



Marshall–Olkin Copulas Revisited

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Abstract. Almost 70-year-old Marshall–Olkin copulas, then wider Marshall copulas, and finally even wider shock model (SM) copulas constitute a substantial part of nowadays copula theory due to numerous applications. Recently, Christian Genest with some coauthors introduced a new stochastic model for a special subclass of SM copulas which gives not only a new angle on these copulas but also widens the range of applications. In this paper, we extend this type of stochastic model to all known subclasses of SM copulas. We also introduce a novel class of SM copulas and extend the new stochastic model to this class as well.

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1. Introduction

Copula is simply a joint distribution with uniform margins, but when we insert arbitrary univariate distributions as margins into it, we can get any bivariate distribution. This seminal 1959 result of Sklar [24] has made them the most important tool of dependence modeling [2, 10, 20]. Most of what we do in this paper has an extension to its multivariate version, but we will restrict ourselves to the bivariate case to make intuitively clearer the stochastic model that we are introducing.

If one wants to build a stochastic model for describing the dependence among two (or more) lifetimes, i.e., positive random variables, one comes to shock models. In engineering applications, joint models of lifetimes may serve to estimate the expected lifetime of a system composed by several components. In a related situation like portfolio credit risk, instead, the lifetimes may have the interpretation of time-to-default of firms, or generally financial entities, while a stochastic model may estimate the price/risk of a related derivative contract (e.g., CDO). In both cases, it is of interest to estimate the probability of the occurrence of a joint default. This approach is a modern practitioners' view on what was started analytically in the 1967 key paper of

Marshall and Olkin [18] and was extended later by Marshall in Ref. [17]. (Perhaps we should be aware that in the original paper [18], the word copula was not even mentioned.) An important portion of copula theory (here is some recent publications [3–5, 9, 16, 19]) is based on these two papers: The first one allows only exponential shocks, i.e., lifetimes, while the second one allows for arbitrary distributions. In 2016, Omladič and Ružić [21] propose to study Marshall type of models which result in slightly different kind of copulas they call maxmin copulas. That paper started a series of papers that introduced some classes of copulas obtained in similar ways, including RMM (reflected maxmin) copulas, proposed by Košir and Omladič in 2020 [14], cf. also [6, 11–13, 15]. Many authors are calling copulas of all these types *Marshall–Olkin copulas*, but most of them keep this term only for the type introduced in Ref. [18], i.e., the ones modeled by exponential shocks, and call the general class *Marshall copulas*. We will call the whole class *shock model copulas* and add the term *exponential*, respectively, *general* if our shocks will be distributed exponentially resp. according to more general distributions.

The main goal of this paper is twofold. First, we follow a new stochastic model that gives rise to the usual shock model copulas and has some advantages over the original one. This kind of model was first introduced by Genest et al. in 2018 [8] and later by Bentoumi et al. [1] and by El Ktaibi et al. [7]. They are developing the model only for very special classes of exponential models. We show that this approach extends to all known cases of shock model copulas. Moreover, we introduce a novel class we call the *survival RMM copula (SMM for short)*. The proposal of this class is the second goal of this paper. Of course, we extend the new stochastic model to this class as well.

The paper is organized as follows: Sect. 2 provides preliminaries on the Marshall, Maxmin, and RMM copulas. Section 3 exhibits the new stochastic model for the Marshall case. Section 4 presents the new model for the RMM case and Sect. 5 introduces the new class of SMM copulas. Finally, Sect. 6 gives the new stochastic model for the SMM copulas.

2. Preliminaries on Marshall, Maxmin, and RMM Copulas

A function $C : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is called a *copula* if conditions (a) and (b) below are fulfilled.

- (a) C is grounded and 1 is the neutral element of C ;
- (b) C -volume of every rectangle is positive.

Here, condition (a) means that $C(x, 0) = C(0, x) = 0$ for all $x \in [0, 1]$ and $C(x, 1) = C(1, x) = x$ for all $x \in [0, 1]$. Condition (b) means that

$$(P) \quad V_C([x_1, x_2] \times [y_1, y_2]) = C(x_2, y_2) - C(x_1, y_2) - C(x_2, y_1) + C(x_1, y_1) \geq 0$$

for all $x_1, x_2, y_1, y_2 \in [0, 1]$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$.

A set of the form $[x_1, x_2] \times [y_1, y_2] \subseteq [0, 1] \times [0, 1]$, $x_1 \leq x_2$, $y_1 \leq y_2$, is called a *rectangle* and condition (P) may be seen as the definition of its C -volume denoted by V_C , together with the requirement that it is positive.

The copulas presented here are sometimes called bivariate copulas as opposed to more general multivariate copulas, where the bivariate domain $(x, y) \in [0, 1]^2$ is replaced by $(x_1, x_2, \dots, x_n) \in [0, 1]^n$ for any integer $n \geq 2$. However, we will not study copulas in this generality. It turns out that for any copula C , we have

$$W(x, y) \leq C(x, y) \leq M(x, y),$$

where

$$W(x, y) = \max\{0, x + y - 1\} \quad \text{respectively} \quad M(x, y) = \min\{x, y\}$$

are copulas called, respectively, Fréchet–Hoeffding lower bound and upper bound of all bivariate copulas.

For a bivariate distribution $F(x, y)$ of a random vector (X, Y) , it is easy to find the distributions of its components. For simplicity, we assume that the domains of X and Y are contained in the interval $[0, 1]$. Then

$$F_X(x) = F(x, 1) \quad \text{and} \quad F_Y(y) = F(1, y)$$

are the respective distributions of the components. We call them the *marginal distributions*. (For instance, copulas may be seen as bivariate distributions whose marginal distributions are both uniform on the interval $[0, 1]$.) In his seminal 1959 paper [24], A. Sklar defined copulas and proved the following theorem. (Actually, he proved it in the multivariate setting.)

Sklar Theorem. *For any bivariate distribution $F(x, y)$, there exists a copula $C(x, y)$ such that*

$$F(x, y) = C(F_X(x), F_Y(y))$$

where F_X, F_Y are the corresponding marginal distributions. Conversely, if $C(x, y)$ is any copula and F_X, F_Y are any univariate distributions, then

$$H(x, y) = C(F_X(x), F_Y(y))$$

is a bivariate distribution whose copula is $C(x, y)$.

We refer to Refs. [2, 20] for further details on general theory on copulas.

Marshall [17] showed that shock models governed by

$$U = \max\{X, Z\}, \quad V = \max\{Y, Z\},$$

where X, Y , and Z are independent random variables, are described by copulas of the form

$$C(u, v) = \min\{ug(v), f(u)v\},$$

where generating functions f and g satisfy the following conditions (here and in what follows we are using notation \mathbb{I} for the closed interval $[0, 1]$)

- (M1) $f(0) = 0, f(1) = 1, g(0) = 0, g(1) = 1,$
- (M2) f is nondecreasing on \mathbb{I}, g is nondecreasing on $\mathbb{I},$
- (M3) function $f^*(u) = \frac{f(u)}{u}$ is nonincreasing on $(0, 1],$ function $g^*(u) = \frac{g(u)}{u}$ is nonincreasing on $(0, 1].$

Functions f and g that satisfy conditions (M1)–(M3) have certain additional properties that we collect in the following Lemma. (See Property (Fc) and Lemma 3 in [21].)

Lemma 1. *Assume that properties (M1)–(M3) hold for functions f and g . Then f, f^*, g and g^* are continuous on $(0, 1]$. For each $x \in I$, we have $f(x) \geq x$ and $g(x) \geq x$.*

The joint distribution function H of (U, V) is then given by

$$H(x, y) = C(F_U(x), F_V(y)),$$

where F_U and F_V are the marginal distribution functions of (U, V) , i.e., distributions of U and V , respectively, given by

$$F_U(x) = F_X(x)F_Z(x) \quad \text{and} \quad F_V(x) = F_Y(x)F_Z(x).$$

Omladič and Ružić [21] introduced a new class of copulas modeling shocks given by

$$U = \max\{X, Z\}, \quad V = \min\{Y, Z\}.$$

where X, Y , and Z are independent random variables. A copula for (U, V) is then of the form

$$C(u, v) = \min\{u, \phi(u)(v - \psi(v)) + u\psi(v)\}, \tag{2.1}$$

where generating function ϕ satisfies conditions (M1)–(M3) with $\phi = f$, while ψ has the following properties

- (F1) $\psi(0) = 0, \psi(1) = 1$,
- (F2) ψ is nondecreasing on \mathbb{I} ,
- (F3) function $\psi_*(v) = \frac{1 - \psi(v)}{v - \psi(v)}$ is nonincreasing on $[0, 1)$. Here, $\psi_*(v) = \infty$ if $\psi(v) = v$.

Functions of the form (2.1) are called *maxmin copulas*.

Košir and Omladič [14] studied reflected version of the maxmin copulas that are called *reflected maxmin copulas* (RMM for short). Their generators f and g satisfy the following conditions. (Here, we introduce additionally functions $f^*(u) = \frac{f(u)}{u}, g^*(u) = \frac{g(u)}{u}, \hat{f}(u) = f(u) + u$, and $\hat{g}(u) = g(u) + u$ in order to make these conditions easier to state and understand.)

- (G1) $f(0) = g(0) = 0, f(1) = g(1) = 0$, and $f^*(0) = g^*(0) = 0$;
- (G2) functions \hat{f} and \hat{g} are nondecreasing on \mathbb{I} ;
- (G3) functions f^* and g^* are nonincreasing on \mathbb{I} .

An RMM copula is then of the form

$$C(u, v) = \max\{0, uv - f(u)g(v)\}.$$

Functions that satisfy conditions (F1)–(F3) or (G1)–(G3) have certain additional properties that we collect in the following Lemma. (See Lemma 3 in Ref. [21] and Lemma 1 of Ref. [15].)

Lemma 2. *Assume that properties (F1)–(F3) hold for functions φ and ψ . Then φ and φ^* are continuous on $(0, 1]$ and functions ψ and ψ^* are continuous on $[0, 1)$.*

If properties (G1)–(G3) hold for functions f and g , then f, f^, \hat{f}, g, g^* and \hat{g} are continuous on \mathbb{I} and f^* and g^* are absolutely continuous on $[\varepsilon, 1]$ for any $\varepsilon > 0$.*

3. Shock Model for Marshall Copulas

We first consider a slight variation of Marshall model [17, Proposition 3.2] that still leads to the same family of copulas. We consider four random variables $X, Y, Z_1,$ and Z_2 representing shocks. The two triples X, Y, Z_1 and X, Y, Z_2 are triples of jointly independent random variables, while Z_1 and Z_2 form a comonotonic pair, so that their dependence is modeled by the Fréchet–Hoeffding upper bound $M(u, v) = \min\{u, v\}$. Denote by F_X and F_Y the distribution functions of X and Y , respectively, and by G_i the distribution functions of Z_i for $i = 1, 2$. We construct two random variables

$$U = \max\{X, Z_1\}, \quad V = \max\{Y, Z_2\},$$

whose distributions are clearly $F_U(x) = F_X(x)G_1(x)$ and $F_V(y) = F_Y(y)G_2(y)$. Before giving the copula connecting U and V , we introduce some further notation.

Let F be a distribution function. For $u \in \mathbb{I}$, we have

$$F^{-1}(u) = \inf\{x \in \mathbb{R} : F(x) \geq u\}.$$

We assume that infimum of an empty set is equal to $+\infty$ and infimum of \mathbb{R} is equal to $-\infty$. Note that F^{-1} is nondecreasing. We have $F(F^{-1}(u)) \geq u$ for all $u \in \mathbb{I}$ and $F^{-1}(F(x)) \leq x$ for all $x \in \mathbb{R}$. Following the notation of Ref. [21], we denote further by $f(x-)$ the left limit of function f if it exists. (Note that all our functions will be monotone, so that our left limits will always exist at x .) For $u \notin \text{im}F \cup \{0, 1\}$, we write $\bar{u} = F(F^{-1}(u))$ and $\underline{u} = F(F^{-1}(u)-)$. Then $\bar{u} \in \text{im}F$ and either $\underline{u} \in \text{im}F$ or $\underline{u} \notin \text{im}F$ and $(\underline{u} - \varepsilon, \underline{u}) \cap \text{im}F \neq \emptyset$ for every small enough $\varepsilon > 0$.

Let us introduce generating functions for our model by

$$\varphi(u) = \begin{cases} 0, & \text{if } u = 0 \\ 1, & \text{if } u = 1 \\ F_X(F_U^{-1}(u)), & \text{if } u \in \text{im}F_U \setminus \{0, 1\} \\ \frac{\varphi(\bar{u}) - \varphi(\underline{u}-)}{\bar{u} - \underline{u}}(u - \underline{u}) + \varphi(\underline{u}-) & \text{otherwise;} \end{cases}$$

and

$$\psi(v) = \begin{cases} 0, & \text{if } v = 0 \\ 1, & \text{if } v = 1 \\ F_Y(F_V^{-1}(v)), & \text{if } v \in \text{im}F_V \setminus \{0, 1\} \\ \frac{\psi(\bar{v}) - \psi(\underline{v}-)}{\bar{v} - \underline{v}}(v - \underline{v}) + \psi(\underline{v}-) & \text{otherwise.} \end{cases}$$

A careful reader will have noticed that the first three lines of each definition are the obvious natural choice while the fourth one is a linear interpolation on the intervals left undefined. Let us give the proof of the following proposition for the sake of completeness.

Proposition 3. *The copula of the pair (U, V) is a Marshall copula*

$$C(u, v) = \min\{u\psi(v), v\varphi(u)\}.$$

Proof. Denote by H the joint distribution function of the random vector (U, V) and compute

$$\begin{aligned} H(x, y) &= \mathbb{P}[U \leq x, V \leq y] = \mathbb{P}[X \leq x, Z_1 \leq x, Y \leq y, Z_2 \leq y] \\ &= F_X(x)F_Y(y)\mathbb{P}[Z_1 \leq x, Z_2 \leq y] = F_X(x)F_Y(y) \min\{G_1(x), G_2(y)\} \\ &= \min\{F_U(x)\psi(F_V(y)), F_V(y)\varphi(F_U(x))\} \\ &= C(F_U(x), F_V(y)), \end{aligned}$$

where C is the Marshall copula with generating functions φ and ψ . □

We now state and prove two results that exhibit a modified version of Marshall’s Theorem [17, Proposition 3.2]. Our results are based on Lemma 5.6 and Theorem 5.9 of Ref. [22].

Lemma 4. *Suppose that U and V are two random variables with distribution functions F_U and F_V , and with the joint distribution function H . Then the following two statements are equivalent.*

- (a) *There are two triples of independent random variables X, Y, Z_1 and X, Y, Z_2 , where Z_1 and Z_2 are comonotonic, such that*

$$U = \max\{X, Z_1\} \quad \text{and} \quad V = \max\{Y, Z_2\}.$$

- (b) *There are distribution functions F_X, F_Y, G_1 , and G_2 such that*

$$\begin{aligned} H(x, y) &= F_X(x)F_Y(y) \min\{G_1(x), G_2(y)\} \\ &= F_X(x)F_Y(y)M(G_1(x), G_2(y)) \end{aligned}$$

If any of (a) or (b) holds, then we also have

$$\begin{aligned} F_U(x) &= F_X(x)G_1(x), \quad F_V(y) = F_Y(y)G_2(y), \\ F_U &\leq \min\{F_X, G_1\}, \quad \text{and} \quad F_V \leq \min\{F_Y, G_2\}. \end{aligned}$$

Proof. Suppose (a) holds. Then we see as in the proof of Proposition 3 that

$$H(x, y) = F_X(x)F_Y(y) \min\{G_1(x), G_2(y)\}.$$

Now, assume (b). Then $H(x, y)$ is the joint distribution function of random variables $U = \max\{X, Z_1\}$ and $V = \max\{Y, Z_2\}$ as above. Since the random vector is uniquely determined by its joint distribution function, (a) follows. □

Remark 5. *Lemma 4 remains correct if we replace the word “comonotonic” in (a) with the word “countermonotonic” and the formula in (b) with*

$$H(x, y) = F_X(x)F_Y(y)W(G_1(x), G_2(y))$$

The proof follows in a similar way as above.

Theorem 6. Let $C_{\varphi,\psi}$ be a Marshall copula, and let F_U and F_V be two distribution functions that satisfy the following assumptions:

- (a) There is an increasing function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ such that if $x \in \mathbb{R}$ is such that $F_U(\chi(x)) > 0$ and $F_V(x) > 0$, then

$$\varphi^*(F_U(\chi(x))) = \psi^*(F_V(x)).$$

- (b) Function φ is either continuous at 0 or

$$x_U = \inf\{x \in \mathbb{R}, F_U(x) > 0\} > -\infty.$$

Function ψ is either continuous at 0 or

$$x_V = \inf\{x \in \mathbb{R}, F_V(x) > 0\} > -\infty.$$

- (c) Function φ is such that either $\varphi^*(0+) = \infty$ or there is $x \in \mathbb{R}$ such that $F_U(x) = 0$. Function ψ is such that either $\psi^*(0+) = \infty$ or there is $x \in \mathbb{R}$ such that $F_V(x) = 0$.

Then there are two triples of independent random variables X, Y, Z_1 , and X, Y, Z_2 , where Z_1 and Z_2 are comonotonic, such that $H(x, y) = C_{\varphi,\psi}(F_U(x), F_V(y))$ is the joint distribution function of

$$U = \max\{X, Z_1\} \quad \text{and} \quad V = \max\{Y, Z_2\}.$$

Proof. We define $F_X(x) = \varphi(F_U(x))$ and $F_Y(y) = \psi(F_V(y))$ for all $x, y \in \mathbb{R}$. Assumption (b) implies that F_X and F_Y are distribution functions. Namely, φ and ψ are continuous by Lemma 1 for $x \in (0, 1]$. If φ is continuous at 0, then F_X is clearly a distribution function. If φ is not continuous at 0, then we have

$$F_X(x_U) = \lim_{x \rightarrow x_U, x > x_U} \varphi(F_U(x))$$

and $F_X(x) = 0$ for $x < x_U$. Again, F_X is a distribution function. A similar argument using x_V if necessary shows that F_Y is a distribution function.

Next we define

$$G_2(x) = \begin{cases} 0, & \text{if } F_U(\chi(x)) = F_V(x) = 0, \\ \frac{F_V(x)}{\psi(F_V(x))}, & \text{if } F_U(\chi(x)) = 0 \text{ and } F_V(x) > 0, \\ \frac{F_U(\chi(x))}{\varphi(F_U(\chi(x)))}, & \text{otherwise.} \end{cases}$$

Note that the second statement of Lemma 1 implies that $\varphi(x) > 0$ and $\psi(x) > 0$ for $x > 0$. By assumption (a), it follows that

$$\frac{F_V(x)}{\psi(F_V(x))} = \frac{F_U(\chi(x))}{\varphi(F_U(\chi(x)))}$$

whenever both $F_U(\chi(x)) > 0$ and $F_V(x) > 0$. We also take $G_1(x) = G_2(\chi^{-1}(x))$ for all $x \in \mathbb{R}$. Here, χ^{-1} is the inverse of χ which is an increasing function since χ is increasing. Functions G_1 and G_2 are well defined because $\varphi(x) \geq 0$ and $\psi(y) \geq 0$ for all $x, y \in \mathbb{I}$. Assumptions (a)–(c) imply that G_1 and G_2 are distribution functions.

If $F_U(x) = 0$, then

$$F_X(x)G_1(x) = \varphi(F_U(x))G_1(x) = 0 = F_U(x).$$

If $F_U(x) > 0$, then

$$F_X(x)G_1(x) = \varphi(F_U(x))G_2(\chi^{-1}(x)) = \frac{\varphi(F_U(x))F_U(x)}{\varphi(F_U(x))} = F_U(x).$$

Therefore, we have $F_U(x) = F_X(x)G_1(x)$ for all $x \in \mathbb{R}$. If $F_V(y) = 0$ then

$$F_Y(y)G_2(y) = 0 = F_V(y).$$

Otherwise, if $F_V(y) > 0$, then

$$F_Y(y)G_2(y) = \frac{\psi(F_V(y))F_V(y)}{\psi(F_V(y))} = F_V(y).$$

Hence, we have $F_V(y) = F_Y(y)G_2(y)$ for all $y \in \mathbb{R}$.

Finally, the definitions of $H, C_{\varphi,\psi}, F_X, F_Y, G_1$, and G_2 imply

$$\begin{aligned} H(x, y) &= C_{\varphi,\psi}(F_U(x), F_V(y)) = \min\{F_U(x)\psi(F_V(y)), \varphi(F_U(x))F_V(y)\} \\ &= \min\{F_U(x)F_Y(y), F_V(y)F_X(x)\} = F_X(x)F_Y(y) \min\{G_1(x), G_2(y)\}. \end{aligned}$$

The implication $(a) \Rightarrow (b)$ of Lemma 4 then completes the proof. □

4. Shock Model for RMM Copulas

Next we present a shock model setup first proposed in Genest et al. [8] (cf. also [1, 7]). We consider two triples (X, Y, Z_1) and (X, Y, Z_2) of independent random variables, so that the dependence of resulting pair (U, V) is measured by the Fréchet–Hoeffding lower bound $W(u, v) = \max\{0, u + v - 1\}$. This time we assume that Z_1 and Z_2 are a *countermonotonic pair*. More precisely, there is a standard uniform random variable Z , independent of X and Y , such that $Z = G_1(Z_1) = 1 - G_2(Z_2)$. Again we construct two random variables

$$U = \max\{X, Z_1\}, \quad V = \max\{Y, Z_2\},$$

whose distributions are clearly $F_U(x) = F_X(x)G_1(x)$ and $F_V(y) = F_Y(y)G_2(y)$ as above. Let us introduce auxiliary generating functions for our model by

$$\widehat{f}(u) = \begin{cases} 0, & \text{if } u = 0 \\ 1, & \text{if } u = 1 \\ F_X(F_U^{-1}(u)), & \text{if } u \in \text{im}F_U \setminus \{0, 1\} \\ \frac{\widehat{f}(\bar{u}) - \widehat{f}(\underline{u})}{\bar{u} - \underline{u}}(u - \underline{u}) + \widehat{f}(\underline{u}) & \text{otherwise;} \end{cases}$$

and

$$\widehat{g}(v) = \begin{cases} 0, & \text{if } v = 0 \\ 1, & \text{if } v = 1 \\ F_Y(F_V^{-1}(v)), & \text{if } v \in \text{im}F_V \setminus \{0, 1\} \\ \frac{\widehat{g}(\bar{v}) - \widehat{g}(\underline{v})}{\bar{v} - \underline{v}}(v - \underline{v}) + \widehat{g}(\underline{v}) & \text{otherwise.} \end{cases}$$

Note that $F_U^{-1}(0) = -\infty$, so $F_X(F_U^{-1}(0)) = 0$ independently of the point 0 being in $\text{im}F_U$ or not, and similarly for \widehat{g} . In the same way, if 1 is not in $\text{im}F_U$, we get that $F_X(F_U^{-1}(1)) = F_X(+\infty) = 1$, and if 1 is in $\text{im}F_U$, then also $F_X(F_U^{-1}(1)) = 1$. We write also $f(u) = \widehat{f}(u) - u$ and $g(v) = \widehat{g}(v) - v$.

Theorem 7. *The copula of the pair (U, V) is an RMM copula with generating functions f and g*

$$C(u, v) = \max\{0, uv - f(u)g(v)\}.$$

Proof. Since X and Z_1 , respectively, Y and Z_2 , are independent, we have

$$F_U(u) = F_X(u)G_1(u), \quad \text{respectively} \quad F_V(v) = F_Y(v)G_2(v). \tag{4.1}$$

The joint distribution function H of the random vector (U, V) is then

$$\begin{aligned} H(u, v) &= \mathbb{P}[U \leq u, V \leq v] \\ &= \mathbb{P}[X \leq u, Y \leq v, G_1^{-1}(Z) \leq u, G_2^{-1}(1 - Z) \leq v], \end{aligned}$$

where Z is a standard uniform random variable, independent of X and Y , such that $Z = G_1(Z_1) = 1 - G_2(Z_2)$. Since X, Y , and Z are independent, we obtain

$$\begin{aligned} H(u, v) &= F_X(u)F_Y(v)\mathbb{P}[1 - G_2(v) \leq Z \leq G_1(u)] \\ &= F_X(u)F_Y(v) \max\{G_1(u) - (1 - G_2(v)), 0\} \\ &= F_X(u)F_Y(v)W(G_1(u), G_2(v)), \end{aligned}$$

where, in the second line, we have used the fact that Z is distributed uniformly on \mathbb{I} and then the definition of the Fréchet–Hoeffding lower bound W . The copula of (U, V) is then given by

$$\begin{aligned} C(u, v) &= H(F_U^{-1}(u), F_V^{-1}(v)) \\ &= F_X(F_U^{-1}(u))F_Y(F_V^{-1}(v))W(G_1(F_U^{-1}(u)), G_2(F_V^{-1}(v))). \end{aligned}$$

We use (4.1) and the definition of the generating functions f and g to conclude

$$C(u, v) = \max\{0, uv - f(u)g(v)\}.$$

It remains to show that functions f and g satisfy conditions (G1)–(G3) for generators of RMM copulas.

(G1): By definition, $\widehat{f}(0) = 0$ and $\widehat{f}(1) = 1$.

(G2): Consider function $\widehat{f}(u) = f(u) + u = F_X(F_U^{-1}(u))$; since distribution functions F_X and F_U are nondecreasing, so that F_U^{-1} is also nondecreasing, and since composite of two nondecreasing functions is nondecreasing, the claim follows for \widehat{f} . The conclusion for \widehat{g} is shown in a similar way.

(G3): We want to show that function $f^*(u) = \frac{f(u)}{u}$ is nonincreasing on $(0, 1]$. It suffices to show that $f^*(u) + 1 = \frac{\widehat{f}(u)}{u}$ is nonincreasing on $(0, 1]$. Choose $u \in \text{im}F_U$ to get (using (4.1))

$$\frac{\widehat{f}(u)}{u} = \frac{F_X(F_U^{-1}(u))}{F_U(F_U^{-1}(u))} = \frac{F_X(F_U^{-1}(u))}{F_X(F_U^{-1}(u))G_1(F_U^{-1}(u))} = \frac{1}{G_1(F_U^{-1}(u))}.$$

This function has a nondecreasing function in its denominator, so that it is nonincreasing on $\text{im}F_U$. Since we defined \widehat{f} outside this region by linear interpolation, the claim follows. Using similar arguments, we prove that g^* is also nonincreasing. □

Theorem 8. *Let U and V be two random variables with respective distribution functions F_U and F_V such that there exists an $x_0 \in \mathbb{R}$ with $F_U(x_0), F_V(x_0) \in (0, 1)$. Furthermore, let the copula of the pair (F_U, F_V) be an RMM copula with generating functions f and g , i.e.,*

$$C(u, v) = \max\{0, uv - f(u)g(v)\},$$

where f and g satisfy $f^*(F_U(x))g^*(1 - F_V(x)) = 1$, for all $x \in \mathbb{R}$ such that $F_U(x) > 0, F_V(x) < 1$. Then there exist random variables X, Y, Z_1, Z_2 such that X, Y, Z_i are independent for $i = 1, 2$, Z_2 is countermonotonic to Z_1 , and function

$$H(x, y) = C(F_U(x), F_V(y))$$

is the joint distribution function of the variables

$$\max\{X, Z_1\} \quad \text{and} \quad \max\{Y, Z_2\}.$$

Proof. For all $x, y \in \mathbb{R}$, define $F_X(x) = \hat{f}(F_U(x))$ and $F_Y(y) = \hat{g}(F_V(y))$, where \hat{f} and \hat{g} are defined in the usual way, i.e., $\hat{f}(u) = f(u) + u$, and $\hat{g}(u) = g(u) + u$. Moreover, let

$$G_1(x) = \begin{cases} \frac{g^*(1 - F_V(x))}{1 + g^*(1 - F_V(x))}, & \text{if } F_U(x) = 0 \\ \frac{F_U(x)}{\hat{f}(F_U(x))}, & \text{otherwise,} \end{cases}$$

and

$$G_2(y) = \begin{cases} \frac{f^*(1 - F_U(y))}{1 + f^*(1 - F_U(y))}, & \text{if } F_V(y) = 0 \\ \frac{F_V(y)}{\hat{g}(F_V(y))}, & \text{otherwise.} \end{cases}$$

Since f and g are nondecreasing, the same holds for \hat{f} and \hat{g} , so that F_X and F_Y are nondecreasing and càdlàg by Lemma 2. Continuity of \hat{f} and \hat{g} implies that

$$\lim_{x \rightarrow -\infty} F_X(x) = 0 \quad \text{and} \quad \lim_{x \rightarrow +\infty} F_X(x) = 1,$$

and similarly for F_Y . We conclude that F_X and F_Y are distribution functions of some random variables we denote by X and Y . The condition given in the theorem tells us that if $F_U(x) = 0$ for some x , then $x < x_0$, so that $F_V(x) \leq F_V(x_0) < 1$. Similarly, $F_V(x) = 1$ implies $x_0 < x$, so that $F_U(x) \geq F_U(x_0) > 0$.

Now, choose $x \in \mathbb{R}$ with $F_U(x) > 0$ and $F_V(x) < 1$ to get

$$\frac{1}{G_1(x)} = \frac{1 + g^*(1 - F_V(x))}{g^*(1 - F_V(x))} = \frac{\hat{f}(F_U(x))}{F_U(x)}$$

and similarly for G_2 . When x goes to $-\infty$, we use the fact that g^* is continuous at 0 (by Lemma 2) to get that G_1 approaches to 0. When x goes to $+\infty$, we use the fact that \hat{f} is continuous at 1 to get that G_1 approaches to 1. Next we prove that G_1 is nondecreasing. If $F_U(x) > 0$, then

$$\frac{1}{G_1(x)} = \frac{f(F_U(x))}{F_U(x)} + 1$$

which is nonincreasing by (G3). If $F_V(x) < 1$, then

$$\frac{1}{G_1(x)} = \frac{1}{g^*(1 - F_V(x))} + 1$$

which is nonincreasing by (G3). So, it remains to consider the case that $F_U(x) = 0$ and $F_V(y) = 1$ for some x and y . Clearly, we have $x < y$ and the point x_0 given in the theorem belongs to the interval (x, y) so that

$$\begin{aligned} G_1(x) &= \frac{g^*(1 - F_V(x))}{1 + g^*(1 - F_V(x))} \leq \frac{g^*(1 - F_V(x_0))}{1 + g^*(1 - F_V(x_0))} \\ &= \frac{F_U(x_0)}{\hat{f}(F_U(x_0))} \leq \frac{F_U(y)}{\hat{f}(F_U(y))} = G_1(y). \end{aligned}$$

The first one of the inequalities above holds because $F_V(x), F_V(x_0) < 1$ and the second one because $F_U(x_0), F_U(y) > 0$. Furthermore, since \hat{f} and g^* are continuous and there exists some x_0 with $F_V(x) < 1$ and $F_U(x_0) > 0$, it follows that G_1 is càdlàg. Similar considerations and conclusions apply to G_2 . Finally, we have shown existence of random variables Z_1 and Z_2 whose distribution functions are, respectively, G_1 and G_2 .

We now want to study the connection of the joint distribution function H of U and V to the copula

$$C(u, v) = \max\{0, uv - f(u)g(v)\} = uv \max\{0, 1 - f^*(u)g^*(v)\}.$$

Indeed, after introducing marginal distributions F_U and F_V into this formula, we get

$$\begin{aligned} C(F_U(x), F_V(y)) &= F_U(x)F_V(y) \max \left\{ 0, 1 - \left(\frac{\hat{f}(F_U(x)) - F_U(x)}{F_U(x)} \right) \left(\frac{\hat{g}(F_V(y)) - F_V(y)}{F_V(y)} \right) \right\} \\ &= F_U(x)F_V(y) \max \left\{ 0, \frac{\hat{f}(F_U(x))}{F_U(x)} + \frac{\hat{g}(F_V(y))}{F_V(y)} - \frac{\hat{f}(F_U(x))}{F_U(x)} \frac{\hat{g}(F_V(y))}{F_V(y)} \right\} \\ &= \hat{f}(F_U(x))\hat{g}(F_V(y)) \max \left\{ 0, \frac{F_U(x)}{\hat{f}(F_U(x))} + \frac{F_V(y)}{\hat{g}(F_V(y))} - 1 \right\} \\ &= F_X(x)F_Y(y)W(G_1(x), G_2(y)). \end{aligned}$$

Choose independent random variables with respective distributions F_X, F_Y , and G_1 . Define $Z_2 = G_2^{-1}(1 - G_1(Z_1))$ to get a countermonotonic random variable whose distribution equals G_2 Fig. 1.

The proof that $H(x, y)$ is the joint distribution function of random variables $\max\{X, Z_1\}$ and $\max\{Y, Z_2\}$ follows by Remark 5. □

Example 9. The so-called *Eyraud–Farlie–Gumbel–Morgenstern (EFGM) distributions* have been considered by many authors (cf. [23]). We will follow the approach of Durante and Sempi [2, Section 6.3]. We want to present shocks X, Y, Z_1 and Z_2 such that the corresponding RMM copula is of the form

$$C_a(u, v) = uv - a^2uv(1 - u)(1 - v), \quad \text{where } 0 \leq a \leq 1.$$

So, C_a is of the EFGM type. (We assume actually $0 < a \leq 1$.)

Proof. Observe that $\widehat{f}_a(t) = (a + 1)t - at^2$ and that in the model $\widehat{f}_a(t) = F_X(F_U^{-1}(t))$. Choose

$$F_X(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ x, & \text{if } 0 \leq x \leq 1 \\ 1, & \text{otherwise,} \end{cases}$$

i.e., $X \sim U(0, 1)$. To obtain F_U and then also G_1 , we have to solve

$$(a + 1)t - at^2 = x$$

for t . We obtain

$$F_U(x) = \begin{cases} 0, & \text{if } x \leq 0 \\ \frac{1}{2a} \left(a + 1 - \sqrt{(a + 1)^2 - 4ax} \right), & \text{if } 0 \leq x \leq 1 \\ 1, & \text{otherwise.} \end{cases}$$

It is easy to verify that F_U is a distribution function of a continuous random variable whose density function is equal to

$$p_U(x) = \begin{cases} \frac{1}{\sqrt{(a + 1)^2 - 4ax}}, & \text{if } 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

To obtain the distribution function of G_1 , we use the relation $F_U(x) = F_X(x)G_1(x)$. Therefore, we get after a simplification that

$$G_1(x) = \begin{cases} \frac{2}{(a + 1) + \sqrt{(a + 1)^2 - 4ax}}, & \text{if } x \leq 1 \\ 1, & \text{otherwise.} \end{cases}$$

With not much more effort, we can also compute the density of this continuous random variable

$$p_1(x) = \begin{cases} \frac{2a}{((a + 1)^2 - 2ax) \sqrt{(a + 1)^2 - 4ax} + (a + 1)^3 - 4(a + 1)ax}, & \text{if } x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

Since $C_a(u, v)$ is symmetric, we take $F_X(x) = F_Y(x)$ and $G_1(x) = G_2(x)$, while Z_1 and Z_2 are related via copula $W(u, v)$. □

Example 10. (Case of exponential shocks) If

$$\begin{aligned} X &\sim \exp(\lambda_1), & Y &\sim \exp(\lambda_2), \\ Z_1 &\sim \exp(\mu_1), & Z_2 &\sim \exp(\mu_2), \end{aligned}$$

then we have a copula for any $\alpha, \beta \in (0, 1]$ by letting

$$C_{\alpha, \beta}(u, v) = \max\{0, uv - (u^\alpha - u)(v^\beta - v)\}, \text{ where } \alpha = \frac{\lambda_1}{\lambda_1 + \mu_1}, \beta = \frac{\lambda_2}{\lambda_2 + \mu_2}.$$

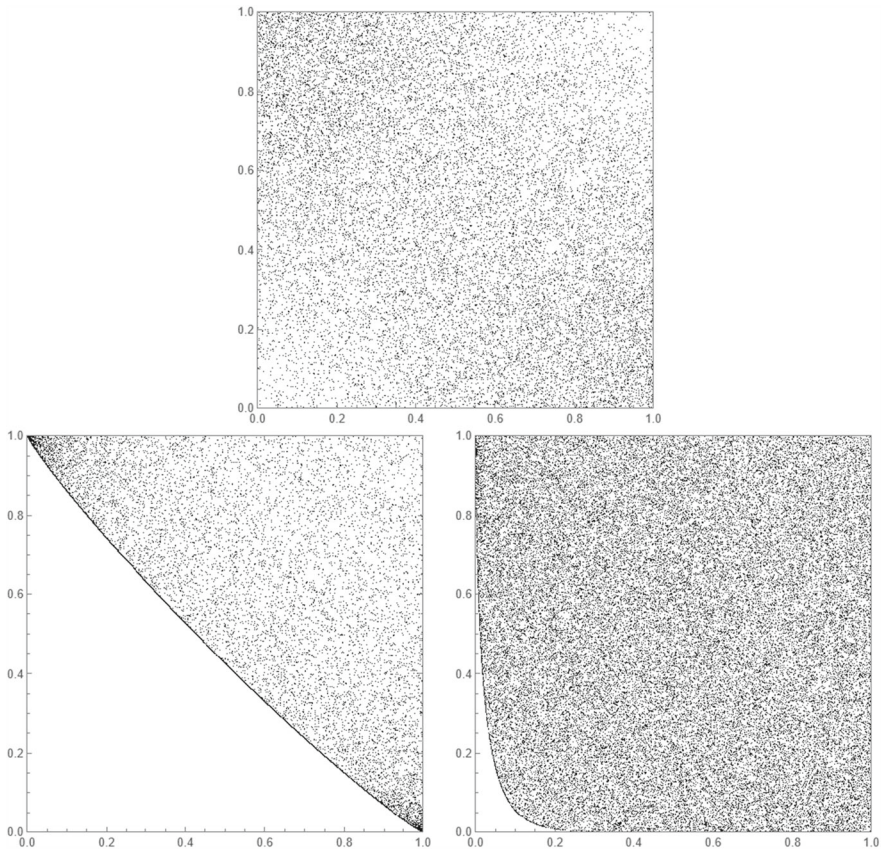


Figure 1. *Up*—Example 9: Copula for the value of the parameter $a = 0.95$. *Down* – Example 10: Copulas for $\alpha = \beta = 0.1$ (left) and $\alpha = 0.4, \beta = 0.9$ (right)

Proof. This can easily be determined by verifying conditions $(G_1) - (G_3)$ for the generating functions of the form

$$f(t) = t^\alpha - t, \quad \text{for } \alpha \in (0, 1].$$

More generally, a function

$$f(t) = t^\alpha(1 - t^\beta),$$

is a generating function f of an RMM copula if $\beta \in (0, 1]$ for $\alpha = 1$ and $\beta \geq 1 - \alpha$ for $\alpha \in (0, 1)$. □

5. Survival Copulas of RMM Copulas

For every RMM copula

$$C(u, v) = \max\{0, uv - f(u)g(v)\},$$

we have its survival copula

$$\widehat{C}(u, v) = u + v - 1 + C(1 - u, 1 - v) = \max\{u + v - 1, uv - f(1 - u)g(1 - v)\}.$$

We will now acquire the properties of functions $h, k : \mathbb{I} \rightarrow \mathbb{I}$ defined by $h(u) = f(1 - u)$ and $k(v) = g(1 - v)$; this time we need auxiliary functions

$$h_{\dagger}(u) = \frac{h(u)}{1 - u}, k_{\dagger}(v) = \frac{k(v)}{1 - v}, \widehat{h}_{\dagger}(u) = u - h(u), \text{ and } \widehat{k}_{\dagger}(v) = v - k(v),$$

defined so that they have positive values. Here are their properties.

(S1) $h(0) = k(0) = 0, h(1) = k(1) = 0,$ and $\widehat{h}_{\dagger}(1) = 0, \widehat{k}_{\dagger}(1) = 0;$

(S2) \widehat{h}_{\dagger} and \widehat{k}_{\dagger} are nondecreasing on $\mathbb{I};$

(S3) functions $h_{\dagger}(u)$ and $k_{\dagger}(u)$ are nondecreasing on $\mathbb{I}.$

Lemma 11. *If $f,$ respectively, g has properties (G1)–(G3), then $h,$ respectively, k has properties (S1)–(S3).*

Proof. We consider only the case of f and h since the case of g and k goes similarly. Property (G1) clearly implies (S1). By (G2), function $\widehat{f}(u) = f(u) + u$ is nondecreasing. So, $h(u) = f(1 - u) = \widehat{f}(1 - u) - 1 + u$ and $\widehat{h}_{\dagger}(u) = u - h(u) = 1 - \widehat{f}(1 - u)$ is nondecreasing. Finally, since $f^*(u) = \frac{f(u)}{u}$ is nonincreasing, it follows that $h_{\dagger}(u) = \frac{h(u)}{1 - u} = \frac{f(1 - u)}{1 - u} = f^*(1 - u)$ is nondecreasing. □

Lemma 12. *If functions h and k satisfy properties (S1)–(S3), then functions $f(u) = h(1 - u)$ and $g(v) = k(1 - v)$ satisfy properties (G1)–(G3).*

Proof. Property (S1) clearly implies (G1). By (S2), we have that $\widehat{h}_{\dagger}(u) = u - h(u)$ is nondecreasing. Then $\widehat{f}(u) = u + f(u) = u + h(1 - u) = 1 - \widehat{h}_{\dagger}(1 - u)$ is nondecreasing as well. In the same way, we prove that since \widehat{k}_{\dagger} is nondecreasing, then \widehat{g} is of the kind. Finally, (S3) implies that

$$f^*(u) = \frac{f(u)}{u} = \frac{h(1 - u)}{u} = h_{\dagger}(1 - u),$$

respectively $g^*(v) = k_{\dagger}(1 - v)$ is nonincreasing. □

We call a copula of the form

$$C_{h,k}(u, v) = \max\{u + v - 1, uv - h(u)k(v)\},$$

where h and k satisfy conditions (S1)–(S3), a *survival RMM copula* or *SMM copula* for short.

Let us now exhibit how the RMM and SMM copulas and their generating functions can be deduced from the original maxmin copula C of Omladič and Ružić [21], and its generating functions. The RMM, respectively, SMM copula can be obtained as

$$C^{\sigma_2}(u, v) = u - C(u, 1 - v) \quad \text{respectively} \quad C^{\sigma_1}(u, v) = v - C(1 - u, v).$$

Choose generating functions ϕ and ψ as in Ref. [21], so that

$$C(u, v) = \min\{u, uv + (\phi(u) - u)(v - \psi(v))\}.$$

Then

$$C^{\sigma_2}(u, v) = \max\{0, uv - (\phi(u) - u)(1 - v - \psi(1 - v))\}$$

and the generating functions of the so obtained RMM copula are clearly

$$f(u) = \phi(u) - u \quad \text{and} \quad g(v) = 1 - v - \psi(1 - v).$$

On the other hand,

$$C^{\sigma_1}(u, v) = \max\{u + v - 1, uv - (\phi(1 - u) - (1 - u))(v - \psi(v))\},$$

so that the generating functions of the so acquired SMM copula are

$$h(u) = \phi(1 - u) - (1 - u) \quad \text{and} \quad k(v) = v - \psi(v).$$

6. Stochastic Model for SMM Copulas

In this section, we present a different model as an outcome of the stochastic setup exhibited in Sect. 3. As before, suppose that (X, Y) and (Z_1, Z_2) are two independent pairs of random variables, where X and Y are independent, while Z_1 and Z_2 are countermonotonic. More precisely, let Z be a random variable distributed uniformly on \mathbb{I} and let $Z_1 = G_1^{-1}(Z)$ and $Z_2 = G_2^{-1}(I - Z)$ for some distribution functions G_1 and G_2 . Unlike in Sect. 3, we assume now that

$$U = \min\{X, Z_1\}, \quad V = \min\{Y, Z_2\}.$$

Theorem 13. *The copula of pair (U, V) is an SMM copula.*

Proof. Suppose that

$$C_{h,k}(u, v) = \max\{u + v - 1, uv - h(u)k(v)\},$$

where h, k satisfy properties (S1)–(S3). By Lemma 12, functions $f(u) = h(1 - u)$ and $g(u) = k(1 - u)$ satisfy conditions (G1)–(G3). Define

$$C_{f,g}(u, v) = \max\{0, uv - f(u)g(v)\},$$

then $C_{f,g}$ is an RMM copula by Ref. [14, Theorem 3]. Let us compute its survival copula

$$\begin{aligned} \widehat{C}_{f,g}(u, v) &= u + v - 1 - C_{f,g}(1 - u, 1 - v) \\ &= \max\{u + v - 1, uv - f(1 - u)g(1 - v)\}. \end{aligned}$$

According to Ref. [20, Section 2.6], this is a copula since it is a survival copula of a copula. We introduce $h(u) = f(1 - u)$ and $k(v) = g(1 - v)$ to conclude

$$\widehat{C}_{f,g}(u, v) = \max\{u + v - 1, uv - h(u)k(v)\} = C_{h,k}(u, v).$$

□

We combine Theorem 8 with Theorem 13 to obtain a shock model for an SMM copula.

Theorem 14. *Let U and V be two random variables with respective distribution functions F_U and F_V such that there exists an $x_0 \in \mathbb{R}$ with $F_U(x_0), F_V(x_0) \in (0, 1)$. Furthermore, let the copula of the pair (F_U, F_V) be an SMM copula with generating functions h and k , i.e.,*

$$C(u, v) = \max\{u + v - 1, uv - h(u)k(v)\},$$

where h and k satisfy $h_{\dagger}(1 - F_U(x))k_{\dagger}(F_V(x)) = 1$, for all $x \in \mathbb{R}$ such that $F_U(x) < 1, F_V(x) > 0$. Then there exist random variables X, Y, Z_1, Z_2 such that X, Y, Z_i are independent for $i = 1, 2$, Z_2 is countermonotonic to Z_1 , and function

$$H(x, y) = C(F_U(x), F_V(y))$$

is the joint distribution function of the variables

$$\min\{X, Z_1\} \quad \text{and} \quad \min\{Y, Z_2\}.$$

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