axioms

MDPI

Article

Upper Bounds for Double Roman Domination and $[k]$-Roman Domination of Cylindrical Graphs $C_m \Box P_n$

Simon Brezovnik [1,2,3,*] and Janez Žerovnik [1,4]

[1] Faculty of Mechanical Engineering, University of Ljubljana, 1000 Ljubljana, Slovenia; janez.zerovnik@fs.uni-lj.si
[2] Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia
[3] Faculty of Mathematics and Physics, University of Ljubljana, 1000 Ljubljana, Slovenia
[4] Rudolfovo–Science and Technology Centre, 8000 Novo Mesto, Slovenia
* Correspondence: simon.brezovnik@fs.uni-lj.si

Abstract

Roman-type domination parameters form an important class of graph invariants that model protection and resource allocation problems on networks. Among them, $[k]$-Roman domination provides a unified framework that generalizes Roman, double Roman, and higher-order variants. In this paper we investigate the $[k]$-Roman domination number of cylindrical grids $C_m \Box P_n$ and derive several new constructive upper bounds. Our approach combines three complementary techniques: linear periodic constructions, uniform ceiling-type labelings, and packing-based refinements. We first analyze the case $C_9 \Box P_n$, where these three families of bounds can be compared explicitly and their relative efficiency is shown to depend on the parameter k. We then extend the linear constructions to cylindrical grids whose circumference is a multiple of one of the values $r \in \{3, 4, 5, \dots, 9\}$, obtaining a unified family of upper bounds for $C_{rt} \Box P_n$. Motivated by the asymptotic behavior of these estimates, we further derive general upper bounds depending only on the residue class of m modulo 5, which apply to all cylindrical grids. As a consequence, we obtain explicit estimates for the double Roman domination number $\gamma_{[2R]}(C_m \Box P_n)$ and compare the resulting multiple-based constructions with the residue-class bounds. This comparison shows that the residue-class construction becomes asymptotically superior for all sufficiently large admissible circumferences, while several exceptional small cases remain better covered by tailored constructions.

Keywords: $[k]$-Roman domination; double Roman domination; cylindrical grids; Cartesian product of graphs

MSC: 05C69; 05C76

Academic Editor: Elena Guardo

Received: 15 April 2026
Revised: 12 May 2026
Accepted: 18 May 2026
Published: 20 May 2026

1. Introduction

Domination-type parameters form one of the central topics of modern graph theory and arise naturally in applications such as facility location, monitoring of communication networks, and deployment of emergency or defense resources [1,2]. Such models are also relevant for large-scale networked systems, where limited resources must be placed at selected nodes in order to provide local protection, monitoring, or recovery capacity. In communication, transportation, and infrastructure networks, failures or attacks at one node may have to be compensated by nearby protected nodes. Although the present paper is theoretical, cylindrical grids provide a regular product graph model in which one can study how local protection rules give rise to global resource requirements. This makes them a useful test class for understanding periodic protection patterns in structured networks. For broader perspectives on large-scale graph processing and modern graph-based learning methods, see for example [3,4]. Among these parameters, Roman domination has received considerable attention since its introduction by Cockayne et al. [5]. A Roman dominating function assigns labels from $\{0, 1, 2\}$ to vertices so that each vertex labeled 0 is protected by a neighboring vertex labeled 2, and the minimum total weight of such a labeling defines the Roman domination number $\gamma_R(G)$.

Several stronger variants of Roman domination have been introduced in order to model more robust protection mechanisms. These include double Roman domination [6,7], triple Roman domination [8], and related higher-order extensions. A unified framework covering these variants was proposed by Abdollahzadeh Ahangar et al. [8] under the name $[k]$-Roman domination. In this setting, vertices receive labels from $\{0, 1, \ldots, k+1\}$ subject to local neighborhood constraints depending on both the assigned weight and the number of positively labeled neighbors. The corresponding minimum total weight defines the $[k]$-Roman domination number $\gamma_{[kR]}(G)$.

Roman-type domination parameters have been investigated on numerous graph classes, including trees, cycles, grids, and Cartesian product graphs (see, for example, refs. [9,10]).

Cartesian products of paths and cycles play a particularly important role in domination theory, since their regular structure supports periodic constructions while still exhibiting nontrivial combinatorial behavior [11–13]. Among such graphs, cylindrical grids $C_m \Box P_n$ form a natural intermediate class between cycles, rectangular grids, and toroidal graphs.

Recent work on $[k]$-Roman domination has focused mainly on general bounds, extremal graph classes, and complexity questions. Valenzuela-Tripodoro et al. [14] studied the $[k]$-multiple Roman domination framework as a common generalization of double, triple, and quadruple Roman domination. They derived sharp bounds and exact values for several graph classes and considered the corresponding decision problem. Khalili et al. [15] further developed the general theory of $[k]$-Roman domination by proving upper and lower bounds for arbitrary graphs and trees, characterizing extremal cases, and establishing NP-completeness for bipartite and chordal graphs.

For double Roman domination, Anu and Lakshmanan [16] studied Cartesian products in detail. They improved a general lower bound for products $G \Box H$ when one factor has an efficient dominating set, derived general upper bounds, and obtained exact values for several products involving complete graphs and cycles.

The present paper follows this direction, but works in the more general $[k]$-Roman domination framework and focuses specifically on cylindrical grids $C_m \Box P_n$. This setting permits more refined constructions than those obtained from general graph bounds alone. In particular, periodic labelings, ceiling-type assignments, and packing-based reductions can be adapted to the geometry of the cylinder and compared asymptotically as functions of k, m, and n.

More recently, $[k]$-Roman domination on cylindrical grids has been studied in [17,18]. In ref. [17], the connection between $[k]$-Roman domination and efficient domination was used to obtain explicit periodic $[k]$-Roman dominating functions for $C_3 \Box P_n$ and $C_4 \Box P_n$. Besides constructive upper bounds, lower bounds based on local neighborhood constraints and efficient domination were also established. In particular, for cylindrical grids one obtains a general lower bound of the form

$$\gamma_{[kR]}(C_m \Box P_n) > (k+1) \left\lceil \frac{mn}{5} \right\rceil,$$

except in the efficient cases where the corresponding local optimum is attained. These lower bounds make it possible to compare the asymptotic behaviour of the constructed upper bounds with the best-known general lower estimates.

The analysis was extended in [18] to cylindrical grids $C_m \Box P_n$ for $m \in \{5, 6, 7, 8\}$. In particular, explicit linear constructions, uniform ceiling-type bounds, and packing-based refinements were developed and systematically compared depending on the parameter k and the length of the path. The obtained lower bounds furthermore showed that several of these constructions are asymptotically close to optimal for fixed small circumferences.

In this paper, we continue this avenue of research by extending the analysis to a general class of cylindrical graphs. Our approach combines three complementary techniques: periodic linear labeling along the cycle direction, uniform ceiling-type constructions based on neighborhood size, and packing-based refinements exploiting disjoint closed neighborhoods.

We first consider the case $C_9 \Box P_n$, where several competing constructions yield different upper bounds depending on the parameter k. A comparison of these bounds shows that linear, uniform, and packing-based constructions become optimal in different parameter regimes. We then extend the linear constructions obtaining general bounds for graphs of the form $C_{rt} \Box P_n$, where $r \in \{3, 4, \ldots, 9\}$.

Motivated by the observation that decompositions into blocks of length five yield the smallest asymptotic slopes with respect to n for $k \geq 2$, we then derive a unified family of upper bounds depending on the residue class of m modulo 5. This leads to explicit estimates for $\gamma_{[kR]}(C_m \Box P_n)$ valid for arbitrary cylindrical grids. In the special case $k = 2$, these results yield new upper bounds for the double Roman domination number, together with a comparison between the multiple-based constructions and the general residue-class construction.

The paper is organized as follows. In Section 2 we introduce notation and basic definitions. Section 3 presents three types of upper bounds for $C_9 \Box P_n$ and compares their efficiency for different values of k. Section 4 extends the linear constructions to multiples of $r \in \{3, 4, \ldots, 9\}$, and develops general modulo-5 upper bounds for arbitrary cylindrical grids. Section 5 specializes these results to the double Roman domination number and compares the resulting estimates. Concluding remarks and directions for future research are given in the final section.

2. Preliminaries

Throughout the paper we consider finite simple graphs $G = (V(G), E(G))$, where $V(G)$ denotes the vertex set and $E(G)$ denotes the edge set of G. For a vertex $v \in V(G)$, the *open neighborhood* of v is defined as $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, while the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$.

For a subset $S \subseteq V(G)$ we write $N[S] = \bigcup_{v \in S} N[v]$.

Let $f : V(G) \to \mathbb{N}_0$ be a function. For any subset $S \subseteq V(G)$ we define $f(S) = \sum_{v \in S} f(v)$. The *weight* of f is given by

$$\omega(f) = f(V(G)) = \sum_{v \in V(G)} f(v).$$

We next recall the definition of a $[k]$-*Roman dominating function* [8]. Let $k \in \mathbb{N}$. A function

$$f : V(G) \to \{0, 1, \ldots, k+1\}$$

is called a $[k]$-*Roman dominating function* if for every vertex $v \in V(G)$ with $f(v) < k$ we have

$$f(N[v]) \geq k + |AN(v)|,$$

where the *active neighborhood* $AN(v)$ of vertex v is defined as

$$AN(v) = \{u \in N(v) \mid f(u) > 0\}.$$

The minimum possible weight of such a function is called the $[k]$*-Roman domination number* of G and is denoted by $\gamma_{[kR]}(G)$.

Intuitively, the condition $f(N[v]) \geq k + |AN(v)|$ can be interpreted in terms of military defense. Vertices with positive labels represent defended cities containing military units, while the value assigned to a vertex corresponds to the defensive strength stationed at that location. A vertex v with $f(v) < k$ is not sufficiently protected on its own and therefore requires reinforcement from its closed neighborhood.

The term $|AN(v)|$ measures the number of neighboring cities that already contain military units. The additional term in the requirement accounts for the fact that active neighboring cities may have to send units to support the attacked city, while still maintaining sufficient protection in the local neighborhood.

For example, let $k = 5$ and suppose that a vertex v has three active neighbors. Then the $[k]$-Roman domination condition requires

$$f(N[v]) \geq 5 + 3 = 8.$$

Thus the cities in the closed neighborhood of v collectively contain at least eight military units: five units provide the basic required level of defense, while the additional three units correspond to the three active neighboring cities that may need to contribute reinforcements without compromising local protection.

The parameter $\gamma_{[kR]}(G)$ generalizes several well-known domination-type invariants. In particular, for $k = 1$ it coincides with the classical Roman domination number, while for $k = 2$ it coincides with the double Roman domination number.

Since the graphs studied in this paper belong to the class of cylindrical grids, we briefly recall the definition of the Cartesian product. For graphs G and H, *Cartesian product* $G \Box H$ has vertex set $V(G) \times V(H)$, where vertices (g_1, h_1) and (g_2, h_2) are adjacent whenever either $g_1 = g_2$ and $h_1 h_2 \in E(H)$, or $h_1 = h_2$ and $g_1 g_2 \in E(G)$.

Let C_m denote the cycle on m vertices and P_n the path on n vertices. The Cartesian product $C_m \Box P_n$ is called a *cylindrical grid*. Throughout the paper we interpret C_m as the cyclic direction and P_n as the longitudinal direction of the cylinder. Vertices of $C_m \Box P_n$ will be denoted by (i, j), where $i \in \{0, \ldots, m-1\}, j \in \{0, \ldots, n-1\}$. For each j, the subgraph induced by $F_j = \{(i, j) \mid 0 \leq i \leq m-1\}$ is called the j-th fibre of $C_m \Box P_n$. We identify each fibre with its vertex set whenever convenient.

Periodic constructions along the cyclic direction play a central role in the analysis of $[k]$-Roman domination on cylindrical grids. In particular, decompositions into blocks of length five will serve as the main building blocks in several later constructions.

We now describe an explicit periodic construction for the base case $m = 5$, which was introduced in [18]. This construction will serve as a reference configuration in Section 5, where it will be used to derive general upper bounds for $\gamma_{[kR]}(C_m \Box P_n)$ for arbitrary cylindrical grids and to obtain corresponding consequences for double Roman domination.

Consider the following periodic labeling pattern:

$$\begin{pmatrix} \cdots & 0 & | & k+1 & 0 & 0 & 0 & 0 & | & k+1 & \cdots \\ \cdots & 0 & | & 0 & 0 & 0 & k+1 & 0 & | & 0 & \cdots \\ \cdots & 0 & | & 0 & k+1 & 0 & 0 & 0 & | & 0 & \cdots \\ \cdots & k+1 & | & 0 & 0 & 0 & 0 & k+1 & | & 0 & \cdots \\ \cdots & 0 & | & 0 & 0 & k+1 & 0 & 0 & | & 0 & \cdots \end{pmatrix}. \quad (1)$$

The entry in row i and column j of Equation (1) represents the weight assigned to the vertex (i, j) of the cylindrical grid. Thus entries equal to 0 correspond to vertices without stationed units, while entries equal to $k+1$ correspond to vertices carrying the maximum defensive strength used in this pattern. The positive entries in Equation (1) form a periodic diagonal arrangement along the path direction. From one fibre to the next, the position of the label $k+1$ is shifted cyclically by two rows modulo 5. Consequently, every fibre in the displayed periodic block contains exactly one vertex with positive weight, and the whole pattern repeats with period five.

Since the pattern is periodic, it is enough to verify the $[k]$-Roman domination condition on one block of five consecutive fibres, together with the boundary corrections described below. With this choice of the shifts, every vertex of weight 0 in the interior of the periodic pattern has exactly one vertex of weight $k+1$ in its closed neighborhood. Moreover, this vertex of weight $k+1$ is the only active vertex in that closed neighborhood. Hence, for every vertex v with $f(v) = 0$ whose closed neighborhood is fully contained in the periodic part of the construction, we have

$$f(N[v]) = k+1 \qquad \text{and} \qquad |AN(v)| = 1.$$

Therefore

$$f(N[v]) = k+1 = k + |AN(v)|,$$

so the $[k]$-Roman domination condition is satisfied for all interior vertices of weight 0.

Vertices in the first and last fibres, as well as vertices whose closed neighborhoods meet these boundary fibres, have to be checked separately, because one neighbouring fibre in the path direction is missing. This is the reason for adding boundary corrections. The following concrete example illustrates this correction for $C_5 \square P_8$, and it will serve as a reference configuration for the modulo-5 constructions used later. It is obtained by taking eight consecutive fibres from the periodic labeling in Equation (1) and then adding two boundary corrections, shown in bold, in the first and last fibres. These bold entries, both equal to k, compensate for the missing neighbouring fibres beyond the two ends of the path and ensure that the $[k]$-Roman domination condition remains valid at the boundary.

$$\begin{pmatrix} 0 & | & k+1 & 0 & 0 & 0 & 0 & | & k+1 & 0 \\ \mathbf{k} & | & 0 & 0 & 0 & k+1 & 0 & | & 0 & 0 \\ 0 & | & 0 & k+1 & 0 & 0 & 0 & | & 0 & k+1 \\ k+1 & | & 0 & 0 & 0 & 0 & k+1 & | & 0 & 0 \\ 0 & | & 0 & 0 & k+1 & 0 & 0 & | & 0 & \mathbf{k} \end{pmatrix}. \tag{2}$$

Finally, we recall a simple extension principle that allows periodic constructions on smaller cylindrical grids to be lifted to larger ones. Since cylindrical graphs are invariant under cyclic shifts along the C_m direction, many constructions of $[k]$-Roman dominating functions can be obtained by repeating a fixed pattern.

More precisely, let r be a divisor of m and suppose that a $[k]$-Roman dominating function is defined on $C_r \square P_n$. By repeating this labeling periodically along the cycle direction, we obtain a $[k]$-Roman dominating function on $C_m \square P_n$, where $m = rt$ and $t \geq 1$. The weight of the resulting labeling is t times the weight of the original one. Consequently,

$$\gamma_{[kR]}(C_{rt} \square P_n) \leq t\, \gamma_{[kR]}(C_r \square P_n).$$

This observation will be used repeatedly in the sequel to extend constructions from small base circumferences to their multiples.

For later reference, we recall that explicit constructive upper bounds for the base circumferences $r \in \{3,4,\ldots,8\}$ were obtained in [17,18]. Together with the new construction for $r = 9$ developed in Section 3, these base estimates are the ingredients used in Theorem 5.

3. Upper Bounds for $m = 9$

We now derive an upper bound for $\gamma_{[kR]}(C_9 \square P_n)$ by constructing a suitable $[k]$-Roman dominating function. Theorem 1 completes the analysis of the base cylindrical grids $C_m \square P_n$ with $3 \le m \le 9$, extending previous work that covered the cases $m \in \{3,4\}$ [17] and $m \in \{5,6,7,8\}$ [18]. The resulting estimate is given below.

Theorem 1. *Let $n \ge 2$. Then*

$$\gamma_{[kR]}(C_9 \square P_n) \le 2n(k+1) + 2k. \tag{3}$$

Proof. We define a periodic labeling using the pattern

$$\begin{pmatrix}
k+1 & | & 0 & k+1 & 0 & 0 & \cdots & k+1 & | & 0 & \cdots \\
0 & | & 0 & 0 & 0 & 0 & \cdots & 0 & | & 0 & \cdots \\
0 & | & k+1 & 0 & k+1 & 0 & \cdots & 0 & | & k+1 & \cdots \\
k & | & 0 & 0 & 0 & 0 & \cdots & 0 & | & 0 & \cdots \\
0 & | & 0 & k+1 & 0 & k+1 & \cdots & 0 & | & 0 & \cdots \\
k+1 & | & 0 & 0 & 0 & 0 & \cdots & k+1 & | & 0 & \cdots \\
0 & | & 0 & 0 & k+1 & 0 & \cdots & 0 & | & 0 & \cdots \\
0 & | & k+1 & 0 & 0 & 0 & \cdots & 0 & | & k+1 & \cdots \\
0 & | & 0 & 0 & 0 & k+1 & \cdots & 0 & | & 0 & \cdots
\end{pmatrix}. \tag{4}$$

Observe that the pattern requires a modification at the boundary. Let $j \in \{0,1,\ldots,8\}$ be such that $f(n-2,j) = f(n-2,j+4) = k+1$. Then we set

$$f(n-1,j-1) = k, \quad f(n-1,j+2) = k+1, \ \text{and} \ f(n-1,j+4) = f(n-1,j-3) = k+1,$$

where the indices are taken to be modulo 9, and all other vertices of the last fibre F_{n-1} are labeled 0.

The matrix, together with the adjustment of the last fibre, gives a $[k]$-RDF of the desired weight; therefore, the proof is complete. □

It turns out that for larger values of k, a more efficient strategy is to distribute the weight uniformly among all vertices. We will show in the subsequent analysis that such a uniform assignment yields a better upper bound compared to more irregular constructions. This observation motivates us to consider weight functions that assign equal contributions across the graph.

Based on this approach, we obtain the following result for the $[k]$-Roman domination number of the Cartesian product $C_9 \square P_n$.

Theorem 2. *For $n \ge 4$,*

$$\begin{aligned}
\gamma_{[kR]}(C_9 \square P_n) &\le 9(n-2)\left\lceil \frac{k+4}{5} \right\rceil + 18\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil & (5)\\
&\le \frac{9nk+81n+6k-6}{5}. & (6)
\end{aligned}$$

Proof. For each interior fibre F_i, $1 \leq i \leq n-2$, assign weight

$$\left\lceil \frac{k+4}{5} \right\rceil$$

to each vertex of the fibre.

On each boundary fibre (i.e., $i = 0$ and $i = n-1$), assign weight

$$\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil$$

to each of its nine vertices. Thus, each boundary fibre receives total weight

$$9\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil.$$

This assignment ensures that every vertex v in an interior fibre satisfies

$$f(N[v]) \geq 5\left\lceil \frac{k+4}{5} \right\rceil \geq k+4,$$

since the closed neighborhood of an interior vertex consists of five vertices, implying $|AN(v)| \leq 4$.

On the other hand, every boundary vertex v satisfies

$$f(N[v]) \geq \left\lceil \frac{k+4}{5} \right\rceil + 3\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil \geq k+3,$$

because the closed neighborhood of a boundary vertex consists of its two neighbors in the same fibre, the vertex itself, and its unique neighbor in the adjacent interior fibre.

Therefore, f is a $[k]$-RDF. Its total weight is at most the contribution of the interior fibres, which equals

$$9(n-2)\left\lceil \frac{k+4}{5} \right\rceil,$$

plus the contribution of the two boundary fibres, which equals

$$18\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil.$$

Hence,

$$\gamma_{[kR]}(C_9 \square P_n) \leq 9(n-2)\left\lceil \frac{k+4}{5} \right\rceil + 18\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil.$$

Using the inequalities $\lceil x \rceil \leq x+1$ and $\lceil x \rceil \geq x$, we obtain

$$\left\lceil \frac{k+4}{5} \right\rceil \leq \frac{k+4}{5} + 1 = \frac{k+9}{5},$$

and

$$\left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil \leq \frac{k+3-\frac{k+4}{5}}{3} + 1 = \frac{4k+26}{15}.$$

Substituting these bounds yields

$$w(f) \leq 9(n-2)\frac{k+9}{5} + 18\frac{4k+26}{15} = \frac{9nk+81n+6k-6}{5}.$$

Therefore,

$$\gamma_{[kR]}(C_9 \square P_n) \leq \frac{9nk+81n+6k-6}{5},$$

which completes the proof. □

The construction described above is not tight, as the neighborhood sums frequently exceed the minimum required by the $[k]$-Roman domination condition. This excess allows for further optimization. In particular, for certain values of k, one can improve the bound by increasing the base weights to $\left\lceil \frac{k+5}{5} \right\rceil$ and then reducing the weight by one on vertices belonging to a suitably chosen packing set. This motivates the following notion.

Definition 1. *A set $D \subseteq V(G)$ is called a* packing *if $N[u] \cap N[v] = \emptyset$ for every pair of distinct vertices $u, v \in D$. The maximum cardinality of a packing in G is called the* packing number *of G and is denoted by $\rho(G)$.*

Packing sets will be used to decrease the weight on selected vertices while preserving the Roman domination condition, which leads to improved upper bounds.

Lemma 1. *Let $n \geq 4$. The packing number of $C_9 \square P_n$ is*

$$\rho(C_9 \square P_n) = 2n - \left\lfloor \frac{n}{3} \right\rfloor.$$

Proof. We first construct a packing of the required size. In fibres with two selected vertices, we choose vertices at cycle distance 4 in C_9. For every intermediate fibre we select one vertex, shifted one position along the cycle, regarding to the previous intermediate fibre. See Figure 1.

For optimality, let S be an arbitrary packing of $C_9 \square P_n$, and write

$$s_i = |S \cap F_i|.$$

The construction form Figure 1 has average density $5/3$, since its periodic pattern selects $2, 2, 1$ vertices in every three consecutive fibres. We show that no packing can have larger density.

Clearly, $s_i \leq 3$ for every fibre F_i. Suppose first that $s_i = 3$ for some i. Then the three selected vertices in F_i must be mutually at cyclic distance 3 in C_9, say in rows $a, a+3, a+6$ modulo 9. Their closed neighborhoods cover the whole fibre F_i. Hence no vertex from either neighbouring fibre F_{i-1} or F_{i+1} can belong to S. Thus a fibre with three selected vertices is isolated between two empty neighbouring fibres. The local density of such a configuration is therefore at most $3/2$, which is smaller than $5/3$. Consequently, fibres with three selected vertices cannot improve on the density of the displayed construction.

It remains to consider the case in which each fibre contains at most two selected vertices. We claim that three consecutive fibres cannot all contain two selected vertices. Suppose, to the contrary, that

$$s_i = s_{i+1} = s_{i+2} = 2.$$

Let $S \cap F_{i+1} = \{(a, i+1), (b, i+1)\}$. Since S is a packing, the two vertices in F_{i+1} must be at a cyclic distance of at least 3. Moreover, any vertex of $S \cap F_i$ must avoid the rows a and b, because vertices in the same row in adjacent fibres are adjacent. It must also avoid the rows adjacent to a and b on the cycle, because otherwise the corresponding closed neighborhoods would intersect at a vertex of F_{i+1}. Thus the two vertices in F_{i+1} forbid the six rows

$$a-1,\ a,\ a+1,\ b-1,\ b,\ b+1$$

of modulo 9 in the neighbouring fibre F_i. Only three rows remain available. Among these three rows, two are necessarily at a cyclic distance of at most 2, so two packing vertices cannot be placed in F_i, which constitutes a contradiction.

Hence, in the case $s_i \leq 2$ for all fibres, every three consecutive fibres contribute at most five vertices. This gives density at most $5/3$, with at most two additional vertices in a remaining block of one or two fibres. Therefore

$$|S| \leq 5\left\lfloor \frac{n}{3} \right\rfloor + 2\left(n - 3\left\lfloor \frac{n}{3} \right\rfloor\right) = 2n - \left\lfloor \frac{n}{3} \right\rfloor.$$

Together with the construction described above, this proves the lemma. □

	F_0	F_1	F_2	F_3	F_4	F_5
0			$\ast$			
1	$\ast$					$\ast$
2				$\ast$		
3		$\ast$				
4					$\ast$	
5	$\ast$					
6				$\ast$		
7		$\ast$				
8					$\ast$	

...

Figure 1. Packing pattern in $C_9 \square P_n$, where each $\ast$ represents a vertex selected in the packing. The fibres follow a periodic pattern, with a cyclic shift along C_9.

The role of Lemma 1 in the sequel is to make it possible to reduce the weight of a suitable labeling on many vertices simultaneously. The reason is that packing vertices have pairwise disjoint closed neighborhoods, so a local reduction at the packing vertices cannot accumulate in any single closed neighborhood.

Remark 1. *The packing-based reductions used below rely on the following simple principle. Suppose that a labeling f satisfies*

$$f(N[v]) \geq k + |AN_f(v)| + 1$$

for every vertex v, where $AN_f(v)$ denotes the active neighborhood with respect to f. If S is a packing, then the closed neighborhoods of distinct vertices of S are pairwise disjoint. Consequently, decreasing the weight of each vertex in S by one affects every closed neighborhood by at most one unit. Moreover, the reduction cannot increase the number of active neighbors. Therefore, the reduction still guarantees the $[k]$-Roman domination condition for the modified labeling. This is the mechanism used in the packing-based estimates below.

We now state an upper bound for $\gamma_{[kR]}(C_9 \square P_n)$ that uses both the uniform ceiling-type labeling and the packing reduction principle described above.

Theorem 3. *For $n \geq 4$,*

$$\gamma_{[kR]}(C_9 \Box P_n) \leq 9(n-2)\left\lceil \frac{k+5}{5} \right\rceil + 18\left\lceil \frac{k+4-\left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil - \left(2n - \left\lfloor \frac{n}{3} \right\rfloor\right) \quad (7)$$

$$\leq \frac{27nk + 245n + 18k}{15}. \quad (8)$$

Proof. For every interior fibre $1 \leq i \leq n-2$, assign to each of its vertices the value

$$\left\lceil \frac{k+5}{5} \right\rceil.$$

For the boundary fibres, namely $i = 0$ and $i = n-1$, assign the same value to all nine vertices in the fibre, where each vertex receives weight

$$\left\lceil \frac{k+4-\left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil.$$

This labeling is defined so that for every vertex v we have

$$f(N[v]) \geq k + |AN(v)| + 1.$$

Let S be a maximum packing of $C_9 \Box P_n$. By Lemma 1, we have

$$|S| = 2n - \left\lfloor \frac{n}{3} \right\rfloor.$$

As shown above, the initial ceiling-type labeling has one unit of slack in every closed neighborhood relevant to the $[k]$-Roman domination condition. Since S is a packing, Remark 1 implies that the weight at each vertex of S can be reduced by one without violating the $[k]$-Roman domination condition. Hence, the modified labeling remains a $[k]$-RDF. Consequently,

$$\gamma_{[kR]}(C_9 \Box P_n) \leq 9(n-2)\left\lceil \frac{k+5}{5} \right\rceil + 18\left\lceil \frac{k+4-\left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil - \left(2n - \left\lfloor \frac{n}{3} \right\rfloor\right),$$

which establishes Inequality (7).
For the final estimate, we use the inequalities

$$\left\lceil \frac{k+5}{5} \right\rceil \leq \frac{k+5}{5} + 1 = \frac{k+10}{5}$$

and

$$\left\lceil \frac{k+4-\left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil \leq \frac{k+4-\frac{k+5}{5}}{3} + 1 = \frac{4k+30}{15}.$$

Moreover,

$$\left\lfloor \frac{n}{3} \right\rfloor \leq \frac{n}{3}.$$

Substituting these bounds yields

$$\gamma_{[kR]}(C_9 \Box P_n) \leq 9(n-2)\frac{k+10}{5} + 18\frac{4k+30}{15} - 2n + \frac{n}{3} = \frac{27nk + 245n + 18k}{15}.$$

This completes the proof. □

https://doi.org/10.3390/axioms15050382

To illustrate the transition from the linear bound to the uniform and packing-based constructions, we present in Figures 2 and 3 the optimal bounds for selected ranges of the parameter k. The first figure focuses on small and intermediate values, namely $k \in \{1, 2, \ldots, 27\}$, where the linear bound dominates for very small k, while uniform Inequality (5) and packing-based Inequality (7) begin to appear in restricted regions as k increases.

The second figure illustrates the situation for larger values of k, namely $k \in \{35, 36, \ldots, 45\}$. In this regime, linear Inequality (3) is no longer competitive, and the comparison is primarily between Inequalities (5) and (7). We observe that the optimal bound depends on both k and n, with the uniform construction typically prevailing for larger values of n, while the packing-based construction may yield improvements for smaller values of n.

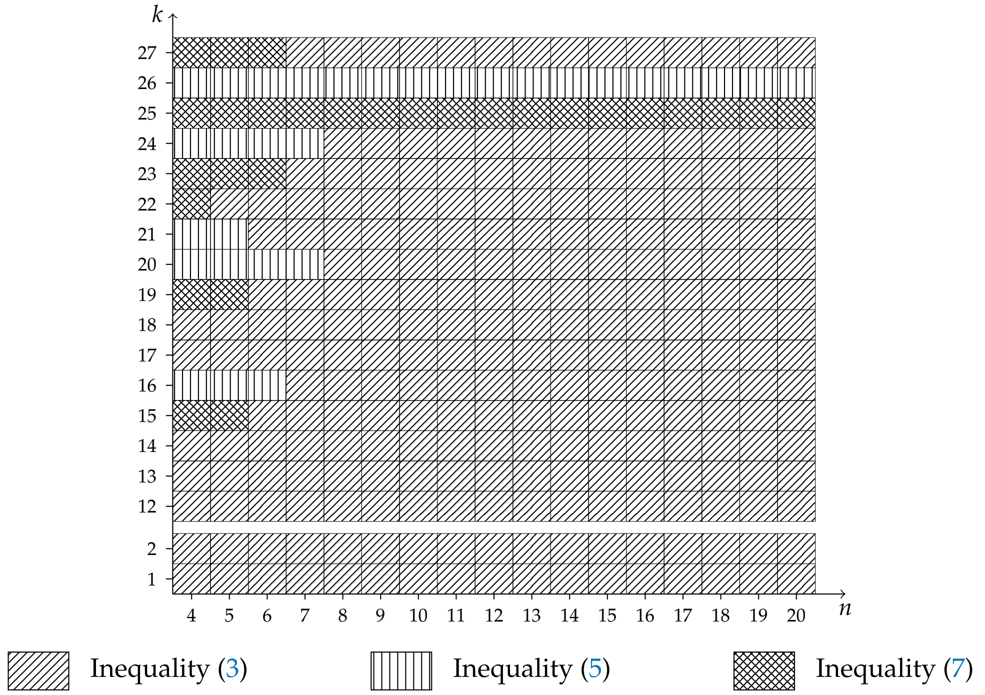

Figure 2. The best bound among Inequalities (3), (5), and (7) for $n = 4, \ldots, 20$ and $k = 1, \ldots, 27$ (with the gap between $k = 2$ and $k = 12$).

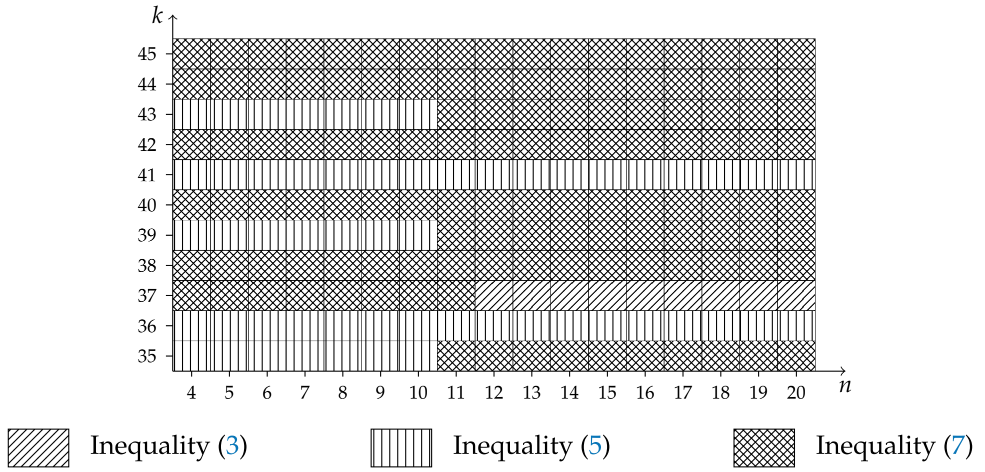

Figure 3. The best bound among Inequalities (3), (5), and (7) for $n = 4, \ldots, 20$ and $k = 35, \ldots, 45$ for $C_9 \square P_n$.

These figures clearly demonstrate the transition thresholds between the different constructions and motivate the precise comparison given in Theorem 4.

Theorem 4. *For sufficiently large k, at least one of the Inequalities (5) or (7) improves the linear Inequality (3). More precisely:*

- *if $k \equiv 1 \pmod 5$, then for $k \geq 26$ and all sufficiently large n, Inequality (5) is the smallest among (3), (5), and (7);*
- *if $k \not\equiv 1 \pmod 5$, then for $k \geq 38$ and all sufficiently large n, Inequality (7) is the smallest among (3), (5), and (7).*

Proof. Fix k and compare the three bounds as functions of n.

Inequality (3) has the form

$$2n(k+1) + O_k(1).$$

Let

$$a = \left\lceil \frac{k+4}{5} \right\rceil \qquad \text{and} \qquad b = \left\lceil \frac{k+3-a}{3} \right\rceil.$$

Then Inequality (5) equals

$$9(n-2)a + 18b = 9an + O_k(1).$$

Let

$$a' = \left\lceil \frac{k+5}{5} \right\rceil \qquad \text{and} \qquad b' = \left\lceil \frac{k+4-a'}{3} \right\rceil.$$

Since

$$2n - \left\lfloor \frac{n}{3} \right\rfloor = \frac{5}{3}n + O(1),$$

Inequality (7) equals

$$9(n-2)a' + 18b' - \left(2n - \left\lfloor \frac{n}{3} \right\rfloor\right) = \left(9a' - \frac{5}{3}\right)n + O_k(1).$$

Therefore, for sufficiently large n, the smallest bound is determined by comparing the linear coefficients

$$2(k+1), \qquad 9a, \qquad 9a' - \frac{5}{3}.$$

Comparing $9a$ with $2(k+1)$ shows that

$$9a < 2(k+1)$$

holds when $k \equiv 1 \pmod 5$ and $k \geq 26$.

In the remaining residue classes, comparing $9a' - \frac{5}{3}$ with $2(k+1)$ shows that

$$9a' - \frac{5}{3} < 2(k+1)$$

holds for all $k \geq 38$.

Finally, if $k \equiv 1 \pmod 5$, then $a < a'$, and hence

$$9a < 9a' - \frac{5}{3},$$

so Inequality (5) is asymptotically smaller than Inequality (7).

If $k \not\equiv 1 \pmod 5$, then $a = a'$, and therefore

$$9a' - \frac{5}{3} < 9a,$$

so Inequality (7) is asymptotically smaller than Inequality (5).

Thus, for sufficiently large n, Inequality (5) is the smallest among (3), (5), and (7) whenever $k \equiv 1 \pmod 5$ and $k \geq 26$, while Inequality (7) is the smallest among these three bounds whenever $k \not\equiv 1 \pmod 5$ and $k \geq 38$. This completes the proof. □

4. General Upper Bound for $\gamma_{[kR]}(C_m \square P_n)$

The constructions derived for the base cases $m = 3, 4, \ldots, 9$ (see Section 3 and [17,18]) are local and periodic along the cycle direction. In each case, the labeling is confined to a fixed number of consecutive rows and depends only on neighborhood conditions inside this block. Consequently, these patterns can be replicated independently along the cycle whenever the circumference is a multiple of $3, 4, \ldots, 9$. This observation allows us to extend all previously obtained bounds to the graphs $C_{3t} \square P_n$, $C_{4t} \square P_n$, …, $C_{9t} \square P_n$.

4.1. Linear Constructions

The periodic nature of the linear constructions allows them to be replicated along the cycle direction. This observation leads to a unified formulation of the bounds for all circumferences that are multiples of $3, 4, \ldots, 9$.

Theorem 5. *Let $t \geq 1$ and $n \geq 2$. For $m \in \{3t, \ldots, 9t\}$ the following bounds hold:*

$$\gamma_{[kR]}(C_m \square P_n) \leq \begin{cases} t(nk+2), & m = 3t, \\ tn(k+1), & m = 4t, \\ t\big(n(k+1)+2k\big), & m = 5t, \\ t\Big(\Big\lceil \frac{4n}{3} \Big\rceil (k+1) + (k+1)\Big), & m = 6t,\ n \equiv 0 \pmod 3, \\ t\Big\lceil \frac{4n}{3} \Big\rceil (k+1), & m = 6t,\ n \equiv 1 \pmod 3, \\ t\Big(\Big\lceil \frac{4n}{3} \Big\rceil (k+1) + k\Big), & m = 6t,\ n \equiv 2 \pmod 3, \\ t\Big((n+1)(k+1) + \frac{n-1}{2}(k+1) + 2k\Big), & m = 7t,\ n \text{ odd}, \\ t\Big(n(k+1) + \frac{n}{2}(k+1) + 2k + 1\Big), & m = 7t,\ n \text{ even}, \\ t\Big(2n(k+1) - \Big(\Big\lfloor \frac{n-2}{5} \Big\rfloor + \Big\lfloor \frac{n}{5} \Big\rfloor\Big) + 2k\Big), & m = 8t, \\ t\big(2n(k+1)+2k\big), & m = 9t. \end{cases}$$

Proof. We partition the cycle of $C_m \square P_n$ into t consecutive blocks of size 3, 4, …, or 9, according to the value of m.

Let $m = rt$, where $r \in \{3, 4, \ldots, 9\}$. Partition the cycle of $C_m \square P_n$ into t consecutive blocks, each inducing a copy of $C_r \square P_n$. On every such block we apply the corresponding construction for $C_r \square P_n$.

For $r = 9$, the construction is given in the present paper, while for $r \in \{3, \ldots, 8\}$ we use the constructions established in our earlier work on cylindrical grids $C_r \square P_n$ (see [17,18]).

Since each construction is defined locally within a block and depends only on adjacency relations inside that block, it remains valid when repeated periodically along the cycle.

Moreover, the blocks are disjoint in the cycle direction, so the total weight equals the sum of the contributions of all t blocks.

Consequently,

$$\gamma_{[kR]}(C_m \Box P_n) \le t \cdot \gamma_{[kR]}(C_r \Box P_n),$$

which yields the stated bounds for $m \in \{3t, \dots, 9t\}$ by substituting the corresponding estimates for $C_r \Box P_n$. $\Box$

As observed above, the same cylindrical graph $C_m \Box P_n$ may admit several distinct decompositions into blocks of admissible sizes $r \in \{3, 4, \dots, 9\}$. Each such decomposition yields a valid upper bound obtained by repeating the corresponding base construction. Since these bounds are not necessarily equal, it is natural to compare them in order to determine which decomposition provides the best estimate.

To compare these bounds, we first consider their asymptotic behavior as functions of n for fixed k. More precisely, we determine the linear coefficient (slope) with respect to n in each case. This comparison identifies which construction is asymptotically strongest as $n \to \infty$.

However, the bounds also contain additive terms depending on k and on residue classes of n. These terms do not affect the asymptotic slope, but they may influence which estimate is numerically best for moderate values of n. For this reason, the asymptotic comparison below is complemented in Section 5 by an explicit finite comparison in the double Roman case. If $m = rt$, then each bound is of the form $t \cdot B_r(n,k)$, and hence its slope equals $\frac{m}{r}$ times the slope of the corresponding base bound for $C_r \Box P_n$.

From the expressions in Theorem 5, we obtain the following slopes:

$$\begin{aligned} r = 3: \ &\frac{mk}{3}, & r = 4: \ &\frac{m(k+1)}{4}, & r = 5: \ &\frac{m(k+1)}{5}, \\ r = 6: \ &\frac{2m(k+1)}{9}, & r = 7: \ &\frac{3m(k+1)}{14}, & r = 8: \ &\frac{m(5k+4)}{20}, \\ r = 9: \ &\frac{2m(k+1)}{9}. \end{aligned}$$

(Note that the slopes for $r = 6$ and $r = 9$ are equal.) Since the factor m is common to all expressions, it suffices to compare the normalized slopes

$$\frac{k}{3}, \quad \frac{k+1}{4}, \quad \frac{k+1}{5}, \quad \frac{2(k+1)}{9}, \quad \frac{3(k+1)}{14}, \quad \frac{5k+4}{20}.$$

A direct comparison shows that for all $k \ge 2$ we have

$$\frac{k+1}{5} < \min\left\{\frac{k}{3}, \frac{k+1}{4}, \frac{2(k+1)}{9}, \frac{3(k+1)}{14}, \frac{5k+4}{20}\right\}.$$

Hence, among all admissible constructions, the decomposition with $r = 5$ (i.e., $m = 5t$) yields the smallest slope for every $k \ge 2$.

For $k = 1$, however, we have

$$\frac{k}{3} = \frac{1}{3} < \frac{2}{5} = \frac{k+1}{5},$$

so in this case the decomposition with $r = 3$ gives the smallest slope.

Consequently, for sufficiently large n, the best linear bound is obtained from the decomposition $m = 5t$ whenever $k \ge 2$, and from $m = 3t$ when $k = 1$.

The case $k = 1$ corresponds to classical Roman domination and has already been extensively studied in the literature [12,13,19]. We therefore focus on the general case

$k \geq 2$ and use the above observations to derive upper bounds for arbitrary cylindrical graphs $C_m \square P_n$.

Motivated by the fact that the decomposition $m = 5t$ yields the smallest slope for all $k \geq 2$, we take the construction for $C_{5t} \square P_n$ as a starting point and extend it to general values of m. More precisely, we modify the periodic pattern of width 5 by locally removing certain rows and adjusting the remaining labels so that the $[k]$-Roman domination condition is preserved. In this way, we obtain constructions for all residue classes of m modulo 5, which allows us to derive upper bounds for all cylindrical graphs $C_m \square P_n$. This leads to the following theorem.

Theorem 6. *Let $k \geq 2$ and $n \geq 4$. For $m \geq 3$,*

$$\gamma_{[kR]}(C_m \square P_n) \leq \begin{cases} \frac{m}{5}\big(n(k+1)+2k\big), & m \equiv 0 \pmod{5}, \\ \frac{m+4}{5}\big(n(k+1)+2k\big) - (4k+1) - 2(k+2)\left\lfloor \frac{n-2}{5} \right\rfloor, & m \equiv 1 \pmod{5}, \\ \frac{m+3}{5}\big(n(k+1)+2k\big) - 3k - (k+3)\left\lfloor \frac{n-2}{5} \right\rfloor, & m \equiv 2 \pmod{5}, \\ \frac{m+2}{5}\big(n(k+1)+2k\big) - 2k - 2\left\lfloor \frac{n-2}{5} \right\rfloor, & m \equiv 3 \pmod{5}, \\ \frac{m+1}{5}\big(n(k+1)+2k\big) - 2k, & m \equiv 4 \pmod{5}. \end{cases}$$

Proof. For $m \equiv 0 \pmod{5}$, the claim follows directly from Theorem 5.

Assume now that $m \not\equiv 0 \pmod{5}$. In each residue class, we obtain a labeling on $C_m \square P_n$ from the periodic labeling on the next larger cylinder $C_{m+r} \square P_n$, where $r \in \{1, 2, 3, 4\}$ is chosen so that $m + r \equiv 0 \pmod{5}$. We then delete $5 - r$ consecutive rows and modify the labels only in a bounded neighborhood of the deleted rows. The explicit local constructions for the residue classes of m and n modulo 5 are given in Appendix A. In each case, only the vertices in the neighborhood of the deleted rows are affected, while the remaining part of the labeling stays periodic. The local corrections guarantee that every zero-labeled vertex still receives desired total weight from its closed neighborhood. Thus the resulting labeling is a valid $[k]$-Roman dominating function on $C_m \square P_n$, and the stated upper bounds follow. □

For clarity, Table 1 summarizes the five residue classes appearing in Theorem 6. The first term in each row corresponds to the construction obtained from the next larger multiple of 5, while the subtracted terms represent the savings obtained by deleting rows and applying local corrections.

As a simple illustration, consider $m = 6$. Since $6 \equiv 1 \pmod{5}$, Theorem 6 gives

$$\gamma_{[kR]}(C_6 \square P_n) \leq 2n(k+1) - 1 - 2(k+2)\left\lfloor \frac{n-2}{5} \right\rfloor.$$

The same residue class also contains the first exceptional circumference not covered by the multiple-based constructions, namely $m = 11$. Since $11 \equiv 1 \pmod{5}$, Theorem 6 yields

$$\gamma_{[kR]}(C_{11} \square P_n) \leq 3n(k+1) + 2k - 1 - 2(k+2)\left\lfloor \frac{n-2}{5} \right\rfloor.$$

Thus the modulo-5 construction provides an explicit bound even for circumferences such as 11, 13, and 17, which are not multiples of any of the base values $3, \ldots, 9$. The optimality of these bounds remains open, but the example illustrates the flexibility of the residue-class method.

Table 1. Summary of the residue-class upper bounds in Theorem 6.

Residue Class of m	Upper Bound for $\gamma_{[kR]}(C_m \square P_n)$
$m \equiv 0 \pmod 5$	$\frac{m}{5}(n(k+1)+2k)$
$m \equiv 1 \pmod 5$	$\frac{m+4}{5}(n(k+1)+2k)-(4k+1)-2(k+2)\left\lfloor\frac{n-2}{5}\right\rfloor$
$m \equiv 2 \pmod 5$	$\frac{m+3}{5}(n(k+1)+2k)-3k-(k+3)\left\lfloor\frac{n-2}{5}\right\rfloor$
$m \equiv 3 \pmod 5$	$\frac{m+2}{5}(n(k+1)+2k)-2k-2\left\lfloor\frac{n-2}{5}\right\rfloor$
$m \equiv 4 \pmod 5$	$\frac{m+1}{5}(n(k+1)+2k)-2k$

4.2. Uniform Constructions

The uniform ceiling-type constructions obtained for the base cases $m = 3, \ldots, 9$ can likewise be extended to general cylindrical graphs. By repeating the corresponding periodic pattern along the cycle direction, we obtain a unified upper bound of the following form.

Theorem 7. *For $n \geq 4$,*

$$\gamma_{[kR]}(C_m \square P_n) \leq m(n-2)\left\lceil\frac{k+4}{5}\right\rceil + 2m\left\lceil\frac{k+3-\left\lceil\frac{k+4}{5}\right\rceil}{3}\right\rceil.$$

Proof. Let

$$a = \left\lceil\frac{k+4}{5}\right\rceil \quad \text{and} \quad b = \left\lceil\frac{k+3-a}{3}\right\rceil.$$

We assign weight a to every vertex in each interior fibre F_i, $1 \leq i \leq n-2$. On the two boundary fibres F_0 and F_{n-1}, we assign weight b to every vertex.

We first verify that this labeling is a $[k]$-RDF. Let v be an interior vertex. If v belongs to a fibre F_i with $2 \leq i \leq n-3$, then all vertices in $N[v]$ have weight a, and hence

$$f(N[v]) = 5a \geq k+4.$$

If v belongs to F_1 or F_{n-2}, then one of its neighbors lies in a boundary fibre and therefore has weight b, while the remaining vertices in $N[v]$ have weight a. Thus

$$f(N[v]) \geq 4a + b.$$

Since

$$4a + b \geq k+4,$$

every interior vertex satisfies the required condition.

For a boundary vertex v, its closed neighborhood contains the vertex itself, its two neighbors on the cycle in the same fibre, and one neighbor in the adjacent interior fibre. Hence

$$f(N[v]) \geq a + 3b.$$

By the definition of b, we have

$$3b \geq k+3-a,$$

and therefore

$$f(N[v]) \geq a + (k+3-a) = k+3.$$

Since a boundary vertex has at most three neighbors, the $[k]$-Roman domination condition also holds at every boundary vertex.

Thus the constructed labeling is a valid $[k]$-RDF on $C_m \square P_n$. Its total weight is the sum of the contributions of the interior and boundary fibres. There are $n-2$ interior fibres, each containing m vertices of weight a, so their total contribution equals

$$m(n-2)a.$$

The two boundary fibres together contain $2m$ vertices of weight b, so their total contribution equals

$$2mb.$$

Consequently,

$$\gamma_{[kR]}(C_m \square P_n) \leq m(n-2)\left\lceil \frac{k+4}{5} \right\rceil + 2m \left\lceil \frac{k+3-\left\lceil \frac{k+4}{5} \right\rceil}{3} \right\rceil,$$

as claimed. □

4.3. Packing Constructions

The packing-based refinements developed in the previous subsections can be extended to larger cylindrical graphs. The key idea is that vertices forming a packing can have their weights reduced simultaneously without violating the $[k]$-Roman domination condition, which leads to an improvement of the basic ceiling-type upper bound. This yields the following general estimate.

Theorem 8. *For $n \geq 4$,*

$$\gamma_{[kR]}(C_m \square P_n) \leq m(n-2)\left\lceil \frac{k+5}{5} \right\rceil + 2m \left\lceil \frac{k+4-\left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil - \rho(C_m \square P_n),$$

where $\rho(C_m \square P_n)$ denotes the packing number of $C_m \square P_n$.

Proof. Let

$$a' = \left\lceil \frac{k+5}{5} \right\rceil \qquad \text{and} \qquad b' = \left\lceil \frac{k+4-a'}{3} \right\rceil.$$

We define a labeling f by assigning weight a' to every vertex of each interior fibre F_i, $1 \leq i \leq n-2$, and weight b' to every vertex of the two boundary fibres.

Let v be an interior vertex. If v belongs to a fibre F_i with $2 \leq i \leq n-3$, then all vertices in $N[v]$ have weight a', and hence

$$f(N[v]) = 5a' \geq k+5.$$

If v belongs to F_1 or F_{n-2}, then one of its neighbors lies in a boundary fibre and has weight b', while the remaining four vertices in $N[v]$ have weight a'. Thus

$$f(N[v]) \geq 4a' + b'.$$

Since

$$4a' + b' \geq k + 5,$$

it follows that every interior vertex satisfies the required condition.

On the other hand, for every boundary vertex v,

$$f(N[v]) \geq a' + 3b' \geq k + 4.$$

Thus the neighborhood sums exceed the minimum requirement by at least one unit at the relevant vertices, which creates the necessary slack.

Let S be a maximum packing of $C_m \square P_n$. By the definition of packing, the closed neighborhoods of distinct vertices of S are pairwise disjoint. The labeling constructed above has one extra unit of slack in every closed neighborhood relevant to the $[k]$-Roman domination condition. Therefore, by Remark 1, decreasing the weight of every vertex of S by one preserves the $[k]$-Roman domination condition.

The total weight of the resulting $[k]$-RDF is obtained from the original ceiling-type construction by subtracting exactly one for each vertex of the packing set S. The original construction has total weight

$$m(n-2)a' + 2mb',$$

and the reduction contributes

$$|S| = \rho(C_m \square P_n).$$

Consequently,

$$\gamma_{[kR]}(C_m \square P_n) \leq m(n-2)\left\lceil \frac{k+5}{5} \right\rceil + 2m \left\lceil \frac{k + 4 - \left\lceil \frac{k+5}{5} \right\rceil}{3} \right\rceil - \rho(C_m \square P_n),$$

which proves the theorem. □

5. Upper Bounds for the Double Roman Domination Number of $C_m \square P_n$

For small values of k, numerical comparison of all derived bounds shows that the linear constructions consistently provide the smallest upper estimates among the three families (linear, uniform, and packing-based). In particular, for $k = 2$ the linear bounds dominate in all relevant parameter ranges considered above. Therefore, we state explicitly only the consequence of the linear construction for the double Roman domination number.

For circumferences that are multiples of $3, \ldots, 9$, the linear constructions from Theorem 5 yield the following bounds.

On the other hand, substituting $k = 2$ into Theorem 6 yields a residue-class bound that applies to all cylindrical grids.

It is natural to compare Corollaries 1 and 2. The first one is based on constructions tailored to special circumferences, whereas the second one applies to all cylindrical grids. Hence the residue-class bound of Corollary 2 has a wider range of applicability, but it is not always numerically smaller for small values of m.

To determine which estimate is stronger, we compare the linear coefficients with respect to n. For large n, the dominant term determines which construction provides the better bound.

Corollary 1. *Let $t \geq 1$ and $n \geq 2$. Then the following bounds hold:*

$$\gamma_{[2R]}(C_m \square P_n) \leq \begin{cases} t(2n+2), & m = 3t, \\ 3tn, & m = 4t, \\ t(3n+4), & m = 5t, \\ t\left(3\left\lceil \frac{4n}{3} \right\rceil + 3\right), & m = 6t,\ n \equiv 0 \pmod 3, \\ 3t\left\lceil \frac{4n}{3} \right\rceil, & m = 6t,\ n \equiv 1 \pmod 3, \\ t\left(3\left\lceil \frac{4n}{3} \right\rceil + 2\right), & m = 6t,\ n \equiv 2 \pmod 3, \\ t\frac{9n+11}{2}, & m = 7t,\ n \text{ odd}, \\ t\frac{9n+10}{2}, & m = 7t,\ n \text{ even}, \\ t\left(6n - \left(\left\lfloor \frac{n-2}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor\right) + 4\right), & m = 8t, \\ t(6n+4), & m = 9t. \end{cases}$$

Corollary 2. *Let $m \geq 3$ and $n \geq 4$. Then*

$$\gamma_{[2R]}(C_m \square P_n) \leq \begin{cases} \frac{m}{5}(3n+4), & m \equiv 0 \pmod 5, \\ \frac{m+4}{5}(3n+4) - 9 - 8\left\lceil \frac{n-2}{5} \right\rceil, & m \equiv 1 \pmod 5, \\ \frac{m+3}{5}(3n+4) - 6 - 5\left\lceil \frac{n-2}{5} \right\rceil, & m \equiv 2 \pmod 5, \\ \frac{m+2}{5}(3n+4) - 4 - 2\left\lceil \frac{n-2}{5} \right\rceil, & m \equiv 3 \pmod 5, \\ \frac{m+1}{5}(3n+4) - 4, & m \equiv 4 \pmod 5. \end{cases}$$

Theorem 9. *Let $m \geq 3$ and $n \geq 4$.*

1. *If $5 \mid m$, then the bounds from Corollaries 1 and 2 coincide.*
2. *For $5 \nmid m$, $3 \leq m \leq 19$, the comparison is as follows:*
   - *The bound from Corollary 1 is smaller for*
     - $m \in \{3, 6, 7, 12\}$;
     - $m = 8$ *and* $4 \leq n \leq 11$, *or* $n \geq 12$ *and* $n \equiv 0, 1 \pmod 5$;
     - $m = 16$ *and* $4 \leq n \leq 6$;
     - $m = 18$ *and* $n \in \{4, \ldots, 11, 13, 16\}$.
   - *the bounds coincide for*
     - $m \in \{4, 9\}$;
     - $m = 8$ *and* $n \geq 12$ *and* $n \not\equiv 0, 1 \pmod 5$;
     - $m = 18$ *and* $n \in \{12, 14, 15, 19\}$.
   - *The bound from Corollary 2 is smaller for*
     - $m = 14$;

  - $m = 16$ *and* $n \geq 7$;
  - $m = 18$ *and* $n \in \{17, 18\}$ *or* $n \geq 20$.

3. *For* $5 \nmid m$, $m \geq 19$ *and for sufficiently large* n *the bound from Corollary 2 is strictly smaller than every applicable bound from Corollary 1.*

Proof. We compare the dominant linear coefficients in n appearing in Corollaries 1 and 2.

For Corollary 1, admissible decompositions $m = rt$ with $r \in \{3, \ldots, 9\}$ yield asymptotic slopes reduced to the set

$$\left\{ \frac{3m}{5}, \frac{9m}{14}, \frac{2m}{3}, \frac{3m}{4} \right\}.$$

For Corollary 2 the asymptotic slopes are from the set

$$\left\{ \frac{3m}{5}, \frac{3m+4}{5}, \frac{3m+3}{5} \right\}.$$

If $5 \mid m$, both constructions yield the same bound. Therefore, assume that $5 \nmid m$. Among the slopes arising from Corollary 2, the largest one equals

$$\frac{3m+4}{5},$$

while among the admissible slopes from Corollary 1 the smallest one equals

$$\frac{9m}{14}.$$

A direct comparison shows that

$$\frac{3m+4}{5} < \frac{9m}{14}$$

if and only if

$$42m + 56 < 45m,$$

that is, if and only if $m > \frac{56}{3}$. Hence for every $m \geq 19$ with $5 \nmid m$, the residue-class construction has strictly smaller slope than any construction from Corollary 1. Consequently, Corollary 2 yields the better bound for all sufficiently large n in this range.

It remains to compare the finitely many cases $3 \leq m \leq 18$ directly. For each admissible value of m in this range, we evaluated the explicit expressions from Corollaries 1 and 2 and compared the resulting bounds as functions of n for all $n \geq 4$.

More precisely, for every fixed m we first determined all applicable decompositions $m = rt$ with $r \in \{3, \ldots, 9\}$ and selected the smallest corresponding bound from Corollary 1. We then compared this bound with the residue-class bound from Corollary 2. Since both expressions are linear functions of n up to periodic floor terms, their difference can be analyzed explicitly. In particular, by comparing the linear coefficients in n we determined the asymptotic ordering of the bounds, and for the remaining cases we computed the exact values of the difference for $n \geq 4$ in order to identify the transition points where the ordering changes.

We illustrate the above comparison procedure in the case $m = 16$. The admissible decompositions $16 = 4 \cdot 4 = 8 \cdot 2$ yield from Corollary 1 the bound

$$\min\left\{ 12n,\ 12n - 2\left(\left\lfloor \frac{n-2}{5} \right\rfloor + \left\lfloor \frac{n}{5} \right\rfloor \right) + 8 \right\}.$$

Since $16 \equiv 1 \pmod 5$, Corollary 2 gives

$$12n + 7 - 8\left\lfloor \frac{n-2}{5} \right\rfloor.$$

Comparing the two expressions for $n \geq 4$ shows that the multiple-based bound is smaller for $4 \leq n \leq 6$, while the residue-class bound is smaller for $n \geq 7$. □

Theorem 9 shows that the multiple-based constructions remain optimal only for a small number of exceptional circumferences, while the residue-class construction provides the strongest general upper bounds for all sufficiently large m and n. Together, the two approaches therefore yield effective estimates for $\gamma_{[2R]}(C_m \square P_n)$ for all cylindrical grids.

6. Conclusions

In this paper, we studied $[k]$-Roman domination on cylindrical grids $C_m \square P_n$ and developed several constructive upper bounds based on three different approaches: linear periodic constructions, uniform ceiling-type assignments, and packing-based refinements. For the special case $C_9 \square P_n$, we compared these three families in detail and showed that their relative performance depends essentially on the parameter k. For small values of k, the linear constructions provide the best bounds, whereas for larger values of k the uniform and packing-based constructions become asymptotically superior.

We then extended the linear constructions to all cylindrical grids whose circumference is a multiple of one of the values $3, \ldots, 9$, obtaining a unified family of explicit upper bounds. Motivated by the fact that blocks of length five give the smallest asymptotic slope for $k \geq 2$, we further derived a general residue-class construction depending on $m \bmod 5$. This yielded upper bounds for $\gamma_{[kR]}(C_m \square P_n)$ for all cylindrical grids and, in particular, explicit consequences for the double Roman domination number.

For $k = 2$, we showed that the linear constructions provide the strongest bounds among the three general families considered here. This allowed us to compare in detail the multiple-based bounds and the modulo-5 construction. The comparison shows that the residue-class construction is asymptotically best for all sufficiently large admissible circumferences, while the multiple-based constructions remain optimal only in several exceptional small cases.

The obtained upper bounds should be viewed in relation to the known density-type lower estimates for cylindrical grids. In earlier work, lower bounds based on local neighborhood constraints and efficient domination show that the natural benchmark for $\gamma_{[kR]}(C_m \square P_n)$ is of the order

$$(k+1)\left\lceil \frac{mn}{5} \right\rceil.$$

Thus the constructions developed in the present paper have the correct linear dependence on both m and n, while the remaining gap is mainly reflected in the precise leading constants and in additive boundary terms. Closing this gap remains one of the main challenges for future work.

In this way, the present paper completes the analysis of the previously untreated base case $C_9 \square P_n$ and closes the remaining gap in the sequence of periodic constructions for cylindrical grids with small fixed circumferences. Combined with the earlier results for $m \in \{3, 4, \ldots, 8\}$, the constructions developed here now provide a unified framework for constructive upper bounds on $[k]$-Roman domination for all cylindrical grids.

The obtained results also refine several previously known bounds for double Roman domination on Cartesian product graphs. In particular, Corollary 1 improves Theorem 3.3 of Anu and Lakshmanan [16] for cylindrical graph families of the form $C_3 \square P_n \cong K_3 \square P_n$.

More generally, the present paper extends such product-based results from the special case $k = 2$ to the full $[k]$-Roman domination framework and shows that constructions adapted to cylindrical grids lead to sharper estimates than those obtained from general Cartesian product techniques alone.

There are several natural directions for future work. First, the general bounds in Theorem 6 are formulated only in terms of the residue class of m modulo 5. However, the terminal corrections also depend on the residue class of n modulo 5, because the periodic pattern is truncated differently at the two ends of the path. The constructions collected in Appendix A illustrate this dependence.

For example, in the case $m \equiv 1 \pmod 5$ and $n \equiv 3 \pmod 5$, the local correction allows a saving of $2k+1$ at both ends of the path. In the uniform statement of Theorem 6, however, we use a correction that is valid for all values of n: one obtains a saving of $2k+1$ at one end and $2k$ at the other end. Thus a separate treatment of the five residue classes of n modulo 5 could slightly improve the additive constants in the main theorem. Such refinements would not change the asymptotic slope in n, but they could sharpen the bounds for fixed and moderate values of n.

Second, the comparison carried out for the double Roman domination number suggests that several small circumferences still deserve special attention. In particular, the graphs $C_{11}\Box P_n$, $C_{13}\Box P_n$, and $C_{17}\Box P_n$ are not multiples of any of the base circumferences $3, \ldots, 9$ considered in the multiple-based construction. Although the modulo-5 construction provides explicit upper bounds for these graphs, it is not clear whether these bounds are optimal, or even asymptotically best among constructions adapted to these particular circumferences.

A possible way to treat these missing cases would be to develop tailored constructions for $C_{11}\Box P_n$, $C_{13}\Box P_n$, and $C_{17}\Box P_n$, and to combine them with packing-based refinements. This would require determining the corresponding packing numbers $\rho(C_{11}\Box P_n)$, $\rho(C_{13}\Box P_n)$, and $\rho(C_{17}\Box P_n)$. Then one could compare the resulting linear coefficients in n with those of the general modulo-5 construction, as was done above for the double Roman case. This remains an interesting open problem.

Finally, another challenging direction is to determine exact values, or at least asymptotically tight formulas, for $\gamma_{[kR]}(C_m\Box P_n)$ for fixed small values of k. The periodic structure of cylindrical grids suggests that sharper characterizations may be possible, especially for double and triple Roman domination. A possible approach to obtain exact formulae is to use the algebraic method as in [19,20]. While the method is universal it is computationaly intensive, and consequently, only some partial results for relative small m are obtained until now.

Author Contributions: Conceptualization, S.B. and J.Ž.; Methodology, S.B. and J.Ž.; Software, S.B. and J.Ž.; Validation, S.B. and J.Ž.; Formal analysis, S.B. and J.Ž.; Investigation, S.B. and J.Ž.; Resources, S.B. and J.Ž.; Data curation, S.B. and J.Ž.; Writing—original draft, S.B. and J.Ž.; Writing—review and editing, S.B. and J.Ž.; Visualization, S.B. and J.Ž.; Supervision, S.B. and J.Ž.; Project administration, S.B. and J.Ž.; Funding acquisition, S.B. and J.Ž. All authors have read and agreed to the published version of the manuscript.

Funding: The first author (S.B.) acknowledges financial support from the Slovenian Research and Innovation Agency (ARIS) through research programme No. P1-0297 and research project No. J1-70016. The second author (J.Ž.) was partially supported by ARIS through the annual work program of Rudolfovo and by the research grants P2-0248 and L1-60136. This work was also supported in part by the Horizon Europe project Quantum Excellence Centre for Quantum-Enhanced Applications (QEC4QEA). All authors have read and agreed to the published version of the manuscript.

Data Availability Statement: During the research no new data were created.

Acknowledgments: The authors wish to sincerely thank to the four anonymous reviewers for their constructive remarks that helped us to considerably improve the manuscript.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A. Local Constructions for the Modulo-5 Residue Classes

This appendix contains the local constructions used in the proof of Theorem 6. They describe how the periodic construction based on blocks of length five can be modified when the circumference m is not divisible by 5. The four cases correspond to the residue classes $m \equiv 4, 3, 2, 1 \pmod 5$, respectively. Within each case, the subcases depend on the residue class of n modulo 5. The gray marks indicate the local row deletions and boundary corrections needed to preserve the $[k]$-Roman domination condition.

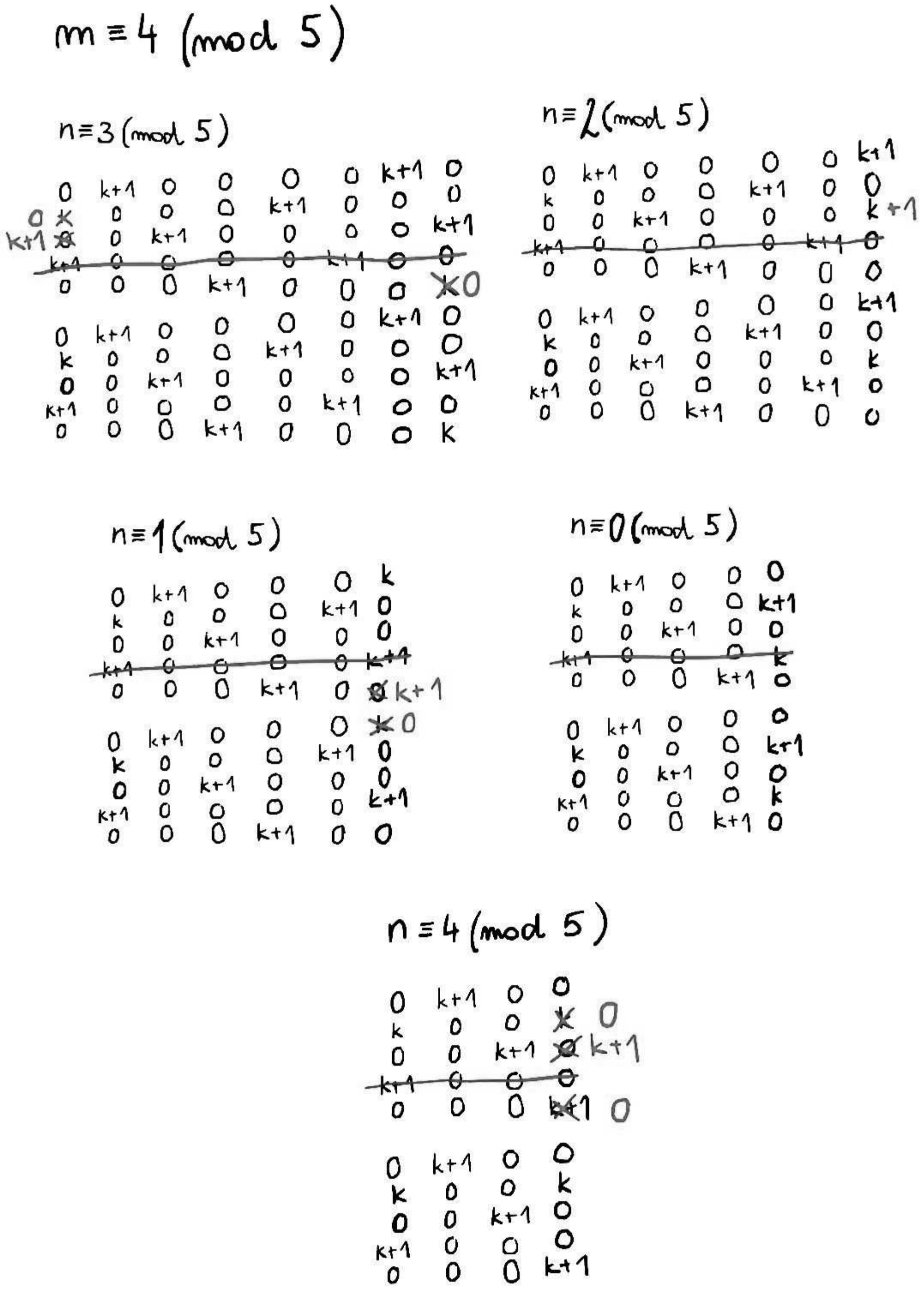

Figure A1. Local constructions for the case $m \equiv 4 \pmod 5$.

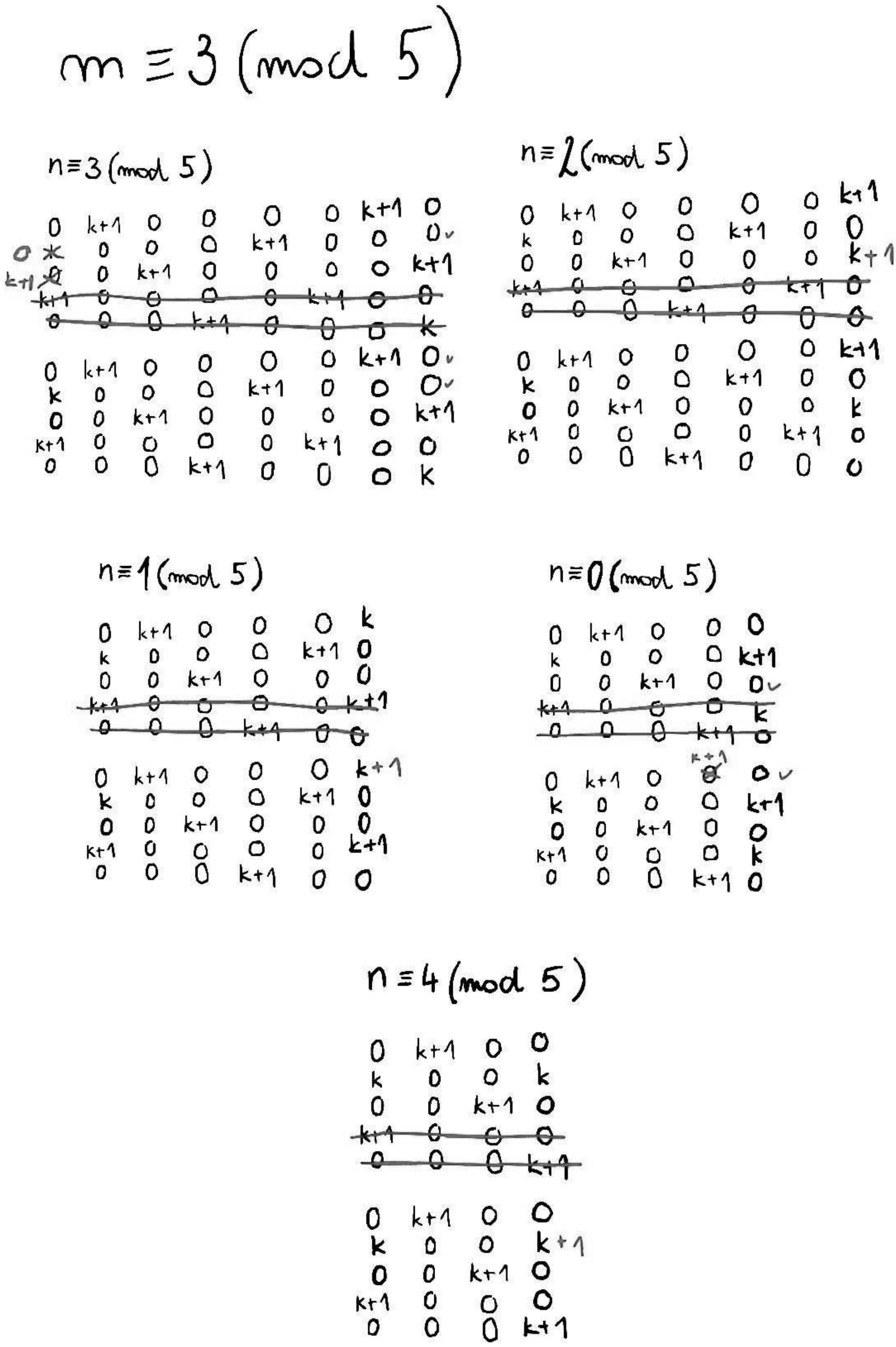

Figure A2. Local constructions for the case $m \equiv 3 \pmod 5$.

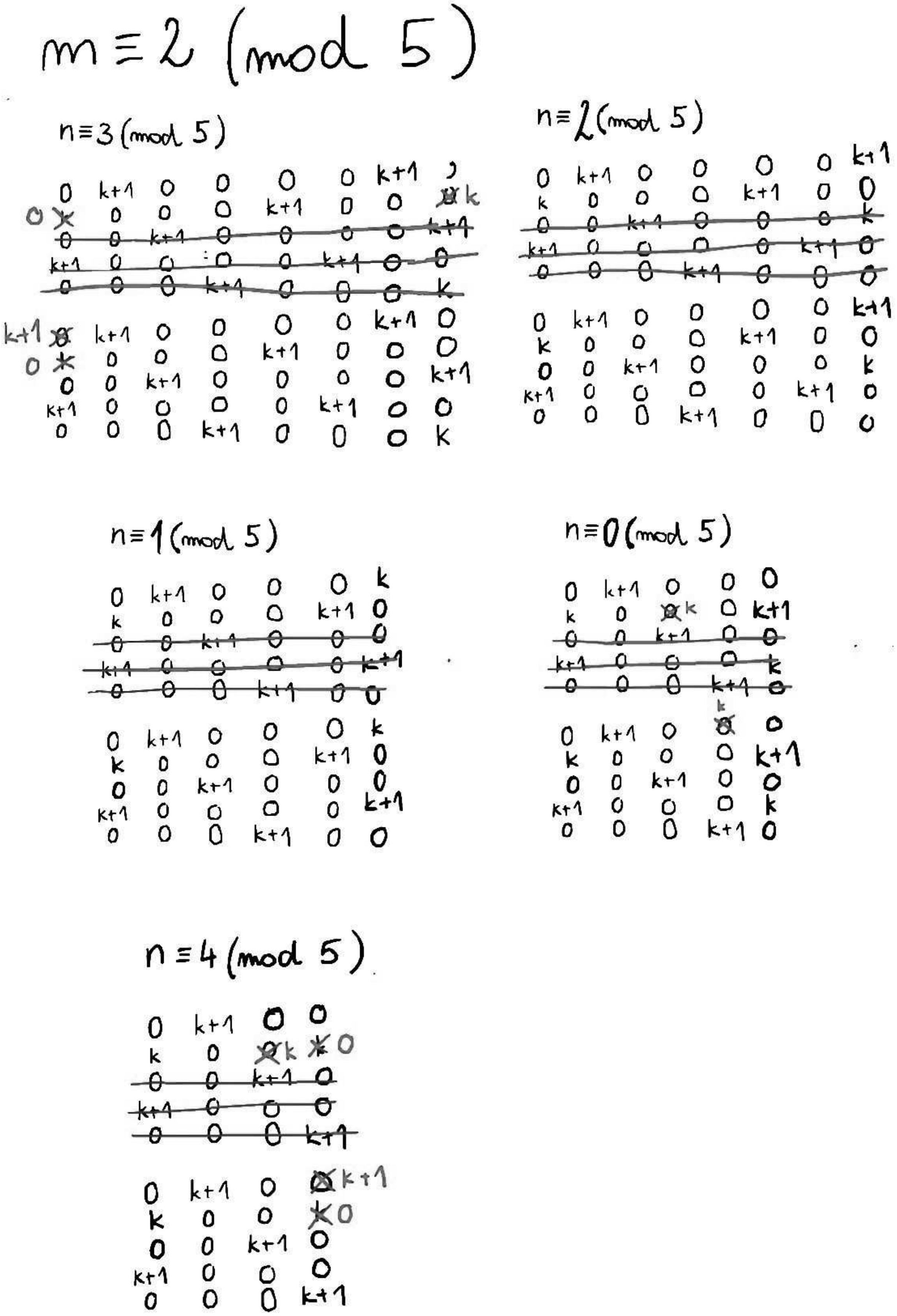

Figure A3. Local constructions for the case $m \equiv 2 \pmod 5$.

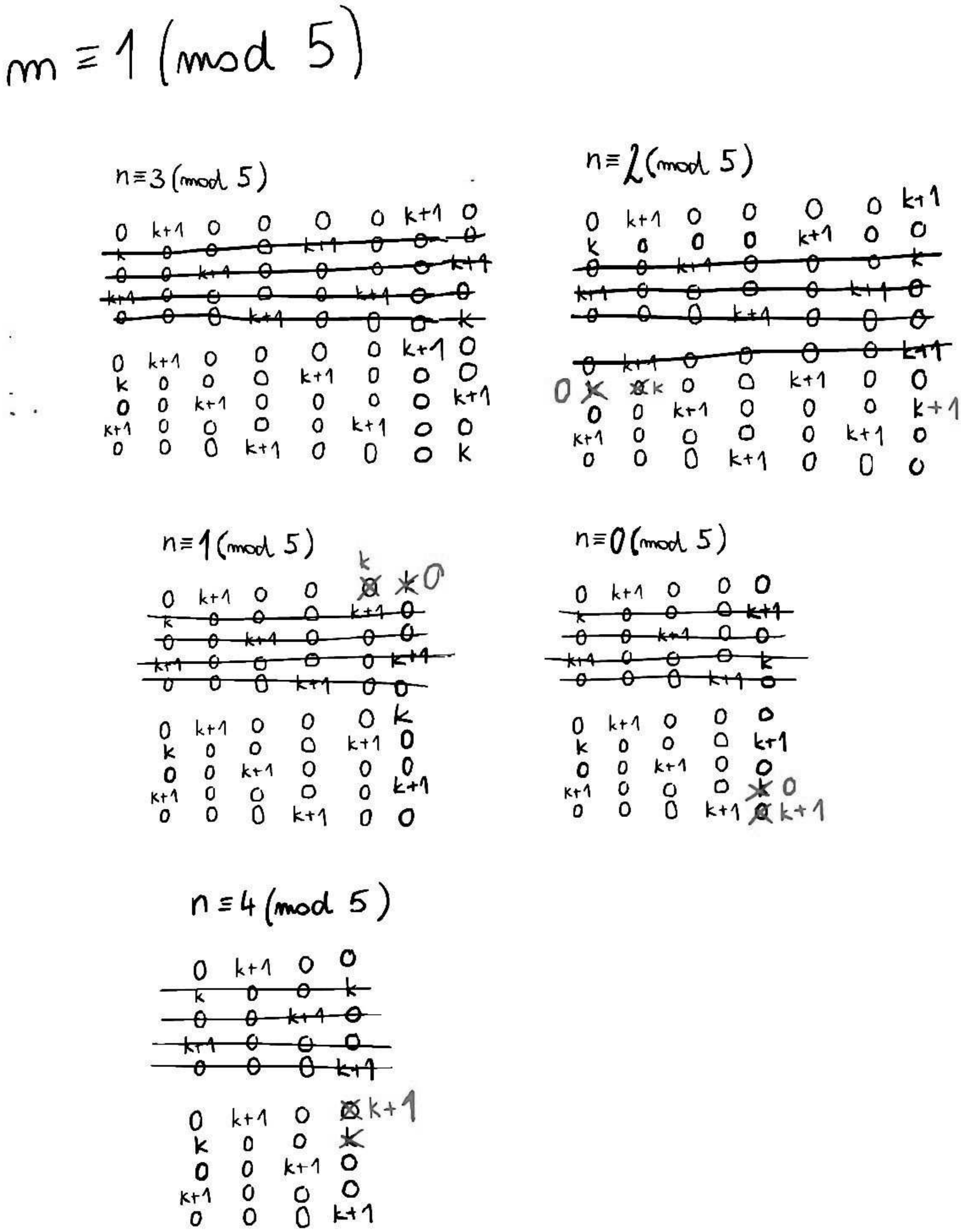

Figure A4. Local constructions for the case $m \equiv 1 \pmod 5$.

References

1. Haynes, T.; Hedetniemi, S.; Slater, P. *Fundamentals of Domination in Graphs*, 1st ed.; CRC Press: Boca Raton, FL, USA, 1998. [CrossRef]
2. Garey, M.R.; Johnson, D.S. *Computers and Intractability: A Guide to the Theory of NP-Completeness*; W. H. Freeman & Co.: New York, NY, USA, 1979.

3. Gan, X.; Li, T.; Gong, C.; Li, D.; Dong, D.; Liu, J.; Lu, K. GraphCSR: A Degree-Equalized CSR Format for Large-scale Graph Processing. *Proc. VLDB Endow.* **2025**, *18*, 4255–4268. [CrossRef]
4. Khemani, B.; Patil, S.; Kotecha, K.; Tanwar, S. A review of graph neural networks: Concepts, architectures, techniques, challenges, datasets, applications, and future directions. *J. Big Data* **2024**, *11*, 18. [CrossRef]
5. Cockayne, E.J.; Dreyer, P.A.; Hedetniemi, S.M.; Hedetniemi, S.T. Roman domination in graphs. *Discret. Math.* **2004**, *278*, 11–22. [CrossRef]
6. Beeler, R.A.; Haynes, T.W.; Hedetniemi, S.T. Double Roman domination. *Discret. Appl. Math.* **2016**, *211*, 23–29. [CrossRef]
7. Rupnik Poklukar, D.; Žerovnik, J. Double Roman Domination: A Survey. *Mathematics* **2023**, *11*, 351. [CrossRef]
8. Abdollahzadeh Ahangar, H.; Álvarez, M.; Chellali, M.; Sheikholeslami, S.; Valenzuela-Tripodoro, J. Triple Roman domination in graphs. *Appl. Math. Comput.* **2021**, *391*, 125444. [CrossRef]
9. Chellali, M.; Jafari Rad, N.; Sheikholeslami, S.M.; Volkmann, L. Varieties of Roman Domination. In *Structures of Domination in Graphs*; Haynes, T.W., Hedetniemi, S.T., Henning, M.A., Eds.; Developments in Mathematics; Springer: Cham, Switzerland, 2021; Volume 66, pp. 273–307. [CrossRef]
10. Chellali, M.; Jafari Rad, N.; Sheikholeslami, S.M.; Volkmann, L. Varieties of Roman domination II. *AKCE Int. J. Graphs Comb.* **2020**, *17*, 966–984. [CrossRef]
11. Imrich, W.; Klavzar, S. *Product Graphs: Structure and Recognition*; Wiley-Interscience: New York, NY, USA, 2000.
12. Chang, T.Y. Domination Numbers of Grid Graphs. Ph.D. Thesis, Department of Mathematics, University of South Florida, Tampa, FL, USA, 1992.
13. Alanko, S.; Crevals, S.; Isopoussu, A.; Ostergard, P.; Pettersson, V. Computing the domination number of grid graphs. *Electron. J. Comb.* **2011**, *18*, 141. [CrossRef] [PubMed]
14. Valenzuela-Tripodoro, J.C.; Mateos-Camacho, M.A.; Cera, M.; Alvarez-Ruiz, M.P. The $[k]$-multiple Roman domination in graphs. *Res. Sq.* **2023**, *preprint*. [CrossRef]
15. Khalili, N.; Amjadi, J.; Chellali, M.; Sheikholeslami, S.M. On [k]-Roman domination in graphs. *AKCE Int. J. Graphs Comb.* **2023**, *20*, 291–299. [CrossRef]
16. Anu, V.; Lakshmanan, S.A. Double Roman Domination in Cartesian Product. *Creat. Math. Inform.* **2024**, *33*, 1–6. [CrossRef]
17. Brezovnik, S.; Žerovnik, J. $[k]$-Roman Domination on Cylindrical Grids $C_m \square P_n$. *arXiv* **2026**, arXiv:2603.02831.
18. Brezovnik, S.; Žerovnik, J. Further results on $[k]$-Roman domination on cylindrical grids $C_m \square P_n$. *arXiv* **2026**, arXiv:2603.25191.
19. Pavlič, P.; Žerovnik, J. A note on the domination number of the Cartesian products of paths and cycles. *Kragujev. J. Math.* **2013**, *37*, 275–285.
20. Martínez, J.A.; Garzón, E.M.; Puertas, M.L. Powers of Large Matrices on GPU Platforms to Compute the Roman Domination Number of Cylindrical Graphs. *IEEE Access* **2021**, *9*, 29346–29355. [CrossRef]