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Transforming Solutions for the Oberwolfach Problem into Solutions for the Spouse-Loving Variant

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ABSTRACT

The Oberwolfach problem OP(F), for a 2-factor F of K_n, asks whether there exists a 2-factorization of K_n (if n is odd) or $K_n - I$ (if n is even) where each 2-factor is isomorphic to F. Here, I denotes any 1-factor of K_n. For even n, the problem OP(F) may also be denoted $\mathrm{OP}^-(F)$, and has been nicknamed the spouse-avoiding variant. Similarly, the spouse-loving variant is denoted $\mathrm{OP}^+(F)$ and asks for a 2-factorization of $K_n + I$ (the complete graph with the edges of a 1-factor I duplicated, rather than deleted) in which each 2-factor is isomorphic to F. To date, many more infinite families of cases of OP and OP^- have been solved than of $\mathrm{OP}^+(F)$. In this paper, we show how certain solutions to $\mathrm{OP}^-(F)$ can be used to construct solutions to $\mathrm{OP}^+(F)$; in particular, when the number of odd cycles in the 2-factor F is not too large. Our technique of *setups* also allows us to completely solve the two-table OP^+; that is, $\mathrm{OP}^+(F)$, where F has exactly two components.

1 | Introduction

The well-known *Oberwolfach problem* (shortly OP) asks the following: Given t round tables of sizes $m_1, \ldots, m_t$ such that $m_1 + \cdots + m_t = n$ and each $m_i \geq 3$, is it possible to seat n people around the t tables for an appropriate number of meals so that every person sits next to every other person exactly once? In graph-theoretic terms, the question is asking whether K_n can be decomposed into 2-factors, each a disjoint union of cycles of lengths $m_1, \ldots, m_t$, whenever $m_1 + \cdots + m_t = n$. Since a graph with odd-degree vertices cannot admit a 2-factorization, Huang, Kotzig, and Rosa [1] proposed the analogous problem for $K_n - I$, the complete graph of even order with a 1-factor removed. They called it the *spouse-avoiding variant* (shortly OP^-) since it models a sitting arrangement of $\frac{n}{2}$ couples, where each person gets to sit next to every other person, except their spouse, exactly once, and never gets to sit next to their spouse. In the literature, both the original Oberwolfach problem and the spouse-avoiding variant with tables of sizes $m_1, \ldots, m_t$ are denoted $\mathrm{OP}(m_1, \ldots, m_t)$, however, in this paper, it will be sometimes helpful to write $\mathrm{OP}^-(m_1, \ldots, m_t)$ to emphasize that we are dealing with the spouse-avoiding variant.

Many infinite families of cases to OP have been solved to date. The only known exceptions—that is, cases when the obvious necessary conditions are satisfied but the problem does not have a solution—are OP(3, 3), OP(4, 5), OP(3, 3, 5), and OP(3, 3, 3, 3). Apart from these exceptions, it is known that $\mathrm{OP}(m_1, \ldots, m_t)$ has a solution when $m_1 = \cdots = m_t$ [2–4], when $m_1, \ldots, m_t$ are all even [5, 6], when $n \leq 100$ [7–12], when $t = 2$ [13, 14], and when n is sufficiently large [15]—to mention just some of the most prominent results. Below, we explicitly state the two results most relevant for this paper.

Theorem 1.1 (Adams and Bryant [7], Deza et al. [8], Franek et al. [9], Franek and Rosa [10], Meszka [11], and Salassa et al. [12]). *Let $3 \leq m_1 \leq \cdots \leq m_t$ and $n = m_1 + \cdots + m_t$. If $n \leq 100$, then $OP(m_1, \ldots, m_t)$ has a solution if and only if $(m_1, \ldots, m_t) \notin \{(3, 3), (4, 5), (3, 3, 5), (3, 3, 3, 3)\}$.*

Theorem 1.2 (Gvozdjak [13] and Traetta [14]). *For $3 \leq m_1 \leq m_2$, $OP(m_1, m_2)$ has a solution if and only and $(m_1, m_2) \notin \{(3, 3), (4, 5)\}$.*

In general, however, OP is still open.

The spouse-avoiding variant of OP can also be viewed as the *maximum packing* variant. The *minimum covering* version of the problem was introduced in [16] and nicknamed the *spouse-loving variant* (shortly OP^+). It models a setting with $\frac{n}{2}$ couples where we wish for each person to sit next to exactly one other person—their spouse—twice, and next to every other person exactly once. In other words, we are interested in decomposing $K_n + I$ (the complete graph of even order with a 1-factor I duplicated) into 2-factors, each a disjoint union of t cycles of lengths $m_1, \ldots, m_t$, where $m_1 + \cdots + m_t = n$. Note that, since $K_n + I$ has exactly $\frac{n}{2}$ pairs of parallel edges, we can have $m_i = 2$ for at most one index i. This problem is denoted by $\mathrm{OP}^+(m_1, \ldots, m_t)$, or $\mathrm{OP}^+(n; m)$ when $m_1 = \cdots = m_t = m$. A lot fewer cases of OP^+ than of OP have been solved to date, and the most relevant results are as follows.

Theorem 1.3 (Bolohan et al. [16], Assaf et al. [17], Lamken and Mills [18], and Shanmuga Vadivu et al. [19]). *$\mathrm{OP}^+(n; m)$ has a solution if and only if n is even, $3 \leq m \leq n$, $m | n$, and $(n, m) \notin \{(6, 3), (12, 3)\}$.*

Theorem 1.4 (Bryant and Danziger [6] and Bolohan et al. [16]). *If m_i is even and $4 \leq m_i \leq n$ for all $i = 1, 2, \ldots, t$, then $\mathrm{OP}^+(m_1, \ldots, m_t)$ has a solution.*

It is worthwhile mentioning two related, recently studied variants of OP^+. In both cases, we have $\frac{n}{2}$ couples; in the honeymoon variant [20], each person sits next to their spouse every time, and next to everybody else exactly once, while in the loving couples variant [21], for a fixed integer r, $0 \leq r \leq n - 2$, each person sits next to their spouse exactly r times, and next to everybody else exactly once.

Recently, a powerful result on both OP (including OP^-) and OP^+ was obtained in [22]. To state it, we first need some terminology.

A 2-regular graph F is said to be of *type* $(m_1, \ldots, m_t)$ if F is a disjoint union of t cycles of lengths $m_1, \ldots, m_t$, respectively. A 2-regular (simple) graph of type $(m_1, \ldots, m_t)$ is said to be *single-flip* if exactly one of the odd elements in $\{m_1, \ldots, m_t\}$ occurs in the sequence $(m_1, \ldots, m_t)$ with odd multiplicity.

A 2-factorization of a graph G is said to be *f-pyramidal* if it admits a group of automorphisms that fixes f vertices of G and acts sharply transitively on the rest. A 2-factorization that is 0-pyramidal (1-pyramidal) is also called *regular* (*1-rotational*, respectively).

Theorem 1.5 (Burgess et al. [22, Theorem 1.2]). *Let F be a 2-regular single-flip graph of type $(m_1, \ldots, m_t)$, where we assume that if its order is odd, then m_1 is the unique odd element that occurs an odd number of times in $(m_1, \ldots, m_t)$.*

Then $OP(m_1, \ldots, m_t)$, $\mathrm{OP}^-(m_1, \ldots, m_t)$, or $\mathrm{OP}^+(m_1, \ldots, m_t)$ admits a 1-rotational, 2-pyramidal, or regular solution, respectively, whenever m_1 satisfies a certain explicit lower bound (which depends on $m_1, \ldots, m_t$).

We invite the reader to check [22, Theorem 1.2] for the exact value of the lower bound.

The goal of this paper is to develop techniques to convert solutions for OP to solutions for OP^+. In Section 4, we introduce a technique of *setups* (Definition 4.6), and in Theorem 4.12, we show that a suitable sequence of setups allows us to convert a solution for $\mathrm{OP}^-(m_1, \ldots, m_t)$ into a solution for $\mathrm{OP}^+(m_1, \ldots, m_t)$. Furthermore, in Theorem 4.14, we give sufficient conditions on the list $(m_1, \ldots, m_t)$ for such a sequence of setups to exist. A more concrete consequence of Theorem 4.12 is presented in Section 5, namely, a complete solution to the two-table OP^+ (Theorem 5.1).

Finally, in Section 6, we apply the technique of setups to convert solutions for OP that are 1-rotational over a cyclic group and that satisfy certain conditions on the lengths of the odd cycles into solutions to OP^+; see Theorem 6.8.

Before we get to the main techniques in Section 4, however, we will start with basic definitions and notation in Section 2, and follow with some easy observations on converting solutions for OP to solutions for OP^+, and vice versa, in Section 3.

2 | Preliminaries

If S and T are disjoint sets, then their union will be denoted as $S \dot{\cup} T$.

For a graph G and $V' \subseteq V(G)$, we use the symbols $G[V']$ and $G \backslash V'$ to denote the subgraph of G induced by V' and the graph obtained from G by deleting the vertices in V', respectively. If $V' = \{u\}$, we may write $G \backslash u$ instead of $G \backslash \{u\}$. Similarly, if $E' \subseteq E(G)$, the symbol $G - E'$ denotes the graph obtained from G by deleting the edges in E', and if E' is a subset of the edge set of a complete graph with vertex set $V(G)$, then $G + E'$ denotes the graph obtained from G by adjoining the edges in E' (possibly duplicating some of them).

As usual, K_n denotes the complete graph with n vertices, and for n even, $K_n - I$ and $K_n + I$ denote the complete graph K_n with the edges of the 1-factor I deleted and duplicated, respectively. A cycle of length m (or m-cycle) is denoted by C_m, and a path of length n (i.e., with n edges) is denoted by P_n.

A set $\{H_1, \ldots, H_k\}$ of subgraphs of a graph G is called a *decomposition* of G if $\{E(H_1), \ldots, E(H_k)\}$ is a partition of $E(G)$. If this is the case, we write $G = H_1 \oplus \cdots \oplus H_k$.

A *2-factor* in a graph G is a spanning 2-regular subgraph of G (not necessarily simple). A 2-regular graph consisting of t disjoint cycles of lengths $m_1, m_2, \ldots, m_t$, respectively, is said to be of *type* $(m_1, m_2, \ldots, m_t)$.

A *2-factorization* of a graph G is a decomposition of G into 2-factors. An *F-factorization* is a 2-factorization consisting of 2-factors isomorphic to F.

As mentioned above, assuming $n = m_1 + \cdots + m_t$, a solution to $\mathrm{OP}(m_1, \ldots, m_t)$ is an F-factorization of K_n (if n is odd) or $K_n - I$ (if n is even) for a 2-factor F of K_n of type $(m_1, \ldots, m_t)$. Similarly, a solution to $\mathrm{OP}^+(m_1, \ldots, m_t)$ is an F-factorization of $K_n + I$. If n is even, $\mathrm{OP}(m_1, \ldots, m_t)$ is also denoted $\mathrm{OP}^-(m_1, \ldots, m_t)$. In all of these cases, n is said to be the *order* of the problem, which may also be denoted $\mathrm{OP}(F)$, $\mathrm{OP}^-(F)$, or $\mathrm{OP}^+(F)$, respectively.

In this paper, the vertex set of K_{2n+1} will often be assumed to be $\{z_i : i \in \mathbb{Z}_{2n}\} \cup \{z_\infty\}$. In this case, the *difference* of an edge of the form $z_i z_j$ (for $i, j \in \mathbb{Z}_{2n}$, $i < j$) is defined as $\min\{j - i, n - (j - i)\}$, and of an edge of the form $z_i z_\infty$ (for $i \in \mathbb{Z}_{2n}$), as ∞. Thus, every edge difference is in the set $\{1, 2, \ldots, n\} \cup \{\infty\}$.

Similarly, the vertex set of $K_{2n+2} + I$ may be assumed to be $\{z_i : i \in \mathbb{Z}_{2n}\} \cup \{z_\infty, z_{\infty'}\}$. If so, the *difference* of an edge of the form $z_i z_j$ (for $i, j \in \mathbb{Z}_{2n}$, $i < j$), $z_i z_\infty$, and $z_i z_{\infty'}$ is defined as $\min\{j - i, n - (j - i)\}$, ∞, and ∞', respectively. (The difference of an edge $z_\infty z_{\infty'}$ remains undefined.)

3 | From OP to OP^+ and Back: Easy Observations

Before we get to the main approach of this paper, we mention a few easy observations on how to get solutions to OP^+ from solutions to OP, or vice versa. First, we need to point out that Theorem 1.4 is an easy corollary of [6, Theorem 2]; namely, the existence of a solution to $\mathrm{OP}^+(m_1, m_2, \ldots, m_t)$ when $m_1, m_2, \ldots, m_t$ are all even and greater than 2 follows easily from the existence of a solution to $\mathrm{OP}^-(m_1, m_2, \ldots, m_t)$; see [16, Theorem 1.3]. In Section 4, we will generalize this result to certain 2-factors with odd cycles.

The following observation is a slight generalization of [22, Theorem 2.7]; see also Remark 6.5.

Theorem 3.1. *Let $n, m_1, m_2, \ldots, m_t$ be integers greater than 2 such that $m_1 + m_2 + \cdots + m_t = 2n + 1$, and suppose that there exists a solution $\{F_i : i \in \mathbb{Z}_n\}$ to $OP(m_1, m_2, \ldots, m_t)$ such that for some vertex u of $G \cong K_{2n+1}$, in every 2-factor F_i, the vertex u lies in an m_1-cycle.*

Then $\mathrm{OP}^+(m_1 - 1, m_2, \ldots, m_t)$ has a solution.

Proof. For each $i \in \mathbb{Z}_n$, let x_i and y_i be distinct vertices of F_i such that $x_i u, y_i u \in E(F_i)$, and obtain F_i' from F_i by replacing the 2-path $x_i u y_i$ with a 1-path $x_i y_i$. Since u lies in an m_1-cycle of F_i, the resulting graph F_i' is 2-regular of type $(m_1 - 1, m_2, ..., m_t)$, and has vertex set $V(G) \backslash \{u\}$. Let $G' = G \backslash u$. Then $G' \cong K_{2n}$, $\{x_i y_i : i \in \mathbb{Z}_n\}$ is the edge set of a 1-factor I of G', and $\{F_i \backslash u : i \in \mathbb{Z}_n\}$ covers each edge of G' exactly once. Hence $\{F_i' : i \in \mathbb{Z}_n\}$ is a 2-factorization of $G' + I$, that is, a solution to $\mathrm{OP}^+(m_1 - 1, m_2, ..., m_t)$. □

Theorem 3.2. *Let $m_1, m_2, ..., m_t$ be integers greater than 2 such that there exists a solution to* $\mathrm{OP}^+(2, m_1, m_2, ..., m_t)$. *Then there exists a solution to* $OP(3, m_1, m_2, ..., m_t)$.

Proof. Let $2 + m_1 + m_2 + \cdots + m_t = 2n$, and let $V = \{z_i : i \in \mathbb{Z}_{2n}\}$ be the vertex set of K_{2n}. Furthermore, let $\{F_i : i \in \mathbb{Z}_n\}$ be a solution to $\mathrm{OP}^+(2, m_1, m_2, ..., m_t)$, and we may assume that for $i = 0, 1, ..., n - 1$, the set $\{z_{2i}, z_{2i+1}\}$ is the vertex set of the 2-cycle in the 2-factor F_i.

Consider the complete graph K_{2n+1} on the vertex set $V' = V \cup \{z_\infty\}$. For $i = 0, 1, ..., n - 1$, obtain F_i' from F_i by replacing the 2-cycle $z_{2i} z_{2i+1} z_{2i}$ with the 3-cycle $z_{2i} z_{2i+1} z_\infty z_{2i}$. Clearly, F_i' is a 2-factor of K_{2n+1} of type $(3, m_1, m_2, ..., m_t)$. Moreover, in $\{F_i' : i \in \mathbb{Z}_n\}$, each edge of K_{2n+1} is covered exactly once, so this set is a solution to $\mathrm{OP}(3, m_1, m_2, ..., m_t)$. □

Since OP(3, 3, 5) has no solution by Theorem 1.1, Theorem 3.2 allows us to conclude the following.

Corollary 3.3. $\mathrm{OP}^+(2, 3, 5)$ *has no solution.*

4 | Techniques

Recall that if the cycle lengths $m_1, m_2, ..., m_t$ are all even and greater than 2, a solution to $\mathrm{OP}^-(m_1, m_2, ..., m_t)$ can easily be extended to a solution to $\mathrm{OP}^+(m_1, m_2, ..., m_t)$; see [16, Theorem 1.3]. In this section, we develop a technique that will allow us to construct a solution to $\mathrm{OP}^+(m_1, m_2, ..., m_t)$ from a suitable solution to $\mathrm{OP}^-(m_1, m_2, ..., m_t)$ in cases where some of the cycle lengths $m_1, m_2, ..., m_t$ are odd. The notions of a 6-setup and 12-setup, introduced in Definitions 4.2 and 4.4, will be central to this technique.

Definition 4.1. Let n be a positive integer. A $\{P_1, P_2\}$*-factor* of K_n is a spanning subgraph of K_n each of whose connected components is isomorphic to either P_1 or P_2. For $i \in \{1, 2\}$, a connected component of a $\{P_1, P_2\}$-factor of K_n that is isomorphic to P_i will be called a P_i*-component*.

Definition 4.2. Let n be a positive integer, and F and H edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. A 6-tuple $(x_0, x_1, ..., x_5)$ of pairwise distinct vertices of G is called a *6-setup for F and H* if F has a cycle $C = u_0 u_1 \dots u_{s-1} u_0$ such that:

i. $u_0 = x_0, u_1 = x_1$;
ii. for some $\ell \in \mathbb{Z}_s$, we have $u_\ell = x_3, u_{\ell+1} = x_4$; and
iii. $x_0 x_3, x_1 x_2, x_4 x_5$ are P_1-components of H.

If $(x_0, x_1, ..., x_5)$ is a 6-setup for F and H, then we define the following subgraphs of G:

$$F' = (F - \{x_0 x_1, x_3 x_4\}) + \{x_0 x_3, x_1 x_4\} \quad \text{and} \quad H' = (H - \{x_0 x_3\}) + \{x_0 x_1, x_3 x_4\}.$$

We say that graphs F' and H' are *derived* from F and H, respectively, using the 6-setup $(x_0, x_1, ..., x_5)$. By the *vertex set* of a 6-setup $X = (x_0, x_1, ..., x_5)$ we mean the set $\{x_0, x_1, ..., x_5\}$, which we denote by $V(X)$.

The following observations follow easily from Definition 4.2 (see Figure 1).

Lemma 4.3. *Let n be a positive integer, and F and H edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. Furthermore, let $X = (x_0, x_1, ..., x_5)$ be a 6-setup for F and H, and F' and H' the subgraphs of G derived from F and H, respectively, using X. Then the following hold:*

i. *F' is a 2-factor of G isomorphic to F;*
ii. *H' is a $\{P_1, P_2\}$-factor of G;*
iii. *F' and H' are edge-disjoint;*

iv. $E(F' \cup H') = E(F \cup H) \dot{\cup} \{x_1x_4\}$;
v. *for* $U = V(G) - \{x_0, x_1, x_3, x_4\}$, *we have* $F'[U] = F[U]$ *and* $H'[U] = H[U]$; *and*
iv. $x_0x_1x_2$ *and* $x_3x_4x_5$ *are* P_2*-components of* H'.

Definition 4.4. Let n be a positive integer, and F and H edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. An ordered pair (X, Y) of 6-tuples $X = (x_0, x_1, ..., x_5)$ and $Y = (y_0, y_1, ..., y_5)$ of pairwise distinct vertices of G is called a *12-setup for F and H* if F has cycles $C_1 = u_0u_1 ... u_{s-1}u_0$ and $C_2 = v_0v_1 ... v_{t-1}v_0$, with $s \le t$, such that:

i. $u_0 = x_0, u_1 = x_1, v_0 = x_4, v_1 = x_3$;
ii. for some $\ell \in \{0, 1, ..., s-1\}$, we have $u_\ell = y_0, u_{\ell+1} = y_1, v_\ell = y_4, v_{\ell+1} = y_3$; and
iii. $x_0x_3, x_1x_2, x_4x_5, y_0y_3, y_1y_2, y_4y_5$ are P_1-components of H.

If (X, Y) is a 12-setup for F and H, then we define the following subgraphs of G:

$$F' = (F - \{x_0x_1, x_3x_4, y_0y_1, y_3y_4\}) + \{x_0x_3, x_1x_4, y_0y_3, y_1y_4\}$$

and

$$H' = (H - \{x_0x_3, y_0y_3\}) + \{x_0x_1, x_3x_4, y_0y_1, y_3y_4\}.$$

We say that graphs F' and H' are *derived* from F and H, respectively, using the 12-setup (X, Y). The *vertex set* of the 12-setup (X, Y) is defined as $V(X, Y) = \{x_0, x_1, ..., x_5, y_0, y_1, ..., y_5\}$.

The following properties of a 12-setup are easy to deduce from Definition 4.4 (see Figure 2).

Lemma 4.5. *Let n be a positive integer, and F and H edge-disjoint* 2*-factor and* $\{P_1, P_2\}$*-factor, respectively, of* $G \cong K_{2n}$. *Furthermore, let* (X, Y) *be a* 12*-setup for F and H, where* $X = (x_0, x_1, ..., x_5)$ *and* $Y = (y_0, y_1, ..., y_5)$, *and let* F' *and* H' *be the subgraphs of G derived from F and H, respectively, using* (X, Y). *Then the following hold:*

i. F' *is a* 2*-factor of G isomorphic to F;*
ii. H' *is a* $\{P_1, P_2\}$*-factor of G;*
iii. F' *and* H' *are edge-disjoint;*

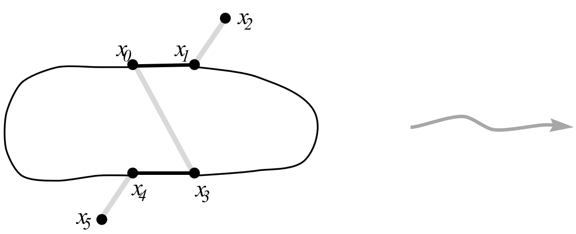

FIGURE 1 | A 6-setup for F and H, and the derived graphs F' and H'.

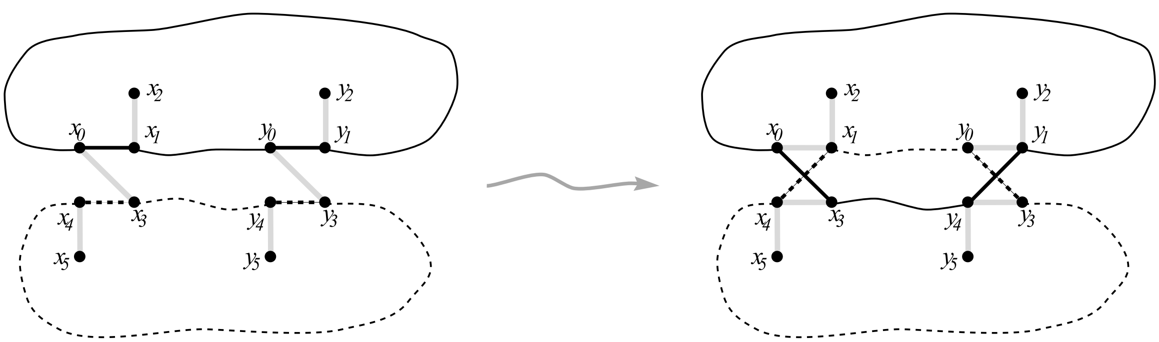

FIGURE 2 | A 12-setup for F and H, and the derived graphs F' and H'.

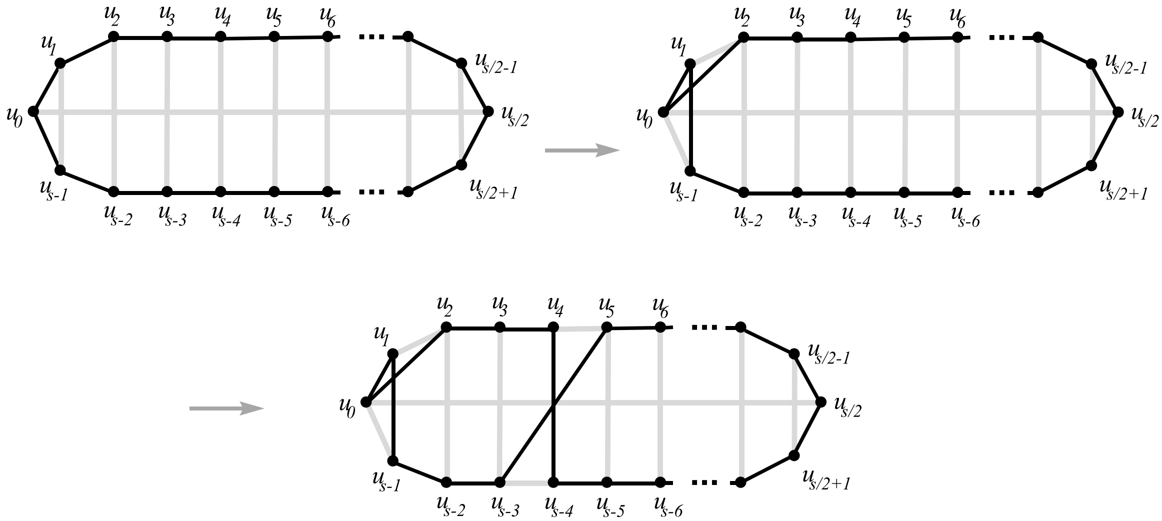

FIGURE 3 | F_0 and H_0, F_1 and H_1, and F_2 and H_2 (all restricted to $V(C)$).

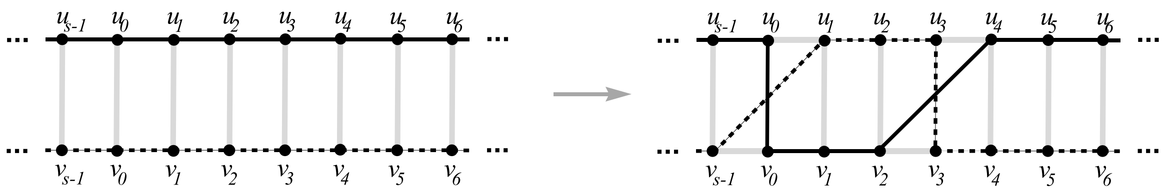

FIGURE 4 | F_0 and H_0, and F_1 and H_1 (all restricted to $V(C_1) \cup V(C_2)$).

iv. $E(F' \cup H') = E(F \cup H) \dot{\cup} \{x_1x_4, y_1y_4\}$;
v. *for* $U = V(G) - \{x_0, x_1, x_3, x_4, y_0, y_1, y_3, y_4\}$, *we have* $F'[U] = F[U]$ *and* $H'[U] = H[U]$; *and*
vi. $x_0x_1x_2, x_3x_4x_5, y_0y_1y_2, y_3y_4y_5$ *are* P_2*-components of* H'.

Definition 4.6. Let n be a positive integer, and F and H edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. By a *setup* for F and H we mean either a 6-setup or a 12-setup for F and H.

As we shall see, a 6-setup allows us to accommodate two odd cycles, while a 12 setup accommodates four odd cycles in the 2-factor. In cases with more odd cycles, several setups will be necessary. We next discuss how to sequence setups to achieve the desired effect.

Definition 4.7. Let n and k be positive integers, and F_0 and H_0 edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. A sequence $(X_0, X_1, ..., X_{k-1})$ of setups for F_0 and H_0 is called *proper* if the following hold:

i. X_0 is a setup for F_0 and H_0;
ii. for all $i = 0, 1, ..., k-2$, if we define $F_{i+1} = F_i'$ and $H_{i+1} = H_i'$, where F_i' and H_i' are the subgraphs of G derived from F_i and H_i using the setup X_i, then X_{i+1} is a setup for F_{i+1} and H_{i+1}.

The *vertex set* of a sequence $(X_0, X_1, ..., X_{k-1})$ of setups is defined as $\cup_{i=0}^{k-1} V(X_i)$.

If $S_1 = (X_0, X_1, ..., X_{k-1})$ and $S_2 = (Y_0, Y_1, ..., Y_{\ell-1})$ are two sequences of setups, then their *concatenation* is the sequence $S_1S_2 = (X_0, X_1, ..., X_{k-1}, Y_0, Y_1, ..., Y_{\ell-1})$.

Lemma 4.8. *Let* n *and* k *be positive integers, and* F_0 *and* H_0 *edge-disjoint* 2*-factor and* $\{P_1, P_2\}$*-factor, respectively, of* $G \cong K_{2n}$. *Let* $(X_0, X_1, ..., X_{k-1})$ *be a proper sequence of setups for* F_0 *and* H_0. *Then* $V(X_0), V(X_1), ..., V(X_{k-1})$ *are pairwise disjoint.*

Proof. As in Definition 4.7, for all $i = 0, 1, ..., k-1$, let $F_{i+1} = F_i'$ and $H_{i+1} = H_i'$, where F_i' and H_i' are the subgraphs of G derived from F_i and H_i using the setup X_i.

Claim 1. Each vertex in $V(X_i)$ belongs to a P_1-component of H_i and to a P_2-component of H_{i+1}.

Proof of Claim 1. This follows easily from Definitions 4.2(iii) and 4.4(iii), and Lemmas 4.3(vi) and 4.5(vi). □

Claim 2. Each vertex in $V(G) - V(X_i)$ lies in a P_1-component of H_i if and only if it lies in a P_1-component of H_{i+1}.

Proof of Claim 2. Take any $u \in V(G) - V(X_i)$, and let P be the connected component of H_i that contains u. By Definitions 4.2(iii) and 4.4(iii), we know $V(P)$ is disjoint from $V(X_i)$, and hence by Lemmas 4.3(v,vi) and 4.5(v,vi), P is also a connected component of H_{i+1}. Thus u lies in a P_1-component of H_i if and only if it lies in a P_1-component of H_{i+1}. □

Claim 3. For all $j, 0 \leq j \leq k-1$, each vertex in $V(X_j)$ belongs to a P_1-component of H_i for all $i \leq j$, and to a P_2-component of H_i for all $i > j$.

Proof of Claim 3. By Claim 1, we know that each vertex in $V(X_0)$ belongs to a P_2-component of H_1, while each vertex in $V(X_1)$ belongs to a P_1-component of H_1. Hence $V(X_0)$ is disjoint from $V(X_1)$, and so by Claim 2, each vertex of $V(X_0)$ lies in a P_2-component of H_2 as well. Repeating this argument, we can see that each vertex of $V(X_0)$ lies in a P_2-component of H_i for all $i > 0$. □

Suppose that for some $j, 0 \leq j \leq k-2$, each vertex in $V(X_j)$ belongs to a P_1-component of H_i for all $i \leq j$, and to a P_2-component of H_i for all $i > j$. Consider any $u \in V(X_{j+1})$. Since u lies in a P_1-component of H_{j+1}, by Claim 1, we know $u \notin V(X_j)$. Hence by Claim 2, vertex u lies in P_1-component of H_j. Repeating this argument we can see that u lies in P_1-component of H_i for all $i \leq j+1$. Similarly, since u lies in a P_2-component of H_{j+2}, by Claim 1, we know $u \notin V(X_{j+2})$, and by Claim 2, it follows that u lies in P_2-component of H_{j+3}. Repeating this argument we can see that u lies in P_2-component of H_i for all $i > j+1$.

Claim 3 follows by induction.

It is easy to see that Claim 3 implies that the sets $V(X_0), V(X_1), \ldots, V(X_{k-1})$ are pairwise disjoint. □

In the next example we show that a sequence of two setups is not necessarily proper, and hence the concatenation of two proper sequences of setups is not necessarily proper.

Example 4.9. Let F_0 be a 2-factor in $G \cong K_{2n}$ with a cycle $C = u_0u_1 \ldots u_{s-1}u_0$, where $s \geq 12$ is even. Let H_0 be a 1-factor of G with

$$\{u_0u_{\frac{s}{2}}, u_1u_{s-1}, u_2u_{s-2}, \ldots, u_{\frac{s}{2}-1}u_{\frac{s}{2}+1}\} \subseteq E(H_0).$$

Define the following 6-tuples of vertices of C:

$$\begin{aligned} X_0 &= (u_0, u_1, u_{s-1}, u_{\frac{s}{2}}, u_{\frac{s}{2}+1}, u_{\frac{s}{2}-1}), \\ X_1 &= (u_1, u_2, u_{s-2}, u_{s-1}, u_0, u_{\frac{s}{2}}), \\ X_2 &= (u_4, u_5, u_{s-5}, u_{s-4}, u_{s-3}, u_3). \end{aligned}$$

It is not difficult to see that each of X_0, X_1, X_2 is a 6-setup for F_0 and H_0, and that (X_1, X_2) is a proper sequence of setups (see Figure 3), while neither (X_0, X_1) nor (X_0, X_2) is a proper sequence of setups for F_0 and H_0.

Lemma 4.10. *Let n be a positive integer, and let F_0 and H_0 be edge-disjoint 2-factor and $\{P_1, P_2\}$-factor, respectively, of $G \cong K_{2n}$. Let S_1 and S_2 be two proper sequences of setups for F_0 and H_0. If no cycle of F_0 contains vertices from both $V(S_1)$ and $V(S_2)$, then S_1S_2 is a proper sequence of setups for F_0 and H_0.*

Proof. Let $S_1 = (X_0, X_1, \ldots, X_{k-1})$ and $S_2 = (Y_0, Y_1, \ldots, Y_{\ell-1})$. For $i = 1, 2$, let $\mathcal{C}_i$ be the set of cycles of F_0 that contain vertices from $V(S_i)$, and let $V_i = \cup_{C \in \mathcal{C}_i} V(C)$. By assumption, $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, and hence $V_1 \cap V_2 = \emptyset$.

For all $i = 0, 1, \ldots, k-1$, let $F_{i+1} = F'_i$ and $H_{i+1} = H'_i$, where F'_i and H'_i are the subgraphs of G derived from F_i and H_i using the setup X_i. Note that $F_k[V_2] = F_0[V_2]$ and $H_k[V_2] = H_0[V_2]$. Hence Y_0 is also a setup for F_k and H_k, and $(X_0, X_1, \ldots, X_{k-1}, Y_0)$ is a proper sequence of setups for F_0 and H_0.

Suppose that for some $j, 0 \leq j \leq \ell - 2$, the sequence $(X_0, X_1, \ldots, X_{k-1}, Y_0, \ldots, Y_j)$ is a proper sequence of setups for F_0 and H_0. For $i = 0, 1, \ldots, j$, let $F_{k+i+1} = F'_{k+i}$ and $H_{k+i+1} = H'_{k+i}$, where F'_{k+i} and H'_{k+i} are the subgraphs of G derived from F_{k+i} and H_{k+i} using the setup Y_i.

On the other hand, since $(Y_0, Y_1, \ldots, Y_j)$ is a proper sequence of setups for F_0 and H_0, we can let $\bar{F}_0 = F_0$ and $\bar{H}_0 = H_0$, and for $i = 0, 1, \ldots, j$, we let $\bar{F}_{i+1} = \bar{F}'_i$ and $\bar{H}_{i+1} = \bar{H}'_i$, where $\bar{F}'_i$ and $\bar{H}'_i$ are the subgraphs of G derived from $\bar{F}_i$ and $\bar{H}_i$ using the setup Y_i.

Since $F_k[V_2] = F_0[V_2]$ and $H_k[V_2] = H_0[V_2]$, we can see that $\bar{F}_{j+1}[V_2] = F_{k+j+1}[V_2]$ and $\bar{H}_{j+1}[V_2] = H_{k+j+1}[V_2]$. Since by assumption, Y_{j+1} is a setup for $\bar{F}_{j+1}$ and $\bar{H}_{j+1}$, it follows that Y_{j+1} is a setup for F_{k+j+1} and H_{k+j+1}, and $(X_0, X_1, \ldots, X_{k-1}, Y_0, Y_1, \ldots, Y_j, Y_{j+1})$ is a proper sequence of setups for F_0 and H_0.

By induction, we conclude that $(X_0, X_1, \ldots, X_{k-1}, Y_0, Y_1, \ldots, Y_{\ell-1})$ is a proper sequence of setups for F_0 and H_0. □

Next, we show that any $\{P_1, P_2\}$-factor with the correct number of P_2-components can be augmented to a 2-factor of the desired type.

Lemma 4.11. *Let n and k be positive integers, F a 2-factor of $G \cong K_{2n}$ with exactly $2k$ odd cycles, and H a $\{P_1, P_2\}$-factor with exactly $2k$ components isomorphic to P_2. Furthermore, let T be the set of vertices of degree 2 in H. Then there exists a 1-factor L of $G \backslash T$ such that $H \cup L$ is a 2-factor of G isomorphic to F.*

Proof. Let $(m_1, m_2, \ldots, m_{2k}, m_{2k+1}, \ldots, m_\ell)$ be the type of F, where $m_1, \ldots, m_{2k}$ are all of the odd cycle lengths. Note that $|T| = 2k$, so $|V(G) - T| = 2n - 2k$.

Let H_1 be a graph obtained from H by replacing each 2-path xyz with a 1-path xz. Then H_1 is a 1-factor of $G \backslash T$, and since $m_1 - 1, m_2 - 1, \ldots, m_{2k} - 1, m_{2k+1}, \ldots, m_\ell$ are all even and $\sum_{i=1}^{2k}(m_i - 1) + \sum_{i=2k+1}^{\ell} m_i = |V(G) - T|$, there clearly exists a 1-factor L of $G \backslash T$ such that $H_1 \cup L$ is a 2-regular graph of type $(m_1 - 1, m_2 - 1, \ldots, m_{2k} - 1, m_{2k+1}, \ldots, m_\ell)$, with vertex set $V(G) - T$. Replacing each 1-path xz of H_1 arising from a 2-path xyz of H with the 2-path xyz results in the 2-factor $H \cup L$ of G of type $(m_1, m_2, \ldots, m_{2k}, m_{2k+1}, \ldots, m_\ell)$. □

Collecting our tools, we now explain how to convert a solution for $\mathrm{OP}^-(F)$ into a solution for $\mathrm{OP}^+(F)$, given a suitable proper sequence of setups.

Theorem 4.12. *Let n be a positive integer, and F and I edge-disjoint 2-factor and 1-factor, respectively, of $G \cong K_{2n}$. Assume that*

i. *$G - I$ admits an F-factorization;*
ii. *F has exactly $2k$ odd cycles; and*
iii. *there exists a proper sequence of setups for F and I with k_1 6-setups and k_2 12-setups, and $k_1 + 2k_2 = k$.*

Then there exists a 1-factor J of G such that $G + J$ admits an F-factorization.

Proof. Let $\mathcal{F}$ be a F-factorization of $G - I$ containing F. Denote $F_0 = F$ and $H_0 = I$, and let $(X_0, X_1, \ldots, X_{\ell-1})$ be a proper sequence of setups for F_0 and H_0 with k_1 6-setups and k_2 12-setups, so $\ell = k_1 + k_2$. As in Definitions 4.2 and 4.4, for all $i \in \mathbb{Z}_\ell$, we let $F_{i+1} = F'_i$ and $H_{i+1} = H'_i$, where F'_i and H'_i are the subgraphs of G derived from F_i and H_i using the setup X_i. Furthermore, let $\pi(H_i)$ be the number of components of H_i isomorphic to P_2, and observe that $\pi(H_0) = 0$.

By repeatedly applying Lemmas 4.3 and 4.5, we observe the following. For $i = 1, \ldots, \ell$, the graphs F_i and H_i are an edge-disjoint 2-factor (isomorphic to F) and $\{P_1, P_2\}$-factor of G, respectively. If X_{i-1} is a 6-setup, then $\pi(H_i) = \pi(H_{i-1}) + 2$, and if X_{i-1} is a 12-setup, then $\pi(H_i) = \pi(H_{i-1}) + 4$. Hence

$$\pi(H_\ell) = \pi(H_0) + 2k_1 + 4k_2 = 2k.$$

Take any $i \in \mathbb{Z}_\ell$. If X_i is a 6-setup, say $X_i = (x_0, x_1, \ldots, x_5)$, then let $T_i = \{x_1, x_4\}$ and $M_i = \{x_1x_4\}$. If X_i is a 12-setup, say $X_i = (X, Y)$, where $X_i = (x_0, x_1, \ldots, x_5)$ and $Y_i = (y_0, y_1, \ldots, y_5)$, then let $T_i = \{x_1, x_4, y_1, y_4\}$ and $M_i = \{x_1x_4, y_1y_4\}$. Using Lemmas 4.3 and 4.5, we can see that in both cases $E(F_{i+1} \cup H_{i+1}) = E(F_i \cup H_i) \,\dot{\cup}\, M_i$ and that T_i is the set of vertices that are of degree 1 in H_i and of degree 2 in H_{i+1}.

Let $M = \cup_{i \in \mathbb{Z}_\ell} M_i$. By Lemma 4.8, the vertex sets of the ℓ setups are pairwise disjoint, and so M is a matching of G. Moreover, $E(F_\ell \cup H_\ell) = E(F_0 \cup H_0) \dot{\cup} M$. Since H_0 is a 1-factor of G, the set of vertices of H_ℓ of degree 2 is precisely $T = \cup_{i \in \mathbb{Z}_\ell} T_i$. By Lemma 4.11, there exists a 1-factor J' of $G \backslash T$ such that $F^* = H_\ell \cup J'$ is a 2-factor of G isomorphic to F. Note that, since M is a perfect matching of $G[T]$, we have that $J = J' + M$ is a 1-factor of G. Finally, since $\mathcal{F}$ is an F-factorization of $G - I$, and

$$F \oplus I \oplus J = F_0 \oplus H_0 \oplus (J' + M) = F_\ell \oplus (H_\ell \oplus J') = F_\ell \oplus F^*,$$

we conclude that $(\mathcal{F} - \{F\}) \cup \{F_\ell, F^*\}$ is an F-factorization of $G + J$. □

In general, to find a sufficiently long proper sequence of setups as required in Theorem 4.12, the combined structure of a 2-factor F and 1-factor I must be well understood. In the case of just two odd cycles, however, a single 6-setup suffices, and in Theorem 4.14, we are able to establish some sufficient conditions on the 2-factor type alone that guarantee the existence of a 6-setup.

Lemma 4.13. *Let F and I be an edge-disjoint 2-factor and 1-factor of $G \cong K_{2n}$, and let $C = u_0 u_1 \ldots u_{m-1} u_0$ be an m-cycle of F. If there exists $\ell \in \mathbb{Z}_m$ such that $u_0 u_\ell \in E(I)$ and $u_1 u_{\ell+1} \notin E(I)$, then there exists a 6-setup for F and I.*

Proof. Denote $x_0 = u_0, x_1 = u_1, x_3 = u_\ell$, and $x_4 = u_{\ell+1}$, and let $x_2, x_5 \in V(G)$ be such that $x_1x_2, x_4x_5 \in E(I)$. Observe that $x_0, x_1, \ldots, x_5$ are necessarily pairwise distinct, and since x_0x_3, x_1x_2, x_4x_5 are all P_1-components of I, we can see that $(x_0, x_1, \ldots, x_5)$ is a 6-setup for F and I. □

Theorem 4.14. *Let $(m_1, m_2, \ldots, m_t)$ be a list of positive integers satisfying the following:*

i. $\mathrm{OP}^-(m_1, m_2, \ldots, m_t)$ *has a solution;*
ii. *exactly two integers on the list $(m_1, m_2, \ldots, m_t)$ are odd; and*
iii. *there exists an integer s, $2 \le s \le t$, such that m_s is odd and $m_j > \sum_{i=1}^{j-1} m_i$ for all $j \ge s$.*

Then $\mathrm{OP}^+(m_1, m_2, \ldots, m_t)$ *has a solution.*

Proof. Let $m_1 + m_2 + \cdots + m_t = 2n$, and let F be a 2-factor of $G \cong K_{2n}$ of type $(m_1, m_2, \ldots, m_t)$. By assumption, there exists an F-factorization of $G - I$, for a 1-factor I of G, and we may assume that F and I are edge-disjoint. Since F has exactly two odd cycles, by Theorem 4.12 (with $k = 1$), if suffices to find one 6-setup for F and I.

Let $C_1, \ldots, C_t$ be the pairwise disjoint cycles of F, of lengths $m_1, m_2, \ldots, m_t$, respectively. We will say that an edge $e \in E(I)$ is an *I-cord* of the cycle C_i if both endpoints of e are in $V(C_i)$. Suppose F has no cycle satisfying the assumptions of Lemma 4.13. This implies that each cycle C_i has either no I-cords or exactly $\frac{m_i}{2}$ I-cords.

Suppose that for some j, $s \le j \le t$, each of the cycles C_i for $j < i \le t$ has exactly $\frac{m_i}{2}$ I-cords. (Note that this is vacuously true for $j = t$.) Let $e = xy$ be an edge of I with $x \in V(C_j)$. By the supposition, we have that $y \notin V(C_i)$ for all $i > j$. Then either $y \in V(C_j)$ as well (and e is an I-cord of C_j), or else $y \in V(C_i)$ for some $i < j$. However, since $m_j > \sum_{i=1}^{j-1} m_i$, it must be that C_j has at least one I-cord, and hence exactly $\frac{m_j}{2}$ I-cords by the above observation. Repeating the argument, we thus deduce that $m_t, m_{t-1}, \ldots, m_s$ are all even—a contradiction since m_s was assumed to be odd.

It follows that F has a cycle satisfying the assumptions of Lemma 4.13, and hence there exists a 6-setup for F and I. □

In Table 1, we show the number (N_1) of instances of OP^+ of order n, $14 \le n \le 100$, that are solved by applying Theorems 1.1 and 4.14.

5 | Results: The Two-Table Problem

In this section, we present a complete solution to the two-table case of the spouse-loving variant of the OP. Note that in the case of two odd cycles, the technique of Section 4 (in the form of Theorem 4.14) will be used.

Theorem 5.1. *Let m_1 and m_1 be integers such that $2 \le m_1 \le m_2$ and $m_1 + m_2 = 2n$ is even. Then $\mathrm{OP}^+(m_1, m_2)$ has a solution if and only if $(m_1, m_2) \notin \{(2, 2), (3, 3)\}$.*

TABLE 1 | The number of instances of order n, $14 \leq n \leq 100$, of OP^+ without 2-cycles. Here: N is the number of all instances without a 2-cycle; of these, N_0 instances are bipartite (i.e., all cycle lengths are even), N_u are non-bipartite uniform (all cycle lengths are equal and odd), N_1 satisfy the assumptions of Theorem 4.14, and of the latter, N_2 have exactly two (necessarily odd) cycles.

n	N	N_0	N_u	N_1	N_2	n	N	N_0	N_u	N_1	N_2
14	13	4	1	2	2	58	25,519	847	1	357	13
16	21	7	0	4	3	60	33,581	1039	3	458	14
18	33	8	2	5	3	62	44,004	1238	1	483	14
20	49	12	1	8	4	64	57,480	1507	0	618	15
22	73	14	1	9	4	66	74,831	1794	3	654	15
24	110	21	1	14	5	68	97,084	2167	1	830	16
26	158	24	1	16	5	70	125,577	2573	3	870	16
28	230	34	1	23	6	72	161,955	3094	2	1101	17
30	331	41	3	25	6	74	208,233	3660	1	1156	17
32	468	55	0	36	7	76	267,016	4378	1	1453	18
34	660	66	1	40	7	78	341,474	5170	3	1513	18
36	927	88	2	55	8	80	435,525	6153	1	1898	19
38	1284	105	1	60	8	82	554,102	7245	1	1980	19
40	1775	137	1	82	9	84	703,263	8591	3	2470	20
42	2438	165	3	90	9	86	890,414	10,087	1	2560	20
44	3323	210	1	120	10	88	1,124,831	11,914	1	3187	21
46	4510	253	1	129	10	90	1,417,812	13,959	5	3307	21
48	6095	320	1	171	11	92	1,783,200	16,424	1	4099	22
50	8182	383	2	185	11	94	2,238,095	19,196	1	4228	22
52	10,945	478	1	241	12	96	2,803,342	22,519	1	5230	23
54	14,578	574	3	257	12	98	3,504,321	26,252	2	5401	23
56	19,323	708	1	334	13	100	4,372,211	30,701	2	6656	24

Proof. $\mathrm{OP}^+(3, 3)$ has no solution by Theorem 1.3, and it is easy to see that $\mathrm{OP}^+(2, 2)$ has no solution either. Hence assume $(m_1, m_2) \notin \{(2, 2), (3, 3)\}$.

Case 1. m_1 and m_2 are both even. If $m_1 \geq 4$, then $\mathrm{OP}^+(m_1, m_2)$ has a solution by Theorem 1.4. Hence let $m_1 = 2$, and denote $m = \frac{m_2}{2}$, so $n = m + 1$.

Let G be the graph isomorphic to $K_{2n} + I$ with vertex set $V = \{z_i : i \in \mathbb{Z}_{2m}\} \cup \{z_\infty, z_{\infty'}\}$, where $I = \{z_i z_{i+m} : i = 0, 1, \ldots, m-1\} \cup \{z_\infty z_{\infty'}\}$. Furthermore, let ρ be the permutation on V defined by $\rho = (z_0\ z_1\ \ldots\ z_{2m-1})$, so ρ fixes z_∞ and $z_{\infty'}$.

Define a $2m$-cycle C of G as follows. If m is even, then let

$$C = z_0\ z_2\ z_{-1}\ z_3\ z_{-2}\ \ldots\ z_{\frac{m}{2}}\ z_{-(\frac{m}{2}-1)}\ z_\infty\ z_{\frac{m}{2}+1}\ z_{-\frac{m}{2}}\ z_{\frac{m}{2}+2}\ \ldots\ z_{m-1} z_{-(m-2)}\ z_m\ z_{\infty'}\ z_0$$

and if m is odd, then let

$$C = z_0\ z_2\ z_{-1}\ z_3\ z_{-2}\ \ldots\ z_{-\frac{m-3}{2}}\ z_{\frac{m+1}{2}}\ z_\infty\ z_{-\frac{m-1}{2}}\ z_{\frac{m+3}{2}}\ z_{-\frac{m+1}{2}}\ \ldots\ z_{m-1} z_{-(m-2)}\ z_m\ z_{\infty'}\ z_0.$$

In both cases, the cycle uses the following differences, in order

$$2, 3, 4, \dots, m-1, \infty, \infty, m-1, m-2, \dots, 2, \infty', \infty'.$$

Next, we define a 2-cycle of G as follows:

$$C' = z_1 z_{m+1} z_1.$$

Observe that C' is disjoint from C and contains two edges of difference m. Hence $F = C \cup C'$ is a 2-factor of G that contains exactly two edges of each difference in the set $\{2, 3, \dots, m-1, m, \infty, \infty'\}$, and these two edges are of the form e and $\rho^m(e)$.

Finally, let C_0 be the $2m$-cycle of G containing all edges of difference 1, and let $C_0' = z_\infty\, z_{\infty'}\, z_\infty$. Then $F_0 = C_0 \cup C_0'$ is a 2-factor of G, and it is easy to see that

$$\{\rho^i(F) : i = 0, 1, \dots, m-1\} \cup \{F_0\}$$

is an F-factorization of G, where F is of type $(2, 2m)$. Hence $\mathrm{OP}^+(2, 2m)$ has a solution.

Case 2. m_1 and m_2 are both odd. If $m_1 = m_2$, then the result follows from Theorem 1.3. Hence assume $m_1 < m_2$. By Theorem 1.2, $\mathrm{OP}^-(m_1, m_2)$ has a solution, and hence $\mathrm{OP}^+(m_1, m_2)$ has a solution by Theorem 4.14.

We conclude that whenever $(m_1, m_2) \notin \{(2, 2), (3, 3)\}$, the two-table problem $\mathrm{OP}^+(m_1, m_2)$ has a solution. □

6 | Results: Using 1-Rotational Solutions to OP

In this section, we show how the technique of Section 4 can be used to obtain solutions to OP^+ of order $2n+2$ from a suitable (cyclically) 1-rotational solution to OP of order $2n+1$, and hence also from a solution to OP^- of order $2n+2$ that is derived from it (see Lemma 6.3). We first recall the definition of a 2-starter of a 1-rotational solution to OP of odd order—that is, a 2-factor that generates such a solution—and its properties [23].

Throughout this section, n will be a positive integer, the graph K_{2n+1} is assumed to have vertex set $V = \{z_i : i \in \mathbb{Z}_{2n}\} \cup \{z_\infty\}$, and ρ is the permutation on V defined by $\rho = (z_0\ z_1\ \dots\ z_{2n-1})$, which fixes z_∞.

Definition 6.1 (Buratti and Rinaldi [23]). Let F be a 2-factor of K_{2n+1}. Then F is called a *2-starter* for K_{2n+1} if the following hold:

i. for each $d \in \{1, 2, \dots, n\} \cup \{\infty\}$, the 2-factor F has an edge of difference d; and
ii. $\rho^n(F) = F$.

The following properties of a 2-starter are easy to see and have been observed in previous work.

Lemma 6.2 (Buratti and Rinaldi [23]). *Let F be a 2-starter for K_{2n+1}. Then the following hold.*

i. *For each $e \in E(F)$, we have $\rho^n(e) \in E(F)$.*
ii. *F has exactly two edges of difference d for each $d \in \{1, 2, \dots, n-1\} \cup \{\infty\}$.*
iii. *F has exactly one edge of difference n.*
iv. *Let C be a cycle of F such that $z_\infty \in V(C)$. Then C is of the form*

$$C = z_\infty\, z_{i_1}\, z_{i_2}\, \dots\, z_{i_s}\, \rho^n(z_{i_s}) \dots \rho^n(z_{i_2})\, \rho^n(z_{i_1})\, z_\infty.$$

In particular, C is of odd length, contains an edge of difference n, and $\rho^n(C) = C$.

v. *Let C be a cycle of F such that $\rho^n(C) = C$ and $z_\infty \notin V(C)$. Then C is of the form*

$$C = z_{i_1}\, z_{i_2}\, \dots\, z_{i_s}\, \rho^n(z_{i_1})\, \rho^n(z_{i_2}) \dots \rho^n(z_{i_s})\, z_{i_1}.$$

In particular, C is of even length.

iv. *Let C be a cycle of F such that $\rho^n(C) \neq C$. Then $\rho^n(C)$ is also a cycle of F.*

Lemma 6.3 (Buratti and Rinaldi [23] and Buratti and Traetta [24]). *Let F be a 2-starter for K_{2n+1}.*

1. *Then $\{\rho^i(F) : i \in \mathbb{Z}_n\}$ is a solution to OP(F).*
2. *Let F' be a 2-factor of K_{2n+2} with vertex set $V \cup \{z_{\infty'}\}$ obtained from F by subdividing the edge of difference n using a new vertex $z_{\infty'}$. Assuming ρ fixes vertex $z_{\infty'}$, we have that $\{\rho^i(F') : i \in \mathbb{Z}_n\}$ is a solution to* $\mathrm{OP}^-(F')$.

Remark 6.4. Recall from the Introduction that a 2-factorization $\mathcal{F}$ of a graph G is said to be *f-pyramidal* under a group $\mathcal{G}$ if $\mathcal{G}$ preserves $\mathcal{F}$ and, acting on $V(G)$, it fixes f vertices of G and acts sharply transitively on the rest. Moreover, 1-pyramidal 2-factorizations are more commonly called *1-rotational*. Note that 1-rotational and 2-pyramidal 2-factorizations (over various groups) were explored in depth in [23, 24], respectively. However, many papers use the term 1-rotational in the more restricted way, that is, with the assumption that G is cyclic; see, for example, [12, 25, 26]. Moreover, the term *cyclic* has also been used to describe 2-factorizations that are 1-rotational under a cyclic group; see, for example, [27].

Hence, we observe that the solutions to OP(F) and $\mathrm{OP}^-(F')$ described in Lemma 6.3 are 1-rotational and 2-pyramidal, respectively, under a cyclic group. In this paper, a solution to OP(F) described in Lemma 6.3(1) will simply be called 1-rotational, and we will say that the 2-factor F' of K_{2n+2} defined in Lemma 6.3(2) is obtained from F *by subdividing the diameter*. Analogously, we will say that the corresponding solution to $\mathrm{OP}^-(F')$ is obtained from a 1-rotational solution to OP(F) *by subdividing the diameter.*

Remark 6.5. In [22, Theorem 2.7], the authors also show how to obtain a (regular, or cyclic) solution to $\mathrm{OP}^+(m_0 - 1, m_1, \ldots, m_t)$ from a 1-rotational solution to OP($m_0, m_1, \ldots, m_t$)—that is, from a 2-starter F for K_{2n+1} of type $(m_0, m_1, \ldots, m_t)$—assuming that the cycle of F containing an edge of difference n is of length m_0. Our Theorem 3.1 is a slight generalization of that result.

In the next two lemmas, we exploit the symmetry of a 2-starter to establish the existence of a proper sequence of setups that will ultimately allow us to construct a solution to certain cases of OP^+.

Lemma 6.6. *Let F be a 2-starter for K_{2n+1}, and let F_0 be the 2-factor of $G \cong K_{2n+2}$ with vertex set $V' = V \cup \{z_{\infty'}\}$ obtained from F by subdividing the diameter. Let H_0 be the 1-factor of G with edge set $E(H_0) = \{z_i z_{i+n} : i = 0, 1, \ldots, n-1\} \cup \{z_\infty z_{\infty'}\}$.*

Furthermore, let C be the cycle of F_0 containing vertices z_∞ and $z_{\infty'}$, and s the length of C. Then there exists a proper sequence of $\lfloor \frac{s}{6} \rfloor$ 6-setups for F_0 and H_0 all of whose vertices are in $V(C)$.

Proof. By Lemma 6.3(2), we know $\{\rho^i(F_0) : i = 0, 1, \ldots, n-1\}$ is an F_0-factorization of $G - H_0$. Let $C = u_0 u_1 \ldots u_{s-1} u_0$, and note that s is even. By Lemma 6.2(iv), the cycle C is of the form

$$C = z_\infty\, z_{i_1}\, z_{i_2}\, \ldots\, z_{i_{\frac{s}{2}-1}}\, z_{\infty'}\, \rho^n(z_{i_{\frac{s}{2}-1}})\, \ldots\, \rho^n(z_{i_2})\, \rho^n(z_{i_1})\, z_\infty,$$

so we may assume $u_0 = z_\infty, u_1 = z_{i_1}, \ldots, u_{\frac{s}{2}-1} = z_{i_{\frac{s}{2}-1}}, u_{\frac{s}{2}} = z_{\infty'}, u_{\frac{s}{2}+1} = \rho^n(z_{i_{\frac{s}{2}-1}}), \ldots, u_{s-1} = \rho^n(z_{i_1})$. Observe that

$$u_0 u_{\frac{s}{2}}, u_1 u_{s-1}, u_2 u_{s-2}, \ldots, u_{\frac{s}{2}-1} u_{\frac{s}{2}+1} \in E(H_0).$$

Let

$$X_0 = (x_0, x_1, \ldots, x_5) = (u_1, u_2, u_{s-2}, u_{s-1}, u_0, u_{\frac{s}{2}}).$$

Then X_0 is a 6-setup for F_0 and H_0. The derived 2-factor $F_1 = F'_0$ contains the s-cycle

$$C' = u_0 u_2 u_3 \ldots u_{s-1} u_1 u_0,$$

and $H_1 = H_0'$ contains P_2-components $u_1 u_2 u_{s-2}$ and $u_{s-1} u_0 u_{\frac{s}{2}}$. See Figure 3.

Next, let

$$X_1 = (u_4, u_5, u_{s-5}, u_{s-4}, u_{s-3}, u_3).$$

It is not difficult to see that X_1 is a 6-setup for F_1 and H_1. The derived 2-factor $F_2 = F_1'$ contains the s-cycle

$$C'' = u_0 u_1 u_{s-1} u_{s-2} u_{s-3} u_5 u_6 \ldots u_{s-5} u_{s-4} u_4 u_3 u_2 u_0,$$

and $H_2 = H_1'$ contains additional P_2-components $u_4 u_5 u_{s-5}$ and $u_{s-4} u_{s-3} u_3$. See Figure 3.

Continuing in this fashion we can construct a proper sequence $(X_0, X_1, \ldots, X_{k-1})$ of 6-setups for F_0 and H_0, where $k = \lfloor \frac{s}{6} \rfloor$. Note that for $i = 1, 2, \ldots, k-1$, we have

$$X_i = (u_{3i+1}, u_{3i+2}, u_{s-(3i+2)}, u_{s-(3i+1)}, u_{s-3i}, u_{3i}).$$

Hence all of the vertices of the proper sequence of 6-setups are in $V(C)$. □

Lemma 6.7. *Let F be a 2-starter for K_{2n+1}, and let F_0 be the 2-factor of $G \cong K_{2n+2}$ with vertex set $V' = V \cup \{z_{\infty'}\}$ obtained from F by subdividing the diameter. Let H_0 be the 1-factor of G with edge set $E(H_0) = \{z_i z_{i+n} : i = 0, 1, \ldots, n-1\} \cup \{z_\infty z_{\infty'}\}$.*

Furthermore, let C_1 be a cycle of F such that $\rho^n(C_1) \neq C_1$, let $C_2 = \rho^n(C_1)$, and let s be the length of C_1. Then there exists a proper sequence of $\lfloor \frac{s}{6} \rfloor$ 12-setups for F_0 and H_0 all of whose vertices are in $V(C_1) \cup V(C_2)$.

Proof. Since F is a 2-starter and $C_2 = \rho^n(C_1)$, cycles C_1 and C_2 are disjoint. By Lemma 6.2, they do not contain vertex z_∞, and hence they are also cycles of F_0. Let $C_1 = u_0 u_1 \ldots u_{s-1} u_0$. Then, without loss of generality, $C_2 = v_0 v_1 \ldots v_{s-1} v_0$, where $v_i = \rho^n(u_i)$ for all $i \in \mathbb{Z}_s$. Moreover, $u_i v_i \in E(H_0)$ for all $i \in \mathbb{Z}_s$.

Let

$$X_0 = (x_0, x_1, \ldots, x_5) = (u_0, u_1, v_1, v_0, v_{s-1}, u_{s-1})$$

and

$$Y_0 = (y_0, y_1, \ldots, y_5) = (u_3, u_4, v_4, v_3, v_2, u_2).$$

Then (X_0, Y_0) is a 12-setup for F_0 and H_0. The derived 2-factor $F_1 = F_0'$ contains the s-cycles

$$C_1' = u_0 v_0 v_1 v_2 u_4 u_5 \ldots u_0 \quad \text{and} \quad C_2' = v_{s-1} u_1 u_2 u_3 v_3 v_4 \ldots v_{s-1}.$$

Observe that $H_1 = H_0'$ contains P_2-components $u_0 u_1 v_1$, $v_0 v_{s-1} u_{s-1}$, $u_3 u_4 v_4$, and $v_3 v_2 u_2$. See Figure 4.

Next, let

$$X_1 = (u_6, u_7, v_7, v_6, v_5, u_5) \quad \text{and} \quad Y_1 = (u_9, u_{10}, v_{10}, v_9, v_8, u_8).$$

Then (X_1, Y_1) is a 12-setup for F_1 and H_1. The derived 2-factor $F_2 = F_1'$ contains the s-cycles

$$C_1'' = u_0v_0v_1v_2u_4u_5u_6v_6v_7v_8u_{10}u_{11}\dots u_0 \quad \text{and} \quad C_2'' = v_{s-1}u_1u_2u_3v_3v_4v_5u_7u_8u_9v_9v_{10}\dots v_{s-1},$$

and $H_2 = H_1'$ contains new P_2-components $u_6u_7v_7$, $v_6v_5u_5$, $u_9u_{10}v_{10}$, and $v_9v_8u_8$.

Continuing in this fashion we can construct a proper sequence $((X_0, Y_0), (X_1, Y_1), \dots, (X_{k-1}, Y_{k-1}))$ of 12-setups for F_0 and H_0, where $k = \lfloor \frac{s}{6} \rfloor$. Note that for all $i = 0, 1, \dots, k-1$, we have

$$X_i = (u_{6i}, u_{6i+1}, v_{6i+1}, v_{6i}, v_{6i-1}, u_{6i-1}) \quad \text{and} \quad Y_i = (u_{6i+3}, u_{6i+4}, v_{6i+4}, v_{6i+3}, v_{6i+2}, u_{6i+2}).$$

Hence all of the vertices of the proper sequence of 12-setups are in $V(C_1) \cup V(C_2)$. □

Finally, we show that a solution to OP^+ of order $2n+2$ can be constructed using a suitable 2-starter F for K_{2n+1} as long as the number of odd cycles in F is not too large.

Theorem 6.8. *Let F be a 2-starter for K_{2n+1} of type $(m_0, m_1, m_2, \dots, m_t)$. Assume that*

a. *the cycle of F containing vertex z_∞ is of length m_0; and*
b. *if C is a cycle of F such that $\rho^n(C) = C$, then its length is in the set $\{m_0, m_{2s+1}, m_{2s+2}, \dots, m_t\}$, for some s with $0 \le s \le \frac{t}{2}$.*

Let ω be the number of odd cycle lengths in $(m_1, m_2, \dots, m_t)$. If

$$\omega \le 2\left\lfloor \frac{m_0+1}{6} \right\rfloor + 2\sum_{i=1}^{2s} \left\lfloor \frac{m_i}{6} \right\rfloor,$$

then $\mathrm{OP}^+(m_0+1, m_1, m_2, \dots, m_t)$ has a solution.

Proof. Let F_0 be the 2-factor of $G \cong K_{2n+2}$ with vertex set $V' = V \cup \{z_{\infty'}\}$ obtained from F by subdividing the diameter. Note that by Lemmas 6.2(iv) and 6.3(2), F_0 is of type $(m_0+1, m_1, m_2, \dots, m_t)$. Let H_0 be the 1-factor of G with edge set $E(H_0) = \{z_iz_{i+n} : i = 0, 1, \dots, n-1\} \cup \{z_\infty z_{\infty'}\}$.

We may assume $m_1 \le m_2 \le \dots \le m_{2s}$. For any cycle C of F with $\rho^n(C) \ne C$, there is a cycle C' of F such that $C' = \rho^n(C)$. Moreover, by Lemma 6.2, both C and C' are cycles of F_0, and by assumption, their length is in $\{m_1, m_2, \dots, m_{2s}\}$. Hence we may assume that $m_{2i-1} = m_{2i}$ for all $i = 1, 2, \dots, s$.

Observe that ω is even, and let $k = \frac{\omega}{2}$. Since

$$k \le \left\lfloor \frac{m_0+1}{6} \right\rfloor + \sum_{i=1}^{2s} \left\lfloor \frac{m_i}{6} \right\rfloor = \left\lfloor \frac{m_0+1}{6} \right\rfloor + 2\sum_{i=1}^{s} \left\lfloor \frac{m_{2i}}{6} \right\rfloor,$$

there exist integers $k_0, k_1, \dots, k_s$ such that

$$0 \le k_0 \le \left\lfloor \frac{m_0+1}{6} \right\rfloor, \quad 0 \le k_i \le \left\lfloor \frac{m_{2i}}{6} \right\rfloor \quad \text{for } i = 1, 2, \dots, s, \quad \text{and} \quad k = k_0 + 2\sum_{i=1}^{s} k_i.$$

First, let C_0 be the cycle of F_0 obtained by subdividing the diameter in the appropriate cycle of F. Note that C_0 has length m_0+1. By Lemma 6.6, since $k_0 \le \lfloor \frac{m_0+1}{6} \rfloor$, there exists a proper sequence S_0 of k_0 6-setups for F_0 and H_0 all of whose vertices are in $V(C_0)$.

Next, let $C_1, C_2, \dots, C_{2s}$ be the pairwise distinct cycles of F (and hence of F_0) such that, for $i = 1, 2, \dots, s$, the length of C_{2i} is m_{2i} and $\rho^n(C_{2i}) = C_{2i-1}$. By Lemma 6.7, for each $i \in \{1, 2, \dots, s\}$, since $k_i \le \lfloor \frac{m_{2i}}{6} \rfloor$, there is a proper sequence S_i of k_i 12-setups for F_0 and H_0 all of whose vertices are in $V(C_{2i-1}) \cup V(C_{2i})$.

Note that the cycles $C_0, C_1, \ldots, C_{2s}$ are pairwise distinct, and the proper sequence S_i has vertices only in $V(C_i)$ for $i = 0$, and in $V(C_{2i-1}) \cup V(C_{2i})$ for $i = 1, 2, \ldots, s$. Hence, by repeatedly applying Lemma 4.10, we can see that $S_0 S_1 \ldots S_s$ is a proper sequence of setups for F_0 and H_0 that contains k_0 6-setups and $\sum_{i=1}^{s} k_i$ 12-setups. Since $k = k_0 + 2\sum_{i=1}^{s} k_i$ and F_0 has exactly $2k$ odd cycles, by Theorem 4.12, for some 1-factor J of G, the graph $G + J$ admits an F_0-factorization. Therefore, $\mathrm{OP}^+(m_0 + 1, m_1, m_2, \ldots, m_t)$ has a solution. □

Corollary 6.9. *Let F be a 2-starter for K_{2n+1} of type $(m_0, m_1, m_2, \ldots, m_t)$. Assume that*

a. *the cycle of F containing vertex z_∞ is of length m_0; and*
b. *at least half of the odd entries in $(m_1, m_2, \ldots, m_t)$ are greater than 6.*

Then $\mathrm{OP}^+(m_0 + 1, m_1, m_2, \ldots, m_t)$ has a solution.

Proof. Let C be an odd cycle of F. By Lemma 6.2, either its length is m_0, or else $\rho^n(C) \neq C$. We may therefore assume that there exists an integer s such that for any m_i-cycle C satisfying $\rho^n(C) = C$, we have that $m_i \in \{m_0, m_{2s+1}, \ldots, m_t\}$.

Let ω be the number of odd entries in $(m_1, m_2, \ldots, m_t)$. Then $\omega \le 2s$, and since at least $\frac{\omega}{2}$ of the odd cycle lengths are at least 7, we have

$$2\left\lfloor \frac{m_0 + 1}{6} \right\rfloor + 2\sum_{i=1}^{2s} \left\lfloor \frac{m_i}{6} \right\rfloor \ge 2\sum_{i=1}^{\omega} \left\lfloor \frac{m_i}{6} \right\rfloor \ge 2\frac{\omega}{2} = \omega.$$

Hence F satisfies the assumptions in Theorem 6.8. We conclude that $\mathrm{OP}^+(m_0 + 1, m_1, m_2, \ldots, m_t)$ has a solution. □

Remark 6.10. Note that the assumptions of Theorem 6.8 and Corollary 6.9 are more restrictive than those of [22, Theorem 2.7]—see Remark 6.5—however, they may nevertheless result in solutions to OP^+ that (presently) cannot be obtained using [22, Theorem 2.7].

7 | Conclusion

In this paper, we developed the technique of setups, which allowed us to convert certain solutions for $\mathrm{OP}^-(F)$ into solutions for $\mathrm{OP}^+(F)$. Such a conversion is possible if a suitable proper sequence of setups can be found (Theorem 4.12); in particular, if F has exactly two odd cycles and its type satisfies some additional arithmetic conditions (Theorem 4.14). These arithmetic conditions are automatically satisfied if F consists of two odd cycles.

A suitable proper sequence of setups is also guaranteed if the number of odd cycles in F is not too large and the solution to $\mathrm{OP}^-(F)$, where F is of type $(m_0 + 1, m_1, \ldots, m_t)$, is derived from a 1-rotational solution to $\mathrm{OP}(m_0, m_1, \ldots, m_t)$—in other words, if there exists a 2-starter for K_{2n+1} of type $(m_0, m_1, \ldots, m_t)$ with vertex z_∞ in an m_0-cycle (Theorem 6.8). Table 1 in [12] shows that many solutions to OP of order n, for $n \equiv 1 \pmod 4$, $40 \le n \le 60$, are indeed 1-rotational. However, those found for $n \equiv 3 \pmod 4$, and also many of those for $n \equiv 1 \pmod 4$, are not 1-rotational. Moreover, the solutions to OP of order n, for n odd, in [8] (for $18 \le n \le 40$) and [11] (for $61 \le n \le 100$) are decidedly not 1-rotational. At best, these solutions are *2-rotational*; that is, they admit a group of automorphisms that fixes one vertex and has two equal-size orbits on the remaining vertices. Can these solutions be converted to solutions for OP^+? They have less symmetry than 1-rotational solutions, so there does not seem to be an easy answer. We will leave it as an open problem.

Finally, we point out that none of the solutions to $\mathrm{OP}^+(F)$ obtained using the techniques of this paper involve a 2-factor F with a 2-cycle. It is not difficult to see that a solution to $\mathrm{OP}^+(2, m_1, \ldots, m_t)$ is equivalent to a decomposition of $K_n - I$, where $n = 2 + m_1 + \cdots + m_t$, into 2-regular graphs of type $(m_1, \ldots, m_t)$. In [28], it was shown that $K_n - I$ indeed admits a decomposition into isomorphic *bipartite* 2-regular graphs of order $n - 2$, however, we are not aware of any results involving non-bipartite 2-regular graphs of order $n - 2$. We thus conclude with another open problem: if $2 < m_1 \le \cdots \le m_t$ and $n = 2 + m_1 + \cdots + m_t$ is even, does there exist a decomposition of $K_n - I$ into 2-regular graphs of type $(m_1, \ldots, m_t)$ for all $(m_1, \ldots, m_t) \neq (3, 5)$?

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The authors have nothing to report.

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