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Diagonally-reduced tensor-product polynomials: Dimensions and boundary–diagonal matching

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ABSTRACT

We study subspaces of tensor-product polynomials of bi-degree (n, n) on the unit square whose restrictions to the domain diagonals have degree reduced by an integer $k \leq n$. For maximal reduction $k = n$, these diagonally-reduced spaces coincide with the Bézier-Smart surface spaces and have dimension $n^2 + 2$. We derive exact dimension formulas and solve the boundary–diagonal matching problem: when can such a polynomial be replaced by a total-degree polynomial that agrees on the boundary and both diagonals? The answer is a sharp threshold on the degree parameters, with a complete characterisation of existence, uniqueness, and non-uniqueness. For the first non-trivial reduction parameters we provide explicit matching formulas and closed-form error estimates; when the solution is non-unique, the remaining freedom is resolved by L^2 optimality. At the endpoint $k = n$, where matching always fails, we show for $n = 3$ that the L^2-best approximation from the total-degree subspace is also the L^∞-best approximation.

1. Introduction

Tensor-product polynomial spaces are fundamental building blocks in both computer-aided geometric design [1] and finite element analysis [2]. A bivariate polynomial of bi-degree (n, n) on the unit square has $(n+1)^2$ coefficients and can represent rich surface geometry, but its restriction to a domain diagonal is generically of degree $2n$, i.e., far exceeding the degree n of the boundary curves. This degree elevation along the diagonals is an obstacle whenever the domain is to be subdivided along those diagonals, for instance when converting between quadrilateral and triangular patch representations.

In computer-aided geometric design, this observation has motivated the study of Bézier-Smart surfaces: tensor-product patches whose diagonal curves are constrained to have the same degree n as the boundary curves, thus enabling seamless quad-to-triangle conversion [3,4]. Arnal and Monterde [5,6] recently computed the dimension of the space of Bézier-Smart surfaces of arbitrary degree, obtaining $n^2 + 2$.

A different approach to reducing the tensor-product space arises in the finite element literature through serendipity elements [7]. For a C^0-conforming element on a quadrilateral mesh, the trace of the element space on each edge must be the full polynomial space of degree n, so that matching with an adjacent element is always possible. The total-degree space $\mathcal{P}_n$ does not satisfy this requirement: the total-degree bound couples the traces on opposite edges, so that the boundary data cannot be prescribed independently on all four sides of the square. The serendipity space $\mathcal{S}_n = \mathcal{P}_n \oplus \operatorname{span}\{x^n y,\, x y^n\}$, of dimension $\frac{1}{2}(n+1)(n+2) + 2$, is the minimal extension of $\mathcal{P}_n$ that restores full boundary independence [8]; robust nodal bases were subsequently constructed by Floater and Gillette [9].

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The Bézier-Smart and serendipity constructions thus represent two different strategies for reducing the tensor-product space: one constrains the diagonal trace degree, the other ensures boundary unisolvency. The two resulting spaces have comparable but distinct dimensions, and understanding the interplay between diagonal-trace reduction and boundary independence is one motivation for the present work.

In this paper we study the family of diagonally-reduced spaces $\mathcal{Q}_{n,k}$ of tensor-product polynomials of bi-degree (n, n), parameterised by an integer $0 \le k \le n$ that controls the degree-reduction on the diagonals to $2n - k$. Moreover, we investigate the associated boundary–diagonal matching problem: given a polynomial $q \in \mathcal{Q}_{n,k}$, when does there exist a polynomial $p \in \mathcal{P}_{n,k}$ of total degree at most $2n - k$ that agrees with q to order $r \ge 0$ on the boundary of $[0, 1]^2$ and in value on both diagonals (see Section 4 for the precise formulation)? This question arises naturally in spline constructions on mixed quadrilateral-triangular meshes, where the diagonals of each quadrilateral cell become shared edges of triangular sub-cells, and inter-element continuity requires the boundary and diagonal data of the tensor-product piece to be reproducible by a total-degree polynomial on each triangle.

The contributions of the paper are the following.

1. Exact dimension formulas for $\mathcal{Q}_{n,k}$, $\mathcal{P}_{n,k}$, and the orthogonal complement $\mathcal{V}_{n,k}$ (Section 3). At $k = n$, the formula $\dim \mathcal{Q}_{n,k} = n^2 + 2$ recovers the Bézier-Smart count.
2. A complete characterisation of boundary–diagonal matching (Section 4): for $k \ge 3$, matching is achievable if and only if $n \ge 2r + 4$ and $k \le n - 2r - 1$, and is unique precisely when $(n, k) = (2r + 4, 3)$.
3. For the first non-trivial reduction parameters, explicit matching formulas with L^2-optimal free parameters and closed-form error estimates (Section 6).
4. At the endpoint $k = n = 3$, where matching fails, the L^2-projection onto $\mathcal{P}_{3,3}$ is simultaneously the best L^∞ approximation (Section 5).

The remainder of the paper is organized as follows. Section 2 introduces the spaces and notation. Section 3 derives the dimension formulas. Section 4 formulates and solves the matching problem. Section 5 treats the endpoint $k = n$. Section 6 works out explicit examples for small n and r. Section 7 provides some concluding remarks.

2. Definitions

In this section we introduce the three polynomial spaces central to this work: the diagonally-reduced space $\mathcal{Q}_{n,k}$, its total-degree subspace $\mathcal{P}_{n,k}$, and their L^2-orthogonal complement $\mathcal{V}_{n,k} = \mathcal{Q}_{n,k} \cap (\mathcal{P}_{n,k})^\perp$. Throughout, $n \ge 1$ is a fixed polynomial degree and all L^2 inner products are taken over $[0, 1]^2$: $\langle f, g\rangle = \int_{[0,1]^2} f(x, y)\, g(x, y)\, \mathrm{d}x\, \mathrm{d}y$.

Definition 2.1 (Full tensor-product space). Let

$$\mathcal{Q}_n \;=\; \Big\{\, q(x, y) = \textstyle\sum_{0\le i,j\le n} a_{i,j}\, x^i y^j \;\Big|\; a_{i,j} \in \mathbb{R} \Big\}$$

denote the space of bivariate polynomials of bi-degree (n, n). Its dimension is $\dim \mathcal{Q}_n = (n + 1)^2$.

Definition 2.2 (Total-degree subspace). Let

$$\mathcal{P}_n \;=\; \Big\{\, p \in \mathcal{Q}_n \;\Big|\; \deg p \le n \Big\} \;=\; \operatorname{span}\big\{x^i y^j \;:\; i + j \le n,\; 0 \le i, j \le n\big\}.$$

Its dimension is $\dim \mathcal{P}_n = \binom{n+2}{2} = \frac{1}{2}(n + 1)(n + 2)$.

We introduce two diagonal trace operators.

Definition 2.3 (Diagonal and anti-diagonal traces). For $q \in \mathcal{Q}_n$, $\tau q(t) = q(t, t)$ is the diagonal trace, and $\sigma q(t) = q(t, 1 - t)$ is the anti-diagonal trace. Both are univariate polynomials of degree at most $2n$.

For an integer $0 \le k \le n$, set $N = 2n - k$.

Definition 2.4 (Diagonally-reduced space). Let

$$\mathcal{Q}_{n,k} \;=\; \Big\{\, q \in \mathcal{Q}_n \;\Big|\; \deg(\tau q) \le N \;\text{ and }\; \deg(\sigma q) \le N \Big\}.$$

When $k = 0$ there is no constraint and $\mathcal{Q}_{n,0} = \mathcal{Q}_n$.

Definition 2.5 (Restricted total-degree space). Let

$$\mathcal{P}_{n,k} \;=\; \Big\{\, p \in \mathcal{Q}_n \;\Big|\; \deg p \le N \Big\} \;=\; \operatorname{span}\big\{x^i y^j \;:\; i + j \le N,\; 0 \le i, j \le n\big\}.$$

When $k = 0$ we have $\mathcal{P}_{n,0} = \mathcal{Q}_n$, and when $k = n$ we recover $\mathcal{P}_{n,n} = \mathcal{P}_n$.

Remark 2.6. The inclusion $\mathcal{P}_{n,k} \subseteq \mathcal{Q}_{n,k}$ holds for every k since, clearly, if $\deg p \le N$ then $\deg(\tau p) \le N$ and $\deg(\sigma p) \le N$.

3. Dimensions

We derive closed-form dimension formulas for $\mathcal{P}_{n,k}$, $\mathcal{Q}_{n,k}$, and the orthogonal complement $\mathcal{V}_{n,k}$. While the dimension of $\mathcal{P}_{n,k}$ can be argued trivially, the dimension of $\mathcal{Q}_{n,k}$ requires a more involved analysis.

3.1. Dimension of $\mathcal{P}_{n,k}$

The dimension of $\mathcal{P}_{n,k}$ follows from the dimension of its complement in $\mathcal{Q}_n$.

Proposition 3.1. *For* $0 \le k \le n$,

$$\dim \mathcal{P}_{n,k} \;=\; (n+1)^2 - \frac{k(k+1)}{2}.$$

Proof. Let

$$\mathcal{H}_{n,k} \;=\; \operatorname{span}\left\{x^i y^j \;:\; N+1 \le i+j,\ 0 \le i,j \le n\right\}.$$

Since $\mathcal{P}_{n,k} \oplus \mathcal{H}_{n,k} = \mathcal{Q}_n$, we have $\dim \mathcal{P}_{n,k} = \dim \mathcal{Q}_n - \dim \mathcal{H}_{n,k}$. Replacing the indices i, j in the definition of $\mathcal{H}_{n,k}$ with $n-\bar{\imath}$, $n-\bar{\jmath}$, respectively, gives

$$\mathcal{H}_{n,k} \;=\; \operatorname{span}\left\{x^{n-\bar{\imath}} y^{n-\bar{\jmath}} \;:\; N+1 \le 2n-(\bar{\imath}+\bar{\jmath}),\ 0 \le \bar{\imath},\bar{\jmath} \le n\right\},$$

where the condition $N+1 \le 2n-(\bar{\imath}+\bar{\jmath})$ is equivalent to $0 \le \bar{\imath}+\bar{\jmath} \le k-1$. Therefore $\dim \mathcal{H}_{n,k} = \binom{k+1}{2}$, and $\dim \mathcal{P}_{n,k} = (n+1)^2 - \frac{1}{2}k(k+1)$. $\square$

3.2. Dimension of $\mathcal{Q}_{n,k}$

The dimension of $\mathcal{Q}_{n,k}$ is determined by counting the independent linear constraints that the diagonal trace conditions impose on the Legendre expansion of q.

The shifted Legendre polynomials on $[0,1]$ are defined by

$$L_i(t) \;=\; \sum_{\ell=0}^{i} (-1)^{i-\ell} \binom{i}{\ell} \binom{i+\ell}{\ell} t^{\ell}, \qquad i = 0, 1, \ldots \tag{1}$$

They satisfy $\int_0^1 L_i(t)\, L_j(t)\, \mathrm{d}t = \delta_{i,j}/(2i+1)$, so $\{L_i(x)\, L_j(y)\}_{0 \le i,j \le n}$ is an orthogonal basis for $\mathcal{Q}_n$.

Lemma 3.2 (Parity relation)**.** $L_i(1-t) = (-1)^i\, L_i(t)$ *for all* $i \ge 0$.

Proof. The standard Legendre polynomials P_i considered on $[-1,1]$ satisfy $P_i(-x) = (-1)^i P_i(x)$. Since $L_i(t) = P_i(2t-1)$, we have $L_i(1-t) = P_i(1-2t) = P_i(-(2t-1)) = (-1)^i P_i(2t-1) = (-1)^i L_i(t)$. $\square$

For a univariate polynomial $f(t) = \sum_j a_j\, t^j$, we write $[t^s]\, f$ for the coefficient a_s.

Lemma 3.3 (Spectral constraint structure)**.** *A polynomial* $q \in \mathcal{Q}_n$ *belongs to* $\mathcal{Q}_{n,k}$ *if and only if*

$$[t^s]\, \tau q(t) = [t^s]\, \sigma q(t) = 0 \qquad \textit{for each } s \in \{2n,\, 2n-1,\, \ldots,\, 2n-k+1\}.$$

For $s = 2n$ *the two conditions coincide. For each* $s < 2n$ *they are two linear constraints on the Legendre coefficients. Furthermore, for each* $s < 2n$ *the linear functionals* $q \mapsto [t^s]\, \tau q$ *and* $q \mapsto [t^s]\, \sigma q$ *are linearly independent of all functionals* $\{q \mapsto [t^{\ell}]\, \tau q \mid s < \ell \le 2n\} \cup \{q \mapsto [t^{\ell}]\, \sigma q \mid s < \ell \le 2n\}$.

Proof. Write q in the shifted Legendre basis, i.e.,

$$q(x,y) = \sum_{i,j=0}^{n} c_{i,j}\, L_i(x)\, L_j(y).$$

Group the Legendre products by spectral level $s = i+j$. Since L_i has exact degree i with positive leading coefficient $\binom{2i}{i}$, the product $L_i(t)\, L_{s-i}(t)$ has degree exactly s. Denote by $H_s(x,y) = \sum_i c_{i,s-i}\, L_i(x)\, L_{s-i}(y)$ the sum of all Legendre products at level s, so that $q = p + \sum_{s=2n-k+1}^{2n} H_s$ for some $p \in \mathcal{P}_{n,k}$. Each $H_s(t,t)$ has degree at most s, and $[t^s]\, H_s(t,t) = \sum_i c_{i,s-i}\, \alpha_i$ with every $\alpha_i = \binom{2i}{i}\binom{2(s-i)}{s-i} > 0$. The same holds for $H_s(t, 1-t)$. Since $\deg p \le 2n-k$, both $[t^s]\, \tau q$ and $[t^s]\, \sigma q$ for $s > 2n-k$ depend only on $H_s, H_{s+1}, \ldots, H_{2n}$.

In case $s = 2n$ there is a single Legendre coefficient $c_{n,n}$. Using $[t^{2n}](L_n(t)^2) = \binom{2n}{n}^2 > 0$ and Lemma 3.2 it follows that

$$[t^{2n}]\, \tau q = c_{n,n} \binom{2n}{n}^2, \qquad [t^{2n}]\, \sigma q = (-1)^n\, c_{n,n} \binom{2n}{n}^2.$$

Hence $[t^{2n}]\, \tau q = 0$ if and only if $[t^{2n}]\, \sigma q = 0$, and both reduce to $c_{n,n} = 0$.

Now we consider $s < 2n$ and argue the independence from higher levels. Since L_i has exact degree i with positive leading coefficient $\binom{2i}{i}$, the product $L_i(t)\, L_{s-i}(t)$ has degree exactly s, and its leading coefficient $[t^s](L_i\, L_{s-i}) = \binom{2i}{i}\binom{2(s-i)}{s-i}$ is positive. Hence the coefficients $c_{i,j}$ with $i+j = s$ enter $[t^s]\, \tau q$ with nonzero weights but do not appear in any constraint at level $\ell > s$ (since $L_i\, L_{s-i}$ has degree exactly s). Therefore $[t^s]\, \tau q$ is linearly independent of all higher-level constraints. The same argument applies to $[t^s]\, \sigma q$. $\square$

Proposition 3.4. *For* $1 \le k \le n$,

$$\dim \mathcal{Q}_{n,k} \;=\; (n+1)^2 - 2k + 1.$$

Proof. By Lemma 3.3, the constraints on $\mathcal{Q}_{n,k}$ come from levels $s = 2n, 2n-1, \ldots, 2n-k+1$. Level $s = 2n$ contributes 1 constraint (the two conditions coincide). Each of the remaining $k-1$ levels adds 2 new conditions that are by Lemma 3.3 linearly independent of all higher-level constraints. It remains to show that at each such level $s < 2n$, the two constraints $q \mapsto [t^s]\,\tau q$ and $q \mapsto [t^s]\,\sigma q$ are independent of each other.

Set $\alpha_i = [t^s](L_i(t)\,L_{s-i}(t))$ and $\beta_i = [t^s](L_i(t)\,L_{s-i}(1-t))$. By Lemma 3.2, $\beta_i = (-1)^{s-i}\,\alpha_i$. Each α_i is a product of positive leading coefficients and hence positive. Suppose $q \mapsto [t^s]\,\tau q$ and $q \mapsto [t^s]\,\sigma q$ are linearly dependent, i.e., there exist constants A, B, not both zero, with $A \cdot [t^s]\,\tau q + B \cdot [t^s]\,\sigma q = 0$ for all q. In particular, $A\,\alpha_i + B\,\beta_i = 0$ for every valid i. Since $2n > s \geq 2n-k+1 \geq n+1$, both $i_1 = n$ and $i_2 = n-1$ are valid indices at level s (i.e., $0 \leq i,\ s-i \leq n$) and have different parity, which gives $\beta_{i_1} = (-1)^{s-i_1}\alpha_{i_1}$ and $\beta_{i_2} = -(-1)^{s-i_1}\alpha_{i_2}$. The two equations become

$$A\alpha_{i_1} + B(-1)^{s-i_1}\alpha_{i_1} = 0, \qquad A\alpha_{i_2} - B(-1)^{s-i_1}\alpha_{i_2} = 0.$$

Since α_{i_1} and α_{i_2} are positive, this forces $A = B = 0$, which is a contradiction. Hence $q \mapsto [t^s]\,\tau q$ and $q \mapsto [t^s]\,\sigma q$ are linearly independent.

The total number of independent constraints is $1 + 2(k-1) = 2k-1$, giving $\dim \mathcal{Q}_{n,k}$ as stated. □

Corollary 3.5. *Define $\mathcal{V}_{n,k} = \mathcal{Q}_{n,k} \cap (\mathcal{P}_{n,k})^\perp$ (the L^2-orthogonal complement of $\mathcal{P}_{n,k}$ in $\mathcal{Q}_{n,k}$). For $1 \leq k \leq n$,*

$$\dim \mathcal{V}_{n,k} = \dim \mathcal{Q}_{n,k} - \dim \mathcal{P}_{n,k} = \frac{(k-1)(k-2)}{2}.$$

In particular, for $k \leq 2$, $\mathcal{V}_{n,k} = \{0\}$ and $\mathcal{Q}_{n,k} = \mathcal{P}_{n,k}$.

Remark 3.6. At the maximal reduction $k = n$, the formula gives $\dim \mathcal{Q}_{n,n} = n^2 + 2$. The space of Bézier-Smart surfaces of degree n, as studied by Arnal and Monterde [5,6], is defined as the subspace of bi-degree (n,n) polynomials whose diagonal traces have degree at most n; this is precisely the constraint $\deg(\tau q), \deg(\sigma q) \leq 2n - n = n$ in Definition 2.4 with $k = n$. Hence $\mathcal{Q}_{n,n}$ coincides with the Bézier-Smart surface space, and our formula recovers the dimension $n^2 + 2$ established in [6]. The decomposition $\dim \mathcal{Q}_{n,n} = \frac{1}{2}(n+1)(n+2) + \frac{1}{2}(n-1)(n-2)$ makes the split between total-degree and diagonal-constrained components explicit.

Remark 3.7 (Relationship to serendipity elements)**.** The serendipity space [8]

$$\mathcal{S}_n \;=\; \mathcal{P}_n \oplus \operatorname{span}\{x^n y,\ xy^n\}$$

has dimension $\frac{1}{2}(n+1)(n+2) + 2$, which differs from $\dim \mathcal{Q}_{n,n} = n^2 + 2$. The two serendipity monomials $x^n y$ and xy^n each have diagonal trace t^{n+1}, so neither belongs to $\mathcal{Q}_{n,n}$ individually. However, since the threshold degree for $k = n-1$ is $N = n+1$, both monomials satisfy the diagonal constraints at this level, giving $\mathcal{S}_n \subseteq \mathcal{P}_{n,k} \subseteq \mathcal{Q}_{n,k}$ for all $k \leq n-1$. In particular,

$$\mathcal{P}_{n,n-1} \;=\; \mathcal{S}_n \oplus \operatorname{span}\{x^{n-1}y^2,\ x^{n-2}y^3,\ \ldots,\ x^2 y^{n-1}\},$$

so $\mathcal{P}_{n,n-1}$ extends $\mathcal{S}_n$ by the $n-2$ remaining monomials of total degree $n+1$. A special case is $\mathcal{S}_2 = \mathcal{P}_{2,1} = \mathcal{Q}_{2,1}$.

4. Boundary–diagonal matching

We now study when the "trace data" of any $q \in \mathcal{Q}_{n,k}$ on the boundary, diagonal, and anti-diagonal of $[0,1]^2$ can be reproduced by some $p \in \mathcal{P}_{n,k}$.

4.1. Setup

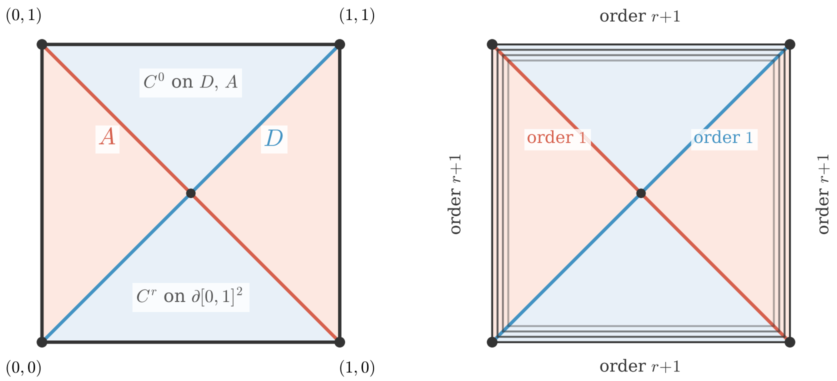
Let $\Sigma = \partial[0,1]^2 \cup D \cup A$ where $D = \{(t,t) : t \in [0,1]\}$ is the diagonal and $A = \{(t,1-t) : t \in [0,1]\}$ is the anti-diagonal. For an integer $r \geq 0$, the (Σ, r)-matching problem is defined as follows: given $q \in \mathcal{Q}_{n,k}$, find $p \in \mathcal{P}_{n,k}$ such that

(M1) p and q agree to order r on $\partial[0,1]^2$ (i.e., all partial derivatives up to order r coincide on each edge), and
(M2) $p(t,t) = q(t,t)$ and $p(t,1-t) = q(t,1-t)$ for all $t \in [0,1]$.

We say that (Σ, r)-matching is achievable for $\mathcal{Q}_{n,k}$ if for every $q \in \mathcal{Q}_{n,k}$ such $p \in \mathcal{P}_{n,k}$ exists.

Remark 4.1. Condition (M2) requires only C^0 matching (value agreement) on the diagonals. Condition (M1) requires C^r matching on the boundary, i.e., agreement of the function and all derivatives through order r.

Remark 4.2. The (Σ, r)-matching problem is formulated in terms of the trace data of a given $q \in \mathcal{Q}_{n,k}$, rather than in the data-driven form customary in computer-aided geometric design, where boundary data and a pair of compatible diagonal curves are prescribed directly. The two formulations are closely related. The pairs of curves that can appear as diagonals of a tensor-product patch are characterized in [5] for general diagonals and in [6] for diagonals of the reduced degree n, where the sole requirement is that the two curves share their midpoint; both works also describe the families of patches realizing prescribed boundary and diagonal curves. Since every patch whose diagonal curves have degree at most N belongs to $\mathcal{Q}_{n,k}$ by definition, admissible data with diagonal curves of degree at most N is realized by some $q \in \mathcal{Q}_{n,k}$, so the corresponding cases of the results below apply to the data-driven formulation as well.



(a) Matching domain Σ (b) Zero set of F_r

Fig. 1. (a) The matching domain $\Sigma = \partial[0,1]^2 \cup D \cup A$: conditions (M1) and (M2) require C^r agreement on the boundary and C^0 agreement on the diagonal D and anti-diagonal A. (b) Zero set of the frame polynomial F_r: it vanishes to order r+1 on each edge and to order 1 on each diagonal, generating the kernel $\mathcal{K}_r = F_r \cdot \mathcal{Q}_{n-2r-4}$ (Lemma 4.3).

4.2. Kernel analysis

Write $f = q - p$. The matching conditions require f to vanish to order $r+1$ on each edge and to vanish (in value) on both diagonals. Let us define the frame polynomial

$$F_r(x, y) = \left[x(1-x)\right]^{r+1} \left[y(1-y)\right]^{r+1} (y-x)(x+y-1), \tag{2}$$

which has bi-degree $(2r+4, 2r+4)$ and total degree $4r+6$. It vanishes to order $r+1$ on each edge of $[0,1]^2$ and to order 1 on each diagonal.

Lemma 4.3 (Structure of the kernel). *Let $\mathcal{K}_r$ denote the space of $f \in \mathcal{Q}_n$ that vanish to order $r+1$ on $\partial[0,1]^2$ and vanish on both diagonals D and A. Then $\mathcal{K}_r = F_r \cdot \mathcal{Q}_{n-2r-4}$. In particular,*

$$\dim \mathcal{K}_r = (n-2r-3)^2 \qquad \textit{for } n \geq 2r+4,$$

and $\mathcal{K}_r = \{0\}$ for $n \leq 2r+3$. Moreover, $\mathcal{K}_r \subseteq \mathcal{Q}_{n,k}$ for every k, since both diagonal traces of every $f \in \mathcal{K}_r$ vanish identically.

Proof. Vanishing to order $r+1$ on the edge $\{y = 0\}$ means $y^{r+1} \mid f$ as a polynomial in y. The four edges give the factor $[x(1-x)]^{r+1}[y(1-y)]^{r+1}$, which has bi-degree $(2r+2, 2r+2)$. Write $f = [x(1-x)]^{r+1}[y(1-y)]^{r+1}\, s(x, y)$ with s of bi-degree $(n-2r-2, n-2r-2)$.

Vanishing on D: $f(t,t) = [t(1-t)]^{2r+2}\, s(t,t) = 0$ forces $s(t,t) = 0$ identically, hence $(y-x) \mid s$ in $\mathbb{R}[x,y]$. Vanishing on A: $f(t, 1-t) = [t(1-t)]^{2r+2}\, s(t, 1-t) = 0$ forces $s(t, 1-t) = 0$, hence $(x+y-1) \mid s$. Since $(y-x)$ and $(x+y-1)$ are coprime, $s = (y-x)(x+y-1)\, u$ with u of bi-degree $(n-2r-4, n-2r-4)$.

Both diagonal traces vanish because $(y-x)\big|_{y=x} = 0$ and $(x+y-1)\big|_{y=1-x} = 0$. $\square$

Lemma 4.4 (Kernel restricted to $\mathcal{P}_{n,k}$). *Define $\mathcal{K}_r^P = \mathcal{K}_r \cap \mathcal{P}_{n,k}$. An element $f = F_r \cdot u$ lies in $\mathcal{P}_{n,k}$ if and only if u satisfies*

$$\deg u \leq N - (4r+6) = 2n - k - 4r - 6, \qquad \deg_x u,\ \deg_y u \leq n - 2r - 4. \tag{3}$$

Moreover, in relation to $\mathfrak{D} = 2n - k - 4r - 6$, $\mathfrak{d} = n - 2r - 4$, and $\mathfrak{g} = \mathfrak{D} - \mathfrak{d} = n - k - 2r - 2$, $\mathcal{K}_r^P$ has the following properties.

1. *If $\mathfrak{D} < 0$ then $\mathcal{K}_r^P = \{0\}$.*
2. *If $0 \leq \mathfrak{D} \leq \mathfrak{d}$ (equivalently $\mathfrak{g} \leq 0$ and $\mathfrak{D} \geq 0$), the total-degree constraint dominates and $\dim \mathcal{K}_r^P = \binom{\mathfrak{D}+2}{2}$.*
3. *If $\mathfrak{D} > \mathfrak{d}$ (equivalently $\mathfrak{g} \geq 1$), the variable-degree constraint excludes $\mathfrak{g}(\mathfrak{g}+1)$ monomials, giving $\dim \mathcal{K}_r^P = \binom{\mathfrak{D}+2}{2} - \mathfrak{g}(\mathfrak{g}+1)$.*

Proof. The bi-degree of F_r is $(2r+4, 2r+4)$: the boundary factor contributes $(2r+2, 2r+2)$ and each linear diagonal factor contributes $(1,1)$. The total degree of F_r is at most $(2r+2)+(2r+2)+2 = 4r+6$ (and this bound is sharp). For $f = F_r \cdot u \in \mathcal{P}_{n,k}$ we need $\deg f \leq N = 2n-k$ and $\deg_x f, \deg_y f \leq n$, which implies the constraints in (3).

Now let us argue the second part of the statement.

1. When $\mathfrak{D} < 0$, there are no polynomials of degree $\leq \mathfrak{D}$, so $\mathcal{K}_r^P = \{0\}$.
2. When $0 \leq \mathfrak{D} \leq \mathfrak{d}$, every monomial $x^i y^j$ with $i + j \leq \mathfrak{D}$ automatically satisfies $i, j \leq \mathfrak{D} \leq \mathfrak{d}$, so $\dim \mathcal{K}_r^P = \binom{\mathfrak{D}+2}{2}$.
3. When $\mathfrak{D} > \mathfrak{d}$ (i.e., $\mathfrak{g} \geq 1$), monomials $x^i y^j$ with $i + j \leq \mathfrak{D}$ but $i > \mathfrak{d}$ are excluded. For each value $i = \mathfrak{d}+1, \ldots, \mathfrak{D}$ we have $j \leq \mathfrak{D} - i$, contributing $\sum_{a=1}^{\mathfrak{g}} a = \frac{1}{2}\mathfrak{g}(\mathfrak{g}+1)$ excluded monomials. By symmetry in j, the total exclusion is $\mathfrak{g}(\mathfrak{g}+1)$. (No double-counting occurs since $\mathfrak{D} < 2\mathfrak{d} + 2$ for $k \geq 1$.)

This concludes the proof. $\square$

4.3. The main matching theorem

We now combine the kernel analysis of Section 4.2 with the dimension formulas of Section 3 to determine exactly when matching is achievable, and whether the solution is unique.

Theorem 4.5 (Boundary–diagonal matching). *Let $n \geq 1$, $0 \leq k \leq n$, and $r \geq 0$.*

(a) For $k \leq 2$, matching is trivially achievable ($\mathcal{Q}_{n,k} = \mathcal{P}_{n,k}$, take $p = q$).
(b) For $k \geq 3$, (Σ, r)-matching is achievable for $\mathcal{Q}_{n,k}$ if and only if

$$n \geq 2r + 4 \quad \text{and} \quad k \leq n - 2r - 1.$$

(c) When matching is achievable and $k \geq 3$, the polynomial p is unique if and only if $n = 2r + 4$ and $k = 3$.

Proof. Let ρ denote the linear map that sends $q \in \mathcal{Q}_n$ to its trace data on Σ: the r-jets $(\partial^\alpha q|_e)_{|\alpha| \leq r}$ on each boundary edge e of $[0,1]^2$ and the values $q|_D$, $q|_A$ on the two diagonals. Matching is achievable if and only if $\rho(\mathcal{P}_{n,k}) \supseteq \rho(\mathcal{Q}_{n,k})$. Since $\mathcal{P}_{n,k} \subseteq \mathcal{Q}_{n,k}$, this is equivalent to $\rho(\mathcal{P}_{n,k}) = \rho(\mathcal{Q}_{n,k})$, i.e., the images have the same dimension:

$$\dim \mathcal{Q}_{n,k} - \dim(\mathcal{K}_r \cap \mathcal{Q}_{n,k}) \;=\; \dim \mathcal{P}_{n,k} - \dim(\mathcal{K}_r \cap \mathcal{P}_{n,k}). \tag{4}$$

Since $\mathcal{K}_r \subseteq \mathcal{Q}_{n,k}$ (Lemma 4.3), this simplifies to the kernel difference satisfying

$$\dim \mathcal{K}_r - \dim \mathcal{K}_r^P = \dim \mathcal{Q}_{n,k} - \dim \mathcal{P}_{n,k} = \dim \mathcal{V}_{n,k} = \tfrac{1}{2}(k-1)(k-2). \tag{5}$$

1. For $k \leq 2$, Corollary 3.5 implies $\mathcal{Q}_{n,k} = \mathcal{P}_{n,k}$, so $p = q$ satisfies all conditions.
2. Suppose $k \geq 3$. Then $\frac{1}{2}(k-1)(k-2) > 0$. When $n \leq 2r+3$, we have $\mathcal{K}_r = \mathcal{K}_r^P = \{0\}$, so (5) does not hold. Let us assume $n \geq 2r+4$ and set $\mathfrak{p} = n - 2r - 3 \geq 1$. Then $\dim \mathcal{K}_r = \mathfrak{p}^2$. Define the excess $\mathfrak{g} = n - k - 2r - 2 = \mathfrak{p} + 1 - k$ and $\mathfrak{D} = \mathfrak{p} + \mathfrak{g} - 1$.
   Case $\mathfrak{g} \geq 1$ *(i.e., $k \leq n - 2r - 3$)*: Here $\mathfrak{D} \geq \mathfrak{p} \geq 1$, and Lemma 4.4(iii) gives

$$\dim \mathcal{K}_r^P = \frac{1}{2}(\mathfrak{p} + \mathfrak{g} + 1)(\mathfrak{p} + \mathfrak{g}) - \mathfrak{g}(\mathfrak{g} + 1).$$

   The kernel difference is

$$\begin{aligned} \mathfrak{p}^2 - \left[\tfrac{1}{2}(\mathfrak{p}+\mathfrak{g}+1)(\mathfrak{p}+\mathfrak{g}) - \mathfrak{g}(\mathfrak{g}+1)\right] &= \tfrac{1}{2}\left[2\mathfrak{p}^2 - (\mathfrak{p}+\mathfrak{g})^2 - (\mathfrak{p}+\mathfrak{g}) + 2\mathfrak{g}(\mathfrak{g}+1)\right] \\ &= \tfrac{1}{2}\left[\mathfrak{p}^2 - (2\mathfrak{g}+1)\mathfrak{p} + \mathfrak{g}(\mathfrak{g}+1)\right] \\ &= \tfrac{1}{2}(\mathfrak{p} - \mathfrak{g})(\mathfrak{p} - \mathfrak{g} - 1). \end{aligned}$$

   Since $\mathfrak{p} - \mathfrak{g} = k - 1$, this equals $\frac{1}{2}(k-1)(k-2) = \dim \mathcal{V}_{n,k}$, so (5) holds.
   Case $\mathfrak{g} \in \{-1, 0\}$ *(i.e., $k \in \{n-2r-2,\, n-2r-1\}$):* Here $\mathfrak{D} = \mathfrak{p} + \mathfrak{g} - 1 \geq \mathfrak{p} - 2 \geq -1$, so $\mathfrak{D} \geq 0$ when $\mathfrak{p} \geq 2$, and $\mathfrak{D} \in \{-1, 0\}$ when $\mathfrak{p} = 1$. In every case

$$\dim \mathcal{K}_r^P = \max\left(0, \tbinom{\mathfrak{D}+2}{2}\right) = \tfrac{1}{2}(\mathfrak{p}+\mathfrak{g}+1)(\mathfrak{p}+\mathfrak{g})$$

   since $(\mathfrak{p}+\mathfrak{g}+1)(\mathfrak{p}+\mathfrak{g}) = 0$ whenever $\mathfrak{D} \leq -1$. The kernel difference is

$$\mathfrak{p}^2 - \tfrac{1}{2}(\mathfrak{p}+\mathfrak{g}+1)(\mathfrak{p}+\mathfrak{g}) = \tfrac{1}{2}\left[\mathfrak{p}^2 - (2\mathfrak{g}+1)\mathfrak{p} - \mathfrak{g}(\mathfrak{g}+1)\right].$$

   For $\mathfrak{g} = 0$ (where $k = \mathfrak{p} + 1$) this is $\frac{1}{2}\mathfrak{p}(\mathfrak{p}-1) = \frac{1}{2}(k-1)(k-2)$, and for $\mathfrak{g} = -1$ (where $k = \mathfrak{p} + 2$) it is $\frac{1}{2}\mathfrak{p}(\mathfrak{p}+1) = \frac{1}{2}(k-1)(k-2)$. In both cases (5) holds.
   Case $\mathfrak{g} \leq -2$ *(i.e., $k \geq n - 2r$):* Let us write $\mathrm{gap}(k) = \dim \mathcal{V}_{n,k} - (\dim \mathcal{K}_r - \dim \mathcal{K}_r^P)$. We show the gap is positive by monotonicity. At $\mathfrak{g} = -1$ (i.e., $k = \mathfrak{p} + 2$) we showed that gap is zero. When k increases by 1, $\dim \mathcal{V}_{n,k}$ increases by $k-1$, while $\dim \mathcal{K}_r^P$ decreases by at most $\mathfrak{d} + 1 = \mathfrak{p}$ (the number of monomials at any single total degree within the bi-degree bound). Since $k - 1 \geq \mathfrak{p} + 2 > \mathfrak{p}$ for $k \geq \mathfrak{p} + 3$, the gap strictly increases at each step. Hence $\mathrm{gap}(k) > 0$ for all $k \geq \mathfrak{p} + 3$, i.e., $\mathfrak{g} \leq -2$, and matching fails.
   Combining: matching holds for $k \geq 3$ if and only if $n \geq 2r + 4$ and $\mathfrak{g} \geq -1$, i.e., $k \leq n - 2r - 1$.
3. When matching is achievable, p is unique if and only if $\mathcal{K}_r^P = \{0\}$. For $k \leq 2$ it holds that $p = q$ and is trivially unique. For $k \geq 3$ when matching is achievable (so $\mathfrak{g} \geq -1$ and $n \geq 2r+4$), the identity (5) gives $\dim \mathcal{K}_r^P = \mathfrak{p}^2 - \frac{(k-1)(k-2)}{2}$. This vanishes if and only if $2\mathfrak{p}^2 = (k-1)(k-2)$, i.e.,

$$k = \tfrac{1}{2}\left(3 + \sqrt{1 + 8\mathfrak{p}^2}\right).$$

   For $\mathfrak{p} = 1$ we notice that $k = 3$ and $k = 3 \leq n - 2r - 1 = \mathfrak{p} + 2 = 3$, so matching is achievable and unique. For $\mathfrak{p} \geq 2$ it holds that $1 + 8\mathfrak{p}^2 > (2\mathfrak{p}+1)^2$, so $k > \mathfrak{p} + 2 = n - 2r - 1$, which violates the achievability constraint of part (b).
   Hence uniqueness holds if and only if $n = 2r + 4$ and $k = 3$.

This proves the statement in full. □

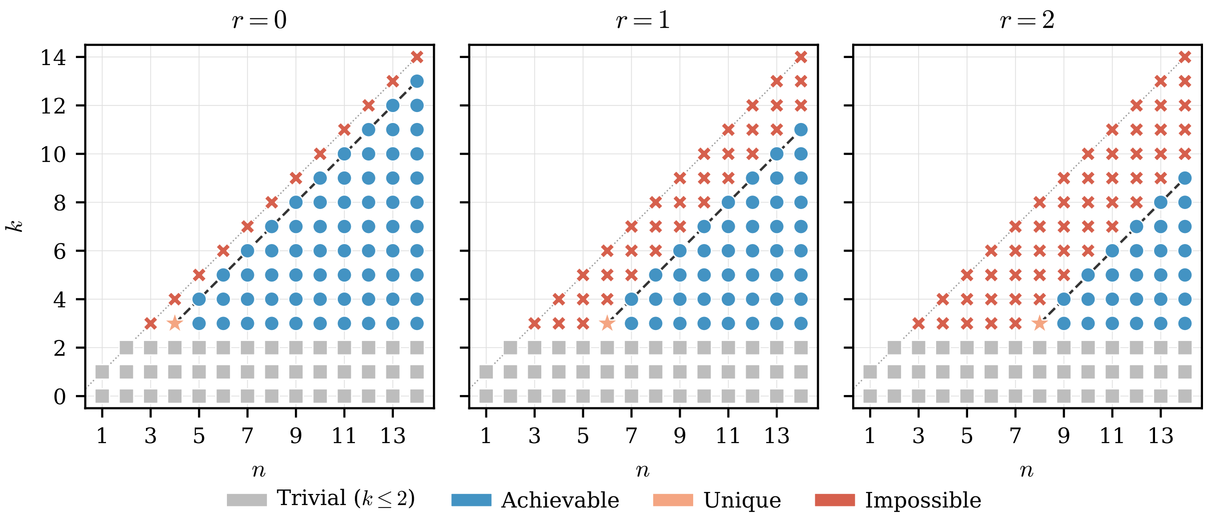


Fig. 2. Achievability of (Σ, r)-matching in the (n, k) lattice for $r = 0, 1, 2$. Blue circles: achievable (non-unique). Star: unique matching ($(n, k) = (2r+4,\ 3)$). Red crosses: impossible ($k \geq 3$ outside the achievable region). Grey squares: trivial ($k \leq 2$, where $\mathcal{Q}_{n,k} = \mathcal{P}_{n,k}$). The dashed line marks the boundary $k = n - 2r - 1$.

5. The endpoint $k = n$

When $k = n$ we have $\mathcal{P}_{n,k} = \mathcal{P}_n$ (total degree $\leq n$) and the matching problem asks whether $\mathcal{P}_n$ can reproduce the trace data of every $q \in \mathcal{Q}_{n,n}$.

Proposition 5.1. *For $k = n \geq 3$, (Σ, r)-matching fails for every $r \geq 0$.*

Proof. By Theorem 4.5(b), matching for $k \geq 3$ requires $k \leq n - 2r - 1$. Since $k = n > n - 2r - 1$ for all $r \geq 0$, matching always fails. ☐

Remark 5.2 (Boundary trace obstruction). The matching failure at $k = n$ can be understood through the boundary trace map itself. For $p \in \mathcal{P}_n$, the total-degree bound forces the leading coefficients on opposite edges to agree: since $x^n y^j \in \mathcal{P}_n$ only for $j = 0$, the x^n-coefficient of $p(x, 0)$ equals that of $p(x, 1)$, and likewise the y^n-coefficient of $p(0, y)$ equals that of $p(1, y)$. These two constraints reduce the rank of the C^0 boundary trace of $\mathcal{P}_n$ to $4n - 2$ inside the $4n$-dimensional boundary data space.

Conversely, elements of $\mathcal{Q}_{n,n}$ vanishing on $\partial[0, 1]^2$ factor as $x(1-x)\, y(1-y)\, g$ with g of bi-degree $(n-2, n-2)$. The inherited diagonal constraints $\deg(\tau g) \leq n - 4$ and $\deg(\sigma g) \leq n - 4$ impose $2n - 1$ independent conditions (by the argument of Proposition 3.4), leaving $(n-1)^2 - (2n-1) = (n-2)^2 - 2$ such elements for $n \geq 4$. The boundary trace of $\mathcal{Q}_{n,n}$ therefore has rank $n^2 + 2 - \left((n-2)^2 - 2\right) = 4n$, and the gap with $\mathcal{P}_n$ is 2—independently of r and of any diagonal condition.

For $n = 3$ the same factoring gives g of bi-degree $(1, 1)$, but the diagonal constraints force $g = 0$, so the boundary trace is injective on $\mathcal{Q}_{3,3}$ (rank 11). The gap with $\mathcal{P}_n$ (rank 10) is 1, carried by the complement element $v \in \mathcal{V}_{3,3}$ (see (6)).

Remark 5.3. For $k \leq n - 1$ the constraint $k \leq n - 2r - 1$ is met at $r = 0$ and matching succeeds.

5.1. The L^2-projection and L^∞ optimality for $n = k = 3$

Although matching fails at $k = n = 3$, the L^2-orthogonal projection onto $\mathcal{P}_{3,3}$ is still well defined and turns out to enjoy a remarkable additional optimality.

Every $q \in \mathcal{Q}_{3,3}$ decomposes orthogonally as $q = p + v$ with $p \in \mathcal{P}_{3,3}$ and $v \in \mathcal{V}_{3,3}$. Since $\dim \mathcal{V}_{3,3} = 1$, the remainder is a scalar multiple of a single function: $\mathcal{V}_{3,3}$ is spanned by $L_3(x)\, L_1(y) - L_1(x)\, L_3(y)$, so $v = c_1\left(L_3(x)\, L_1(y) - L_1(x)\, L_3(y)\right)$ for some $c_1 \in \mathbb{R}$ depending on q. Substituting the explicit shifted Legendre polynomials and factoring gives the closed form

$$v(x, y) \;=\; 10\, c_1\, (x - y)(2x - 1)(2y - 1)(x + y - 1). \tag{6}$$

Note that v is anti-symmetric ($v(y, x) = -v(x, y)$) and vanishes on both diagonals $\{y = x\}$ and $\{x + y = 1\}$, as well as on the midlines $\{x = \frac{1}{2}\}$ and $\{y = \frac{1}{2}\}$.

Proposition 5.4 ($L^2 = L^\infty$ optimality). *Let p denote the L^2-orthogonal projection of $q \in \mathcal{Q}_{3,3}$ onto $\mathcal{P}_{3,3}$. Then p is simultaneously the best L^∞ approximation on $[0, 1]^2$: $\|q - p\|_{L^\infty} \leq \|q - \tilde{p}\|_{L^\infty}$ for all $\tilde{p} \in \mathcal{P}_{3,3}$.*

Proof. The residual $v = q - p$ is of the form (6). We may assume $c_1 \neq 0$. By the Kolmogorov criterion [10], it suffices to show that for every $h \in \mathcal{P}_{3,3}$ there is a point in the extremal set

$$E = \{(x, y) \in [0, 1]^2 \;:\; |v(x, y)| = \|v\|_\infty\}$$

where $\mathrm{sign}(v) \cdot h \geq 0$.

Let us introduce centered coordinates $s = 2x - 1$, $t = 2y - 1$ on $[-1, 1]^2$, so that $v = 10\, c_1 \cdot \frac{1}{4}\, st(s^2 - t^2) = \frac{5}{2}\, c_1\, st(s^2 - t^2)$. The interior gradient vanishes only at the origin. On the edge $s = 1$, $v \propto t(1 - t^2)$ attains its maximum at $t = \pm 1/\sqrt{3}$; by symmetry the same holds on all four edges. This gives exactly eight extremal points, partitioned by sign into

$$\text{Group A } (v = -M): \quad (\pm 1, \mp \tfrac{1}{\sqrt{3}}),\ (\pm \tfrac{1}{\sqrt{3}}, \pm 1),$$

$$\text{Group B } (v = +M): \quad (\pm 1, \pm \tfrac{1}{\sqrt{3}}),\ (\mp \tfrac{1}{\sqrt{3}}, \pm 1),$$

where $M = \frac{5}{3\sqrt{3}}\, |c_1|$. Group B is the image of Group A under the reflection $(s, t) \mapsto (t, s)$, which sends $v \mapsto -v$.

Suppose some $h \in \mathcal{P}_{3,3}$ violates the criterion, i.e., $h > 0$ on Group A and $h < 0$ on Group B. The anti-symmetrisation $h_a(s, t) = h(s, t) - h(t, s) \in \mathcal{P}_{3,3}$ then satisfies $h_a > 0$ on Group A. The anti-symmetric subspace of $\mathcal{P}_{3,3}$ (i.e., the subspace of $\mathcal{P}_{3,3}$ anti-symmetric under $(s, t) \mapsto (t, s)$ in the (s, t) coordinates) has basis $\{t - s,\ s^2 - t^2,\ s^3 - t^3,\ s^2 t - st^2\}$. Let us write $h_a = \alpha(t - s) + \beta(s^2 - t^2) + \gamma(s^3 - t^3) + \delta(s^2 t - st^2)$. Note that $s^2 - t^2$ is even under $(s, t) \mapsto (-s, -t)$ while the remaining three basis elements are odd. Summing h_a over the pairs $(1, -1/\sqrt{3})$ and $(-1, 1/\sqrt{3})$ (both in Group A) kills the odd terms and gives $h_a + h_a = 4\beta/3 > 0$, hence $\beta > 0$. Summing over $(1/\sqrt{3}, 1)$ and $(-1/\sqrt{3}, -1)$ gives $-4\beta/3 > 0$, hence $\beta < 0$, which is a contradiction. □

Remark 5.5. The $L^2 = L^\infty$ property does not extend to $n = k = 4$. In centered coordinates $(s, t) \in [-1, 1]^2$, the complement $\mathcal{V}_{4,4}$ contains the element $v_0(s, t) = P_4(s)\, P_2(t) - P_2(s)\, P_4(t)$ (standard Legendre polynomials), whose supremum $\|v_0\|_\infty = 7/8$ is attained at exactly four boundary points: $(\pm 1, 0)$ and $(0, \pm 1)$. The polynomial $h(s, t) = s^2 - t^2 \in \mathcal{P}_{4,4}$ satisfies $\mathrm{sign}(v_0) \cdot h < 0$ at all four points, violating the Kolmogorov criterion. Concretely, $\|v_0 + \varepsilon\, h\|_\infty = 7/8 - \varepsilon + O(\varepsilon^2) < \|v_0\|_\infty$ for small $\varepsilon > 0$, so the L^2-remainder is not L^∞-optimal. The structural reason is that $\dim \mathcal{V}_{4,4} = 3$: individual complement elements have too few extremal points to block all perturbation directions from $\mathcal{P}_{4,4}$.

6. Explicit matching formulas

We now make the matching theorem concrete. We first quantify the solution freedom (Section 6.1), then tabulate the dimensions for small r (Section 6.2), derive general matching formulas valid for all r (Section 6.3), and finally select the L^2-optimal matching polynomial when the solution is non-unique (Section 6.4).

6.1. Matching freedom

When matching is achievable ($k \geq 3$, $n \geq 2r + 4$, $k \leq n - 2r - 1$), the gap Eq. (5) determines the dimension of the solution freedom.

Corollary 6.1. *For $k \geq 3$ with $n \geq 2r + 4$ and $k \leq n - 2r - 1$,*

$$\dim \mathcal{K}_r^P \;=\; \mathfrak{p}^2 - \frac{(k-1)(k-2)}{2}, \qquad \mathfrak{p} = n - 2r - 3.$$

In particular, $\dim \mathcal{K}_r^P$ *depends on n and r only through $\mathfrak{p}$.*

This means that increasing n by 2 while increasing r by 1 preserves the matching freedom exactly. The structural reason is that the frame polynomial F_r gains bi-degree $(2, 2)$ per unit increase in r, which costs exactly 2 units of polynomial degree.

The first non-trivial cases are $\mathfrak{p} = 1$ ($n = 2r + 4$) and $\mathfrak{p} = 2$ ($n = 2r + 5$). Table 1 shows the matching status for these values.

Table 1
Matching status for the first non-trivial effective degrees. The matching status depends only on $\mathfrak{p}$ and k, not on r; the degree is $n = \mathfrak{p} + 2r + 3$.

$\mathfrak{p}$	k	$\dim \mathcal{V}_{n,k}$	$\dim \mathcal{K}_r^P$	matching
1	1–2	0	—	$p = q$ (trivial)
1	3	1	0	unique
1	≥ 4	≥ 3	—	impossible
2	1–2	0	—	$p = q$ (trivial)
2	3	1	3	3-dim freedom
2	4	3	1	1-dim freedom
2	≥ 5	≥ 6	—	impossible

6.2. Dimension tables for $r = 0, 1, 2$

We record the dimensions and matching status for the first three values of r in Tables 2–4. For each r, the threshold degree is $n = 2r + 4$ ($\mathfrak{p} = 1$); the next degree $n = 2r + 5$ ($\mathfrak{p} = 2$) is the first with non-unique matching.

Table 2
C^0 boundary matching ($r = 0$). $F_0 = x(1-x)\,y(1-y)\,(y-x)(x+y-1)$ (total degree 6).

n	k	$\dim \mathcal{P}_{n,k}$	$\dim \mathcal{Q}_{n,k}$	$\dim \mathcal{V}_{n,k}$	status
4	1–2	—	—	0	$p = q$
4	3	19	20	1	unique
4	4	15	18	3	impossible
5	1–2	—	—	0	$p = q$
5	3	30	31	1	3-dim freedom
5	4	26	29	3	1-dim freedom
5	5	21	27	6	impossible

Table 3
C^1 boundary matching ($r = 1$). $F_1 = [x(1-x)]^2[y(1-y)]^2(y-x)(x+y-1)$ (total degree 10).

n	k	$\dim \mathcal{P}_{n,k}$	$\dim \mathcal{Q}_{n,k}$	$\dim \mathcal{V}_{n,k}$	status
6	1–2	—	—	0	$p = q$
6	3	43	44	1	unique
6	4–6	—	—	≥ 3	impossible
7	1–2	—	—	0	$p = q$
7	3	58	59	1	3-dim freedom
7	4	54	57	3	1-dim freedom
7	5–7	—	—	≥ 6	impossible

Table 4
C^2 boundary matching ($r = 2$). $F_2 = [x(1-x)]^3[y(1-y)]^3(y-x)(x+y-1)$ (total degree 14).

n	k	$\dim \mathcal{P}_{n,k}$	$\dim \mathcal{Q}_{n,k}$	$\dim \mathcal{V}_{n,k}$	status
8	1–2	—	—	0	$p = q$
8	3	75	76	1	unique
8	4–8	—	—	≥ 3	impossible
9	1–2	—	—	0	$p = q$
9	3	94	95	1	3-dim freedom
9	4	90	93	3	1-dim freedom
9	5–9	—	—	≥ 6	impossible

6.3. General matching formulas

The tables above confirm a uniform pattern. By Lemma 4.3, the matching correction has the form $f = q - p = F_r \cdot u$ for a low-degree polynomial u, where F_r is the frame polynomial (2). Writing $q = \sum a_{i,j}\, x^i y^j$, the coefficients $a_{i,j}$ with $i + j > N$ lie above the total-degree threshold of $\mathcal{P}_{n,k}$; these excess coefficients determine the polynomial u through the spectral constraints of Lemma 3.3. We give the resulting formulas for the first non-trivial values of $\mathfrak{p}$ and k.

1. *Unique matching* ($\mathfrak{p} = 1$, i.e., $n = 2r + 4$, $k = 3$). Here $\mathcal{K}_r = \operatorname{span}\{F_r\}$ and $\mathcal{K}_r^P = \{0\}$. The spectral constraints of Lemma 3.3 at levels $2n$, $2n-1$, and $2n-2$ force $a_{n,n} = a_{n-1,n} = a_{n,n-1} = a_{n-1,n-1} = 0$ and $a_{n,n-2} = -a_{n-2,n}$. Thus the sole excess coefficient is $c = a_{n-2,\,n}$. Since $\deg F_r = 4r + 6 = 2n - 2$ and the leading monomial of the boundary factor $[x(1-x)]^{r+1}[y(1-y)]^{r+1}$ is $x^{2r+2}y^{2r+2}$, multiplying by the leading form $y^2 - x^2$ of $(y-x)(x+y-1)$ gives $x^{n-2}y^n - x^n y^{n-2}$ as the leading form of F_r. The polynomial
$$p = q - c \cdot F_r \tag{7}$$
has total degree at most $N = 2n - 3$ and hence lies in $\mathcal{P}_{n,k}$.
2. *Three-dimensional freedom* ($\mathfrak{p} = 2$, $k = 3$, i.e., $n = 2r + 5$). Here $\mathcal{K}_r^P = \operatorname{span}\{F_r,\ F_r\,x,\ F_r\,y\}$. The single excess parameter is $c = a_{n-2,\,n}$, and
$$p = q - c \cdot F_r\,xy - F_r(c_0 + c_1\,x + c_2\,y), \qquad c_0, c_1, c_2 \text{ free.} \tag{8}$$
3. *One-dimensional freedom* ($\mathfrak{p} = 2$, $k = 4$, i.e., $n = 2r + 5$). Here $\mathcal{K}_r^P = \operatorname{span}\{F_r\}$ and $\dim \mathcal{V}_{n,4} = 3$. The spectral constraints at levels $2n$ through $2n-2$ are the same as in case (i): $a_{n,n} = a_{n-1,n} = a_{n,n-1} = a_{n-1,n-1} = 0$ and $a_{n,n-2} = -a_{n-2,n}$. Setting $c = a_{n-2,n}$, the additional constraints at level $2n-3$ are
$$a_{n-3,n} + a_{n-1,n-2} = c, \qquad a_{n-2,n-1} + a_{n,n-3} = -c. \tag{9}$$
The correction $f = F_r \cdot (d_0 + d_1\,x + d_2\,y + d_3\,xy)$ has
$$d_3 = c, \quad d_2 = a_{n-3,n} + (r+1)c, \quad d_1 = a_{n-2,n-1} + (r+2)c, \quad d_0 \text{ free.} \tag{10}$$

6.4. L^2-optimal parameter selection

When the matching polynomial is not unique, the set of valid matching polynomials forms an affine coset $p_0 + \mathcal{K}_r^P$ (any particular solution p_0 plus the kernel restricted to $\mathcal{P}_{n,k}$). A natural selection criterion is to minimise $\|q-p\|^2_{L^2([0,1]^2)}$ over this coset. Since the correction $f = q - p$ lies in $\mathcal{K}_r = F_r \cdot \mathcal{Q}_{n-2r-4}$, the minimisation reduces to an orthogonality condition:

$$q - p \;\perp\; \mathcal{K}_r^P \qquad \text{(orthogonality in } L^2([0,1]^2)\text{)}. \tag{11}$$

Remark 6.2. One might hope for the stronger condition $q - p \perp \mathcal{P}_{n,k}$, which would make p the L^2-projection of q onto $\mathcal{P}_{n,k}$. This holds if and only if $\mathcal{V}_{n,k} \subseteq \mathcal{K}_r$. However, this fails already in the simplest nontrivial case: for $n = 4$, $k = 3$, both $\mathcal{K}_0$ and $\mathcal{V}_{4,3}$ are one-dimensional, but $\mathcal{K}_0 \neq \mathcal{V}_{4,3}$ since $F_0 \notin (\mathcal{P}_{4,3})^\perp$ ($\langle F_0, xy^3\rangle_{L^2} = 1/16800 \neq 0$). Condition (11) is the correct (weaker) optimality within the matching constraint.

The optimality conditions turn out to be independent of the boundary smoothness order r. To see why, note that every $f \in \mathcal{K}_r$ has the form $f = F_r \cdot u$. The inner products

$$\langle F_r u_1, F_r u_2\rangle_{L^2} = \int_{[0,1]^2} F_r^2\, u_1\, u_2\, \mathrm{d}x\, \mathrm{d}y$$

are governed by the probability measure $\mathrm{d}\mu = F_r^2\, \mathrm{d}x\, \mathrm{d}y/\|F_r\|^2$; we write $\mathbb{E}_\mu[g] = \int_{[0,1]^2} g\, \mathrm{d}\mu$ for expectation with respect to μ. Since

$$F_r^2 = [x(1-x)]^{2r+2}[y(1-y)]^{2r+2}(y-x)^2(x+y-1)^2,$$

this measure is symmetric under $(x,y) \mapsto (1-x, y)$, $(x,y) \mapsto (x, 1-y)$, $(x,y) \mapsto (1-x, 1-y)$, and $(x,y) \mapsto (y,x)$. In particular,

$$\mathbb{E}_\mu[x] = \tfrac{1}{2}, \qquad \mathbb{E}_\mu[y] = \tfrac{1}{2}, \qquad \mathbb{E}_\mu[xy] = \tfrac{1}{4},$$

and every centered moment $\mathbb{E}_\mu[(x-\frac{1}{2})^a(y-\frac{1}{2})^b]$ with $a+b$ odd vanishes. These symmetries are independent of r, and as we verify below, they alone determine the optimal free parameters in each case. Consequently, the optimal parameters are independent of r.

We now apply this to the two non-unique cases from Section 6.3.

1. *Three-dimensional freedom ($k = 3$).* From (8), the correction is $f = q - p = c\, F_r \cdot u$ with $u = xy + (c_0/c) + (c_1/c)\, x + (c_2/c)\, y$. The optimality condition $\langle f, F_r g_j\rangle = 0$ for the basis $g_j \in \{1, x, y\}$ of the free-parameter space reduces to

   $$\mathbb{E}_\mu[u\, g_j] = 0, \qquad j = 1,2,3.$$

   We claim that $u^*(x,y) = (x-\frac{1}{2})(y-\frac{1}{2})$ solves this system. Indeed:
   - $\mathbb{E}_\mu[u^* \cdot 1] = \mathbb{E}_\mu[(x-\frac{1}{2})(y-\frac{1}{2})] = 0$, since the integrand is odd under $(x,y) \mapsto (1-x,\ y)$.
   - $\mathbb{E}_\mu[u^* \cdot x] = \mathbb{E}_\mu[(x-\frac{1}{2})^2(y-\frac{1}{2})] + \frac{1}{2}\,\mathbb{E}_\mu[u^*] = 0$, since the first term is odd under $(x,y) \mapsto (x,\ 1-y)$ and the second vanishes by the previous line.
   - $\mathbb{E}_\mu[u^* \cdot y] = 0$ by symmetry.

   Therefore, for all r,

   $$p^* \;=\; q \;-\; a_{n-2,n} \cdot F_r\left(x-\tfrac{1}{2}\right)\left(y-\tfrac{1}{2}\right). \tag{12}$$

2. *One-dimensional freedom ($k = 4$).* Here $\mathcal{K}_r^P = \mathrm{span}\{F_r\}$, and the condition $\langle f, F_r\rangle = 0$ yields $d_0 = -\frac{1}{2}\, d_1 - \frac{1}{2}\, d_2 - \frac{1}{4}\, d_3$, i.e.,

   $$d_0^* \;=\; -\tfrac{1}{2}\, a_{n-2,n-1} \;-\; \tfrac{1}{2}\, a_{n-3,n} \;-\; \tfrac{4r+7}{4}\, a_{n-2,n}\,. \tag{13}$$

   Equivalently, setting $s = x - \frac{1}{2}$, $t = y - \frac{1}{2}$, the optimal u^* has zero constant term in centered coordinates: $u^*\left(\frac{1}{2}+s,\ \frac{1}{2}+t\right) = (d_1 + \frac{d_3}{2})\, s + (d_2 + \frac{d_3}{2})\, t + d_3\, st$. (The r-dependence in (13) enters through d_1, d_2, d_3 from (10); the optimality condition $d_0 = -\frac{1}{2}d_1 - \frac{1}{2}d_2 - \frac{1}{4}d_3$ itself is independent of r.)

We now estimate the squared matching error $\mathcal{E} = \|q-p\|^2_{L^2([0,1]^2)}$. Since the correction $f = q - p$ lies in $\mathcal{K}_r$, we have $f = F_r \cdot u$ and $\mathcal{E}$ reduces to integrals of F_r^2 over $[0,1]^2$. In centered coordinates $s = x - \frac{1}{2}$, $t = y - \frac{1}{2}$ (note the different scaling from Section 5.1) the density separates as $F_r^2 = (\frac{1}{4} - s^2)^{2r+2}(\frac{1}{4} - t^2)^{2r+2}(t^2 - s^2)^2$, reducing all required integrals to products of Beta functions. Writing $M = 2r+3$ and $C_M = \binom{2M}{M}$,

$$\|F_r\|^2 \;=\; \frac{2}{C_M^2\, M\,(2M+1)^2\,(2M+3)}\,. \tag{14}$$

The closed-form error for each case is as follows.

1. *Unique matching* ($\mathfrak{p} = 1$, $k = 3$): here $f = c\, F_r$ and

   $$\mathcal{E} \;=\; c^2\, \|F_r\|^2\,. \tag{15}$$

2. *L^2-optimal,* $k = 3$ ($\mathfrak{p} = 2$): here $f = c\,F_r\,(x-\frac{1}{2})(y-\frac{1}{2})$ and

$$\mathcal{E}^* \;=\; \frac{3\,c^2\,\|F_r\|^2}{16\,(2M+3)(2M+5)}\,. \tag{16}$$

3. *L^2-optimal,* $k = 4$ ($\mathfrak{p} = 2$): in centered coordinates, write $\alpha = d_1 + \frac{d_3}{2}$, $\beta = d_2 + \frac{d_3}{2}$, $\gamma = d_3$ for the linear and bilinear parts of u^*. The reflective symmetries of F_r^2 eliminate all cross-terms ($\int F_r^2\, s\,t = \int F_r^2\, s^2 t = \int F_r^2\, s\,t^2 = 0$), giving

$$\mathcal{E}^* \;=\; \frac{3\,(\alpha^2+\beta^2)\,\|F_r\|^2}{4\,(2M+5)} \;+\; \frac{3\,\gamma^2\,\|F_r\|^2}{16\,(2M+3)(2M+5)}\,. \tag{17}$$

7. Conclusion

We have introduced the family of diagonally-reduced tensor-product spaces $\mathcal{Q}_{n,k}$ and established a complete picture of their structure. The dimension formulas $\dim\mathcal{Q}_{n,k} = (n+1)^2 - 2k + 1$ and $\dim\mathcal{P}_{n,k} = (n+1)^2 - \frac{1}{2}k(k+1)$ yield the orthogonal complement dimension $\dim\mathcal{V}_{n,k} = \frac{1}{2}(k-1)(k-2)$, which vanishes for $k \le 2$ and grows quadratically thereafter. At the maximal reduction $k = n$, the formula gives $\dim\mathcal{Q}_{n,n} = n^2 + 2$, recovering the Bézier-Smart count [6].

The boundary–diagonal matching theorem (Theorem 4.5) provides a sharp threshold: for $k \ge 3$, a total-degree polynomial that reproduces the C^r boundary data and C^0 diagonal data of any $q \in \mathcal{Q}_{n,k}$ exists if and only if $n \ge 2r + 4$ and $k \le n - 2r - 1$. In particular, C^0 matching succeeds for all $k \le n - 1$ but always fails at $k = n$. The failure gap at $k = n$ is small (just 1 for $n = 3$ and 2 for $n \ge 4$) and for $n = k = 3$ the L^2-orthogonal projection is simultaneously L^∞-optimal (Proposition 5.4). When matching is achievable but non-unique, the explicit L^2-optimal formulas of Section 6 select a canonical matching polynomial whose error is expressed in closed form.

Several directions remain open. The matching problem studied here governs inter-element compatibility when tensor-product patches are subdivided along their diagonals into triangular sub-cells; a full spline-theoretic treatment of such mixed quadrilateral-triangular meshes (including global smoothness conditions across cell boundaries) is a natural next step. On the algebraic side, extending the analysis to higher-order diagonal smoothness (C^s matching on the diagonals with $s \ge 1$) and to multivariate tensor-product spaces in dimension $d \ge 3$ would broaden the scope of the theory.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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