proof. If $c > 0$ is small enough then for every $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{(k,\alpha)} < c$ the function $f(\mu) \in \mathcal{C}^{(k,\alpha)}(\Omega)$, furnished by Theorem 4.1, is so close to the holomorphic immersion $f(0) = z|_\Omega : \Omega \rightarrow \mathbb{C}$ in $\mathcal{C}^{(k+1,\alpha)}(\Omega)$ that it is an immersion. If $f(\mu)$ is sufficiently close to $f(0) = z|_\Omega$, we can lift it with respect to the holomorphic immersion $z : X \rightarrow \mathbb{C}$ to a unique diffeomorphism $\Phi_\mu : \Omega \rightarrow \Phi_\mu(\Omega) \subset X$ in $\mathcal{C}^{(k+1,\alpha)}(\Omega, X)$, close to $\Phi_0 = \text{Id}_\Omega$, such that

$$z \circ \Phi_\mu = f(\mu) \quad \text{holds on } \Omega. \quad (4.5)$$

To see this, pick $r > 0$ such that for any $q \in \overline{\Omega}$ the immersion $z : X \rightarrow \mathbb{C}$ is injective on the disc $U_r(q) \subset X$ of radius r around q in the metric $|dz|^2$. If $f(\mu)(q) \in \mathbb{C}$ is close enough to $z(q) \in \mathbb{C}$ (which holds if $c > 0$ is small enough), there is a unique point $p \in U_r(q)$ such that $z(p) = f(\mu)(q)$, and we set $\Phi_\mu(q) = p$. Thus, $\Phi_\mu(q)$ is the unique closest point to q among the points in the closed discrete set $z^{-1}(f(\mu)(q)) \subset X$, so Φ_μ is well-defined on $\overline{\Omega}$. This implies $z(\Phi_\mu(q)) = z(p) = f(\mu)(q)$, so (4.5) holds. Since Φ_μ is locally obtained by postcomposing the immersion $f(\mu) : \Omega \rightarrow \mathbb{C}$ with a local inverse of the J -holomorphic immersion $z : X \rightarrow \mathbb{C}$, Φ_μ is an immersion, its Beltrami coefficient is the same as that of $f(\mu)$, which is μ , and the regularity properties remain unchanged. It is easily seen that Φ_μ is injective if $f(\mu)$ is close enough to z , which holds if $c > 0$ is small enough.

This shows that $\Phi_\mu : \Omega \rightarrow \Phi_\mu(\Omega)$ is a family of μ -conformal diffeomorphisms in $\mathcal{C}^{(k+1,\alpha)}(\Omega, X)$ depending analytically on μ . The interpolation conditions $\Phi_\mu(a_j) = a_j$

are achieved as follows. For every $j = 1, \dots, m$ we choose a holomorphic vector field v_j on X which is nonzero at the point a_j and it vanishes at the points a_i for $i \in \{1, \dots, m\} \setminus \{j\}$. Let $t \rightarrow \psi_{j,t}$ denote the local flow of v_j for complex time t . If $c > 0$ is small enough, there is an open relatively compact domain $\Omega' \Subset X$ such that $\Phi_\mu(\Omega) \subset \Omega'$ holds for all $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{(k,\alpha)} < c$. Choose a bigger domain $\Omega'' \Subset X$ such that $\overline{\Omega'} \subset \Omega''$. Since $\overline{\Omega''}$ is compact, there is a $t_0 > 0$ such that the holomorphic map $\Psi_t := \psi_{1,t_1} \circ \dots \circ \psi_{m,t_m} : \Omega'' \rightarrow X$ is well-defined for all $t = (t_1, \dots, t_m) \in \mathbb{C}^m$ in the polydisc $\Delta_{t_0}^m = \{|t_j| < t_0, j = 1, \dots, m\}$. For every $t \in \Delta_{t_0}^m$ the map Ψ_t , being a composition of flows of holomorphic vector fields, is biholomorphic onto its image $\Psi_t(\Omega'') \subset X$. The choice of the vector fields v_j ensures, by the inverse function theorem, that for every m -tuple of points $a' = \{a'_1, \dots, a'_m\} \subset \Omega$ such that a'_j is close enough to a_j for $j = 1, \dots, m$ there is a unique $t = t(a') \in \Delta_{t_0}^m$ close to the origin such that $\Psi_t(a_j) = a'_j$ for $j = 1, \dots, m$, and the map $a' \mapsto t(a')$ is holomorphic. Let $a'(\mu) = (\Phi_\mu(a_1), \dots, \Phi_\mu(a_m))$. Then, for all $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{(k,\alpha)} < c$ for a small enough $c > 0$, the injective holomorphic map $\Psi_{t(a'(\mu))}^{-1} : \Omega' \rightarrow X$ sends the point $\Phi_\mu(a_j)$ back to a_j for $j = 1, \dots, m$. Replacing Φ_μ by $\Psi_{t(a'(\mu))}^{-1} \circ \Phi_\mu$ completes the proof. $\square$

Remark 4.4. Note that every diffeomorphism $\Phi_\mu : \Omega \rightarrow \Phi_\mu(\Omega) \subset X$ in Theorem 4.3 is homotopic to the identity map on Ω by the homotopy $[0, 1] \ni t \mapsto \Phi_{t\mu}$. In particular, if the domain Ω is Runge in X then so is $\Phi_\mu(\Omega)$. In fact, a domain Ω in a Riemann surface X is Runge if and only if the inclusion-induced homomorphism $H_1(\Omega, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z})$ of the first homology groups is injective, and this condition is clearly invariant under homotopies.

Corollary 4.5. Let B , X , and $\mathcal{J} = \{J_b\}_{b \in B}$ be as in Theorem 1.1, with $\mathcal{J}$ of class $\mathcal{C}^{l,(k,\alpha)}$ ($k \in \mathbb{Z}_+$, $l \leq k+1$, $0 < \alpha < 1$). Given $b_0 \in B$ and a smoothly bounded domain $\Omega \Subset X$, there are a neighbourhood $B_0 \subset B$ of b_0 and a map $\Phi : B_0 \times \Omega \rightarrow B_0 \times X$ of the form

$$\Phi(b, x) = (b, \Phi_b(x)), \quad b \in B_0, x \in \Omega \quad (4.6)$$

such that Φ_{b_0} is the identity on Ω and $\Phi_b : \Omega \rightarrow \Phi_b(\Omega)$ is a (J_b, J_{b_0}) -biholomorphic map in $\mathcal{C}^{(k+1,\alpha)}(\Omega, X)$ which is of class $\mathcal{C}^l$ with respect to $b \in B_0$. If A is a finite subset of Ω , we can choose Φ such that $\Phi_b(a) = a$ holds for all $a \in A$ and $b \in B_0$.

Proof. Choose a J_{b_0} -holomorphic immersion $z : X \rightarrow \mathbb{C}$. Theorem 2.2 and Proposition 2.3 imply that the family $\mathcal{J}$ is also of class $\mathcal{C}^{l,(k,\alpha)}$ in the smooth structure on X determined by J_{b_0} . Hence there is a family of Beltrami multipliers $\mu_b : X \rightarrow \mathbb{D}$ ($b \in B$) of class $\mathcal{C}^{l,(k,\alpha)}$, with $\mu_{b_0} = 0$, such that μ_b represents J_b (see (2.2) and (2.4)). Pick $c > 0$ such that Theorem 4.3 applies to all $\mu \in \mathcal{C}^{(k,\alpha)}(\Omega)$ with $\|\mu\|_{(k,\alpha)} < c$. By continuity of the map $B \ni b \mapsto \mu_b$ there is a neighbourhood $B_0 \subset B$ of b_0 such that $\|\mu_b|_\Omega\|_{(k,\alpha)} < c$ for all $b \in B_0$. Let $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ for $b \in B_0$ be a family of μ_b -conformal diffeomorphisms furnished by Theorem 4.3. The map Φ in (4.6) then satisfies the corollary. $\square$

5. Runge and Mergelyan theorems on families of open Riemann surfaces

In this section we prove Theorems 1.1 and 1.3. We begin with the former.

Proof of Theorem 1.1. We first consider the basic case $l = 0$ and arbitrary $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$. By slightly increasing K and L and adding to K small pairwise disjoint discs around the finitely many points in $A \cap (L \setminus K)$, we may assume that $A \cap L$ is contained in the interior of K . Given a continuous function $\epsilon : B \rightarrow (0, +\infty)$, we shall prove that for any compact Runge set $L \subset X$ containing K in its interior there exist an open set $\Omega \subset X$ containing L and a continuous function $F \in \mathcal{C}(B \times \Omega)$ satisfying the following conditions for every $b \in B$.

- (a) The function $F_b = F(b, \cdot) : \Omega \rightarrow \mathbb{C}$ is J_b -holomorphic.
- (b) $\sup_{x \in K} |F_b(x) - f_b(x)| < \epsilon(b)$.
- (c) $F_b - f_b$ vanishes in the points of the finite set $A' = A \cap L = \{a_1, \dots, a_m\}$.

Approximation in the fine $\mathcal{C}^{(k+1, \alpha)}$ -topology will follow from condition (b) in view of the Cauchy estimates. A function $B \times X \rightarrow \mathbb{C}$ satisfying Theorem 1.1 for $l = 0$ is then obtained by induction with respect to an exhaustion of X by an increasing family of compact Runge sets.

Given a point $b_0 \in B$, it suffices to find an open neighbourhood $B_0 \subset B$ of b_0 and a function $F : B_0 \times \Omega \rightarrow \mathbb{C}$ satisfying conditions (a)–(c) for all $b \in B_0$. Since B is Hausdorff and paracompact, this will give a locally finite cover of B by open sets B_j and functions $F_j : B_j \times \Omega \rightarrow \mathbb{C}$ satisfying conditions (a)–(c) for $b \in B_j$. Choose a partition of unity $1 = \sum_j \chi_j$ with $\text{supp } \chi_j \subset B_j$ for every j . The function $F : B \times \Omega \rightarrow \mathbb{C}$, defined by

$$F(b, x) = \sum_j \chi_j(b) F_j(b, x) \quad \text{for } b \in B \text{ and } x \in \Omega, \quad (5.1)$$

then clearly satisfies conditions (a)–(c).

With these reductions in mind, we consider the problem near a parameter value $b_0 \in B$. We endow X with the Riemann surface structure determined by J_{b_0} . By Theorem 2.2, the smooth structure on X , induced by J_{b_0} , is $\mathcal{C}^{(k+1, \alpha)}$ -compatible with the given smooth structure on X . Choose a relatively compact, smoothly bounded domain $\Omega \Subset X$ with $L \subset \Omega$. By Corollary 4.5 there are a neighbourhood $B'_0 \subset B$ of b_0 and a continuous family of (J_b, J_{b_0}) -biholomorphic maps $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ ($b \in B'_0$) in $\mathcal{C}^{(k+1, \alpha)}(\Omega, X)$ such that $\Phi_{b_0} = \text{Id}_\Omega$ and $\Phi_b(a) = a$ holds for all $a \in A$ and $b \in B'_0$. Choose a compact Runge set $K' \subset X$ containing K in its interior such that f_{b_0} is holomorphic on a neighbourhood of K' . Pick a neighbourhood $B_0 \subset B'_0$ of b_0 such that $\Phi_b(K) \subset \overset{\circ}{K}'$ holds for all $b \in B_0$. By Runge theorem in open Riemann surfaces [18], we can approximate f_{b_0} uniformly on K' by a J_{b_0} -holomorphic function $F_{b_0} : X \rightarrow \mathbb{C}$. The function $F_b = F_{b_0} \circ \Phi_b : \Omega \rightarrow \mathbb{C}$ ($b \in B_0$) is then J_b -holomorphic, it depends continuously on $b \in B_0$, and F_b is as close

as desired to f_b uniformly on K if b is close enough to b_0 and F_{b_0} is close enough to f_{b_0} on K' . Indeed, for $x \in K$ we have

$$\begin{aligned} |F_b(x) - f_b(x)| &\leq |F_{b_0} \circ \Phi_b(x) - f_{b_0} \circ \Phi_b(x)| \\ &\quad + |f_{b_0} \circ \Phi_b(x) - f_{b_0}(x)| + |f_{b_0}(x) - f_b(x)|, \end{aligned}$$

and each term on the right hand side is as small as desired if b is close enough to b_0 and F_{b_0} is close enough to f_{b_0} on K' . Hence, shrinking the neighbourhood B_0 around b_0 if necessary, the family $\{F_b\}_{b \in B_0}$ satisfies conditions (a) and (b). Furthermore, as the family of J_{b_0} -holomorphic functions $f_b \circ \Phi_b^{-1}$ ($b \in B_0$) is uniformly close to F_{b_0} on the family of compact sets $\Phi_b^{-1}(K')$, approximation of f_b by F_b on K in the $\mathcal{C}^{(k+1, \alpha)}$ -topology follows from uniform approximation on a bigger compact Runge set, containing K in its interior, in view of the Cauchy estimates.

This proves Theorem 1.1 in the case $l = 0$, except for the interpolation condition (c) which will be dealt with later. The same proof applies to variable families of compact Runge sets in a family of open Riemann surfaces as in the following definition. For later purposes (see in particular in Lemma 5.3), we introduce this notion in the bigger generality of families of complex manifolds.

Definition 5.1. Assume that B is a topological space, X is a smooth manifold of real dimension $2n \geq 2$, and $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of integrable complex structures on X . A closed subset K of $B \times X$ is proper over B if the following two conditions hold.

- (1) For every $b \in B$ the fibre $K_b = \{x \in X : (b, x) \in K\}$ is compact. (K_b may be empty.)
- (2) For every $b_0 \in B$ and open set $U \subset X$ containing K_{b_0} there is a neighbourhood $B_0 \subset B$ of b_0 such that $K_b \subset U$ for all $b \in B_0$.

The set K is Runge in $B \times X$ if it is proper over B and the fibre K_b is holomorphically convex in X with respect to the complex structure J_b for every $b \in B$.

Condition (2) means that the compact fibres $K_b \subset X$ of K are upper semicontinuous with respect to $b \in B$. It is easily seen that if B is Hausdorff and locally compact then a closed subset $K \subset B \times X$ is proper over B if and only if the restriction of the projection $\pi : B \times X \rightarrow B$ to K is a proper map $\pi|_K : K \rightarrow B$ (that is, the inverse image of any compact set in B is compact).

Our proof of Theorem 1.1 in the case $l = 0$ gives the following more general result concerning approximation on Runge sets in families of open Riemann surfaces. For the interpolation statement on $Q \times X$, see Remark 1.7.

Corollary 5.2. Assume that B is a paracompact Hausdorff space, X is a smooth open orientable surface, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^{0, (k, \alpha)}$ for some $k \geq 0$ and $0 < \alpha < 1$, $K \subset B \times X$ is a closed Runge subset (see Definition 5.1),

and Q is a closed subset of B . Given an open subset $U \subset B \times X$ containing K and a continuous $\mathcal{J}$ -holomorphic function $f : U \cup (Q \times X) \rightarrow \mathbb{C}$, we can approximate f in the fine $\mathcal{C}^{0,(k+1,\alpha)}$ topology on K by $\mathcal{J}$ -holomorphic functions $F : B \times X \rightarrow \mathbb{C}$ of class $\mathcal{C}^{0,(k+1,\alpha)}$ satisfying $F = f$ on $Q \times X$.

Next, we consider approximation in the $\mathcal{C}^{l,(k+1,\alpha)}$ -topology for any pair of integers $0 \leq l \leq k+1$. As before, locally in the parameter we can use Corollary 4.5 to reduce the approximation problem for a variable family of complex structures to the case of a moving family of compact Runge sets in a fixed complex structure. With future applications in mind, we consider a more general situation for a family of compact holomorphically convex sets in a Stein manifold X of arbitrary dimension.

Lemma 5.3. *Assume that B is a paracompact Hausdorff space if $l = 0$ and a manifold of class $\mathcal{C}^l$ if $l > 0$, X is a Stein manifold, K is a Runge subset of $B \times X$ (see Definition 5.1), $U \subset B \times X$ is an open set containing K , and $f : U \rightarrow \mathbb{C}$ is a function of class $\mathcal{C}^{l,0}(U)$ such that for every $b \in B$ the function $f_b = f(b, \cdot) : U_b = \{x \in X : (b, x) \in U\} \rightarrow \mathbb{C}$ is holomorphic. Then, $f \in \mathcal{C}^{l,\infty}(U)$ and for any $s \in \mathbb{Z}_+$, f can be approximated in the fine $\mathcal{C}^{l,s}$ topology on K by $\mathcal{C}^{l,\infty}$ functions $F : B \times X \rightarrow \mathbb{C}$ such that $F_b = F(b, \cdot) \in \mathcal{O}(X)$ for every $b \in B$. If B is a topologically closed $\mathcal{C}^l$ submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ (possibly with boundary), or a closed subset of $\mathbb{R}^n$ when $l = 0$, then f can be approximated in the fine $\mathcal{C}^{l,s}$ topology on K by holomorphic functions $F : \mathbb{C}^n \times X \rightarrow \mathbb{C}$.*

Remark 5.4. If B is a subset of $\mathbb{R}^n$ then a compact set $K \subset B \times X$ with $\mathcal{O}(X)$ -convex fibres K_b is $\mathcal{O}(\mathbb{C}^n \times X)$ -convex (see Remark 1.3 and Proposition 1.4 in [48]).

Proof of Lemma 5.3. The assumptions on the function f in the lemma clearly imply that it is of class $\mathcal{C}^{l,\infty}(U)$, so we may talk of $\mathcal{C}^{l,s}$ approximation for any $s \in \mathbb{Z}_+$. Fix a point $b_0 \in B$. Since K is Runge in $B \times X$, there are an open neighbourhood $B_0 \subset B$ of b_0 and neighbourhoods $K' \subset U' \subset X$ of K_{b_0} , where K' is a compact $\mathcal{O}(X)$ -convex set and U' is an open set, such that

$$K \cap (B_0 \times X) \subset B_0 \times \overset{\circ}{K}' \subset B_0 \times U' \subset U \cap (B_0 \times X). \quad (5.2)$$

Choose a smoothly bounded strongly pseudoconvex domain D in X with $K' \subset D \Subset U'$. On D , there is a Henkin–Ramirez type kernel $\omega(x, \zeta)$, which is holomorphic in $x \in D$ for every $\zeta \in bD$, such that every $h \in \mathcal{O}(\overline{D})$ can be represented on D by the integral $h(x) = \int_{\zeta \in bD} h(\zeta) \omega(x, \zeta)$, $x \in D$. (See Henkin and Leiterer [59] or Lieb and Michel [75]. When X is an open Riemann surface, we can use a Cauchy kernel (3.1) on X .) Applying this to the function f in the lemma, we thus have

$$f(b, x) = \int_{\zeta \in bD} f(b, \zeta) \omega(x, \zeta) \quad \text{for all } x \in D \text{ and } b \in B_0. \quad (5.3)$$

Fix $\epsilon > 0$. Approximating the integral for $b = b_0$ by Riemann sums and shrinking B_0 around b_0 if necessary gives a finite set of points $\zeta_i \in bD$ and functions $g_i \in \mathcal{O}(D)$, which come from the kernel $\omega(x, \zeta_i)$, such that

$$\left| f(b, x) - \sum_i f(b, \zeta_i) g_i(x) \right| < \epsilon \quad \text{for all } b \in B_0 \text{ and } x \in K'. \quad (5.4)$$

By the Oka–Weil theorem, we can approximate each g_i uniformly on K' by holomorphic functions $g_i : X \rightarrow \mathbb{C}$. This gives uniform approximation of f on $B_0 \times K'$ by continuous functions $F : B_0 \times X \rightarrow \mathbb{C}$ which are holomorphic on the fibres $\{b\} \times X$. It follows from (5.2) and the Cauchy estimates that $F - f$ can be arbitrarily small in $\mathcal{C}^{0,s}(K \cap (B_0 \times X))$ for a given $s \in \mathbb{Z}_+$.

In the case $l = 0$ and with B a paracompact Hausdorff space, the proof is completed just as above. Assume now that $l > 0$, so B is a manifold of class $\mathcal{C}^l$. As before, we cover K by open sets of the form $B_j \times U_j \subset B \times X$ such that the cover $\{B_j\}_j$ of B is locally finite, the set $\overline{B}_j$ is compact with $\mathcal{C}^l$ boundary for every j , and we have that

$$K_j := K \cap (\overline{B}_j \times X) \subset \overline{B}_j \times U_j \subset U.$$

Choose a $\mathcal{C}^l$ partition of unity $\{\chi_j\}_j$ on B subordinate to the cover $\{B_j\}_j$. Fix a j and choose the sets $K' \subset D$ as in the special case described above with $j = 0$. We represent f by an integral of the form (5.3) for $b \in \overline{B}_j$. Given $\epsilon > 0$, there are finitely many points $\zeta_i \in bD$ and functions $g_i \in \mathcal{O}(D)$ satisfying (5.4) for all $b \in \overline{B}_j$ and $x \in K'$. Furthermore, for any linear differential operator L of order $\leq l$ in the variable $b \in B_j$ we have that $Lf(b, x) = \int_{\zeta \in bD} Lf(b, \zeta) \omega(x, \zeta)$. By adding more points $\zeta_i \in bD$ to the Riemann sum if necessary, we approximate $Lf(b, x)$ uniformly on $\overline{B}_j \times K'$ by functions $\sum_i Lf(b, \zeta_i) g_i(x) = L \sum_i f(b, \zeta_i) g_i(x)$. Applying the Cauchy estimates in the x -variable, we see that f can be approximated in $\mathcal{C}^{l,s}(K_j)$ by functions $F_j : \overline{B}_j \times X \rightarrow \mathbb{C}$ of the form

$$F_j(b, x) = \sum_i h_{j,i}(b) g_{j,i}(x), \quad b \in \overline{B}_j, \quad x \in X,$$

where $h_{j,i} \in \mathcal{C}^l(\overline{B}_j)$ and $g_{j,i} \in \mathcal{O}(X)$. We define the function $F : B \times X \rightarrow \mathbb{C}$ by

$$F(b, x) = \sum_j \chi_j(b) F_j(b, x) = \sum_{j,i} \chi_j(b) h_{j,i}(b) g_{j,i}(x) \quad \text{for } b \in B \text{ and } x \in X. \quad (5.5)$$

Clearly, F approximates f to a given precision in the fine $\mathcal{C}^{l,s}$ topology on K provided that $F_j|_{K_j}$ is sufficiently close to $f|_{K_j}$ in $\mathcal{C}^{l,s}(K_j)$ for every j . (When differentiating $F(b, x)$ on the variable b , the functions χ_j get differentiated as well, so it is important to keep them fixed when approximating f by F_j in the $\mathcal{C}^{l,s}(K_j)$ topology.)

Finally, if B is a closed $\mathcal{C}^l$ submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$, possibly with boundary, we can apply [89, Theorem 1] by Range and Siu to approximate each function $\chi_j h_{j,i} \in \mathcal{C}^l(B)$ in

(5.5) (which has compact support contained in B_j) in the fine $\mathcal{C}^l(B)$ topology by an entire function $\tilde{h}_{j,i} \in \mathcal{O}(\mathbb{C}^n)$. (Another argument is to extend $\chi_j h_{j,i}$ from the submanifold $B \subset \mathbb{R}^n$ to a $\mathcal{C}^l$ function on $\mathbb{R}^n$ and then approximate it in the fine $\mathcal{C}^l(\mathbb{R}^n)$ topology by entire functions using Carleman's theorem [23].) The function $\tilde{F} = \sum_{j,i} \tilde{h}_{j,i} g_{j,i}$ is then holomorphic on $\mathbb{C}^n \times X$ and it approximates f in the fine $\mathcal{C}^{l,s}$ topology on K . For $l = 0$, the same holds if B is any closed subset of $\mathbb{R}^n$, which is seen by combining Tietze's extension theorem with Carleman's approximation theorem. $\square$

We now prove Theorem 1.1 for arbitrary pair of integers $k \geq 0$ and $0 \leq l \leq k+1$. Let $\mathcal{J} = \{J_b\}_{b \in B}$ and $K \subset X$ be as in the theorem, and pick a compact Runge set $L \subset X$ containing K in its interior. By the argument in the beginning of the proof, we may assume that K contains the finite set $A \cap L$ in its interior. Choose a smoothly bounded Runge domain $\Omega \Subset X$ such that $L \subset \Omega$. Fix a point $b_0 \in B$. By Corollary 4.5 there are a neighbourhood $B_0 \subset B$ of b_0 and a map

$$\Phi : B_0 \times \Omega \rightarrow \Phi(B_0 \times \Omega) \subset B_0 \times X, \quad \Phi(b, x) = (b, \Phi_b(x)) \quad (5.6)$$

in $\mathcal{C}^{l, (k+1, \alpha)}(B_0 \times \Omega, X)$ (hence, of class $\mathcal{C}^l$ jointly in both variables (b, x)) such that for every $b \in B_0$ the map $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ is a biholomorphism from (Ω, J_b) onto $(\Phi_b(\Omega), J_{b_0})$ satisfying

$$\Phi_b(a) = a \quad \text{for all } a \in A \cap L, \text{ and } \Phi_{b_0} = \text{Id}_\Omega. \quad (5.7)$$

Clearly, Φ has a continuous inverse $\Phi^{-1}(b, z) = (b, \psi(b, z))$, and if $l > 0$ then Φ^{-1} and hence ψ are of class $\mathcal{C}^l$ by the inverse function theorem. The observations in the following lemma are simple consequences of the chain rule and we leave the proof to the reader.

Lemma 5.5.

- (a) If Φ (5.6) is of class $\mathcal{C}^{l, k+1}$ and $l \leq k+1$, then Φ^{-1} is of class $\mathcal{C}^{l, k+1-l}$.
- (b) If $z = f(b, x)$ is of class $\mathcal{C}^{l, k}$ and $g(b, z)$ is of class $\mathcal{C}^{l, l+k}$, then $g(b, f(b, x))$ is of class $\mathcal{C}^{l, k}$.

Lemma 5.6. Assume that $0 \leq l \leq k+1$, the function $f : B_0 \times \Omega \rightarrow \mathbb{C}$ is of class $\mathcal{C}^{l, 0}$, $f(b, \cdot) : \Omega \rightarrow \mathbb{C}$ is J_b -holomorphic for every $b \in B_0$, and Φ is as above (5.6). Then, the function $F = f \circ \Phi^{-1} : \Phi(B_0 \times \Omega) \rightarrow \mathbb{C}$ is of class $\mathcal{C}^{l, \infty}$ in the smooth structure on X determined by J_{b_0} , $F(b, \cdot) : \Phi_b(\Omega) \rightarrow \mathbb{C}$ is J_{b_0} -holomorphic for every $b \in B_0$, and f is of class $\mathcal{C}^{l, (k+1, \alpha)}$. The analogous result holds for maps to any complex manifold in place of $\mathbb{C}$.

Proof. Clearly, F is continuous. Since the function $F(b, \cdot) = f(b, \psi(b, \cdot))$ is a composition of the (J_{b_0}, J_b) -holomorphic map $\psi(b, \cdot)$ and the J_b -holomorphic function $f(b, \cdot)$, $F(b, \cdot)$ is J_{b_0} -holomorphic for every $b \in B_0$. It follows that F is of class $\mathcal{C}^{0, \infty}$ in the complex

structure J_{b_0} on X , and hence of class $\mathcal{C}^{0,(k+1,\alpha)}$ in the original smooth structure on X (see Theorem 2.2). Since $f = F \circ \Phi$ and Φ is of class $\mathcal{C}^{l,(k+1,\alpha)}$, we infer that f is of class $\mathcal{C}^{0,(k+1,\alpha)}$. This proves the lemma for $l = 0$. Suppose now that $l > 0$. Then, ψ is of class $\mathcal{C}^l$. We shall prove that F is of class $\mathcal{C}^l$ in the variable $b \in B_0$, and hence of class $\mathcal{C}^{l,\infty}$ (since it is J_{b_0} -holomorphic in the space variable). We make the calculation in a local coordinate b of class $\mathcal{C}^l$ on B_0 , and we assume for simplicity of exposition that $B_0 = [0, 1] \subset \mathbb{R}$. On X , we use a J_{b_0} -holomorphic coordinate z . Differentiating the equation $F(b, z) = f(b, \psi(b, z))$ on b and denoting the partial derivatives by the lower case indices gives

$$F_b(b, z) = f_b(b, \psi(b, z)) + f_x(b, \psi(b, z))\psi_b(b, z). \quad (5.8)$$

Here, f_x denotes the total derivative of f with respect to a smooth local coordinate $x = (u, v)$ on X . This shows that $F_b(b, z)$ exists and is continuous in (b, z) . Since F is also holomorphic in z , it follows that $F \in \mathcal{C}^{1,\infty}$ and therefore $f = F \circ \Phi \in \mathcal{C}^{1,(k+1,\alpha)}$ (see Lemma 5.5 (b)). Suppose now that $l \geq 2$, so $k+1 \geq l \geq 2$, $\psi \in \mathcal{C}^2$, and $f \in \mathcal{C}^{2,0} \cap \mathcal{C}^{1,2}$. Differentiating the equation (5.8) on b gives

$$F_{bb} = f_{bb} + 2f_{bx}\psi_b + f_{xx}(\psi_b)^2 + f_x\psi_{bb},$$

and F_{bb} is continuous in (b, z) . Since it is holomorphic in z , it follows that $F \in \mathcal{C}^{2,\infty}$ and therefore $f \in \mathcal{C}^{2,(k+1,\alpha)}$. This process can be continued up to $l = k+1$ but not beyond. $\square$

We continue with the proof of Theorem 1.1. With Φ as in (5.6), the set

$$\tilde{K} = \Phi(B_0 \times K) \subset B_0 \times X$$

is Runge in $B_0 \times X$. Indeed, its fibre $\tilde{K}_b = \Phi_b(K)$ is compact and Runge in $\Phi_b(\Omega)$ for every $b \in B_0$, and since $\Phi_b(\Omega)$ is Runge in X (see Remark 4.4), $\tilde{K}_b$ is Runge in X as well. Furthermore, the fibres $\tilde{K}_b$ depend continuously on $b \in B_0$. Recall that U is an open neighbourhood of $B \times K$ and $f : U \rightarrow \mathbb{C}$ is a $\mathcal{J}$ -holomorphic function of class $\mathcal{C}^{l,0}$. Pick an open subset $V \subset B_0 \times X$ such that $\bar{V} \cap (B_0 \times X) \subset U \cap (B_0 \times \Omega)$ and set $\tilde{V} = \Phi(V) \subset B_0 \times X$. We have $f = \tilde{f} \circ \Phi$ where by Lemma 5.6 the function $\tilde{f} = f \circ \Phi^{-1} : \tilde{V} \rightarrow \mathbb{C}$ is fibrewise J_{b_0} -holomorphic and of class $\mathcal{C}^{l,\infty}$. By Lemma 5.3, for any given $s \in \mathbb{Z}_+$ we can approximate $\tilde{f}$ in $\mathcal{C}^{l,s}(\tilde{K})$ by a fibrewise J_0 -holomorphic function $\tilde{F} : B_0 \times X \rightarrow \mathbb{C}$ of class $\mathcal{C}^{l,\infty}$. If s is chosen big enough and the approximation is close enough then the function $F = \tilde{F} \circ \Phi : B_0 \times \Omega \rightarrow \mathbb{C}$ is $\mathcal{J}$ -holomorphic, of class $\mathcal{C}^{l,(k+1,\alpha)}$, and it approximates f in the $\mathcal{C}^{l,(k+1,\alpha)}$ topology on $B_0 \times K$. This gives a locally finite open cover B_j of B such that f can be approximated as closely as desired in the $\mathcal{C}^{l,(k+1,\alpha)}$ topology on $B_j \times K$ by $\mathcal{J}$ -holomorphic functions $F_j : B_j \times \Omega \rightarrow \mathbb{C}$ of class $\mathcal{C}^{l,(k+1,\alpha)}$. Choose a $\mathcal{C}^l$ partition of unity $\{\chi_j\}_j$ on B with $\text{supp } \chi_j \subset B_j$ for every j . Assuming that F_j is close enough to f in $\mathcal{C}^{l,(k+1,\alpha)}(B_j \times K)$ for every j , the

$\mathcal{J}$ -holomorphic function $F : B \times \Omega \rightarrow \mathbb{C}$ defined by (5.1) is of class $\mathcal{C}^{l,(k+1,\alpha)}$ and it satisfies the required approximation condition.

It remains to obtain the interpolation conditions (c) at the points of $A' = A \cap L = \{a_1, \dots, a_m\} \subset \mathring{K}$. It suffices to explain this in the local situation given by Lemma 5.3; the subsequent steps in the proof preserve this condition up to order $k+1$. In view of (5.6) and (5.7), the points of A' are fixed under the maps Φ_b . Choose $r \in \mathbb{Z}_+$ and set $n = m(r+1)$; this is the complex dimension of the space of complex r -jets of holomorphic functions on X at the points of A' . By the classical function theory on open Riemann surfaces, we can find a family of J_{b_0} -holomorphic functions $\xi_t : X \rightarrow \mathbb{C}$, depending holomorphically on $t \in \mathbb{C}^n$, such that $\xi_0 = 0$ and for every collection of r -jets at the points of A' there is precisely one member ξ_t of this family which assumes these r -jets at the given points. Let F be a function in (5.5) which approximates f to a given precision in the fine $\mathcal{C}^{l,(k+1,\alpha)}$ topology on $B \times K$. Hence, the r -jets of the function $F_b = F(b, \cdot)$ at the points of A' are close to the respective r -jets of $f_b = f(b, \cdot)$ for any $b \in B$. We subtract from each F_b the appropriate uniquely determined member of the family ξ_t so that the r -jets of the new function at the points of A' agree with those of f_b (i.e., the interpolation condition (c) holds.) This does not affect the approximation condition (b) very much since the jets of ξ_t in question are close to those of the zero function, and hence ξ_t is close to zero on K . This completes the proof of Theorem 1.1. $\square$

Proof of Theorem 1.3. Let $f : B \times K \rightarrow \mathbb{C}$ be as in the theorem. Set $f_b = f(b, \cdot) : K \rightarrow \mathbb{C}$ for any $b \in B$. Fix $b_0 \in B$. Choose a smoothly bounded domain $\Omega \Subset X$ containing K . Let $B_0 \subset B$ be a neighbourhood of b_0 and

$$\Phi : B_0 \times \Omega \rightarrow \Phi(B_0 \times \Omega) \subset B_0 \times X$$

be a map of class $\mathcal{C}^{0,(k+1,\alpha)}$ furnished by Corollary 4.5. Thus, $\Phi(b, x) = (b, \Phi_b(x))$ where $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ is a biholomorphism from (Ω, J_b) onto $(\Phi_b(\Omega), J_{b_0})$ satisfying $\Phi_b(a) = a$ for all $a \in A \cap L$, and $\Phi_{b_0} = \text{Id}_\Omega$. Applying the Bishop–Mergelyan theorem [20] we can approximate the function $f_{b_0} : K \rightarrow \mathbb{C}$ uniformly on K by functions $\tilde{f}_{b_0}$ defined on open neighbourhoods $U \subset X$ of K . For b close enough to b_0 the function $\tilde{f}_b := \tilde{f}_{b_0} \circ \Phi_b$ is J_b -holomorphic on a neighbourhood of K and it approximates f_b to a given precision uniformly on K . This gives local approximation near any given point of B , and the proof can be concluded by applying a continuous partition of unity on B as in the proof of Theorem 1.1. Interpolation in the points of the finite set $A \subset \mathring{K}$ is handled as in the proof of Theorem 1.1. $\square$

The same proof gives the following version of Theorem 1.3 with fine $\mathcal{C}^0$ approximation on proper subsets of $B \times X$; compare with the Runge approximation in Corollary 5.2.

Corollary 5.7. Assume that B is a paracompact Hausdorff space, X is a smooth open orientable surface, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^\alpha$ for

some $0 < \alpha < 1$, and $K \subset B \times X$ is a closed Runge subset (see Definition 5.1). Given a continuous function $f : K \cup (Q \times X) \rightarrow \mathbb{C}$ such that $f(b, \cdot)$ is J_b -holomorphic on $\mathring{K}_b$ for all $b \in B$ and is J_b -holomorphic on X for all $b \in Q$, we can approximate f in the fine $\mathcal{C}^0$ topology on K by $\mathcal{J}$ -holomorphic functions $F : B \times X \rightarrow \mathbb{C}$ satisfying $F = f$ on $Q \times X$.

6. The Oka principle for maps from families of open Riemann surfaces to Oka manifolds

In this section we prove Theorem 1.6. The same proof gives the generalisation in Theorem 6.4. We then obtain a couple of Mergelyan-type approximation theorems for manifold-valued maps from families of open Riemann surfaces; see Theorems 6.5 and 6.7. With future applications in mind, the technical results in Lemmas 6.2 and 6.3 are obtained in the bigger generality when X is a Stein manifold of arbitrary dimension.

Recall that a compact set K in a complex manifold X is said to be a *Stein compact* if it admits a basis of open Stein neighbourhoods. Every compact $\mathcal{O}(X)$ -convex set in a Stein manifold X is a Stein compact (see [60, Theorem 5.1.6]). Given a complex manifold Y , we denote by

$$\overline{\mathcal{O}}(K, Y)$$

the space of continuous maps $K \rightarrow Y$ which are uniform limits of holomorphic maps on open neighbourhoods of K in X . Furthermore, we denote by

$$\overline{\mathcal{O}}_{\text{loc}}(K, Y)$$

the space of continuous maps $f : K \rightarrow Y$ with the property that every point $x \in K$ has an open neighbourhood $U \subset X$ such that $f|_{K \cap \overline{U}} \in \overline{\mathcal{O}}(K \cap \overline{U}, Y)$. Clearly, we have the inclusions

$$\{f|_K : f \in \mathcal{O}(K, Y)\} \subset \overline{\mathcal{O}}(K, Y) \subset \overline{\mathcal{O}}_{\text{loc}}(K, Y) \subset \mathcal{A}(K, Y),$$

where $\mathcal{O}(K, Y)$ is the space of holomorphic maps $U \rightarrow Y$ on open neighbourhoods $U \subset X$ of K and $\mathcal{A}(K, Y)$ is the space of continuous maps $K \rightarrow Y$ which are holomorphic in $\mathring{K}$. The importance of the space $\overline{\mathcal{O}}_{\text{loc}}(K, Y)$ lies in the following result of Poletsky [88, Theorem 3.1].

Theorem 6.1 (Poletsky [88]). *If K is a Stein compact in a complex manifold X , Y is a complex manifold and $f \in \overline{\mathcal{O}}_{\text{loc}}(K, Y)$, then the graph of f on K is a Stein compact in $X \times Y$.*

We consider $\mathbb{R}^n$ as the standard real subspace of $\mathbb{C}^n$. Using Theorem 6.1 we prove the following.

Lemma 6.2. Assume that X is a Stein manifold, $\pi : \mathbb{C}^n \times X \rightarrow \mathbb{C}^n$ is the projection, $K \subset \mathbb{C}^n \times X$ is a compact set such that $B := \pi(K) \subset \mathbb{R}^n$ and $K_b = \{x \in X : (b, x) \in K\}$ is $\mathcal{O}(X)$ -convex for every $b \in B$, U is an open neighbourhood of K in $B \times X$, Y is a complex manifold with a distance function dist_Y inducing its manifold topology, and $f : U \rightarrow Y$ is a continuous map such that for every $b \in B$ the map $f_b = f(b, \cdot) : U_b = \{x \in X : (b, x) \in U\} \rightarrow Y$ is holomorphic. Then, the graph

$$G_f = \{(b, x, f(b, x)) : (b, x) \in K\} \subset \mathbb{C}^n \times X \times Y \quad (6.1)$$

of f on K is a Stein compact in $\mathbb{C}^n \times X \times Y$ and $f \in \overline{\mathcal{O}}(K, Y)$. Furthermore, given $\epsilon > 0$ there are a neighbourhood V of K in $\mathbb{C}^n \times X$, a neighbourhood $\tilde{U} \subset U \cap V$ of K in $B \times X$, a holomorphic map $\tilde{f} : V \rightarrow Y$, and a homotopy $g_t : \tilde{U} \rightarrow Y$ ($t \in I = [0, 1]$) satisfying the following conditions.

- (a) $g_0 = f|_{\tilde{U}}$ and $g_1 = \tilde{f}|_{\tilde{U}}$.
- (b) $g_t(b, \cdot) : \tilde{U}_b \rightarrow Y$ is holomorphic for every $b \in B$ and $t \in I$.
- (c) $\sup_K \text{dist}_Y(g_t, f) < \epsilon$ for all $t \in I$.

If in addition B is a $\mathcal{C}^l$ submanifold of $\mathbb{R}^n$ (possibly with boundary) for some $l \in \mathbb{N}$ and $f \in \mathcal{C}^{l,0}(U, Y)$, then for any $s \in \mathbb{Z}_+$ and after shrinking $U \supset K$, the homotopy f_t can be chosen such that, in addition to the above, $f_t \in \mathcal{C}^{l,s}(U, Y)$ for all $t \in I$ and the approximation in (c) holds in $\mathcal{C}^{l,s}(K)$.

Proof. By Remark 5.4, the set K is $\mathcal{O}(\mathbb{C}^n \times X)$ -convex, whence a Stein compact. We shall now verify that the map f satisfies the conditions in Theorem 6.1, and hence G_f (6.1) is a Stein compact.

Fix a point $b_0 \in B$. Since K is compact, given a neighbourhood $V \subset X$ of the fibre K_{b_0} we have

$$K_b \subset V \quad \text{for all } b \in B \text{ sufficiently close to } b_0. \quad (6.2)$$

Since the map $f_{b_0} : U_{b_0} \rightarrow Y$ is holomorphic and K_{b_0} is a Stein compact, the graph $G_{b_0} = \{(b_0, x, f_{b_0}(x)) : x \in K_{b_0}\}$ has an open Stein neighbourhood $\Gamma \subset \mathbb{C}^n \times X \times Y$ by Siu's theorem [93] (see also [26], [28, Theorem 1], and [38, Theorem 3.1.1]). Choose a holomorphic embedding $\Theta : \Gamma \hookrightarrow \mathbb{C}^N$. By a theorem of Docquier and Grauert [29] (see also [38, Theorem 3.3.3]) there are a Stein neighbourhood $O \subset \mathbb{C}^N$ of $\Theta(\Gamma)$ and a holomorphic retraction $\rho : O \rightarrow \Theta(\Gamma)$. In view of (6.2) there is a compact neighbourhood $B_0 \subset B$ of b_0 such that, setting

$$S := \{(b, x) : b \in B_0, x \in K_b\} \subset U,$$

we have that $\tilde{S} := \{(b, x, f(b, x)) : (b, x) \in S\} \subset \Gamma$. Hence, the map

$$h(b, x) := \Theta(b, x, f(b, x)) \in O \subset \mathbb{C}^N \quad (6.3)$$

is well-defined on a neighbourhood of S in $B \times X$, and $h(b, \cdot)$ is holomorphic on a neighbourhood of K_b in X for every $b \in B_0$. By Lemma 5.3 we can approximate h as closely as desired uniformly on S by a holomorphic map $\tilde{h} : W \rightarrow \mathbb{C}^N$ from a neighbourhood $W \subset \mathbb{C}^n \times X$ of S . Assuming that the approximation is close enough and the neighbourhood $W \supset S$ is small enough, we have that $\tilde{h}(W) \subset O$. Let $\tau : \mathbb{C}^n \times X \times Y \rightarrow Y$ denote the projection. The map

$$\tilde{f} := \tau \circ \Theta^{-1} \circ \rho \circ \tilde{h} : W \rightarrow Y \quad (6.4)$$

is then well defined and holomorphic, and it approximates f uniformly on S . Since this holds for every $b_0 \in B$, we see that $f \in \overline{\mathcal{O}}_{\text{loc}}(K, Y)$. Hence, Theorem 6.1 implies that G_f (6.1) is a Stein compact.

To prove that $f \in \overline{\mathcal{O}}(K, Y)$ and the last statement in the lemma, we apply the same argument with the entire parameter space B . Choose a Stein neighbourhood $\Gamma \subset \mathbb{C}^n \times X \times Y$ of G_f (6.1), a holomorphic embedding $\Theta : \Gamma \hookrightarrow \mathbb{C}^N$, and a holomorphic retraction $\rho : O \rightarrow \Theta(\Gamma)$ from a neighbourhood $O \subset \mathbb{C}^N$ of $\Theta(\Gamma)$. The map h given by (6.3) is now defined on a neighbourhood $\tilde{U} \subset (B \times X) \cap U$ of K , and the map $h(b, \cdot)$ is holomorphic on $\tilde{U}_b$ for every $b \in B$. By Lemma 5.3 we can approximate h as closely as desired uniformly on K by a holomorphic map $\tilde{h} : V \rightarrow \mathbb{C}^N$ from a neighbourhood $V \subset \mathbb{C}^n \times X$ of K . We may assume that $\tilde{h}(V) \subset O$. The map $\tilde{f} : V \rightarrow Y$ given by (6.4) is then holomorphic and approximates f on K . Furthermore, if $\tilde{h}$ is close enough to h on K and shrinking the neighbourhood $\tilde{U} \supset K$ if necessary, the family of convex combinations

$$h_t = (1 - t)h + t\tilde{h} : \tilde{U} \rightarrow \mathbb{C}^N, \quad t \in I \quad (6.5)$$

assumes values in O . The family of maps

$$g_t = \tau \circ \Theta^{-1} \circ \rho \circ h_t : \tilde{U} \rightarrow Y, \quad t \in I \quad (6.6)$$

is then a homotopy from $g_0 = f|_{\tilde{U}}$ to $g_1 = \tilde{f}|_{\tilde{U}}$ with the stated properties. The last statement of the lemma follows by the same argument, using Lemma 5.3 with approximation in $\mathcal{C}^{l,s}(K)$. $\square$

The next result is a version of Lemma 5.3 for maps with values in an Oka manifold and with homotopies added to the picture. This is the main ingredient in the proof of Theorem 1.6.

Lemma 6.3. *Assume that $B'' \subset \mathbb{R}^n$ is a neighbourhood retract and $B_0 \subset B_1 \subset B \subset B'$ are compact subsets of B'' , each contained in the relative interior of the next one. Let X be a Stein manifold, $\pi : \mathbb{C}^n \times X \rightarrow \mathbb{C}^n$ be the projection, and $K \subset \mathbb{C}^n \times X$ be a compact*

subset such that $\pi(K) \subset B$ and the fibre $K_b = \{x \in X : (b, x) \in K\}$ is $\mathcal{O}(X)$ -convex for every $b \in B$. Assume that U is an open neighbourhood of K in $B' \times X$, Y is an Oka manifold, and $f : B' \times X \rightarrow Y$ is a continuous map such that for every $b \in B$ the map $f_b = f(b, \cdot) : X \rightarrow Y$ is holomorphic on $U_b = \{x \in X : (b, x) \in U\}$. Fix $\epsilon > 0$ and $s \in \mathbb{Z}_+$. After shrinking the open set $U \supset K$, there is a homotopy $f_t : B \times X \rightarrow Y$ ($t \in I = [0, 1]$) with the following properties.

- (a) $f_0 = f|_{B \times X}$.
- (b) $f_t(b, \cdot) : X \rightarrow Y$ is holomorphic on U_b for every $b \in B$ and $t \in I$.
- (c) f_t approximates f in $\mathcal{C}^{0,s}(K)$ to precision ϵ .
- (d) $f_t(b, \cdot) = f(b, \cdot)$ for all $b \in B \setminus B_1$ and $t \in I$.
- (e) The map $f_1(b, \cdot) : X \rightarrow Y$ is holomorphic for every b in a neighbourhood of B_0 .

If in addition B' is a $\mathcal{C}^l$ submanifold of $\mathbb{R}^n$ for some $l \in \mathbb{N}$ and $f \in \mathcal{C}^{l,0}(B' \times X, Y)$, then for any $s \in \mathbb{Z}_+$ the homotopy f_t can be chosen such that, in addition to the above, $f_t \in \mathcal{C}^{l,s}(B \times X, Y)$ for all $t \in I$ and the approximation in (c) holds in $\mathcal{C}^{l,s}(K)$.

Proof. We focus on the case $l = 0$, $s = 0$. It will be clear that the same proof gives the corresponding results in the general case by using the corresponding versions of Lemmas 5.3 and 6.2.

By the assumption, there are a neighbourhood $V'' \subset \mathbb{C}^n$ of B'' and a retraction $\rho : V'' \rightarrow B''$ onto B'' . The conditions imply that B' is a neighbourhood of B in B'' . Since $\rho|_{B'}$ is the identity map, it follows that there is an open neighbourhood $V \subset \mathbb{C}^n$ of B such that $V \subset V''$ and $\rho(V) \subset B'$. Replacing $f(b, x)$ by $f(\rho(b), x)$ extends f to a continuous map $V \times X \rightarrow Y$, still denoted f . Since every compact subset B of $\mathbb{R}^n$ is polynomially convex in $\mathbb{C}^n$ [95, p. 3], V may be chosen Stein. If B' is a $\mathcal{C}^l$ submanifold of $\mathbb{R}^n$ then the retraction ρ as above always exists and can be chosen of class $\mathcal{C}^l$.

We claim that there are an open neighbourhood $W \subset V \times X$ of K in $\mathbb{C}^n \times X$ and a homotopy $g_t : W \rightarrow Y$ ($t \in I = [0, 1]$) connecting $g_0 = f$ to a holomorphic map $g_1 : W \rightarrow Y$ such that

$$\sup_{(b,x) \in K} \text{dist}_Y(g_t(b, x), f(b, x)) < \epsilon/2 \quad \text{holds for all } t \in I \quad (6.7)$$

and the map $g_t(b, \cdot) : W_b \rightarrow Y$ is holomorphic for every $b \in B$ and $t \in I$. Note that Lemma 6.2 furnishes a homotopy g_t with the desired properties on a neighbourhood of K in $B \times X$; see (6.3), (6.5), and (6.6). In the present situation, all maps in the construction of g_t are defined on a neighbourhood of K in $\mathbb{C}^n \times X$. Hence, the same argument, using convex combinations as in (6.5) and defining g_t by (6.6), yields a desired homotopy on a neighbourhood $W \subset V \times X$ of K in $\mathbb{C}^n \times X$. Applying the same argument on a somewhat bigger compact set $K' \subset U \cap (B \times K)$ with $\mathcal{O}(X)$ -convex fibres and containing K in its relative interior, the Cauchy estimates show that (6.7) can be upgraded to approximation

in the $\mathcal{C}^{0,s}$ topology on K for any given $s \in \mathbb{Z}_+$. Furthermore, if $l > 0$ then the same arguments give approximation in the $\mathcal{C}^{l,s}$ topology on K (cf. Lemma 6.2).

Pick a smooth function $\chi : \mathbb{C}^n \times X \rightarrow [0, 1]$ with support in W such that $\chi = 1$ on a smaller neighbourhood $W' \Subset W$ of K . With g_t as above, consider the map $h_0 : V \times X \rightarrow Y$ given by

$$h_0(z, x) = g_{\chi(z, x)}(z, x) \quad \text{for } z \in V \text{ and } x \in X. \quad (6.8)$$

For $(z, x) \in W'$ we have $\chi = 1$ and hence $h_0|_{W'} = g_1|_{W'}$, which is a holomorphic map. On $(V \times X) \setminus W$ we have $\chi = 0$ and hence $h_0 = g_0 = f$. Furthermore, h_0 is homotopic to f by the homotopy $I \ni t \mapsto g_{t\chi}$, and every map in this homotopy has the same properties as f .

Since V is Stein, the set $K \subset V \times X$ is $\mathcal{O}(\mathbb{C}^n \times X)$ -convex (see Remark 5.4) and Y is an Oka manifold, the main result of Oka theory (see [38, Theorem 5.4.4]) furnishes a homotopy $h_t : V \times X \rightarrow Y$ ($t \in I$) from h_0 to a holomorphic map $h_1 : V \times X \rightarrow Y$ such that the homotopy $f_t : V \times X \rightarrow Y$ ($t \in I$) given by

$$f_t = \begin{cases} g_{2t\chi}, & 0 \leq t \leq 1/2, \\ h_{2t-1}, & 1/2 \leq t \leq 1 \end{cases}$$

satisfies conditions (a)–(c) in the lemma (where we take $s = 0$ in (c)), and it satisfies condition (e) for all $b \in B$ since $f_1 = h_1$. To obtain (d), choose a smooth function $\xi : \mathbb{R}^n \rightarrow [0, 1]$ which equals 1 on a neighbourhood of B_0 and vanishes on $B \setminus B_1$, and replace $f_t(b, \cdot)$ by $f_{t\xi(b)}(b, \cdot)$ for $b \in B$ and $t \in I$.

This proves the lemma for $l = s = 0$. The same arguments apply when $s > 0$, and also for $l > 0$ when B is a $\mathcal{C}^l$ submanifold of $\mathbb{R}^n$, noting that the approximation by holomorphic functions in $\mathcal{C}^{l,s}(K)$ is furnished by Lemma 5.3 and the existence of a homotopy $\{g_t\}_{t \in I}$ with approximation in $\mathcal{C}^{l,s}(K)$ is given by Lemma 6.2. $\square$

Proof of Theorem 1.6. We consider the case $l = k = 0$ with approximation in the fine $\mathcal{C}^0$ -topology. The arguments in the general case are similar by using the corresponding version of Lemma 6.3. As pointed out in the proof of Theorem 1.1, approximation in the fine $\mathcal{C}^{(k+1, \alpha)}$ -topology follows from the fine $\mathcal{C}^0$ approximation on a somewhat bigger Runge set in view of the Cauchy estimates.

Let $\mathcal{J} = \{J_b\}_{b \in B}$ be a family of complex structures on X as in the theorem. Recall (see Definition 5.1) that a closed subset $K \subset B \times X$ is Runge if the projection $\pi|_K : K \rightarrow B$ is proper and every fibre $K_b = \{x \in X : (b, x) \in K\}$ ($b \in B$) is Runge in X . Thus, the set K in the theorem is Runge.

We first explain the proof in the case when the parameter space B is compact. Let $K^0 = K \subset B \times X$ and $f^0 = f : B \times X \rightarrow Y$ be as in the theorem, so f^0 is $\mathcal{J}$ -holomorphic on a neighbourhood of K^0 . Choose an increasing sequence of compact Runge sets $K'_1 \subset K'_2 \subset \cdots \subset \bigcup_{j=1}^\infty K'_j = X$ such that every set is contained in the interior of

the next one, and let $K^j = B \times K'_j \subset B \times X$ for $j = 1, 2, \dots$. We choose K'_1 big enough such that $K^0 \subset K^1$. Given a decreasing sequence $\epsilon_j > 0$, we shall find a sequence of maps $f^j : B \times X \rightarrow Y$ and homotopies $f_t^j : B \times X \rightarrow Y$ ($t \in I = [0, 1]$) satisfying the following conditions for every $j = 1, 2, \dots$

- (i) f^j is $\mathcal{J}$ -holomorphic on a neighbourhood of K^j . (By the assumption, this also holds for $j = 0$.)
- (ii) $f_0^j = f^{j-1}$ and $f_1^j = f^j$.
- (iii) f_t^j is $\mathcal{J}$ -holomorphic on a neighbourhood of K^{j-1} for all $t \in I$.
- (iv) $\max_{K^{j-1}} \text{dist}_Y(f_t^{j-1}, f_t^j) < \epsilon_j$ for all $t \in I$.
- (v) The homotopy $f_t^j(b, \cdot)$ is fixed for all b in a neighbourhood of Q in B .

Assuming that the sequence $\epsilon_j > 0$ is chosen to converge to 0 sufficiently fast, the limit map $F = \lim_{j \rightarrow \infty} f^j : B \times X \rightarrow Y$ exists and is $\mathcal{J}$ -holomorphic, it approximates f as closely as desired uniformly on K , and $F(b, \cdot) = f(b, \cdot)$ holds for all $b \in Q$. Furthermore, the homotopies f_t^j ($j \in \mathbb{N}$, $t \in I$) can be assembled into a single homotopy $f_t : B \times X \rightarrow Y$ ($t \in I$) from $f_0 = f$ to $f_1 = F$ such that f_t is $\mathcal{J}$ -holomorphic on a neighbourhood of K , it approximates f on K for every $t \in I$, and it is fixed for all $b \in Q$.

Every step in the induction is of the same kind, so it suffices to explain the initial step, that is, the construction of a homotopy $f_t^1 : B \times X \rightarrow Y$ ($t \in I$) which is $\mathcal{J}$ -holomorphic on a neighbourhood of K^0 , it approximates $f = f^0$ on K^0 , it is fixed for b in a neighbourhood of the subset $Q \subset B$ (see condition (v) in the theorem), and such that the map $f_1^1 = f^1$ is $\mathcal{J}$ -holomorphic on a neighbourhood of K^1 . This is accomplished by a finite induction with respect to an increasing family of compact subsets of the parameter space B , which we now explain.

Recall that $K^0 \subset K^1 = B \times L$, where L is a compact Runge set in X . If the subset $Q \subset B$ in condition (v) is nonempty, it has a compact neighbourhood $\tilde{Q} \subset B$ such that the map f_b^0 is holomorphic on a neighbourhood of L for every $b \in \tilde{Q}$. Let $\tilde{K}^0 \subset B \times X$ denote the compact set with fibres

$$\tilde{K}_b^0 = \begin{cases} L, & b \in \tilde{Q}; \\ K_b^0, & b \in B \setminus \tilde{Q}. \end{cases} \quad (6.9)$$

Clearly, $K^0 \subset \tilde{K}^0 \subset K^1$ and $\tilde{K}^0$ is Runge in $B \times X$. If $Q = \emptyset$, we take $\tilde{Q} = \emptyset$ and hence $K^0 = \tilde{K}^0$. Pick a smoothly bounded domain $\Omega \Subset X$ and domains $V, V' \subset X$ such that

$$L \subset V \subset V' \subset \Omega \quad (6.10)$$

and the closure of each of these sets is contained in the interior of the next one. Fix a point $b_0 \in B$. The conditions on B imply that there is a neighbourhood $P'' \subset B$ of b_0 which is an ENR (see Definition 1.5). We may therefore consider P'' as a neighbourhood retract in some $\mathbb{R}^n \subset \mathbb{C}^n$. By Corollary 4.5 there are a compact neighbourhood P'

of b_0 , contained in the interior of P'' , and a continuous family of biholomorphic maps $\Phi_b : (\Omega, J_b) \rightarrow (\Phi_b(\Omega), J_{b_0})$ ($b \in P'$) such that

$$\Phi_b(V) \subset V' \subset \Phi_b(\Omega) \quad \text{holds for every } b \in P'. \quad (6.11)$$

Pick a compact neighbourhood $P \subset B$ of b_0 contained in the interior of P' . Let $K' \subset L'$ be compact subsets of $P \times X$ whose fibres over any point $b \in P$ are given by

$$K'_b = \Phi_b(\tilde{K}_b^0), \quad L'_b = \Phi_b(L).$$

By (6.9)–(6.11) we have that

$$K'_b \subset L'_b \subset \Phi_b(V) \subset V' \quad \text{for all } b \in P.$$

Consider the family of maps

$$f'_b = f_b \circ \Phi_b^{-1} : \Phi_b(\Omega) \rightarrow Y, \quad b \in P.$$

Since f_b is J_b -holomorphic on a neighbourhood of $\tilde{K}_b^0$ and the map $\Phi_b : (\Omega, J_b) \rightarrow (\Phi_b(\Omega), J_{b_0})$ is biholomorphic, f'_b is J_{b_0} -holomorphic on a neighbourhood of K'_b for every $b \in P$. Pick a pair of smaller compact neighbourhoods $P_0 \subset P_1 \subset P$ of b_0 , each of them contained in the interior of the next one. Lemma 6.3, applied with X replaced by $V' \subset X$, furnishes a homotopy of maps

$$f'_{t,b} : V' \rightarrow Y \quad \text{for } b \in P \text{ and } t \in I$$

satisfying conditions (a)–(e) in the lemma with the sets $B_0 \subset B_1 \subset B$ replaced by $P_0 \subset P_1 \subset P$. In particular, $f'_{t,b} = f'_{0,b} = f'_b$ holds for $b \in P \setminus P_1$, the map $f'_{t,b}$ approximates f'_b on K'_b for $b \in P$, and $f'_{1,b}$ is J_{b_0} -holomorphic on V' for b in a neighbourhood of P_0 . By (6.11) we have $\Phi_b(V) \subset V'$. Hence,

$$f_{t,b} := f'_{t,b} \circ \Phi_b : V \rightarrow Y \quad \text{for } b \in P \text{ and } t \in I \quad (6.12)$$

is a homotopy of maps which are J_b -holomorphic on a neighbourhood of $\tilde{K}_b^0$, they approximate f_b uniformly on $\tilde{K}_b^0$, they agree with $f_{0,b} = f_b$ for $b \in P \setminus P_1$ (so we can extend the family to all $b \in B$), and the map $f_b^1 := f_{1,b} : V \rightarrow Y$ is J_b -holomorphic for all b in a neighbourhood of P_0 . By using a cut-off function in the parameter of the homotopy, we can extend the maps $f_{t,b}$ to X without changing their values on a neighbourhood of L (compare with (6.8)).

If the sets $Q \subset \tilde{Q}$ are nonempty, we make another modification to the above homotopy in order to ensure condition (v) in the theorem. Choose a function $\chi : B \rightarrow [0, 1]$ such that $\chi = 1$ on $B \setminus \tilde{Q}$ and $\chi = 0$ on a neighbourhood of Q . With $f_{t,b}$ as in (6.12) we set

$$\tilde{f}_{t,b} = f_{t\chi(b),b} \quad \text{for } b \in B \text{ and } t \in I.$$

For $b \in B$ in a neighbourhood of Q we then have $\tilde{f}_{t,b} = f_{0,b} = f_b$ as desired, and the other required properties still hold.

What was just explained serves as a step in a finite induction which we now describe.

The assumptions imply that there is a finite family of triples $P_0^j \subset P_1^j \subset P^j$ ($j = 1, 2, \dots, m$) of compact sets in B such that $\bigcup_{j=1}^m P_0^j = B$ and the above construction can be performed on each of these triples with the same sets in (6.10). The induction proceeds as follows.

In the first step, we perform the procedure explained above on the first triple (P_0^1, P_1^1, P^1) with the set K^0 and the map $g^0 := f^0 = f$. The resulting map $g^1 : B \times X \rightarrow Y$ is $\mathcal{J}$ -holomorphic on a neighbourhood of the compact set

$$S^1 := [(P_0^1 \times X) \cap K^1] \cup K^0 \subset B \times X, \quad (6.13)$$

and $g_b^1 = f_b^0$ holds for all b in a neighbourhood of Q . Note that the fibre S_b^1 of S^1 over any point $b \in B$ is Runge in X . Indeed, we have $S_b^1 = L$ for $b \in P_0^1$ and $S_b^1 = K_b^0$ for $b \in B \setminus P_0^1$. Since $K_b^0 \subset L$ for every $b \in B$, the set S^1 is compact and Runge in $B \times X$, and we clearly have that $K^0 \subset S^1 \subset K^1 = B \times L$. Furthermore, we obtain a homotopy from $f^0 = g^0$ to g^1 such that every map in the homotopy is $\mathcal{J}$ -holomorphic on a neighbourhood of K^0 and it approximates f^0 there, and the homotopy is fixed for b in a neighbourhood of $(B \setminus P_1^1) \cup Q$.

In the second step, the same argument is applied to the map g^1 on the triple (P_0^2, P_1^2, P^2) but with K^0 replaced by the set S^1 in (6.13). The resulting map $g^2 : B \times X \rightarrow Y$ is $\mathcal{J}$ -holomorphic on a neighbourhood of the compact Runge set

$$S^2 = [(P_0^2 \cup P_0^1) \times X) \cap K^1] \cup K^0 \subset B \times X. \quad (6.14)$$

Note that $S_b^2 = L$ for $b \in P_0^1 \cup P_0^2$ and $S_b^2 = S_b^1 = K_b^0$ for $b \in B \setminus (P_0^1 \cup P_0^2)$. We also obtain a homotopy from g^1 to g^2 consisting of maps which are $\mathcal{J}$ -holomorphic on a neighbourhood of S^1 , they approximate g^1 there, and the homotopy is fixed for b in a neighbourhood of $(B \setminus P_1^2) \cup Q$.

Proceeding inductively, we obtain after m steps a map $g^m : B \times X \rightarrow Y$ which is $\mathcal{J}$ -holomorphic on a neighbourhood of $S^m = K^1 = B \times L$. We define $f^1 := g^m$. Furthermore, the individual homotopies between the subsequent maps g^j and g^{j+1} for $j = 0, 1, \dots, m-1$ can be assembled into a homotopy f_t^1 ($t \in I$) from $f_0^1 = f^0 = g^0$ to $f_1^1 = f^1 = g^m$ such that f_t^1 is $\mathcal{J}$ -holomorphic on a neighbourhood of K^0 for all $t \in I$ and the homotopy is fixed for $b \in B$ in a neighbourhood of Q . This completes the proof of the theorem if the parameter space B is compact.

In the general case when B is only σ -compact we choose a normal exhaustion $B_1 \subset B_2 \subset \dots \subset \bigcup_{j=1}^\infty B_j = B$ by compact sets (i.e., such that every set B_j is contained in the interior of the next one) and a normal exhaustion $L_1 \subset L_2 \subset \dots \subset \bigcup_{j=1}^\infty L_j = X$ by compact Runge subsets of X such that

$$(B_j \times X) \cap K^0 \subset B_j \times L_j \quad \text{holds for all } j = 1, 2, \dots$$

Define the increasing sequence of subsets $K = K^0 \subset K^1 \subset \dots \subset \bigcup_{j=0}^{\infty} K^j = B \times X$ by

$$K^j = (B_j \times L_j) \cup K^0, \quad j = 1, 2, \dots$$

Note that every K^j is a closed Runge set in $B \times X$. Applying the special case proved above gives a sequence of maps $f^j : B \times X \rightarrow Y$ ($j = 0, 1, \dots$) with $f^0 = f$ such that for every $j = 1, 2, \dots$ the map f^j is $\mathcal{J}$ -holomorphic on a neighbourhood of K^j , it approximates f^{j-1} in the fine topology on K^{j-1} , it is homotopic to f^{j-1} by a homotopy of maps which are $\mathcal{J}$ -holomorphic on a neighbourhood of K^{j-1} and approximate f^{j-1} on K^{j-1} , and the homotopy is fixed for $b \in Q$. Assuming that the approximation is close enough at every step, we obtain a limit map $F = \lim_{j \rightarrow \infty} f^j : B \times X \rightarrow Y$ which is $\mathcal{J}$ -holomorphic, it approximates the initial map f as closely as desired in the fine topology on $K = K^0$, it agrees with f on $Q \times X$, and it is homotopic to f by maps having the same properties. $\square$

The following result generalises Theorem 1.6. The proof is essentially the same and is omitted.

Theorem 6.4. *Let B , X , $\{J_b\}_{b \in B}$, $K \subset B \times X$, and Y be as in Theorem 1.6. Assume that Z is a Stein manifold and L is a compact $\mathcal{O}(Z)$ -convex set in Z . For every $b \in B$ let $\tilde{J}_b$ be the almost complex structure on $X \times Z$ which equals J_b on TX and equals the given almost complex structure on TZ . Assume that $f : B \times X \times Z \rightarrow Y$ is a continuous map, and there is an open set $U \subset B \times X \times Z$ containing $K \times L$ such that $f_b = f(b, \cdot, \cdot) : X \times Z \rightarrow Y$ is $\tilde{J}_b$ -holomorphic on $U_b = \{(x, z) \in X \times Z : (b, x, z) \in U\}$ for every $b \in B$. Given a continuous function $\epsilon : B \rightarrow (0, +\infty)$, there is a homotopy $f_t : B \times X \times Z \rightarrow Y$ ($t \in I$) satisfying the following.*

- (i) $f_0 = f$.
- (ii) The map $f_{t,b} = f_t(b, \cdot, \cdot) : X \times Z \rightarrow Y$ is $\tilde{J}_b$ -holomorphic near $K_b \times L$ for every $b \in B$.
- (iii) $\sup_{(x,z) \in K_b \times L} \text{dist}_Y(f_{t,b}(x, z), f_b(x, z)) < \epsilon(b)$ for every $b \in B$ and $t \in I$.
- (iv) The map $F = f_1$ is such that $F_b = F(b, \cdot, \cdot) : X \times Z \rightarrow Y$ is $\tilde{J}_b$ -holomorphic for every $b \in B$.

If in addition $0 \leq l \leq k+1$, B is a $\mathcal{C}^l$ manifold if $l > 0$, the family $\{J_b\}_{b \in B}$ is of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$ ($0 < \alpha < 1$) and $f|_U$ is of class $\mathcal{C}^{l,0}$, then $f|_U$ is of class $\mathcal{C}^{l,(k+1,\alpha)}$ and the homotopy $\{f_t\}_{t \in I}$ can be chosen such that it is of class $\mathcal{C}^{l,(k+1,\alpha)}$ on a neighbourhood of $K \times L$, it approximates f in the fine $\mathcal{C}^{l,(k+1,\alpha)}$ -topology on $K \times L$, and $F = f_1$ is class $\mathcal{C}^{l,(k+1,\alpha)}$ on $B \times X \times Z$.

By using the techniques in the proof of Theorem 1.6, we can also extend Mergelyan approximation for functions in Theorem 1.3 to manifold-valued maps as in the following theorem. A similar result in the nonparametric case is [36, Corollary 5, p. 176].

Theorem 6.5. *Assume that X is a smooth open surface, B is as in Theorem 1.6, $\mathcal{J} = \{J_b\}_{b \in B}$ is a continuous family of complex structures on X of class $\mathcal{C}^{(k, \alpha)}$ ($k \in \mathbb{Z}_+$, $0 < \alpha < 1$), $r \in \{0, 1, \dots, k+1\}$, $K \subset X$ is a compact Runge set, $A \subset \mathring{K}$ is a finite set, Y is a complex manifold, and $f : B \times K \rightarrow Y$ is a continuous map which is $\mathcal{J}$ -holomorphic on $B \times \mathring{K}$. Given a continuous function $\epsilon : B \rightarrow (0, +\infty)$, there are a neighbourhood $U \subset B \times X$ of $B \times K$ and a continuous $\mathcal{J}$ -holomorphic map $F : U \rightarrow Y$ such that for every $b \in B$ we have $\sup_{x \in K} \text{dist}_Y(F_b(x), f_b(x)) < \epsilon(b)$ and F_b agrees with f_b to order r in every point $a \in A$.*

Proof. Since the open set $X \setminus K$ has no relatively compact connected components, there are arbitrarily small open coordinate discs $U_1, \dots, U_N \subset X$ and compact discs $D_j \subset U_j$ for $j = 1, \dots, N$ such that $K \subset \bigcup_{j=1}^N \mathring{D}_j$ and $U_j \setminus (K \cap D_j)$ is connected for every j . Fix $b_0 \in B$. We may assume that the discs U_j are chosen small enough so that $f_{b_0}(K \cap D_j) \subset Y$ is contained in a coordinate chart of Y for each j . Hence, by Theorem 1.3 we can approximate f_{b_0} uniformly on $K \cap D_j$ by holomorphic maps from open neighbourhoods of $K \cap D_j$ to Y for $j = 1, \dots, N$. This shows that the hypotheses of Theorem 6.1 hold, so the graph of f_{b_0} on K is a Stein compact in $X \times Y$. By the argument in the proof of Lemma 6.2 we reduce the Mergelyan approximation problem for maps $f_b : K \rightarrow Y$, with $b \in B$ close enough to b_0 , to the scalar-valued case furnished by Theorem 1.3. The local J_b -holomorphic approximants of f_b can be glued together by finding homotopies as in the proof of Lemma 6.2 (see (6.5) and (6.6)) and using cut-off functions in the parameter of the homotopy. The inductive procedure is similar to the one in the proof of Theorem 1.6 and will not be repeated. $\square$

Before stating our next result, we recall the following notion; see [12, p. 69].

Definition 6.6. Let X be a smooth surface. An *admissible set* in X is a compact set of the form $S = K \cup E$, where K is a (possibly empty) finite union of pairwise disjoint compact domains with piecewise smooth boundaries in X and $E = S \setminus \mathring{K}$ is a union of finitely many pairwise disjoint smooth Jordan arcs and closed smooth Jordan curves meeting K only at their endpoints (if at all) such that their intersections with the boundary ∂K of K are transverse.

Admissible sets arise in handlebody decompositions of surfaces; see [12, Sect. 1.4]. For this reason, approximation on such sets plays a major role in constructions of directed holomorphic maps, minimal surfaces and related objects, as is evident from the results in [12]. The basic case for continuous functions follows from Theorem 1.3. In Section 8 we shall also use the following version of the Mergelyan theorem on admissible sets in

families of open Riemann surfaces. The main novelty with respect to Theorem 1.3 is smoothness of the complex structures and the maps in the parameter.

Theorem 6.7. *Assume that X is a smooth open surface, $1 \leq l \leq k+1$ are integers, B is a manifold of class $\mathcal{C}^l$, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$ for some $0 < \alpha < 1$, $S = K \cup E$ is a Runge admissible set in X , $U \subset X$ is an open set containing K , A is a finite subset of $\mathring{K}$, $r \in \{0, 1, \dots, k+1\}$, and $f : B \times (U \cup E) \rightarrow \mathbb{C}$ is a function of class $\mathcal{C}^l$ which is $\mathcal{J}$ -holomorphic on $B \times U$. Then, f can be approximated in the fine $\mathcal{C}^l$ topology on $B \times S$ by $\mathcal{J}$ -holomorphic functions $F : B \times X \rightarrow \mathbb{C}$ of class $\mathcal{C}^{l,(k+1,\alpha)}$ such that for every $b \in B$, $F_b = F(b, \cdot) : X \rightarrow \mathbb{C}$ agrees with f_b to order r in every point $a \in A$. The analogous result holds for maps to any complex manifold Y , where the approximating maps F are defined on small open neighbourhoods of $B \times S$ in $B \times X$. If Y is an Oka manifold then there are maps $F : B \times X \rightarrow Y$ satisfying the same conclusion, except for the interpolation conditions in the points of A .*

Proof. It suffices to prove the result locally in the parameter. Thus, fix a point $b_0 \in B$, a smoothly bounded domain $\Omega \Subset X$ containing S , and a compact neighbourhood $B_0 \subset B$ of b_0 for which Corollary 4.5 applies and gives a family of (J_b, J_{b_0}) -biholomorphic maps $\Phi_b : \Omega \rightarrow \Phi_b(\Omega) \subset X$ ($b \in B_0$) of class $\mathcal{C}^{l,(k+1,\alpha)}$. We may assume that B_0 is a $\mathcal{C}^l$ submanifold of $\mathbb{R}^n \subset \mathbb{C}^n$ for some $n \in \mathbb{N}$. Recall that the function f_b in the theorem is J_b -holomorphic on a neighbourhood $U \subset X$ of K for every $b \in B_0$. Since B_0 is compact, we may choose U independent of $b \in B_0$.

As in the proof of Theorem 1.6, this reduces the approximation problem for $b \in B_0$ to the situation in Lemma 6.3 where the compact sets $\tilde{K}, \tilde{E}, \tilde{S}$ in $B_0 \times X \subset \mathbb{C}^n \times X$ have fibres $K_b = \Phi_b(K)$, $E_b = \Phi_b(E)$, and $S_b = \Phi_b(S) = K_b \cup E_b$, respectively. Note that $\tilde{K}$ and $\tilde{S}$ are holomorphically convex in $\mathbb{C}^n \times X$ (see Remark 5.4). Let $\tilde{U} \subset B_0 \times X$ be the set with fibres $U_b = \Phi_b(U)$. The function $\tilde{f}_b : U_b \cup E_b \rightarrow \mathbb{C}$, defined by $\tilde{f}_b \circ \Phi_b = f_b$ on $U \cup E$ ($b \in B_0$), is J_{b_0} -holomorphic on U_b for every $b \in B_0$. Let $\tilde{f} : \tilde{U} \cup \tilde{E} \rightarrow \mathbb{C}$ be given by $\tilde{f}(b, \cdot) = \tilde{f}_b$ for $b \in B_0$. Note that $\tilde{f}$ is of class $\mathcal{C}^l$. Choose a compact $\mathcal{O}(\mathbb{C}^n \times X)$ -convex set $L \subset \tilde{U}$ containing $\tilde{K}$ in its relative interior. By Lemma 5.3 we can approximate $\tilde{f}$ in $\mathcal{C}^l(L)$ by a function $h : V \rightarrow \mathbb{C}$ on an open neighbourhood $V \subset \mathbb{C}^n \times X$ of L which is holomorphic with respect to the standard complex structure on $\mathbb{C}^n$ and the complex structure J_{b_0} on X . By smoothly gluing h with $\tilde{f} : \tilde{E} \rightarrow \mathbb{C}$ on the set $L \setminus \tilde{K}$, we may assume that h is unchanged (and hence holomorphic) on a neighbourhood $\tilde{V} \subset V$ of $\tilde{K}$ in $\mathbb{C}^n \times X$, it is of class $\mathcal{C}^l$ on $\tilde{E}$, it agrees with $\tilde{f}$ on $\tilde{E} \setminus L$, and it approximates $\tilde{f}$ to a desired precision in $\mathcal{C}^l(\tilde{E})$. Note that $\tilde{E}$ is a totally real submanifold of class $\mathcal{C}^l$ in $\mathbb{C}^n \times X$, and the set $\tilde{S} = \tilde{K} \cup \tilde{E}$ is admissible in the sense of [36, Definition 5 (a), p. 156]. Hence, by [36, Theorem 20, p. 161] we can approximate h in $\mathcal{C}^l(\tilde{S})$ by holomorphic functions on $\mathbb{C}^n \times X$. By the argument in the proof of Lemma 5.3 this gives functions F , defined and $\mathcal{J}$ -holomorphic on open neighbourhoods of $B_0 \times S$ in $B_0 \times X$, which approximate f in $\mathcal{C}^l(B_0 \times S)$. The interpolation conditions in the points of A are obtained

as in the proof of Theorem 1.1. The proof of the theorem for functions is completed by using $\mathcal{C}^l$ partitions of unity on B and Theorem 1.1. For maps to manifolds, we follow the argument in the proof Lemma 6.3. The statement for maps to an Oka manifold Y follows from Theorem 1.6. $\square$

7. Trivialisation of canonical bundles of families of open Riemann surfaces

Every open Riemann surface X has trivial holomorphic cotangent bundle $K_X = T^*X$, trivialised by a nowhere vanishing holomorphic 1-form θ on X . (In fact, every holomorphic vector bundle on an open Riemann surface is holomorphically trivial by the Oka–Grauert principle; see Oka [86], Grauert [53], and [38, Theorem 5.3.1].) We prove the following generalisation to families of complex structures. See also Corollary 8.3, which extends the theorem of Gunning and Narasimhan [56].

Theorem 7.1. *Given a smooth open surface X and a family $\mathcal{J} = \{J_b\}_{b \in B}$ of complex structures of class $\mathcal{C}^{l,(k,\alpha)}$ on X as in Theorem 1.6 (with $l \leq k+1$), there exists a family $\{\theta_b\}_{b \in B}$ of nowhere vanishing holomorphic 1-forms on (X, J_b) of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$.*

Note that a family of holomorphic 1-forms $\{\theta_b\}_{b \in B}$ as in the theorem, which is of class $\mathcal{C}^l$ in $b \in B$, is necessarily of class $\mathcal{C}^{l,(k,\alpha)}(B \times X)$ by Lemma 5.6. Theorem 7.1 is used in Section 8.

Proof. Write $X_b = (X, J_b)$ for $b \in B$. We see as in the proof of Corollary 4.5 that there is a family $\{\theta_b\}_{b \in B}$ of nowhere vanishing $(1,0)$ -forms θ_b on X_b of class $\mathcal{C}^{l,(k,\alpha)}$. We will deform it to a family of nowhere vanishing J_b -holomorphic 1-forms $\{\tilde{\theta}_b\}_{b \in B}$ of class $\mathcal{C}^l$ in the parameter $b \in B$.

Note that $\theta = \{\theta_b\}_{b \in B}$ is a section of the complex line bundle $E \rightarrow B \times X$ whose restriction to the fibre X_b over $b \in B$ equals T^*X_b , the complex cotangent bundle of X_b . We shall inductively deform θ so as to make it $\mathcal{J}$ -holomorphic on larger and larger subsets of $B \times X$. We follow the scheme in the proof of Theorem 1.6. Assuming that $K \subset L$ are compact Runge sets in X and θ is $\mathcal{J}$ -holomorphic on a neighbourhood of $B \times K$, we shall find a multiplier $f : B \times X \rightarrow \mathbb{C}^*$ which is homotopic to the constant function 1 and is $\mathcal{J}$ -holomorphic and close to 1 on a neighbourhood of $B \times K$ (thereby ensuring that $f\theta$ is close to θ on $B \times K$), such that $f\theta$ is $\mathcal{J}$ -holomorphic on a neighbourhood of $B \times L$. We will then conclude the proof by an induction on a normal exhaustion of X by an increasing family of compact Runge sets.

It remains to explain the basic case described above. As in the proof of Theorem 1.6, we proceed by induction with respect to a normal exhaustion of B by compact subsets. For the inductive step, assume that L is a compact Runge set in X and $K^0 \subset B \times L$ is a closed Runge subset (see Definition 5.1). We allow for the possibility that some fibres are empty. Assume that θ as above is $\mathcal{J}$ -holomorphic on a neighbourhood $U \subset B \times X$ of K^0 . Pick a smoothly bounded domain $\Omega \Subset X$ with $L \subset \Omega$. Fix a point $b_0 \in B$. Corollary 4.5

furnishes a compact neighbourhood $P \subset B$ of b_0 and a family of biholomorphic maps $\Phi_b : (\Omega, J_b) \rightarrow (\Phi_b(\Omega), J_{b_0})$ ($b \in P$) of class $\mathcal{C}^{l,k+1}$. By the Oka–Grauert principle there is a function $g : X \rightarrow \mathbb{C}^*$, homotopic to the constant $X \rightarrow 1$ through functions $X \rightarrow \mathbb{C}^*$, such that the 1-form $g\theta_{b_0}$ is J_{b_0} -holomorphic on X . Hence,

$$\phi_b := \Phi_b^*(g\theta_{b_0}) = f_b\theta_b, \quad b \in P$$

is a family of nowhere vanishing J_b -holomorphic 1-forms on Ω . Since $g\theta_{b_0}$ is independent of $b \in P$, the family $\{\phi_b\}_{b \in P}$ is of class $\mathcal{C}^l(P)$. Hence, the same holds for the family of functions $f_b = \phi_b/\theta_b : \Omega \rightarrow \mathbb{C}^*$. By shrinking $U \supset K^0$ if necessary we may assume that $U \subset B \times \Omega$. Since θ_b is J_b -holomorphic on $U_b \supset K_b^0$ for every $b \in P$, the function $f_b = \phi_b/\theta_b$ is also J_b -holomorphic on U_b for every $b \in P$. Theorem 1.6 applied with the Oka manifold $Y = \mathbb{C}^*$ furnishes a homotopy of functions $f_{t,b} : \Omega \rightarrow \mathbb{C}^*$ ($b \in P$, $t \in I$) of class $\mathcal{C}^{l,k}$ satisfying the following conditions:

- (i) $f_{0,b} = f_b$ for all $b \in P$,
- (ii) $f_{1,b}$ is J_b -holomorphic on Ω for all $b \in P$ and of class $\mathcal{C}^l$ in b , and
- (iii) $f_{t,b}$ is J_b -holomorphic on a neighbourhood of K_b^0 and it approximates f_b on K_b as closely as desired for all $b \in P$ and $t \in I$. (In fact, the approximation is in the $\mathcal{C}^{l,k}$ topology.)

The homotopy of 1-forms $\theta'_{t,b} = \phi_b/f_{t,b}$ ($b \in P$, $t \in I$) on Ω is of class $\mathcal{C}^{l,k}$ and satisfies

- (i') $\theta'_{0,b} = \phi_b/f_{0,b} = \phi_b/f_b = \theta_b$ for all $b \in P$,
- (ii') $\theta'_{1,b} = \phi_b/f_{1,b}$ is J_b -holomorphic on Ω for every $b \in P$, and
- (iii') $\theta'_{t,b}$ is J_b -holomorphic on a neighbourhood of K_b^0 and it approximates θ_b on K_b^0 for all $b \in P$. (The approximation is in the $\mathcal{C}^{l,k}$ topology.)

Pick a pair of neighbourhoods $P_0 \subset P_1 \subset P$ of b_0 , each contained in the interior of the next one, and a function $\xi : B \rightarrow [0, 1]$ of class $\mathcal{C}^l$ which equals 1 on a neighbourhood of P_0 and vanishes on $B \setminus P_1$. We define a new homotopy of 1-forms on Ω of class $\mathcal{C}^{l,k}(B \times \Omega)$ by

$$\theta_{t,b} = \theta'_{t\xi(b),b} \quad \text{for every } b \in B \text{ and } t \in I.$$

Then, $\theta_{t,b} = \theta'_{t,b}$ holds for b in a neighbourhood of P_0 (where $\xi = 1$), and $\theta_{t,b} = \theta_{0,b} = \theta_b$ holds for all $b \in B \setminus P_1$ (where $\xi = 0$) and $t \in I$. It follows that

- (i'') $\theta_{0,b} = \theta'_{0,b} = \theta_b$ for all $b \in B$,
- (ii'') $\theta_{t,b}$ is J_b -holomorphic on a neighbourhood of K_b^0 and it approximates θ_b on K_b^0 for all $b \in B$ and $t \in I$ (the approximation is in the fine $\mathcal{C}^{l,k}$ topology), and
- (iii'') $\theta_{1,b} = \theta'_{1,b}$ is J_b -holomorphic on Ω for all $b \in P_0$.

By using another cut-off function in the parameter of the homotopy, we can extend $\theta_{t,b}$ for $p \in P$ to all of X_b without changing its values on a neighbourhood of $(\mathring{P} \times L) \cup ((B \setminus \mathring{P}) \times X)$.

Using this device inductively with respect to an exhaustion of B as in the proof of Theorem 1.6, we can approximate θ in the fine $\mathcal{C}^{l,k}$ topology on K^0 by a family of nowhere vanishing 1-forms $\{\tilde{\theta}_b\}_{b \in X}$ of class $\mathcal{C}^{l,k}(B \times X)$ which are $\mathcal{J}$ -holomorphic on a neighbourhood of $B \times L$. Theorem 7.1 then follows by an obvious induction with respect to a normal exhaustion of X by compact Runge sets. $\square$

8. Families of directed holomorphic immersions and of conformal minimal immersions

In this section, we illustrate how the results of this paper can be used to construct families of directed holomorphic immersions and of conformal minimal immersions from a family of open Riemann surfaces as in Theorem 1.6. There are many further problems of this kind which may possibly or even likely be treated by these new methods, and we indicate a few of them in Problem 8.7.

A connected compact complex submanifold Y of the complex projective space $\mathbb{CP}^{n-1}$, $n \in \mathbb{N}$, determines the punctured complex cone

$$A = \{(z_1, \dots, z_n) \in \mathbb{C}_*^n : [z_1 : \dots : z_n] \in Y\}. \quad (8.1)$$

Note that A is smooth and connected, and its closure $\overline{A} = A \cup \{0\} \subset \mathbb{C}^n$ is an algebraic subvariety of $\mathbb{C}^n$ by Chow's theorem [25]. By [38, Theorem 5.6.5], A is an Oka manifold if and only if Y is an Oka manifold. By [12, Lemma 3.5.1], the convex hull of A is the smallest complex subspace of $\mathbb{C}^n$ containing A , and we shall assume without loss of generality that this hull is all of $\mathbb{C}^n$.

Let X be a connected open Riemann surface and θ be a nowhere vanishing holomorphic 1-form on X . A holomorphic immersion $h : X \rightarrow \mathbb{C}^n$ is said to be *directed by A* , or an *A -immersion*, if its complex derivative with respect to any local holomorphic coordinate on X takes its values in A . Clearly, this holds if and only if the holomorphic map $f = dh/\theta : X \rightarrow \mathbb{C}^n$ assumes values in A . Conversely, a holomorphic map $f : X \rightarrow A$ satisfying the period vanishing conditions

$$\int_C f\theta = 0 \quad \text{for all closed curves } C \subset X \quad (8.2)$$

integrates to a holomorphic A -immersion $h : X \rightarrow \mathbb{C}^n$ by setting

$$h(x) = v + \int_{x_0}^x f\theta, \quad x \in X$$

for any $x_0 \in X$ and $v \in \mathbb{C}^n$. Since $f\theta$ is a holomorphic 1-form, it suffices to verify conditions (8.2) on a basis of the homology group $H_1(X, \mathbb{Z}) \cong \mathbb{Z}^r$, a free abelian group of some rank $r \in \mathbb{Z}_+ \cup \{\infty\}$.

Note that a map directed by the cone $A = \mathbb{C}_*^n$ is simply an immersion. Another case of major interest is the *null quadric*

$$\mathbf{A} = \{(z_1, \dots, z_n) \in \mathbb{C}_*^n = \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \dots + z_n^2 = 0\}, \quad n \geq 3. \quad (8.3)$$

Holomorphic immersions directed by $\mathbf{A}$ are called *holomorphic null curves* in $\mathbb{C}^n$. The real and the imaginary part of a holomorphic null immersion $X \rightarrow \mathbb{C}^n$ are conformal harmonic immersions $X \rightarrow \mathbb{R}^n$. Such immersions parameterize minimal surfaces, hence are called conformal minimal immersions. Conversely, every conformal minimal immersion $X \rightarrow \mathbb{R}^n$ is locally (on any simply connected domain) the real part of a holomorphic null curve. See [12,87] for more information.

Effective methods for construction directed holomorphic immersions were developed by Alarcón and Forstnerič in [7]. Assuming that A is an Oka manifold, they proved an Oka principle with Runge and Mergelyan approximation for holomorphic A -immersions [7, Theorems 2.6 and 7.2]. They also showed that every holomorphic A -immersion can be approximated by holomorphic A -embeddings when $n \geq 3$, and by proper holomorphic A -embeddings under an additional assumption on the cone [7, Theorem 8.1]. Alarcón and Castro-Infantes [6] added interpolation to the picture. A parametric Oka principle for A -immersions was proved in [44, Theorem 5.3]. Algebraic A -immersions from affine Riemann surfaces are studied in [13] under the assumption that A is algebraically elliptic in the sense of Gromov [55] (see also [38, Definition 5.6.13]). Several cones arising in geometric applications, in particular the null quadric $\mathbf{A}$ (8.3), are algebraically elliptic. Recently, Alarcón et al. [13,10] obtained h-principles for algebraic immersions directed by cones which are flexible in the sense of Arzhantsev et al. [14]. Minor variations of these results for the null cone (8.3) yield similar results for conformal minimal immersions of open Riemann surfaces in Euclidean spaces; see the monograph [12].

The main advantage of the techniques in the mentioned papers, when compared to the previously known results, is that they allow a complete control of the conformal structure of the resulting directed curves or minimal surfaces. By using the approximation results developed in the present paper, one can go substantially further and construct families of such objects with a control of the conformal structure of every member of the family, which may depend continuously or smoothly on a parameter. We now present a few specific results in this direction, which are only the tip of an iceberg of possibilities.

In the following, X is a connected, smooth, open oriented surface, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X as in Theorem 1.6, and $\{\theta_b\}_{b \in B}$ is a family of nowhere vanishing J_b -holomorphic 1-forms on X furnished by Theorem 7.1. Recall that a map $f : B \times X \rightarrow Y$ is said to be $\mathcal{J}$ -holomorphic if $f(b, \cdot) : X \rightarrow Y$ is J_b -holomorphic for every $b \in B$. The first two items in the following definition come from [7, Definition 2.2] and apply to any open Riemann surface X .

Definition 8.1. Let $A \subset \mathbb{C}_*^n$ be a smooth connected punctured complex cone of the form (8.1).

- (i) A holomorphic map $f : X \rightarrow A$ is nondegenerate if the tangent spaces $T_{f(x)}A \subset T_{f(x)}\mathbb{C}^n \cong \mathbb{C}^n$ over all points $x \in X$ span $\mathbb{C}^n$.
- (ii) A holomorphic A -immersion $h : X \rightarrow \mathbb{C}^n$ is nondegenerate if the map $f = dh/\theta : X \rightarrow A$ is such, where θ is any nowhere vanishing holomorphic 1-form on X .
- (iii) A $\mathcal{J}$ -holomorphic map $f : B \times X \rightarrow A$ is nondegenerate if $f_b = f(b, \cdot) : X \rightarrow A$ is nondegenerate for every $b \in B$.
- (iv) A map $h : B \times X \rightarrow \mathbb{C}^n$ is an A -immersion if $h_b = h(b, \cdot) : X \rightarrow \mathbb{C}^n$ is a J_b -holomorphic A -immersion for every $b \in B$, and is nondegenerate if $dh_b/\theta_b : X \rightarrow A$ is such for every $b \in B$.
- (v) Let $S = K \cup E \subset X$ be an admissible set (see Definition 6.6). A map $h : B \times S \rightarrow \mathbb{C}^n$ of class $\mathcal{C}^{l,k}$ ($l \geq 0, k \geq 1$) is a *generalized A -immersion* if for every $b \in B$ the map $h_b = h(b, \cdot) : S \rightarrow \mathbb{C}^n$ is an immersion whose derivative assumes values in A , and h_b is J_b -holomorphic on $\dot{S} = \dot{K}$. Such an h is nondegenerate if for every $b \in B$, h_b is nondegenerate on every connected component of K and of E . (See [12, Definition 3.1.2].)

Assuming that A is not contained in any hyperplane of $\mathbb{C}^n$, we show in Lemma 8.5 that for any $r \in \mathbb{N}$ the set of r -tuples of points in A at which the tangent spaces to A fail to span $\mathbb{C}^n$ is a closed algebraic subvariety of A^r , the Cartesian product of r copies of A . From this and the identity principle it follows that a holomorphic map $X \rightarrow A$ from a connected open Riemann surface X is nondegenerate if and only if its restriction to any nonempty open subset $U \subset X$ is such. Nondegenerate holomorphic maps $X \rightarrow A$ clearly form an open subset of the space $\mathcal{O}(X, A)$ of all holomorphic maps in the compact-open topology. By Lemma 8.5 this set is also dense in $\mathcal{O}(X, A)$, and this also holds for families of maps depending continuously on a parameter in a suitable topological space.

A holomorphic map $f : X \rightarrow \mathbf{A} \subset \mathbb{C}^n$ from a connected open Riemann surface X to the null cone (8.3) is nondegenerate if and only if $f(X)$ is not contained in a ray of $\mathbf{A}$ (see [12, Lemma 3.1.1]).

We shall prove the following h-principle for families of directed holomorphic immersions from a family of open Riemann surfaces. Compare with the h-principles for maps from a fixed open Riemann surface in [7, Theorem 2.6] and [44, Theorem 5.3].

Theorem 8.2. Assume that $A \subset \mathbb{C}^n$ is a connected smooth punctured Oka cone (8.1) which is not contained in any hyperplane, X is a connected smooth open surface, B is a parameter space as in Theorem 1.6, $\mathcal{J} = \{J_b\}_{b \in B}$ is a family of complex structures on X of class $\mathcal{C}^{l,(k,\alpha)}$ ($k \geq 1, 0 \leq l \leq k+1, 0 < \alpha < 1$), and $\{\theta_b\}_{b \in B}$ is a family of nowhere vanishing J_b -holomorphic 1-forms on X furnished by Theorem 7.1. Given a continuous map $f_0 : B \times X \rightarrow A$, there is a nondegenerate A -immersion $h : B \times X \rightarrow \mathbb{C}^n$ of class

$\mathcal{C}^{l,k+1}(B \times X)$ such that the map $f : B \times X \rightarrow A$ defined by $f(b, \cdot) = dh_b/\theta_b$ for all $b \in B$ is homotopic to f_0 .

The analogue of Theorem 8.2 also holds with approximation on a Runge subset $K \subset B \times X$ (see Definition 5.1) and interpolation on $Q \times X$ (for a closed subset $Q \subset B$) as in Theorem 1.6 and Remark 1.7. Furthermore, it holds with approximation on $B \times S$ where $S = K \cup E$ is a Runge admissible subset of X (see Definition 6.6); cf. Theorem 6.7. These additions will be evident from the proof.

Taking $A = \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ we obtain the following corollary to Theorem 8.2, which extends the Gunning–Narasimhan theorem [56] to families of complex structures on a smooth open surface. This result also holds with the addition of approximation conditions described above.

Corollary 8.3. *Given a smooth open surface X and a family $\{J_b\}_{b \in B}$ of complex structures on X as in Theorem 8.2, there is a function $h : B \times X \rightarrow \mathbb{C}$ of class $\mathcal{C}^{l,k+1}$ such that $h(b, \cdot) : X \rightarrow \mathbb{C}$ is a J_b -holomorphic immersion for every $b \in B$.*

If h is as in the corollary then $|h(b, \cdot)|^2$ is a smooth strongly subharmonic function on the Riemann surface (X, J_b) for every $b \in B$. By using Theorem 1.1 it is easy to find a function $\rho : B \times X \rightarrow \mathbb{R}_+$ of the form $\rho = \sum_i |f_i|^2$, where each $f_i : B \times X \rightarrow \mathbb{C}$ is $\mathcal{J}$ -holomorphic, satisfying the following.

Corollary 8.4. *Given a smooth open oriented surface X and a family $\{J_b\}_{b \in B}$ of complex structures on X as in Theorem 8.2, there is a function $\rho : B \times X \rightarrow \mathbb{R}_+$ of class $\mathcal{C}^{l,k+1}$ such that $\rho(b, \cdot) : X \rightarrow \mathbb{R}_+$ is a smooth strongly J_b -subharmonic exhaustion function for every $b \in B$.*

Proof of Theorem 8.2. We shall adapt the proof of the parametric h-principle for directed holomorphic immersions from an open Riemann surface, given in [44, Secs. 4–5]. (See especially the proofs of Theorems 4.1 and 5.3 in [44], where the reader can find further details.) For the nonparametric case, see [7, Theorem 2.6] and [12, Theorem 3.6.1].

For simplicity of exposition, we assume that the parameter space B is a compact neighbourhood retract in a Euclidean space $\mathbb{R}^m$. The general case requires an additional induction with respect to a covering of B by such compacts, which proceeds as in the proof of Theorem 1.6.

By Theorem 1.6 we can deform f_0 to a $\mathcal{J}$ -holomorphic map $f_1 : B \times X \rightarrow A$ of class $\mathcal{C}^{l,k+1}$. The following lemma shows that we can choose f_1 to be nondegenerate in the sense of Definition 8.1.

Lemma 8.5. *(Assumptions as in Theorem 8.2.) Every $\mathcal{J}$ -holomorphic map $f : B \times X \rightarrow A$ of class $\mathcal{C}^{l,k+1}$ can be approximated in the $\mathcal{C}^{l,k+1}$ topology on compacts by nondegenerate $\mathcal{J}$ -holomorphic maps of class $\mathcal{C}^{l,k+1}$ homotopic to f . If in addition Q is*

a closed subset of B and the map $f(b, \cdot) : X \rightarrow A$ is nondegenerate for every $b \in Q$ then the deformation with the stated properties can be chosen to agree with f on $Q \times X$.

Proof. Since the cone A is not contained in any hyperplane of $\mathbb{C}^n$, there is an integer $q \in \mathbb{N}$ such that for a generic q -tuple of points $z_1, \dots, z_q$ in A we have $\sum_{i=1}^q T_{z_i} A = \mathbb{C}^n$. (We consider the tangent space $T_z A$ as a subspace of $\mathbb{C}^n$.) To deform a map $f : X \rightarrow A$ to a nondegenerate one, it suffices to push a finite subset of $f(X)$ to such a generic position. A procedure for a single map is given in [7, Theorem 2.3 (a)]; for a family of maps from an open Riemann surface, see [44, Theorem 5.4]. Since the proof of the latter result does not include the details on how to apply the general position argument, I take this occasion to provide the details also for a variable family of complex structures on X .

Assume as before that B is a compact subset of $\mathbb{R}^m$; the general case will follow by using cutoff functions in the parameter $b \in B$ as will be clear from the proof. For any

integer $r \in \mathbb{N}$ we define a subvariety $I_r \subset A^r = \overbrace{A \times \cdots \times A}^{r \text{ times}}$ by

$$I_r = \left\{ z = (z_1, \dots, z_r) \in A^r : \sum_{i=1}^r T_{z_i} A \neq \mathbb{C}^n \right\}. \quad (8.4)$$

Note that I_r is invariant under the action of the symmetric group S_r on r elements permuting the components of A^r . Since $A \cup \{0\}$ is an algebraic subvariety of $\mathbb{C}^n$ having the origin as the only singular point, there are finitely many algebraic vector fields $V_1, \dots, V_d$ on $\mathbb{C}^n$ that are tangent to A , vanish in $0 \in \mathbb{C}^n$, and span the tangent space $T_z A$ for every $z \in A$. Then, I_r is the set of points $z = (z_1, \dots, z_r) \in A^r$ at which the $n \times (rd)$ matrix with columns $V_i(z_j)$ for $i = 1, \dots, d$ and $j = 1, \dots, r$ has rank $< n$, so it is an algebraic subvariety of A^r . Since A is not contained in any hyperplane of $\mathbb{C}^n$, there is an integer $q \in \mathbb{N}$ such that I_q is a proper algebraic subvariety of A^q .

We claim that the codimension of I_r in A^r increases to infinity as $r \rightarrow \infty$. This implies that for r big enough, every continuous map $g = (g_1, \dots, g_r) : B \rightarrow A^r$ can be uniformly approximated by maps whose images avoid I_r . For smooth maps on open domains in $\mathbb{R}^m$, this is an immediate consequence of the transversality theorem when $\text{codim}(I_r, A^r) > m$. By the Tietze extension theorem, the continuous map $g : B \rightarrow A^r$ extends to a continuous map $g : \mathbb{R}^m \rightarrow (\mathbb{C}^n)^r$. There is an open neighbourhood $\Omega \subset (\mathbb{C}^n)^r$ of A^r and a smooth retraction $\rho : \Omega \rightarrow A^r$. Pick a neighbourhood $U \subset \mathbb{R}^m$ of B such that $g(U) \subset \Omega$. Then, the map $\rho \circ g : U \rightarrow A^r$ is a continuous extension of g . Approximating this map uniformly on B gives smooth maps $U \rightarrow A^r$ to which the previous argument applies.

The claim that $\lim_{r \rightarrow \infty} \text{codim}(I_r, A^r) = \infty$ is especially simple to prove when A is the null quadric (8.3) in $\mathbb{C}^n$ for $n > 2$. In this case, $I_r = \tilde{I}_r \cap A^r$ where $\tilde{I}_r \subset (\mathbb{C}_*^n)^r$ consists of all r -tuples $z = (z_1, \dots, z_r)$ of $\mathbb{C}$ -collinear vectors $z_i \in \mathbb{C}_*^n$. For $r = 2$, the collinearity condition on $z_1 = (z_{1,1}, \dots, z_{1,n}) \in \mathbb{C}_*^n$ and $z_2 = (z_{2,1}, \dots, z_{2,n}) \in \mathbb{C}_*^n$ is locally expressed by $n - 1$ independent algebraic equations, given by vanishing of certain minors of the $2 \times n$ matrix of coefficients. For $r \geq 2$ we have locally $(r - 1)(n - 1)$ independent equations

for $\tilde{I}_r$, so this is the codimension of $\tilde{I}_r$ in $(\mathbb{C}_*^n)^r$. Since A^r has codimension r in $(\mathbb{C}_*^n)^r$, the codimension of I_r in A^r is at least $(r-1)(n-1) - r = r(n-2) - n + 1$. Since $n > 2$, this grows to infinity with r .

In the general case we argue as follows. Assume that $r \in \mathbb{N}$ is big enough such that I_r is a proper subvariety of A^r . Set $s = \dim I_r$, so $\text{codim}(I_r, A^r) = r \dim A - s$. Let I' be an irreducible component of I_r of maximal dimension s . Since I' is a proper subvariety of A^r , up to a permutation of the components there are points $(z_1, \dots, z_r) \in I'$ and $z'_r \in A \setminus \{z_r\}$ such that $(z_1, \dots, z_{r-1}, z'_r) \in A^r \setminus I_r$. It follows that $(z_1, \dots, z_{r-1}, z_r, z'_r) \in A^{r+1} \setminus I_{r+1}$. Note that the projection $\pi : A^{r+1} \rightarrow A^r$ which deletes the last component maps I^{r+1} to I^r . It follows that $I_{r+1} \cap \pi^{-1}(I')$ is a proper subvariety of $I' \times A$, so its dimension is at most $\dim I' + \dim A - 1 = s + \dim A - 1$. On an irreducible component I'' of I_r of dimension $\dim I'' < s$ we have $\dim(I_{r+1} \cap \pi^{-1}(I'')) \leq s - 1 + \dim A$. This shows that

$$\begin{aligned} \text{codim}(I_{r+1}, A^{r+1}) &\geq (r+1) \dim A - (s-1 + \dim A) = r \dim A - s + 1 \\ &= \text{codim}(I_r, A^r) + 1. \end{aligned}$$

This establishes the claim that $\lim_{r \rightarrow \infty} \text{codim}(I_r, A^r) = \infty$.

Assume now that $f : B \times X \rightarrow A$ is a $\mathcal{J}$ -holomorphic map. Choose $r \in \mathbb{N}$ big enough such that $\text{codim}(I_r, A^r) > m$. Pick a compact Runge set $K \subset X$ with nonempty interior and distinct points $x_1, \dots, x_r \in \mathring{K}$, and set $g_i = f(\cdot, x_i) : B \rightarrow A$ for $i = 1, \dots, r$. By what was said above, we can approximate the maps g_i uniformly on B by maps $\tilde{g}_i : B \rightarrow A$ such that the image of the map $\tilde{g} = (\tilde{g}_1, \dots, \tilde{g}_r) : B \rightarrow A^r$ avoids the subvariety I_r in (8.4). To prove the lemma, it remains to show that if the above approximations are close enough, there is a $\mathcal{J}$ -holomorphic map $\tilde{f} : B \times X \rightarrow A$ approximating f on $B \times K$ such that $\tilde{f}(\cdot, x_i) : B \rightarrow A$ is so close to $\tilde{g}_i : B \rightarrow A$ for $i = 1, \dots, r$ that $\tilde{f}$ is nondegenerate. This type of perturbation can be obtained by flows of vector fields as described in [44, proof of Theorem 5.4]; let us recall the main idea.

Assume first that A is flexible, in the sense that there exist finitely many $\mathbb{C}$ -complete holomorphic vector fields $V_1, \dots, V_d$ on A which span the tangent space of A at every point. (This holds in particular for the null quadric $\mathbf{A}$, cf. [7, Example 4.4]. A partial list of references to examples of flexible varieties can be found in [37, p. 394] preceding Example 6.3 *ibid.*) For each $i = 1, \dots, r$ choose a $\mathcal{J}$ -holomorphic function $\xi_i : B \times X \rightarrow \mathbb{C}$ such that $\xi_i(b, x_j) = \delta_{i,j}$ (the Kronecker delta) for $j = 1, \dots, r$; see Theorem 1.1. Let ϕ_t^j denote the flow of V_j for $j = 1, \dots, d$. For every $i = 1, \dots, r$ we consider the map $\Psi_i : B \times X \times \mathbb{C}^d \times A \rightarrow A$ given by

$$\Psi_i(b, x, \zeta, z) = \phi_{\zeta_1 \xi_i(b, x)}^1 \circ \dots \circ \phi_{\zeta_d \xi_i(b, x)}^d(z), \quad (8.5)$$

where $\zeta = (\zeta_1, \dots, \zeta_d) \in \mathbb{C}^d$. Note that $\Psi_i(b, \cdot, \cdot, \cdot)$ is J_b -holomorphic in $x \in X$ and holomorphic on $\mathbb{C}^d \times A$, and it satisfies the following conditions.

- (i) $\Psi_i(b, x, 0, z) = z$ for all $(b, x) \in B \times X$ and $z \in A$.

- (ii) $\Psi_i(b, x_j, \zeta, z) = z$ for all $b \in B$, $\zeta \in \mathbb{C}^d$, $z \in A$, and $j \in \{1, \dots, r\} \setminus \{i\}$.
- (iii) The differential $\partial_\zeta \Psi_i(b, x_i, \zeta, z)|_{\zeta=0} : \mathbb{C}^d \rightarrow T_z A$ is surjective at $z = f(b, x_i)$ for all $b \in B$.

Let $\zeta_j = (\zeta_{j,1}, \dots, \zeta_{j,d}) \in \mathbb{C}^d$ for $j = 1, \dots, r$. We define the maps $\Phi_i : B \times X \times (\mathbb{C}^d)^i \times A \rightarrow A$ for $i = 1, \dots, r$ by taking $\Phi_1 = \Psi_1$ and

$$\Phi_i(b, x, (\zeta_1, \dots, \zeta_i), z) = \Psi_i(b, x, \zeta_i, \Phi_{i-1}(b, x, (\zeta_1, \dots, \zeta_{i-1}), z)) \quad (8.6)$$

for $i = 2, \dots, r$. Set $\zeta = (\zeta_1, \dots, \zeta_r) \in (\mathbb{C}^d)^r$. The properties (i)–(iii) of Ψ_i imply that the map

$$\Theta_i(b) = \partial_{\zeta_i} \Phi_r(b, x_i, \zeta, f(b, x_i))|_{\zeta=0} : \mathbb{C}^d \rightarrow T_{f(b, x_i)} A \quad (8.7)$$

is surjective for every $b \in B$ and $i = 1, \dots, r$. Assuming that a map $\tilde{g}_i : B \rightarrow A$ is close enough to $g_i = f(\cdot, x_i)$ for every $i = 1, \dots, r$, the implicit function theorem gives functions $\zeta_i : B \rightarrow \mathbb{C}^d$ close to the origin for $i = 1, \dots, r$ such that the map $\tilde{f} : B \times X \rightarrow A$ given by

$$\tilde{f}(b, x) = \Phi_r(b, x, (\zeta_1(b), \dots, \zeta_r(b)), f(b, x)) \in A$$

is $\mathcal{J}$ -holomorphic, close to f on $B \times K$, and satisfies $\tilde{f}(b, x_i) = \tilde{g}_i(b)$ for $i = 1, \dots, r$. (More precisely, for every $i = 1, \dots, r$ we split the trivial bundle in the form $B \times \mathbb{C}^d = E_i \oplus E'_i$, where $E'_i \subset B \times \mathbb{C}^d$ is the complex vector subbundle with fibres $E'_{i,b} = \ker \Theta_i(b)$ (see (8.7)), and apply the implicit function theorem to find sections $\zeta_i : B \rightarrow E_i$ with the above property for $i = 1, \dots, r$.)

If the cone A is not necessarily flexible, the same procedure can be used to obtain a nondegenerate $\mathcal{J}$ -holomorphic map $\tilde{f} : B \times K \rightarrow A$ with the stated properties. Indeed, the maps Ψ_i (8.5) and Φ_i (8.6) are now defined for the values of ζ in a ball around the origin whose radius may be chosen independent of $x \in K$, and the arguments carry over verbatim. The proof is then completed by approximating this map on $B \times K$ by a $\mathcal{J}$ -holomorphic map $B \times X \rightarrow A$, using Theorem 1.6.

The interpolation condition on a closed subset $Q \subset B$ as in the lemma can be fulfilled by using a cutoff function in the parameter b in the above construction of the deformation. $\square$

In the sequel, $f_1 : B \times X \rightarrow A$ is a nondegenerate $\mathcal{J}$ -holomorphic map of class $\mathcal{C}^{l,k+1}$ given by Lemma 8.5. Fix a complex structure J on X and a strongly J -subharmonic Morse exhaustion function $\rho : X \rightarrow \mathbb{R}_+$. There is an exhaustion $\emptyset = K_0 \subset K_1 \subset \dots \subset \bigcup_{i=0}^\infty K_i = X$ by smoothly bounded compact Runge sets $K_i = \{x \in X : \rho(x) \leq c_i\}$ for a sequence of regular values $0 < c_1 < c_2 < \dots$ of ρ , with $\lim_i c_i = +\infty$, such that $K_1 \neq \emptyset$ and for every $i \in \mathbb{Z}_+$ the open set $D_i = \mathring{K}_{i+1} \setminus K_i$ contains at most one critical point of ρ . (See [12, Sect. 1.4].) In particular, K_1 is a disc. Recall that $\{\theta_b\}_{b \in B}$ is a family

of nowhere vanishing J_b -holomorphic 1-forms on X furnished by Theorem 7.1. We shall inductively construct a sequence of open neighbourhoods $U_i \subset B \times X$ of $B \times K_i$, maps $f_i : U_i \rightarrow A$ of class $\mathcal{C}^{l,k+1}$, and numbers $\epsilon_i > 0$ such that the following conditions hold for every $i = 1, 2, \dots$

- (a) The map $f_{i,b} : U_{i,b} \rightarrow A$ is J_b -holomorphic and nondegenerate for every $b \in B$.
- (b) $\int_C f_{i,b} \theta_b = 0$ for every closed curve $C \subset K_i$.
- (c) f_i is homotopic to $f_1|_{U_i}$ through maps $U_i \rightarrow A$.
- (d) $\|f_{i+1} - f_i\|_{\mathcal{C}^l(B \times K_i)} < \epsilon_i$.
- (e) $0 < \epsilon_{i+1} < \epsilon_i/2$, and if $f : B \times X \rightarrow A \cup \{0\}$ is an $\mathcal{J}$ -holomorphic map of class $\mathcal{C}^l$ such that $\|f - f_i\|_{\mathcal{C}^l(B \times K_i)} < 2\epsilon_i$ then f is nondegenerate and $f(B \times K_{i-1}) \subset A$.

Under these conditions, the limit map $f = \lim_{i \rightarrow \infty} f_i : B \times X \rightarrow A$ exists and is of class $\mathcal{C}^l(B \times X)$, it is $\mathcal{J}$ -holomorphic (hence of class $\mathcal{C}^{l,k+1}(B \times X)$ by Lemma 5.6), nondegenerate (see Definition 8.1), homotopic to f_0 , and $\int_C f(b, \cdot) \theta_b = 0$ holds for every closed curve $C \subset X$ and $b \in B$. Fixing a point $x_0 \in X$, the map $h : B \times X \rightarrow \mathbb{C}^n$ given by

$$h(b, x) = \int_{x_0}^x f(b, \cdot) \theta_b, \quad b \in B, \quad x \in X$$

is well-defined and satisfies $dh(b, \cdot) = f(b, \cdot) \theta_b$ ($b \in B$), so it is a nondegenerate A -immersion.

We now explain the induction. The assumptions imply that K_1 is a smoothly bounded compact disc. Let U_1 be an open disc containing K_1 , and let $f_1 : B \times X \rightarrow A$ be the initial nondegenerate map. Assume inductively that $i \in \mathbb{N}$ and we have already found maps f_j with the required properties for $j = 1, \dots, i$, and let us explain how to obtain the next map f_{i+1} . We distinguish two cases.

The noncritical case: The domain $D_i = \mathring{K}_{i+1} \setminus K_i$ does not contain any critical point of ρ .

The critical case: D_i contains a unique (Morse) critical point of ρ .

We begin with the noncritical case. Then, K_i is a strong deformation retract of K_{i+1} and D_i is a finite union of annuli. In particular, the inclusion $K_i \hookrightarrow K_{i+1}$ induces an isomorphism of their homology groups $H_1(K_i, \mathbb{Z}) \cong H_1(K_{i+1}, \mathbb{Z})$. Assume that K_i is connected; the procedure that we shall explain can be performed independently on every connected component. Fix a point $x_0 \in \mathring{K}_i$. There are finitely many smooth Jordan curves $C_1, \dots, C_m \subset K_i$ such that any two of them only intersect at x_0 , they form a basis of the homology group $H_1(K_i, \mathbb{Z})$, and their union $C = \bigcup_{j=1}^m C_j$ is Runge in X . The same curves then form a basis of $H_1(K_{i+1}, \mathbb{Z})$. Consider the period map $\mathcal{P} : B \times \mathcal{C}(B \times C, A) \rightarrow (\mathbb{C}^n)^m$ given for any $b \in B$ and $f \in \mathcal{C}(B \times C, A)$ by

$$\mathcal{P}(b, f) = \left(\oint_{C_j} f(b, \cdot) \theta_b \right)_{j=1, \dots, m} \in (\mathbb{C}^n)^m, \quad (8.8)$$

By condition (b) we have that $\mathcal{P}(b, f_i) = 0$ for all $b \in B$. Since the map $f_i : B \times K_i \rightarrow A$ is nondegenerate, we can apply [7, Lemma 5.1] (see also [12, Lemma 3.2.1]) to find a *period dominating spray* of J_b -holomorphic maps

$$F_i(\zeta, b, \cdot) : K_i \rightarrow A \quad \text{for } b \in B,$$

of class $\mathcal{C}^l(B \times K_i)$, depending holomorphically on $\zeta = (\zeta_1, \dots, \zeta_N)$ in a ball $\mathbb{B} \subset \mathbb{C}^N$, such that $F_i(0, \cdot, \cdot) = f_i$. (Recall that a map is called holomorphic on a compact set if it is holomorphic in an open neighbourhood of the said set.) The period domination property means that the map

$$\mathbb{B} \ni \zeta \mapsto \mathcal{P}(b, F_i(\zeta, b, \cdot)) = \left(\oint_{C_j} F_i(\zeta, b, \cdot) \theta_b \right)_{j=1, \dots, m} \in (\mathbb{C}^n)^m \quad (8.9)$$

is submersive at $\zeta = 0$, i.e., its differential at $\zeta = 0$ is surjective for every $b \in B$. A map F_i with these properties can be chosen to be of the form

$$\Psi(\zeta, b, x) = \phi_{\zeta_1 h_1(b, x)}^1 \circ \dots \circ \phi_{\zeta_N h_N(b, x)}^N(f(b, x)) \quad (8.10)$$

where $\zeta = (\zeta_1, \dots, \zeta_N) \in \mathbb{C}^N$ for a suitable $N \in \mathbb{N}$, $(b, x) \in B \times K_i$, every ϕ^j is the flow of one of the holomorphic vector fields $V_1, \dots, V_d$ on A (possibly with repetitions) which span the tangent space of A at every point, and the functions $h_j \in \mathcal{C}^{l, k+1}(B \times X)$ are $\mathcal{J}$ -holomorphic. We begin by choosing functions $h_j : B \times C \rightarrow \mathbb{C}$ of class $\mathcal{C}^l$ to ensure that the map Ψ in (8.10) is period dominating on the curves in C ; see [12, proof of Lemma 3.2.1] for the details. By the parametric Mergelyan theorem (see Theorem 1.3 when $l = 0$ and Theorem 6.7 when $l > 0$) we can approximate the functions h_j in $\mathcal{C}^l(B \times C)$ by $\mathcal{J}$ -holomorphic functions of class $\mathcal{C}^l(B \times K_i)$ depending holomorphically on ζ . By Lemma 5.6 these approximants are then of class $\mathcal{C}^{l, k+1}(B \times K_i)$. If the approximation is close enough then the resulting map has the stated properties.

For each $b \in B$ let $V_b \subset \mathbb{C}^N$ denote the kernel of the differential of the period map (8.9) at $\zeta = 0$. This is a complex vector subspace of $\mathbb{C}^N$ with $\dim V_b = N - mn$ which is of class $\mathcal{C}^l$ in $b \in B$. Let $W_b \subset \mathbb{C}^N$ denote the orthogonal complement of V_b . Fix a number $0 < r < 1$. Since A is an Oka manifold and K_i is a strong deformation retract of K_{i+1} , Theorem 6.4 allows us to approximate F_i in the $\mathcal{C}^l$ topology on $r\mathbb{B} \times B \times K_i$ by a family of J_b -holomorphic maps $g(\zeta, b, \cdot) : K_{i+1} \rightarrow A$ ($\zeta \in r\mathbb{B}$, $b \in B$) which are holomorphic in ζ and of the same regularity class as F_i . If the approximation is sufficiently close, the implicit function theorem gives a map $\zeta : B \rightarrow \mathbb{B}$ of class $\mathcal{C}^l(B)$, close to the zero map, such that $\zeta(b) \in W_b$ for all $b \in B$ and the J_b -holomorphic map

$$f_{i+1}(b, \cdot) := g(\zeta(b), b, \cdot) : K_{i+1} \rightarrow A$$

satisfies the period vanishing conditions $\mathcal{P}(b, f_{i+1}) = 0$ in (8.9) for every $b \in B$. If the approximations were close enough then the map f_{i+1} is nondegenerate. To complete the induction step, we choose a number ϵ_{i+1} satisfying condition (e).

Next, we consider the critical case. Let $x_i \in D_i$ be the unique critical point of ρ in D_i . Since ρ is strongly subharmonic, the Morse index of ρ at x_i is either 0 or 1.

If the Morse index is 0, the point x_i is a local minimum of ρ , and hence a new connected component of the sublevel set $\{\rho \leq t\}$ appears when t passes the value $\rho(x_i)$. On this new component of K_{i+1} we can take f_{i+1} to be any nondegenerate $\mathcal{J}$ -holomorphic map to A . On the remaining connected components of K_{i+1} we proceed as in the noncritical case explained above.

Assume now that ρ has Morse index 1 at x_i . In this case, there is a smooth embedded arc $x_i \in E_i \subset D_i \cup bK_i$, which is transversely attached with both endpoints $bE_i = \{p_i, q_i\} \subset bK_i$ to K_i and is otherwise disjoint from K_i , such that $S_i = K_i \cup E_i$ is a Runge admissible set in X (see Definition 6.6), and S_i is a strong deformation retract of K_{i+1} . (See [12, pp. 21–22] for the details.) We assume that the Runge admissible set $S_i = K_i \cup E_i$ is connected, since on the remaining components of K_i we are faced with the noncritical case described above. We orient E_i so that p_i is the initial endpoint and q_i is the final endpoint. The inductive hypothesis implies that there is an A -immersion $h_i : B \times K_i \rightarrow \mathbb{C}^n$ satisfying

$$2\partial_{J_b} h_i(b, \cdot) = f_i(b, \cdot)\theta_b \quad \text{for all } b \in B.$$

We now extend f_i from a small open neighbourhood U_i of $B \times K_i$ to a map $f_i : U_i \cup (B \times S_i) \rightarrow A$ of class $\mathcal{C}^l$ such that the extended map is homotopic to f_1 through a homotopy that is fixed on U_i , and for every $b \in B$ the map $f_i(b, \cdot) : E_i \rightarrow A$ is nondegenerate (see Definition 8.1) and satisfies

$$\int_{E_i} f_i(b, \cdot)\theta_b = h_i(b, q_i) - h_i(b, p_i), \quad b \in B. \quad (8.11)$$

An extension with these properties is found as follows. (For more details, see [44, proof of Theorem 4.1] and note that the choice of the complex structure on X does not play any role in these arguments.)

We subdivide the arc E_i in two subarcs such that $E_i = E_{i,1} \cup E_{i,2}$ and $E_{i,1} \cap E_{i,2} = \{x_i\}$ (this point lies in the relative interior of E_i). We begin by extending f_i from U_i to a map $f'_i : B \times E_i \rightarrow A$ satisfying the required homotopy condition and such that it is nondegenerate on $B \times E_{i,2}$; this is possible by the transversality argument in the proof of Lemma 8.5. Choose a number $\epsilon > 0$ to be specified later. Using [44, Lemma 3.1], we modify f'_i on the relative interior of $B \times E_{i,1}$ to a map $f''_i : B \times E_i \rightarrow A$ satisfying

$$\left| \int_{E_i} f''_i(b, \cdot)\theta_b - (h_i(b, q_i) - h_i(b, p_i)) \right| < \epsilon, \quad b \in B. \quad (8.12)$$

In the final step, f''_i is modified on the relative interior of $B \times E_{i,2}$ to obtain a map f_i satisfying (8.11). To do this, we construct a spray of maps with the core f''_i , with the deformation supported on the relative interior of $B \times E_{i,2}$, which is period dominating on the arc E_i (this is possible since f''_i is nondegenerate on $B \times E_{i,2}$). Assuming as we may that $\epsilon > 0$ was chosen small enough, a suitable map in this spray corrects the small error in (8.12) to zero.

We can now proceed as in the noncritical case. Let $\mathcal{C} = \{C_1, \dots, C_m\}$ be a homology basis of the admissible set S_i such that $C = \bigcup_{j=1}^m C_j$ is a Runge set (see [12, Lemma 1.12.10]). If the arc E_i connects two different connected components of K_i , we add it to the family $\mathcal{C}$. (In the opposite case, E_i is a part of a new closed curve in the homology basis of S_i .) As in the noncritical case, we construct a $\mathcal{C}$ -period dominating spray $F_i : \mathbb{B} \times B \times S_i \rightarrow A$ of the form (8.10), where $\mathbb{B} \subset \mathbb{C}^N$ is a ball, such that $F_i(0, \cdot, \cdot) = f_i$ and the map $F_i(\cdot, b, \cdot)$ is holomorphic in the complex structure $J_{\text{st}} \times J_b$ on $\mathbb{B} \times K_i$ for every $b \in B$. By Theorem 6.5 (if $l = 0$) or Theorem 6.7 (if $l > 0$) we can approximate f_i in $\mathcal{C}^l(B \times S_i)$ by $\mathcal{J}$ -holomorphic maps $f'_i : B \times V_i \rightarrow A$, where V_i is a neighbourhood of S_i . Likewise, we can approximate the $\mathcal{J}$ -holomorphic functions ξ_j in (8.10) in $\mathcal{C}^l(B \times S_i)$ by functions ξ'_j which are $\mathcal{J}$ -holomorphic on $B \times V_i$. Pick a number $0 < r < 1$. Inserting these approximants in the expression (8.10) for the spray F_i and shrinking the neighbourhood V_i around S_i if necessary gives a map $G_i : r\mathbb{B} \times B \times V_i \rightarrow A$ approximating F_i in $\mathcal{C}^l(r\mathbb{B} \times B \times S_i)$ such that $G_i(\cdot, b, \cdot)$ is holomorphic in the complex structure $J_{\text{st}} \times J_b$ on $r\mathbb{B} \times X$ for every $b \in B$.

The final step is exactly as in the noncritical case, and we obtain a nondegenerate $\mathcal{J}$ -holomorphic map $\tilde{f}_i : B \times V_i \rightarrow A$ approximating f_i in $\mathcal{C}^l(B \times K_i)$ such that for every $b \in B$, the J_b -holomorphic map $\tilde{f}_i(b, \cdot) : V_i \rightarrow A$ satisfies $\mathcal{P}(b, \tilde{f}_i) = \mathcal{P}(b, f_i)$ (see (8.8)) on all curves in $\mathcal{C}$. (Hence, the periods of $\tilde{f}_i$ vanish on all closed curves in $\mathcal{C}$.) Next, we extend $\tilde{f}_i$ by approximation in $\mathcal{C}^l(B \times S_i)$ to a $\mathcal{J}$ -holomorphic map $f_{i+1} : B \times K_{i+1} \rightarrow A$, keeping the periods on the curves in $\mathcal{C}$ unchanged. This is accomplished by the noncritical case since S_i has a compact neighbourhood $S'_i \subset V_i$ such that $K_{i+1} \setminus S'_i$ is an annulus. We conclude the induction step by choosing ϵ_{i+1} satisfying condition (e). $\square$

The following h-principle for families of conformal minimal immersions is an immediate corollary to Theorem 8.2 applied with the null quadric $\mathbf{A}$ (8.3).

Corollary 8.6. *Let X , $\{J_b\}_{b \in B}$, and $\{\theta_b\}_{b \in B}$ be as in Theorem 8.2. Given a continuous map $f_0 : B \times X \rightarrow \mathbf{A}$ to the punctured null quadric $\mathbf{A} \subset \mathbb{C}^n$ (8.3) for some $n \geq 3$, there is a map $u : B \times X \rightarrow \mathbb{R}^n$ of class $\mathcal{C}^{l,k+1}$ such that $u_b = u(b, \cdot) : (X, J_b) \rightarrow \mathbb{R}^n$ is a nonflat conformal minimal immersion for every $b \in B$, and the $\mathcal{J}$ -holomorphic map $f : B \times X \rightarrow \mathbf{A}$, defined by $f(b, \cdot) = \partial_{J_b} u_b / \theta_b$ for all $b \in B$, is homotopic to f_0 .*

This result can be improved by adding approximation and prescribing the flux homomorphisms. We invite the reader to supply the precise statements and proofs of these generalisations.

By adding various global conditions on the maps in Theorem 8.2 and Corollary 8.6 such as properness, embeddedness, or completeness, the construction methods become more intricate, and we do not know whether they can be made in families. We pose the following problems.

Problem 8.7. Let $\{(X, J_b)\}_{b \in B}$ be a family of open Riemann surfaces as in Theorem 1.6.

- (a) Is there a continuous or a smooth family of proper J_b -holomorphic immersions $X \rightarrow \mathbb{C}^2$ and embeddings $X \hookrightarrow \mathbb{C}^3$? (The basic case is classical, see [38, Theorem 2.4.1] and the references therein. Without the properness condition, the affirmative result is given by Theorem 8.2.)
- (b) Assuming that $A \subset \mathbb{C}^n$ is an Oka cone (8.1), is there a continuous or a smooth family of proper J_b -holomorphic A -immersions or A -embeddings $X \rightarrow \mathbb{C}^n$? (For the basic case, see [7]. Without the properness or embeddedness condition, the affirmative answer is given by Theorem 8.2.)
- (c) Is there a continuous or a smooth family of proper conformal harmonic immersions $(X, J_b) \rightarrow \mathbb{R}^n$ for $n \geq 3$? (For the basic case, see [12, Theorem 3.6.1] and the references therein.)
- (d) Assume that X is a bordered Riemann surface. Is there a family of complete conformal minimal immersions $(X, J_b) \rightarrow \mathbb{R}^n$ ($n \geq 3$) with bounded images, i.e., does the Calabi–Yau phenomenon for minimal surfaces hold in families? (For the non-parametric case, see [12, Chapter 7] and [4].) The analogous question can be asked for holomorphic (directed) immersions $(X, J_b) \rightarrow \mathbb{C}^n$, $n \geq 2$ in the context of the problem asked by Yang [99]; see the survey by Alarcón [5].
- (e) Let $\eta = dz + \sum_{j=1}^n x_j dy_j$ be the standard complex contact form on $\mathbb{C}^{2n+1}$, $n \geq 1$. Is there a continuous or a smooth family of proper J_b -holomorphic Legendrian immersions $f_b : X \rightarrow \mathbb{C}^{2n+1}$, that is, such that $f_b^* \eta = 0$ holds for all $b \in B$? (For the basic case, see [11]. For the parametric case for maps from a single Riemann surface and without the properness condition, see [43].)

Note that problem (a) is a special case of (b) with the cone $A = \mathbb{C}_*^n$ for $n = 2$ and $n = 3$.

After the completion of this paper, a partial affirmative answer to Problem 8.7 (a) was given by Drinovec Drnovšek and Kališnik [31]. They proved that for any smooth open surface X , endowed with a family of complex structures $\{J_b\}_{b \in B}$ depending continuously on the parameter b in a metrisable space B , there is a continuous family of proper holomorphic maps $F_b : (X, J_b) \rightarrow \mathbb{C}^2$, $b \in B$.

It was recently observed by Alarcón and the author [9, p. 3] (May 2026) that, at least for parameter spaces B as in Theorem 1.6, the proof in [31] can be upgraded, using also Corollary 8.3 with approximation on admissible Runge subsets, to obtain a continuous family of proper J_b -holomorphic maps $F_b = (F_{b,1}, F_{b,2}) : X \rightarrow \mathbb{C}^2$ such that every component function $F_{b,i} : X \rightarrow \mathbb{C}$ ($b \in B$, $i = 1, 2$) is an immersion. In particular,

$F_b : X \rightarrow \mathbb{C}^2$ is a proper immersion for every $b \in B$. See also [9, Corollary 1.3] for an explicit result in this direction.

The paper [9] also gives partial affirmative answers to problems (b) and (c) above. It is shown that if B is a compact Euclidean neighbourhood retract and $\mathcal{J} = \{J_b\}_{b \in B}$ is a continuous family of complex structures of local Hölder class $\mathcal{C}^\alpha$ ($0 < \alpha < 1$) on an open surface X , then there is a continuous family of proper J_b -conformal minimal immersions $u_b : X \rightarrow \mathbb{C}^3$, $b \in B$, with an arbitrary given continuous family of flux homomorphisms $\text{Flux}_{u_b} : H_1(X, \mathbb{Z}) \rightarrow \mathbb{R}^3$; see [9, Theorem 1.1]. In particular, there is a $\mathcal{J}$ -holomorphic map $F = (F_1, F_2, F_3) : B \times X \rightarrow \mathbb{C}^3$ such that for every $b \in B$, the map $F_b = F(b, \cdot) = (F_{b,1}, F_{b,2}, F_{b,3}) : X \rightarrow \mathbb{C}^3$ is a J_b -holomorphic null immersion and the map $(\Re F_{b,1}, \Re F_{b,2}) : X \rightarrow \mathbb{R}^2$ is proper; see [9, Corollary 1.2].

Proper embeddings of smooth Cartan manifolds of type (m, n) (with complex leaves of dimension n and real codimension m) into $\mathbb{R}^{2m} \times \mathbb{C}^{m+2n+1}$ were constructed by Jurchescu [65, Sect. 7].

Acknowledgments

The author is supported by the European Union (ERC Advanced grant HPDR, 101053085) and grants P1-0291 and N1-0237 from ARIS, Republic of Slovenia. I wish to thank Oliver Dragičević, Paul Gauthier, Jure Kališnik, Yuta Kusakabe, Finnur Lárusson, Jaka Smrekar, Sašo Strle, and Tjaša Vrhovnik for remarks which helped me to improve the presentation. I also thank Francisco J. López for having asked whether complex analysis could be used to construct families of minimal surfaces of prescribed conformal types in Euclidean spaces. This question was the principal motivation for the paper, and it is answered in part by Theorem 8.2 and Corollary 8.6. Last but not least, I thank an anonymous referee for very helpful and pointed remarks which helped me to improve the presentation, and for drawing my attention to the works of Jurchescu and others on Cartan manifolds.

References

- [1] L.V. Ahlfors, Conformality with respect to Riemannian metrics, *Ann. Acad. Sci. Fenn., Ser. A I* 206 (1955) 22.
- [2] L.V. Ahlfors, *Lectures on Quasiconformal Mappings*, second edition, University Lecture Series, vol. 38, American Mathematical Society, Providence, RI, 2006, With supplemental chapters by C.J. Earle, I. Kra, M. Shishikura and J.H. Hubbard.
- [3] L.V. Ahlfors, L. Bers, Riemann's mapping theorem for variable metrics, *Ann. Math. (2)* 72 (1960) 385–404.
- [4] A. Alarcon, Complete minimal surfaces with Cantor ends in minimally convex domains, <https://arxiv.org/abs/2410.13687>, 2024.
- [5] A. Alarcón, The Yang problem for complete bounded complex submanifolds: a survey, in: *Cutting-Edge Mathematics, RSME 2022*, Ciudad Real, Spain, January 17–21, 2022, Springer, Cham, 2024, pp. 1–25.
- [6] A. Alarcón, I. Castro-Infantes, Interpolation by conformal minimal surfaces and directed holomorphic curves, *Anal. PDE* 12 (2) (2019) 561–604.
- [7] A. Alarcón, F. Forstnerič, Null curves and directed immersions of open Riemann surfaces, *Invent. Math.* 196 (3) (2014) 733–771.

- [8] A. Alarcón, F. Forstnerič, Oka-1 manifolds, *Math. Z.* 311 (2) (2025) 34, Id/No 33.
- [9] A. Alarcón, F. Forstnerič, Families of proper minimal surfaces, <https://arxiv.org/abs/2605.19883>, 2026.
- [10] A. Alarcón, F. Forstnerič, F. Lárusson, Isotopies of complete minimal surfaces of finite total curvature, *J. Lond. Math. Soc.* 114 (1) (2026) e70640.
- [11] A. Alarcón, F. Forstnerič, F.J. López, Holomorphic Legendrian curves, *Compos. Math.* 153 (9) (2017) 1945–1986.
- [12] A. Alarcón, F. Forstnerič, F.J. López, Minimal Surfaces from a Complex Analytic Viewpoint, *Springer Monographs in Mathematics*, Springer, Cham, 2021.
- [13] A. Alarcón, F. Lárusson, Regular immersions directed by algebraically elliptic cones, *J. Reine Angew. Math.* 823 (2025) 113–135.
- [14] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch, M. Zaidenberg, Infinite transitivity on affine varieties, in: *Birational Geometry, Rational Curves, and Arithmetic*, Springer, New York, 2013, pp. 1–13.
- [15] K. Astala, T. Iwaniec, G. Martin, *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, Princeton Mathematical Series, vol. 48, Princeton University Press, Princeton, NJ, 2009.
- [16] M.S. Baouendi, P. Ebenfelt, L.P. Rothschild, *Real Submanifolds in Complex Space and Their Mappings*, Princeton Mathematical Series, vol. 47, Princeton University Press, Princeton, NJ, 1999.
- [17] H. Behnke, F. Sommer, *Theorie der analytischen Funktionen einer komplexen Veränderlichen*. Zweite veränderte Auflage, *Die Grundlehren der mathematischen Wissenschaften*, vol. 77, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1962.
- [18] H. Behnke, K. Stein, Entwicklung analytischer Funktionen auf Riemannschen Flächen, *Math. Ann.* 120 (1949) 430–461.
- [19] G. Bellettini, *Lecture Notes on Mean Curvature Flow, Barriers and Singular Perturbations*, Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) (Lecture Notes. Scuola Normale Superiore di Pisa (New Series)), vol. 12, Edizioni della Normale, Pisa, 2013.
- [20] E. Bishop, Subalgebras of functions on a Riemann surface, *Pac. J. Math.* 8 (1958) 29–50.
- [21] B. Bojarski, P. Hajłasz, P. Strzelecki, Sard's theorem for mappings in Hölder and Sobolev spaces, *Manuscr. Math.* 118 (3) (2005) 383–397.
- [22] B.V. Bojarskij, Verallgemeinerte Lösungen eines Systems von Differentialgleichungen erster Ordnung vom elliptischen Typus mit unstetigen Koeffizienten, *Mat. Sb., Nov. Ser.* 43 (1958) 451–503.
- [23] T. Carleman, Sur un théorème de Weierstraß, *Ark. Mat. Astron. Fys.* 20 (4) (1927) 5.
- [24] S.-S. Chern, An elementary proof of the existence of isothermal parameters on a surface, *Proc. Am. Math. Soc.* 6 (1955) 771–782.
- [25] W.-L. Chow, On compact complex analytic varieties, *Am. J. Math.* 71 (1949) 893–914.
- [26] M. Coltoiu, Complete locally pluripolar sets, *J. Reine Angew. Math.* 412 (1990) 108–112.
- [27] J.B. Conway, *A Course in Functional Analysis*, second edition, Graduate Texts in Mathematics, vol. 96, Springer-Verlag, New York, 1990.
- [28] J.-P. Demailly, Cohomology of q -convex spaces in top degrees, *Math. Z.* 204 (2) (1990) 283–295.
- [29] F. Docquier, H. Grauert, Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten, *Math. Ann.* 140 (1960) 94–123.
- [30] A. Douady, Variétés et espaces mixtes, in: *Séminaire Henri Cartan: Familles d'espaces complexes et fondement de la géométrie analytique*. 13e année 1960/61, École Normale Supérieure, Secrétariat mathématique, Paris, 1962, Fasc. 2, ii, 111 p. (1962).
- [31] B. Drinovec Drnovšek, J. Kališnik, Families of proper holomorphic maps, *J. Geom. Anal.* 36 (3) (2026) 18, Id/No 102.
- [32] J. Dugundji, Note on CW polytopes, *Port. Math.* 11 (1952) 7–10.
- [33] A. Fletcher, V. Marković, Infinite dimensional Teichmüller spaces, in: *Handbook of Teichmüller Theory. Volume II*, European Mathematical Society (EMS), Zürich, 2009, pp. 65–91.
- [34] P. Flondor, E. Pascu, Some results on mixed manifolds, in: *Complex Analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981, Part 2*, in: *Lect. Notes Math.*, vol. 1014, 1983, pp. 17–26.
- [35] H. Florack, Reguläre und meromorphe Funktionen auf nicht geschlossenen Riemannschen Flächen, *Schriftenr. Math. Inst. Univ. Münst.* 1948 (1) (1948) 34.
- [36] J.E. Fornæss, F. Forstnerič, E. Wold, Holomorphic approximation: the legacy of Weierstrass, Runge, Oka-Weil, and Mergelyan, in: *Advancements in Complex Analysis. From Theory to Practice*, Springer, Cham, 2020, pp. 133–192.
- [37] F. Forstnerič, Recent developments on Oka manifolds, *Indag. Math., New Ser.* 34 (2) (2023) 367–417.

- [38] F. Forstnerič, Stein Manifolds and Holomorphic Mappings (the Homotopy Principle in Complex Analysis), second edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 56*, Springer, Cham, 2017.
- [39] F. Forstnerič, The nonhomogeneous Cauchy-Riemann equation on families of open Riemann surfaces, *Complex Anal. Synergies* 12 (1) (2026) 7.
- [40] F. Forstnerič, The universal family of punctured Riemann surfaces is Stein, *Bull. Lond. Math. Soc.* 58 (5) (2026) e70383.
- [41] F. Forstnerič, Y. Kusakabe, Oka tubes in holomorphic line bundles, *Math. Ann.* 391 (4) (2025) 5265–5292.
- [42] F. Forstnerič, F. Lárusson, Survey of Oka theory, *N.Y. J. Math.* 17A (2011) 11–38.
- [43] F. Forstnerič, F. Lárusson, The Oka principle for holomorphic Legendrian curves in $\mathbb{C}^{2n+1}$, *Math. Z.* 288 (1–2) (2018) 643–663.
- [44] F. Forstnerič, F. Lárusson, The parametric h -principle for minimal surfaces in $\mathbb{R}^n$ and null curves in $\mathbb{C}^n$, *Commun. Anal. Geom.* 27 (1) (2019) 1–45.
- [45] F. Forstnerič, F. Lárusson, Every projective Oka manifold is elliptic, *Math. Res. Lett.* 33 (1) (2026) 297–312.
- [46] F. Forstnerič, F. Lárusson, Oka-1 manifolds: new examples and properties, *Math. Z.* 309 (2) (2025) 16, Id/No 26.
- [47] F. Forstnerič, Á.E. Sigurðardóttir, The Oka principle for tame families of Stein manifolds, *Trans. Am. Math. Soc. Ser. B* 13 (2026) 477–511.
- [48] F. Forstnerič, E.F. Wold, Fibrations and Stein neighborhoods, *Proc. Am. Math. Soc.* 138 (6) (2010) 2037–2042.
- [49] R. Fritsch, R. Piccinini, CW-complexes and Euclidean spaces, in: *Topology, Proc. 4th Meet., Sorrento/Italy, 1988*, in: *Suppl. Rend. Circ. Mat. Palermo, II. Ser.*, vol. 24, 1990, pp. 79–95.
- [50] D. Gaier, *Lectures on Complex Approximation*, Birkhäuser Boston, Inc., Boston, MA, 1987, Translated from the German by Renate McLaughlin.
- [51] P.M. Gauthier, Meromorphic uniform approximation on closed subsets of open Riemann surfaces, in: *Approximation Theory and Functional Analysis, Proc. Internat. Sympos. Approximation Theory, Univ. Estadual de Campinas, Campinas, 1977*, in: *North-Holland Math. Stud.*, vol. 35, North-Holland, Amsterdam-New York, 1979, pp. 139–158.
- [52] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, 2nd ed., *Grundlehren Math. Wiss.*, vol. 224, Springer, Cham, 1983.
- [53] H. Grauert, Analytische Faserungen über holomorph-vollständigen Räumen, *Math. Ann.* 135 (1958) 263–273.
- [54] R.E. Greene, S.G. Krantz, Deformation of complex structures, estimates for the (partial $\bar{\partial}$) equation, and stability of the Bergman kernel, *Adv. Math.* 43 (1982) 1–86.
- [55] M. Gromov, Oka’s principle for holomorphic sections of elliptic bundles, *J. Am. Math. Soc.* 2 (4) (1989) 851–897.
- [56] R.C. Gunning, R. Narasimhan, Immersion of open Riemann surfaces, *Math. Ann.* 174 (1967) 103–108.
- [57] R.S. Hamilton, Deformation of complex structures on manifolds with boundary. I: the stable case, *J. Differ. Geom.* 12 (1977) 1–45.
- [58] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
- [59] G.M. Henkin, J. Leiterer, *Theory of Functions on Complex Manifolds*, *Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien (Mathematical Textbooks and Monographs, Part II: Mathematical Monographs)*, vol. 60, Akademie-Verlag, Berlin, 1984.
- [60] L. Hörmander, *An Introduction to Complex Analysis in Several Variables*, third edition, *North-Holland Mathematical Library*, vol. 7, North-Holland Publishing Co., Amsterdam, 1990.
- [61] Y. Iwayoshi, M. Taniguchi, *An Introduction to Teichmüller Spaces*, Springer-Verlag, Tokyo, 1992.
- [62] M. Jurchescu, Variétés mixtes, in: *Romanian-Finnish Seminar on Complex Analysis, Proc., Bucharest 1976*, in: *Lect. Notes Math.*, vol. 743, 1979, pp. 431–448.
- [63] M. Jurchescu, Espaces mixtes, in: *Complex Analysis - Proc. 5th Rom.-Finn. Semin., Bucharest 1981, Part 2*, in: *Lect. Notes Math.*, vol. 1014, 1983, pp. 37–57.
- [64] M. Jurchescu, Variétés mixtes et cohomologie, *Riv. Mat. Univ. Parma, IV. Ser.* 10 (1984) 55–79.
- [65] M. Jurchescu, Coherent sheaves on mixed manifolds, *Rev. Roum. Math. Pures Appl.* 33 (1–2) (1988) 57–81.
- [66] M. Jurchescu, Relative Runge domains, *An. Univ. Bucur., Mat.* 37 (1) (1988) 9–20.
- [67] M. Jurchescu, The Cauchy-Riemann complex on a mixed manifold, *Rev. Roum. Math. Pures Appl.* 39 (10) (1994) 951–971.

- [68] A.A. Kirillov, *Elements of the Theory of Representations*, Grundlehren Math. Wiss., vol. 220, Springer, Cham, 1976, Translated from the Russian by Edwin Hewitt.
- [69] K. Kodaira, D.C. Spencer, On deformations of complex analytic structures. I, II, *Ann. Math. (2)* 67 (1958) 328–466.
- [70] A. Korn, Zwei Anwendungen der Methode der sukzessiven Annäherungen, in: *Schwarz-Festschr.*, 1914, pp. 215–229.
- [71] F. Lárusson, Model structures and the Oka principle, *J. Pure Appl. Algebra* 192 (1–3) (2004) 203–223.
- [72] F. Lárusson, Mapping cylinders and the Oka principle, *Indiana Univ. Math. J.* 54 (4) (2005) 1145–1159.
- [73] O. Lehto, K.I. Virtanen, *Quasiconformal Mappings in the Plane*, second edition, Die Grundlehren der mathematischen Wissenschaften, vol. 126, Springer-Verlag, New York-Heidelberg, 1973, Translated from the German by K.W. Lucas.
- [74] L. Lichtenstein, Zur Theorie der konformen Abbildung nichtanalytischer, singularitätenfreier Flächenstücke auf ebene Gebiete, *Krak. Anz.* 1916 (1916) 192–217.
- [75] I. Lieb, J. Michel, *The Cauchy-Riemann Complex. Integral Formulae and Neumann Problem*, Aspects of Mathematics, vol. E34, Friedr. Vieweg & Sohn, Braunschweig, 2002.
- [76] S. Mardešić, Absolute neighborhood retracts and shape theory, in: *History of Topology*, Elsevier, Amsterdam, 1999, pp. 241–269.
- [77] V. Marković, D. Šarić, The universal properties of Teichmüller spaces, in: *Geometry of Riemann Surfaces and Their Moduli Spaces*, in: *Surveys in Differential Geometry*, vol. 14, International Press, Somerville, MA, 2010, pp. 261–294.
- [78] J.P. May, *A Concise Course in Algebraic Topology*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1999.
- [79] S.N. Mergelyan, On the representation of functions by series of polynomials on closed sets, *Dokl. Akad. Nauk SSSR (N.S.)* 78 (1951) 405–408.
- [80] E. Michael, Another note on paracompact spaces, *Proc. Am. Math. Soc.* 8 (1957) 822–828.
- [81] S. Mongodi, G. Tomassini, 1-complete semiholomorphic foliations, *Trans. Am. Math. Soc.* 368 (9) (2016) 6271–6292.
- [82] C.B.j. Morrey, On the solutions of quasilinear elliptic partial differential equations, *Trans. Am. Math. Soc.* 43 (1938) 126–166.
- [83] J. Mujica, *Complex Analysis in Banach Spaces. Holomorphic Functions and Domains of Holomorphy in Finite and Infinite Dimensions*, North-Holland Math. Stud., vol. 120, Elsevier, Amsterdam, 1986.
- [84] S. Nag, *The Complex Analytic Theory of Teichmüller Spaces*, Can. Math. Soc. Ser. Monogr. Adv. Texts, Wiley, New York etc., 1988.
- [85] A. Norton, A critical set with nonnull image has large Hausdorff dimension, *Trans. Am. Math. Soc.* 296 (1986) 367–376.
- [86] K. Oka, Sur les fonctions analytiques de plusieurs variables. III. Deuxième problème de Cousin, *J. Sci. Hiroshima Univ., Ser. A* 9 (1939) 7–19.
- [87] R. Osserman, *A Survey of Minimal Surfaces*, second edition, Dover Publications, Inc., New York, 1986.
- [88] E.A. Poletsky, Stein neighborhoods of graphs of holomorphic mappings, *J. Reine Angew. Math.* 684 (2013) 187–198.
- [89] R.M. Range, Y.T. Siu, C^k approximation by holomorphic functions and $\bar{\partial}$ -closed forms on C^k submanifolds of a complex manifold, *Math. Ann.* 210 (1974) 105–122.
- [90] M.E. Rudin, A new proof that metric spaces are paracompact, *Proc. Am. Math. Soc.* 20 (1969) 603.
- [91] C. Runge, Zur Theorie der Eindeutigen Analytischen Functionen, *Acta Math.* 6 (1) (1885) 229–244.
- [92] S. Scheinberg, Uniform approximation by functions analytic on a Riemann surface, *Ann. Math. (2)* 108 (2) (1978) 257–298.
- [93] Y.T. Siu, Every Stein subvariety admits a Stein neighborhood, *Invent. Math.* 38 (1) (1976/77) 89–100.
- [94] A.H. Stone, Paracompactness and product spaces, *Bull. Am. Math. Soc.* 54 (1948) 977–982.
- [95] E.L. Stout, *Polynomial Convexity*, Progress in Mathematics, vol. 261, Birkhäuser Boston, Inc., Boston, MA, 2007.
- [96] V. Văjăitu, Mixed manifolds. A survey, in: *Seminari di geometria. 2000*, Università degli Studi di Bologna, Dipartimento di Matematica, Bologna, 2001, pp. 87–93.
- [97] I.N. Vekua, Das Problem der Reduktion von Differentialformen vom elliptischen Typus auf die kanonische Form und das verallgemeinerte Cauchy-Riemannsche System, *Dokl. Akad. Nauk SSSR* 100 (1955) 197–200.

- [98] K. Weierstrass, Ueber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen, Berl. Ber. 1885 (1885) 633–640, 789–806.
- [99] P. Yang, Curvatures of complex submanifolds of $\mathbb{C}^n$, J. Differ. Geom. 12 (4) (1978) 499–511, (1977).