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The universal family of punctured Riemann surfaces is Stein

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Abstract

We show that the universal Teichmüller family $V(g,n)$ of compact Riemann surfaces of genus $g \geqslant 0$ with $n > 0$ punctures is a Stein manifold. We describe its basic function-theoretic properties and pose some challenging questions. We show, in particular, that the space of fibrewise algebraic functions on the universal family is dense in the space of holomorphic functions, and that there is a fibrewise algebraic map of the universal family to a Euclidean space which restricts to a proper embedding on any fibre. We also obtain a relative Oka principle for holomorphic fibrewise algebraic maps of the universal family to any flexible algebraic manifold.

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1 | INTRODUCTION

The notion of a Teichmüller space originates in the papers [50–52] of Oswald Teichmüller, who defined a complex manifold structure on the set of isomorphism classes of marked closed Riemann surfaces of genus g. Ahlfors [2] showed that this complex structure can be defined by periods of holomorphic abelian differentials. In [52], Teichmüller also introduced the universal Teichmüller curve—a space V over a Teichmüller space T whose fibre over $t \in T$ is a Riemann surface (M, J_t) representing that point, also called the universal family of Riemann surfaces over T—and showed that it has the structure of a complex manifold. Teichmüller's theory was developed by Ahlfors and Bers [3] and by Grothendieck, who gave a series of lectures in Cartan's seminar 1960–1961; see the discussion and references in [1]. Grothendieck asked whether every finite-dimensional Teichmüller space is a Stein manifold [32, p. 14]. An affirmative answer was given by Bers and Ehrenpreis [11, Theorem 2] who showed that any finite-dimensional Teichmüller space embeds as a domain of holomorphy in a complex Euclidean space, hence is Stein. (Another proof was given by Wolpert [53]; see also the surveys by Bers [10] and Nag [43].) The Teichmüller space $T(M)$ of a Riemann surface M is finite dimensional if and only if $M = \widehat{M} \setminus \{p_1, \dots, p_n\}$ is a compact Riemann surface $\widehat{M}$ of some genus $g \geqslant 0$ with $n \geqslant 0$ punctures. Such M is said to be of finite conformal type, and its Teichmüller space is denoted $T(g,n)$. The universal family $\pi : \widehat{V}(g,n) \to T(g,n)$ is a holomorphic submersion whose fibre over $t \in T(g,n)$ is the compact surface $\widehat{M}$ endowed with the complex structure J_t determined by t, and with n canonical holomorphic sections $s_1, \dots, s_n : T(g,n) \to \widehat{V}(g,n)$ with pairwise disjoint images representing the punctures. (See Nag [43, pp. 322–323].) The open subset

$$V(g,n) = \widehat{V}(g,n) \setminus \bigcup_{i=1}^{n} s_i(T(g,n)) \tag{1.1}$$

of $\widehat{V}(g,n)$ is the universal family of n-punctured compact Riemann surfaces of genus g. If $2g + n \geqslant 3$ then the Teichmüller family $\pi : V(g,n) \to T(g,n)$ is the universal object in the complex analytic category of topologically marked, holomorphically varying families of n-punctured genus g Riemann surfaces (see [43, Theorem 5.4.3]).

In this paper, we establish several function-theoretic properties of the universal family $V(g,n)$. Our first main result is the following.

Theorem 1.1. *The Teichmüller family $V(g,n)$ for $n \geqslant 1$ is a Stein manifold.*

This is a special case of Theorems 2.1 and 2.2. The result is obvious if $g = 0$ and $n \in \{1,2,3\}$ since $T(0,n)$ is then a singleton and $V(0,n)$ is biholomorphic to $\mathbb{C}$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and $\mathbb{C} \setminus \{0,1\}$, respectively. If $2g + n \geqslant 3$ then $T(g,n)$ is biholomorphic to a bounded topologically contractible Stein domain in $\mathbb{C}^{3g-3+n}$ [43, p. 161], but it cannot be holomorphically realised as a convex domain in $\mathbb{C}^{3g-3+n}$ if $g \geqslant 2$ since the Kobayashi metric and the Carathéodory metric on it differ (see Marković [42]), while they agree on a convex domain (see Lempert [41]). Note also that $\widehat{V}(g,n)$ is holomorphically convex and all holomorphic functions on it come from the base $T(g,n)$.

We provide three proofs of Theorem 1.1. The first one in Section 2 shows that $V(g,n)$ admits a strongly plurisubharmonic exhaustion function, hence is Stein by a theorem of Grauert [28] (see also [36, Theorem 5.2.10]). Since every fibre of $V(g,n)$ is an affine algebraic curve, it is natural to expect that $V(g,n)$ admits holomorphic functions which are algebraic on every fibre. This is indeed the case. The second proof of the Steinness of $V(g,n)$ follows from Theorem 3.1, which implies that holomorphic fibrewise algebraic functions on $V(g,n)$ separate points and provide holomorphic convexity. The proof relies on Grauert's theorem on coherence of direct images of coherent analytic sheaves and other techniques of complex analytic geometry. In Theorem 3.2 we show that the algebra $\mathscr{A}(V(g,n))$ of fibrewise algebraic functions on $V(g,n)$ is dense in the algebra $\mathscr{O}(V(g,n))$ of holomorphic functions in the compact-open topology. A similar argument gives a holomorphic fibrewise algebraic map $V(g,n) \to \mathbb{C}^N$ for some $N \in \mathbb{N}$ which restricts to a proper embedding on every fibre; see Theorem 3.4, whose proof was contributed by Yiran Lin. Since the Teichmüller space $T(g,n)$ is Stein, it follows that $V(g,n)$ admits a proper holomorphic embedding in a Euclidean space which is algebraic on every fibre; see Corollary 3.5. This provides the third proof that $V(g,n)$ is a Stein manifold. In Section 4 we show that fibrewise algebraic maps of $V(g,n)$ to any algebraically special elliptic manifold (in particular, to any flexible algebraic manifold) satisfy a relative Oka principle; see Theorem 4.1.

In the remainder of this introduction, we mention some consequences of Theorem 1.1 and pose a few open problems.

By classical results of Remmert et al. (see [20, Theorem 2.4.1] and the references therein), $V(g,n)$ for $n \geqslant 1$ admits a proper holomorphic embedding in $\mathbb{C}^N$ with $N = 2\dim V(g,n)+1$, which equals $6g-6+2n+1$ if $2g+n \geqslant 3$. When $\dim V(g,n) \geqslant 2$, it also embeds properly holomorphically in $\mathbb{C}^N$ with $N = \left[\frac{3\dim V(g,n)}{2}\right]+1$ (see Eliashberg and Gromov [16] and Schürmann [46], or the exposition in [20, Sections 9.3–9.4]).

Since the Teichmüller space $T(g,n)$ is contractible, the submersion $\pi : V(g,n) \to T(g,n)$ is smoothly trivial, and the inclusion of any fibre of π in $V(g,n)$ is a homotopy equivalence. It follows that for any manifold Y, the restriction of a continuous map $V(g,n) \to Y$ to any fibre of π lies in the same homotopy class. The following is a corollary to this observation, Theorem 1.1, and the main result of Oka theory [20, Theorem 5.4.4]. (See also the surveys [22, 23, 25].)

Corollary 1.2. *Let $\pi : V(g,n) \to T(g,n)$ be as above, $n \geqslant 1$, and let Y be an Oka manifold. There is a holomorphic map $V(g,n) \to Y$ in every homotopy class. Furthermore, a holomorphic map $M_t = \pi^{-1}(t) \to Y$, $t \in T(g,n)$, from any fibre extends to a holomorphic map $V(g,n) \to Y$. More generally, given a closed complex subvariety T' of $T(g,n)$, every continuous map $F_0 : V(g,n) \to Y$ which is holomorphic on $\pi^{-1}(T')$ is homotopic to a holomorphic map $F : V(g,n) \to Y$ by a homotopy which is fixed on $\pi^{-1}(T')$.*

If $n \geqslant 1$ and $(g,n) \neq (0,1)$ then $V(g,n)$ is not simply connected, and its homotopy type is that of a finite bouquet of circles. In this case, homotopically nontrivial maps $V(g,n) \to Y$ exist whenever the manifold Y is not simply connected. This gives the following corollary. The last statement follows by taking the Oka manifold $Y = \mathbb{C}^*$. The result obviously holds for $g=0,\ n=1$ since $V(0,1) = \mathbb{C}$.

Corollary 1.3. *If $n \geqslant 1$ and Y is an Oka manifold which is not simply connected, there is a holomorphic map $V(g,n) \to Y$ which is nonconstant on every fibre. In particular, $V(g,n)$ admits a nowhere vanishing holomorphic function which is nonconstant on every fibre.*

Another way to obtain fibrewise nonconstant holomorphic maps from $V(g,n)$ to an Oka manifold is to inductively use the Oka property with approximation on compact holomorphically convex subsets of the Stein manifold $V(g,n)$; see [20, Theorem 5.4.4]. The Oka principle can be used to obtain many further properties of holomorphic universal families $V(g,n) \to Y$ in any Oka manifold.

The fact that $V(g,n)$ for $n \geqslant 1$ is homotopy equivalent to a bouquet of circles implies that every complex vector bundle on $V(g,n)$ is topologically trivial. Since $V(g,n)$ is Stein, the Oka–Grauert principle (see Grauert [27] or [20, Theorem 3.2.1]) implies the following.

Proposition 1.4. *Every holomorphic vector bundle on $V(g,n)$ for $n \geqslant 1$ is holomorphically trivial.*

This fails in the algebraic category, even on a single fibre. In fact, the tangent bundle of a finitely punctured compact Riemann surface is not algebraically trivial in general.

Corollary 1.5. *Assume that $g \geqslant 0$ and $n \geqslant 1$.*

(a) *There exists a nowhere vanishing holomorphic vector field ξ on $V(g,n)$ which is tangent to the fibres of the projection $\pi : V(g,n) \to T(g,n)$, that is, $d\pi(\xi) = 0$.*
(b) *With ξ as in (a), there exists a holomorphic 1-form θ on $V(g,n)$ satisfying $\langle \theta, \xi \rangle = 1$. In particular, θ is nowhere vanishing on the tangent bundle to any fibre of π.*

Proof. Part (a) follows by applying Proposition 1.4 to the holomorphic line bundle $\ker d\pi \to V(g,n)$, the vertical tangent bundle of the holomorphic submersion $\pi : V(g,n) \to T(g,n)$. To see (b), consider the following short exact sequence of vector bundles on $V(g,n)$:

$$0 \longrightarrow \ker d\pi \hookrightarrow TV(g,n) \stackrel{\alpha}{\longrightarrow} H := TV(g,n)/\ker \pi \longrightarrow 0.$$

Here, $TV(g,n)$ denotes the tangent bundle of $V(g,n)$. By Cartan's Theorem B the sequence splits, that is, there is a holomorphic vector bundle injection $\sigma : H \hookrightarrow TV(g,n)$ such that $\alpha \circ \sigma = \mathrm{Id}_H$. Hence, $TV(g,n) = \ker d\pi \oplus \sigma(H) = \mathbb{C}\xi \oplus \sigma(H)$, where ξ is as in part (a). The unique holomorphic 1-form θ on $V(g,n)$ satisfying $\langle \theta, \xi \rangle = 1$ and $\xi = 0$ on $\sigma(H)$ clearly satisfies part (b). □

By the Gunning and Narasimhan theorem [33], every open Riemann surface M admits a holomorphic immersion $f : M \to \mathbb{C}$. In view of Corollary 1.5 (b), the following is a natural question.

Problem 1.6. Let $g \geqslant 1$ and $n \geqslant 1$. Is there a holomorphic function $f : V(g,n) \to \mathbb{C}$ whose restriction to every fibre of $V(g,n)$ is an immersion?

By [18, Theorem 1] there exists a holomorphic function $f : V(g,n) \to \mathbb{C}$ without critical points. The problem is to find f such that $\ker df_z$ is transverse to $\ker d\pi_z$ at every point $z \in V(g,n)$. Note that [24, Corollary 8.3] gives a smooth function $f : V(g,n) \to \mathbb{C}$ whose restriction to every fibre is a holomorphic immersion. Problem 1.6 is related to the question whether a holomorphic 1-form θ in Corollary 1.5 (b) can be made exact on every fibre of π by multiplying it with a suitably chosen nowhere vanishing holomorphic function on $V(g,n)$. However, this is not the only problem. Since the Teichmüller submersion $\pi : V(g,n) \to T(g,n)$ does not admit a holomorphic section when $g \geqslant 3$ (see Hubbard [37, 38] and Earle and Kra [15, p. 50] for a precise description of sections of $\widehat{V}(g,n)$ and $V(g,n)$), there is no natural way of choosing the initial point for computing the fibrewise integrals of θ, which would give a holomorphic family of immersions on the fibres.

Another interesting question is whether the Riemann surfaces in the Teichmüller family $V(g,n)$, $n \geqslant 1$, admit a representation as a family of conformal minimal surfaces in $\mathbb{R}^k$, $k \geqslant 3$, whose (1,0)-derivatives depend holomorphically on $t \in T(g,n)$. For background, see [44] and [4]. By [24, Corollary 8.6] the answer is affirmative with continuous or smooth dependence on the parameter. It remains an open problem whether minimal surfaces in such families can be chosen to be complete and with finite total curvature. Each single surface in the family can be made such by [6, 7].

It is natural to wonder whether the results of this paper extend to infinite dimensional Teichmüller families. The analogous problem with continuous or smooth dependence of the complex structures and the holomorphic maps on the parameter was studied in [24] for very general families of open Riemann surfaces $\{(M, J_t)\}_{t \in T}$, where M is a smooth open surface and J_t are complex structures on M of some local Hölder class depending continuously or smoothly on the parameter t in a topological space T. The Riemann surfaces in such families need not belong to the same Teichmüller space. For example, a punctured Riemann surface can be a member of a family in which the punctures develop into boundary curves, and vice versa, boundary curves may degenerate to punctures. Under mild assumptions on the family $\{J_t\}_{t\in T}$, J_t-holomorphic functions on (M, J_t) satisfy the Runge approximation theorem with continuous or smooth dependence on $t \in T$ [24, Theorem 1.1]. For a class of parameter spaces including finite CW complexes we also have the Oka principle for continuous or smooth families of holomorphic maps from (M, J_t) to any Oka manifold [24, Theorem 1.6]. These results were extended in [26] to maps from tame families of Stein manifolds of arbitrary dimension to Oka manifolds.

2 | PROOF OF THEOREM 1.1

In view of the description of the Teichmüller submersion $\pi : V(g,n) \to T(g,n)$ (1.1), Theorem 1.1 is an immediate consequence of the following result with an arbitrary Stein manifold as the base.

Theorem 2.1. *Assume that X is a Stein manifold, Z is a complex manifold with* $\dim Z = \dim X + 1$, *$\pi : Z \to X$ is a surjective proper holomorphic submersion with connected fibres, and $s_1, \dots, s_n : X \to Z$ $(n \geqslant 1)$ are holomorphic sections with pairwise disjoint images. Then, the domain $\Omega = Z \setminus \bigcup_{i=1}^n s_i(X)$ is Stein.*

The assumption that the sections $s_1, \dots, s_n$ have pairwise disjoint images is inessential and is imposed only for convenience of the proof; see Theorem 2.2 for a more general result. The conclusion fails if the fibres of π have complex dimension > 1, or if the sections s_i are not holomorphic. In such a case, the domain Ω in the theorem fails to be locally pseudoconvex at some boundary point $s_i(x)$, $x \in X$. Note that Stein complements of smooth complex hypersurfaces in compact Kähler manifolds have recently been studied by Höring and Peternell [35] where the reader can find references to earlier works. In our case, Z is not compact unless X is a point.

Proof of Theorem 2.1. Let $\pi : Z \to X$ be as in the theorem. Note that every fibre $Z_x = \pi^{-1}(x)$, $x \in X$, is a compact connected Riemann surface, and the fibres are diffeomorphic but not necessarily biholomorphic to one another. Hence, $\{Z_x\}_{x\in X}$ is a holomorphic family of compact Riemann surfaces and $\Omega_x = Z_x \setminus \bigcup_{i=1}^n s_i(x)$ $(x \in X)$ is a holomorphic family of n-punctured Riemann surfaces. Each $H_i = s_i(X)$ is a closed complex hypersurface in Z whose ideal sheaf is principal, that is, locally near each point of H_i it is generated by a single holomorphic function.

Recall the following result (see Grauert and Remmert [29, Theorem 5, p. 129]): If Z is a Stein space and H is a closed complex analytic hypersurface in Z (of pure codimension one) whose ideal sheaf is a principal ideal sheaf, then $Z \setminus H$ is also Stein. If Z is nonsingular then the ideal sheaf of any closed complex subvariety of pure codimension one in Z is a principal ideal sheaf (see [29, Chapter A.3.5]). Hence, it suffices to prove the theorem in the case $n = 1$, that is, to show that the complement $Z \setminus s(X)$ of a holomorphic section $s : X \to Z$ is a Stein manifold.

By a theorem of Siu [48], the Stein hypersurface $H = s(X)$ has a basis of open Stein neighbourhoods U in Z. Since $U \setminus H$ is a Stein manifold [29, Theorem 5, p. 129], it admits a strongly plurisubharmonic exhaustion function $\phi : U \setminus H \to \mathbb{R}_+$. To prove the theorem, we shall construct a strongly plurisubharmonic exhaustion function $Z \setminus H \to \mathbb{R}_+$; a theorem of Grauert [28] will then imply that $Z \setminus H$ is Stein.

Fix a point $x_0 \in X$ and set $z_0 = s(x_0) \in H \subset Z$. Since $\pi : Z \to X$ is a holomorphic submersion with compact one dimensional fibres, it is a smooth fibre bundle whose fibre M is a compact smooth surface. In particular, there is a neighbourhood $X_0 \subset X$ of x_0 such that the restricted bundle $Z|X_0 = \pi^{-1}(X_0) \to X_0$ can be smoothly identified with the trivial bundle $X_0 \times M \to X_0$. In this identification, $z_0 = (x_0, p_0)$ with $p_0 \in M$. Since ϕ tends to $+\infty$ along H, there are small smoothly bounded open discs $D \Subset D' \Subset M$ with $p_0 \in D$ such that

$$\inf_{p\in bD} \phi(x_0, p) > \max_{p\in bD'} \phi(x_0, p). \tag{2.1}$$

The set $O = M \setminus D$ is a compact bordered Riemann surface with smooth boundary $bO = bD$, endowed with the complex structure inherited by the identification $M \cong Z_{x_0} = \pi^{-1}(x_0)$. Note that bD' is contained in the interior of O. It follows from (2.1) and standard results that there is a smooth strongly subharmonic function $u_0 : O \to \mathbb{R}_+$ such that

$$u_0 < \phi(x_0, \cdot) \text{ on } bO = bD \text{ and } u_0 > \phi(x_0, \cdot) \text{ on } bD'.$$

Shrinking the neighbourhood $X_0 \subset X$ of x_0 if necessary, the following conditions hold for every $x \in X_0$, where we use the smooth fibre bundle isomorphism $Z|X_0 \cong X_0 \times M$:

(a) $s(x) \in D$,
(b) the function $u(x, \cdot) = u_0$ is strongly subharmonic on O in the complex structure on $Z_x \cong M$,
(c) $u(x, \cdot) < \phi(x, \cdot)$ on $bO = bD$, and
(d) $u(x, \cdot) > \phi(x, \cdot)$ on bD'.

Condition (b) holds since being strongly subharmonic on a compact subset is a stable property under small smooth deformations of the complex structure. We define a function $\rho_0 : (X_0 \times M) \setminus H \to \mathbb{R}_+$ by taking for every $x \in X_0$:

$$\rho_0(x, p) = \begin{cases} \phi(x, p), & p \in D \setminus \{s(x)\}; \\ \max\{\phi(x, p), u(x, p)\}, & p \in D' \setminus D; \\ u(x, p), & p \in M \setminus D'. \end{cases}$$

Note that ρ_0 is well defined, piecewise smooth, strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_0)$, and it agrees with ϕ on $(X_0 \times D) \setminus H$. In particular, ρ_0 is exhausting along $s(X_0)$. By using the regularised maximum in the definition of ρ_0 (see [20, Equation (3.1), p. 69]), we may assume that ρ_0 is smooth and enjoys the other stated properties.

This construction gives an open locally finite cover $\{X_j\}_{j=1}^\infty$ of X with smooth fibre bundle trivialisations $Z|X_j \cong X_j \times M$, discs $D_j \subset M$ such that $s(x) \in D_j$ for all $x \in X_j$, and smooth functions $\rho_j : (Z|X_j) \setminus H \to \mathbb{R}_+$ such that ρ_j is strongly subharmonic on each fibre $Z_x \setminus \{s(x)\}$ $(x \in X_j)$ and it agrees with ϕ on $(X_j \times D_j) \setminus H$. Let $\{\chi_j\}_j$ be a smooth partition of unity on X with compact supports $\mathrm{supp}(\chi_j) \subset X_j$ for each j. Set

$$\rho = \sum_{j=1}^{\infty} \chi_j \rho_j : Z \setminus H \to \mathbb{R}_+.$$

By the construction, the restriction of ρ to each fibre $Z_x \setminus \{s(x)\}$ $(x \in X)$ is strongly subharmonic, and there is an open neighbourhood $U_0 \subset U \subset Z$ of H such that $\rho = \phi$ holds on $U_0 \setminus H$. In particular, ρ is strongly plurisubharmonic on $U_0 \setminus H$. Note that for every compact set $K \subset X$, the set $\pi^{-1}(K) \setminus U_0 \subset Z \setminus H$ is compact. Hence, choosing a strongly plurisubharmonic exhaustion function $\tau : X \to \mathbb{R}_+$ whose Levi form $dd^c\tau$ grows fast enough, we can ensure that

$$\rho + \tau\circ\pi : Z \setminus H \to \mathbb{R}_+$$

is a strongly plurisubharmonic exhaustion function, thus proving the theorem. Indeed, denoting by J the almost complex structure operator on Z and by d^c the conjugate differential defined by

$$(d^c\rho)(z,\xi) = -d\rho(z,J\xi) \ \text{ for } z \in Z \text{ and } \xi \in T_zZ,$$

the function ρ is strongly plurisubharmonic at $z \in Z$ if and only if

$$(dd^c\rho)(z,\xi \wedge J\xi) > 0 \ \text{ for every tangent vector } 0 \neq \xi \in T_zZ.$$

(Up to a positive factor, this equals the Laplaian of ρ on the 2-plane span$(\xi, J\xi) \subset T_zZ$.) Since ρ is strongly subharmonic on every fibre $Z_x \setminus \{s(x)\}$, $x \in X$, we have that

$$(dd^c\rho)(z,\xi \wedge J\xi) > 0 \ \text{ if } z \in Z \setminus H \text{ and } 0 \neq \xi \in \ker d\pi_z.$$

Hence, the eigenvectors of $(dd^c\rho)(z,\cdot)$ associated to nonpositive eigenvalues lie in a closed cone $C_z \subset T_zZ$ which intersects $\ker d\pi_z$ only in the origin. It follows that if $\tau : X \to \mathbb{R}$ is such that $dd^c\tau > 0$ is sufficiently large on T_xX where $x = \pi(z)$, then $dd^c\rho + dd^c(\tau\circ\pi) > 0$ on T_zZ. Furthermore, the estimates are uniform on the compact set $\pi^{-1}(K) \setminus U_0$, where $U_0 \subset Z$ is a neighbourhood of H such that $dd^c\rho > 0$ on $U_0 \setminus H$. To see that τ can be chosen such that $dd^c\tau$ grows as fast as desired, note that if $h : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$ function then for each point $x \in X$ and vector $\xi \in T_xX$ we have that

$$dd^c(h\circ\tau)(x,\xi \wedge J\xi) = h'(\tau(x))\,(dd^c\tau)(x,\xi \wedge J\xi) + h''(\tau(x))\left(|d\tau(x,\xi)|^2 + |d\tau(x,J\xi)|^2\right).$$

Hence, if τ is a strongly plurisubharmonic exhaustion function on X and the function $h : \mathbb{R} \to \mathbb{R}$ is chosen such that $h'' \geqslant 0$ and h' grows sufficiently fast, then $dd^c(h\circ\tau)$ also grows as fast as desired. □

A minor modification of the proof of Theorem 2.1 gives the following more general result.

Theorem 2.2. *Assume that X is a connected Stein manifold, Z is a complex manifold, $\pi : Z \to X$ is a surjective proper holomorphic submersion with purely one-dimensional fibres, and H is a closed complex subvariety of Z of pure codimension one which intersects every connected component of each fibre $Z_x = \pi^{-1}(x)$, $x \in X$, but it does not contain any such component. Then, $Z \setminus H$ is Stein.*

Proof. Since $\pi : Z \to X$ is proper and H is closed in Z, $\pi|_H : H \to X$ is a proper holomorphic map. The conditions imply that H intersects every fibre in a finite set of points, so $\pi|_H : H \to X$ is a finite holomorphic map. The proper mapping theorem of Remmert [45] (see also [12, p. 29]) implies that $\pi(H)$ is a closed complex subvariety of X of pure dimension $\dim X$, hence $\pi(H) = X$ since X is connected. By [29, Theorem 1 (d), p. 125], the subvariety H is Stein. A theorem of Siu [48] implies that H has a basis of open Stein neighbourhoods $U \subset Z$. By the argument in the proof of Theorem 2.1, the ideal sheaf of H is a principal ideal sheaf, so the manifold $U \setminus H$ is Stein for any Stein neighbourhood $U \subset Z$ of H [29, Theorem 5, p. 129]. The proof can now be completed by a similar argument as in the proof of Theorem 2.1, and we leave the details to the reader. □

Remark 2.3. Theorem 2.2 still holds if X is a Stein space, Z is a complex space, $H \subset Z$ is a closed complex hypersurface whose ideal sheaf is principal, and the other conditions on the submersion $\pi : Z \to X$ and H are satisfied.

3 | FIBREWISE ALGEBRAIC FUNCTIONS ON $V(g, n)$

In this section, we show that the universal family $V(g, n)$ of n-punctured compact Riemann surfaces of genus g for $n \geqslant 1$ has a large algebra of holomorphic functions which are algebraic on every fibre of the Teichmüller projection $\pi : V(g, n) \to T(g, n)$. Indeed, such functions are dense in the algebra of holomorphic functions in the compact-open topology; see Theorem 3.2. Furthermore, the universal family $V(g, n)$ is affine; see Corollary 3.5.

We shall consider the more general situation in Theorem 2.2. Thus, assume that X is a connected Stein manifold, Z is a complex manifold, $\pi : Z \to X$ is a surjective proper holomorphic submersion with connected one dimensional fibres, H is a closed complex subvariety of Z of pure codimension one which does not contain any fibre of π, and $\Omega = Z \setminus H$. Every fibre $\Omega_x = \Omega \cap \pi^{-1}(x)$ $(x \in X)$ is an affine complex curve. Let $\mathscr{A}(\Omega)$ denote the subalgebra of $\mathscr{O}(\Omega)$ consisting of functions which are algebraic on Ω_x for every $x \in X$. Note that $\mathscr{A}(\Omega)$ contains the subset $\{f \circ \pi : f \in \mathscr{O}(X)\}$. We have the following result; see also the global version in Theorem 3.4.

Theorem 3.1 (Assumptions as above.). *Given a relatively compact domain $U \Subset X$, there exist finitely many functions $f_1, \dots, f_N \in \mathscr{A}(\Omega)$ such that the map $F : \Omega \to X \times \mathbb{C}^N$ given by*

$$F(z) = (\pi(z), f_1(z), \dots, f_N(z)), \quad z \in \Omega \tag{3.1}$$

induces a proper embedding $\Omega_U := \Omega \cap \pi^{-1}(U) \to U \times \mathbb{C}^N$.

The proof of Theorem 3.1 does not rely on the fact, proved in Theorem 2.2, that $\Omega = Z \setminus H$ is a Stein manifold. It clearly implies that functions in $\mathscr{A}(\Omega)$ separate points and establish holomorphic convexity, so it gives another proof that Ω is Stein. This applies in particular to $\Omega = V(g, n)$ for $n \geqslant 1$. See also Theorem 3.2 and Corollary 3.3 for more precise results on the algebra $\mathscr{A}(\Omega)$.

A map F satisfying the conclusion of Theorem 3.1 is called a *proper embedding over* U. The same result holds if X is a Stein space, Z is a complex space, and the other conditions on the submersion $\pi : Z \to X$ and the hypersurface $H \subset Z$ hold; in particular, the ideal sheaf of H is principal.

Proof. Let $L \to Z$ denote the holomorphic line bundle determined by the divisor $H \subset Z$, and let $\sigma_0 : Z \to L$ be a holomorphic section whose zero divisor equals H. (The existence of such a section is tautological from the construction of L.) Since the fibre $Z_x = \pi^{-1}(x)$ is a projective curve for every $x \in X$, Serre's GAGA principle [47] implies that the restricted line bundle $L_x := L|Z_x \to Z_x$ is algebraic, and every holomorphic section of L_x over Z_x is algebraic. It follows that for any holomorphic section $\sigma : Z \to L$, the quotient $f = \sigma/\sigma_0$ is a function in $\mathscr{A}(\Omega)$. The restriction of f to Ω_x is algebraic and has an effective pole at every end of Ω_x (that is, a point of $H_x = H \cap Z_x$) at which σ does not vanish. Note that the restricted line bundle $L_x = L|Z_x \to Z_x$ is associated to the effective divisor supported on H_x in which the multiplicity of a point $z \in H_x$ is the intersection number of H with Z_x at z. In particular, L_x is an ample line bundle on the compact Riemann surface Z_x whose degree $\deg L_x$ is independent of $x \in X$. Hence, some tensor power $L_x^{\otimes d} \to Z_x$ is very ample, so it admits finitely many holomorphic sections $\sigma_{1,x}, \dots, \sigma_{N,x} : Z_x \to L_x^{\otimes d}$ such that the map

$$Z_x \ni z \longmapsto \left[\sigma_{0,x}^d(z) : \sigma_{1,x}(z) : \cdots : \sigma_{N,x}(z)\right] \in \mathbb{CP}^N \tag{3.2}$$

is a holomorphic embedding. Here, σ_0^d denotes the d-th power of σ_0, a section of the line bundle $L^{\otimes d}$. Note that $\sigma_{i,x}/\sigma_{0,x}^d$ is a regular algebraic function on Ω_x for every $i = 1, \dots, N$, and the map

$$Z_x \setminus H_x = \Omega_x \ni z \longmapsto \frac{1}{\sigma_{0,x}^d}(\sigma_{1,x}, \dots, \sigma_{N,x}) \in \mathbb{C}^N \tag{3.3}$$

is a proper algebraic embedding.

To prove the theorem, we will show that for $d > 0$ big enough, every holomorphic section $\sigma_x : Z_x \to L_x^{\otimes d}$ extends to a holomorphic section $\sigma : Z \to L^{\otimes d}$. Assume for a moment that this holds true. Extending all sections $\sigma_{i,x}$ in (3.2) for $i = 1, \dots, N$ to Z, the map in (3.2) is an embedding for any base point in an open neighbourhood $U_1 \subset X$ of x. Hence, the map F of the form (3.1) with the components $f_i = \sigma_i/\sigma_0^d \in \mathscr{A}(\Omega)$ is a proper embedding over U_1. Indeed, for any pair of points $x \in U_1$ and $z \in H_x$, at least one of the sections $\sigma_{i,x}$ $(i = 1, \dots, N)$ does not vanish at z. Since $\sigma_0(z) = 0$, the function $f_i = \sigma_i/\sigma_0^d$ has an effective pole at z, which implies properness. Repeating the same construction at other points $x \in \overline{U}$ and assembling the resulting functions as components of a map (3.1) yields a proper embedding over U as in the theorem.

It remains to explain why every holomorphic section $\sigma_x : Z_x \to L_x^{\otimes d}$ for $d > 0$ big enough extends to a holomorphic section $\sigma : Z \to L^{\otimes d}$. Let $\mathscr{L} = \mathscr{O}_Z(H) \to Z$ denote the sheaf of holomorphic sections of $L \to Z$; note that $\mathscr{L}$ is a coherent (locally free) $\mathscr{O}_Z$-analytic sheaf. Since the holomorphic projection $\pi : Z \to X$ is surjective and proper, the image sheaf $\pi_*\mathscr{L}$ (see [30, p. 227]) is a coherent analytic sheaf on X by Grauert's direct image theorem [30, p. 207]. A section of $\pi_*\mathscr{L}$ over an open subset $V \subset X$ is the same thing as a section of $\mathscr{L}$ over $Z_V := \pi^{-1}(V)$. The argument given (in the algebraic case) in [49, Tag 0D2M] shows that for $d_0 = d_0(x) > 0$ big enough, the evaluation map

$$E_x : \pi_*(\mathscr{L}^{\otimes d})_x \to H^0(Z_x, \mathscr{L}_x^{\otimes d}) \tag{3.4}$$

is surjective for $d \geqslant d_0$. This means that every holomorphic section $\sigma_x : Z_x \to L_x^{\otimes d}$ of the line bundle $L_x^{\otimes d} \to Z_x$ extends to a holomorphic section σ of $L^{\otimes d}|Z_V \to Z_V$ for an open neighbourhood V of x. Denote by $[\sigma]_x \in \pi_*(\mathscr{L}^{\otimes d})_x$ the associated element of the stalk $\pi_*(\mathscr{L}^{\otimes d})_x$, so $E_x([\sigma]_x) = \sigma_x$. Since the sheaf $\pi_*\mathscr{L}^{\otimes d}$ is $\mathscr{O}_X$-coherent, Cartan's Theorem A [29, p. 124] shows that $\pi_*(\mathscr{L}^{\otimes d})_x$ is generated as an $\mathscr{O}_{X,x}$-module by global sections of $\pi_*(\mathscr{L}^{\otimes d})$. Hence, there are sections $\xi_1, \dots, \xi_m \in H^0(X, \pi_*(\mathscr{L}^{\otimes d}))$ and germs of holomorphic functions $g_1, \dots, g_m \in \mathscr{O}_{X,x}$ such that

$$[\sigma]_x = \sum_{j=1}^{m} g_j[\xi_j]_x \in \pi_*(\mathscr{L}^{\otimes d})_x. \tag{3.5}$$

Since X is Stein, there are functions $\tilde{g}_j \in \mathcal{O}(X)$ such that $\tilde{g}_j(x) = g_j(x)$ for $j = 1, \dots, m$. Then, $\tilde{\sigma} := \sum_{j=1}^m \tilde{g}_j \xi_j \in H^0(X, \pi_*(\mathcal{L}^{\otimes d}))$ is a section whose value at x equals

$$E_x(\tilde{\sigma}) = \sum_{j=1}^{m} \tilde{g}_j(x) E_x(\xi_j) = \sum_{j=1}^{m} g_j(x) E_x(\xi_j) = E_x([\sigma]_x) = \sigma_x$$

(see (3.5)). This completes the proof. □

Recall (see, e.g., [36]) that a compact set K in a complex manifold Ω is said to be $\mathcal{O}(\Omega)$-convex if and only if it equals its $\mathcal{O}(\Omega)$-convex hull

$$\widehat{K}_{\mathcal{O}(\Omega)} = \{z \in \Omega : |f(z)| \leqslant \sup_K |f| \text{ for all } f \in \mathcal{O}(\Omega)\}.$$

Assume now that $\Omega = Z \setminus H$ is as in Theorem 3.1. We define the hull $\widehat{K}_{\mathcal{A}(\Omega)}$ in a similar way by using functions $f \in \mathcal{A}(\Omega)$. Since $\mathcal{A}(\Omega) \subset \mathcal{O}(\Omega)$, we have $\widehat{K}_{\mathcal{O}(\Omega)} \subset \widehat{K}_{\mathcal{A}(\Omega)}$. Theorem 3.1 clearly implies that the hull $\widehat{K}_{\mathcal{A}(\Omega)}$ of any compact set $K \subset \Omega$ is compact. Furthermore, functions in $\mathcal{A}(\Omega)$ separate points of Ω, so they satisfy the axioms of a Stein manifold. The following theorem shows that the algebra $\mathcal{A}(\Omega)$ is dense in $\mathcal{O}(\Omega)$ in the compact-open topology.

Theorem 3.2. *Let $\pi : Z \to X$, $H \subset Z$, and $\Omega = Z \setminus H$ be as in Theorem 3.1. Given a compact $\mathcal{O}(\Omega)$-convex set $K \subset \Omega$, every holomorphic function on a neighbourhood of K is a uniform limit on K of functions in $\mathcal{A}(\Omega)$. This holds in particular for the Teichmüller family $\Omega = V(g, n)$ with $n \geqslant 1$.*

Proof. Choose a relatively compact Runge domain $U \Subset X$ such that $\pi(K) \subset U$. Let $F : \Omega \to X \times \mathbb{C}^N$ be a map of class $\mathcal{A}(\Omega)$, furnished by Theorem 3.1 (see (3.1)), such that $F : \Omega_U \to U \times \mathbb{C}^N$ is a proper embedding. Its image $F(\Omega_U)$ is then a closed complex submanifold of the Stein manifold $U \times \mathbb{C}^N$. It follows that the set $F(K)$ is holomorphically convex in $U \times \mathbb{C}^N$, and hence in $X \times \mathbb{C}^N$ since U is Runge in X. Let f be a holomorphic function on a Stein neighbourhood $V \subset \Omega$ of K. We may assume that $\pi(V) \subset U$. Then, $\tilde{f} = f \circ F^{-1} : F(V) \to \mathbb{C}$ is a holomorphic function on the open subset $F(V)$ in the closed complex submanifold $F(\Omega_U)$ of $U \times \mathbb{C}^N$. By the Cartan–Oka extension theorem, $\tilde{f}$ extends to a holomorphic function on a neighbourhood of $F(K)$ in $U \times \mathbb{C}^N$. (This can also be seen by precomposing $\tilde{f}$ by a holomorphic retraction from a neighbourhood of $F(\Omega_U)$ in $U \times \mathbb{C}^N$ onto $F(\Omega_U)$, given by Docquier and Grauert [14]; see also [20, Theorem 3.3.3, p. 74].) By the Oka–Weil theorem, $\tilde{f}$ can be approximated uniformly on $F(K)$ by functions $g \in \mathcal{O}(X \times \mathbb{C}^N)$. Every such function is of the form $g(x, w) = \sum_{\alpha \in \mathbb{Z}_+^N} g_\alpha(x) w^\alpha$, where $w = (w_1, \dots, w_N)$ are coordinates on $\mathbb{C}^N$ and $g_\alpha \in \mathcal{O}(X)$ for every $\alpha \in \mathbb{Z}_+^N$. By cutting off the power series at finite levels gives polynomials in $w \in \mathbb{C}^N$ depending holomorphically on $x \in X$ which approximate g on $F(K)$. Precomposing such functions with F gives the desired approximation of f by functions in $\mathcal{A}(\Omega)$. □

The following is an immediate corollary to Theorem 3.2.

Corollary 3.3. *For any compact set $K \subset \Omega$ we have that $\widehat{K}_{\mathcal{O}(\Omega)} = \widehat{K}_{\mathcal{A}(\Omega)}$.*

Added in the revision. It was pointed out by a referee that Theorem 3.1 has the following global version, whose proof was proposed (in the special case when $Z = \widehat{V}(g,n)$ and $X = T(g,n)$ for $n \geqslant 1$) by Yiran Lin in a private communication.

Theorem 3.4. *Assume that X is a connected Stein manifold, Z is a complex manifold, $\pi : Z \to X$ is a surjective proper holomorphic submersion with connected one dimensional fibres, and H is a closed complex subvariety of Z of pure codimension one which does not contain any fibre of π. Then there exist an integer $N \in \mathbb{N}$ and a holomorphic map $f : \Omega = Z \setminus H \to \mathbb{C}^N$ which restricts to a proper algebraic embedding $\Omega_x \hookrightarrow \mathbb{C}^N$ on every fibre $\Omega_x = \Omega \cap \pi^{-1}(x)$, $x \in X$.*

Proof. Let g denote the genus of the Riemann surface $Z_x = \pi^{-1}(x)$ and $n \geqslant 1$ the degree of the divisor $[H]_{Z_x}$; these numbers are independent of $x \in X$. Choose $k \in \mathbb{N}$ such that $kn > 2g$ and consider the divisor $D = k[H]$ on Z. By Riemann–Roch we have for any $x \in X$ that

$$h^0(Z_x, \mathcal{O}_Z(D)|_{Z_x}) = kn + 1 - g > g + 1, \qquad h^1(Z_x, \mathcal{O}_Z(D)|_{Z_x}) = 0.$$

(As usual, h^0 and h^1 denote the dimensions of the cohomology groups H^0 and H^1, respectively.) By [30, Theorem, p. 211] it follows that the direct image $\mathcal{E} = \pi_* \mathcal{O}_Z(D)$ is a locally free sheaf on X, that is, the sheaf of germs of holomorphic sections of a holomorphic vector bundle $E \to X$. Since X is Stein, $\mathcal{E}$ is generated by finitely many global sections $\xi_1, \dots, \xi_N$. (Indeed, by an extension of Cartan's Theorem A (see Forster [17, Corollary 4.4] or Kripke [40]), every holomorphic vector bundle on a Stein manifold admits finitely many holomorphic sections which span the fibre over each point.) The restrictions of the sections $\xi_1, \dots, \xi_N$ to any fibre Z_x span the vector space $H^0(Z_x, \mathcal{O}_Z(D)|_{Z_x})$. Since $kn > 2g$, the divisor $\mathcal{O}_Z(D)|_{Z_x}$ is very ample for every $x \in X$ (see [34, IV. Corollary 3.2]). Hence, any basis of $H^0(Z_x, \mathcal{O}_Z(D)|_{Z_x})$ consisting of m elements defines an embedding $\Omega_x = Z_x \setminus H_x \hookrightarrow \mathbb{C}^m$. Thus, the map $f = (f_1, \dots, f_N) : Z \setminus H = \Omega \to \mathbb{C}^N$ of class $\mathscr{A}(\Omega)$, defined by the sections $\xi_1, \dots, \xi_N$ of $\mathcal{E}$ as in (3.2)–(3.3), restricts to a proper algebraic embedding on each fibre Ω_x. ☐

Corollary 3.5. *The manifold Ω in Theorem 3.4 admits a proper holomorphic embedding $\Omega \hookrightarrow \mathbb{C}^m$ for some $m \in \mathbb{N}$ which restricts to an algebraic map on every fibre $\Omega_x = \pi^{-1}(x)$, $x \in X$. This holds in particular for the universal Teichmüller family $V(g,n)$ for any $g \geqslant 0$ and $n \geqslant 1$.*

Proof. Let $f = (f_1, \dots, f_N) : \Omega \to \mathbb{C}^N$ be as in Theorem 3.4. Choose a proper holomorphic embedding $h : X \hookrightarrow \mathbb{C}^{N'}$ for some $N' \in \mathbb{N}$. The map

$$\Omega \ni z \longmapsto (h \circ \pi(z), f(z)) \in \mathbb{C}^{N'+N}$$

is then a proper embedding of class $\mathscr{A}(\Omega)$. ☐

The analogous statement holds in the algebraic setting when the base X is an affine algebraic variety. The following is a special case of a result due to Dobben de Bruyn [13, Lemma].

Theorem 3.6. *Assume that X is an affine algebraic variety, Z is an algebraic variety, $\pi : Z \to X$ is a proper algebraic submersion with one dimensional fibres, and $s_1, \dots, s_n$ for $n \geqslant 1$ are algebraic sections. If all fibres $\Omega_x = Z_x \setminus \bigcup_{i=1}^n s_i(x)$ $(x \in X)$ are affine (equivalently, no connected component* *of Ω_x is compact) then there exists an algebraic morphism $f : \Omega = Z \setminus \bigcup_{i=1}^n s_i(X) \to \mathbb{C}^N$ for some $N \in \mathbb{N}$ which is a proper embedding on every fibre Ω_x.*

4 | A RELATIVE OKA PRINCIPLE FOR FIBREWISE ALGEBRAIC MAPSs

In this section, we prove a relative Oka principle for holomorphic fibrewise algebraic maps from any manifold $\Omega = Z \setminus H$ as in Theorem 3.1 to a class of complex algebraic manifolds which includes all flexible manifolds in the sense of Arzhantsev et al. [9]; see Theorem 4.1. This holds in particular for the Teichmüller family $\Omega = V(g, n)$ for any $g \geqslant 0$ and $n \geqslant 1$. In order to state this result, we recall the following notions; see Gromov [31, 0.5, p. 8.5.5], [20, Definition 5.6.13], or [22, Definition 3.1].

A *dominating holomorphic spray* on a complex manifold Y is a holomorphic map $s : E \to Y$ from the total space of a holomorphic vector bundle $E \to Y$ such that

$$s(0_y) = y \text{ and } ds_{0_y}(E_y) = T_yY \text{ for every } y \in Y. \tag{4.1}$$

Here, 0_y denotes the origin of the fibre E_y of E over the point $y \in Y$, and E_y is considered as a $\mathbb{C}$-linear subspace of the tangent space $T_{0_y}E$. (Identifying Y with the zero section of E, we have a natural direct sum decomposition $TE|Y \cong TY \oplus E$.) The spray is algebraic if the vector bundle $E \to Y$ and the map $s : E \to Y$ are algebraic.

A complex manifold Y is said to be *elliptic* if it admits a dominating holomorphic spray, and to be *special elliptic* if such a spray exists on a trivial bundle $E = Y \times \mathbb{C}^m$ for some $m \in \mathbb{N}$. An algebraic manifold Y is *algebraically (special) elliptic* if these conditions hold with algebraic sprays. Every compact special elliptic manifold is complex homogeneous [21, Proposition 6.2]. On the other hand, there exist many noncompact nonhomogeneous special elliptic manifolds. An important source of examples are *flexible manifolds*. An algebraic manifold Y is said to be flexible [9] if its tangent bundle TY is pointwise spanned by finitely many LNDs, that is, algebraic vector fields V_j with complete algebraic flows ϕ_t^j $(t \in \mathbb{C},\ j = 1, \dots, m)$. The map $s : Y \times \mathbb{C}^m \to Y$ defined by

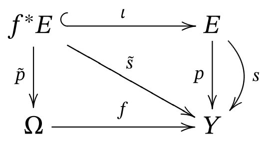
$$s(y, t_1, \dots, t_m) = \phi_{t_1}^1 \circ \cdots \phi_{t_m}^m(y), \quad y \in Y,\ (t_1, \dots, t_m) \in \mathbb{C}^m$$

is then a dominating algebraic spray on Y, so every flexible manifold is algebraically special elliptic. Similarly, a complex manifold is holomorphically flexible if its tangent bundle is pointwise spanned by finitely many $\mathbb{C}$-complete holomorphic vector fields. By the same argument, every such manifold is special elliptic. References to examples of flexible manifolds can be found in [22, p. 394].

It was proved in [19, Theorem 3.1] that a holomorphic map $X \to Y$ from an affine algebraic variety X to an algebraically elliptic manifold Y, which is homotopic to an algebraic map, is a limit of algebraic maps. (See also [20, Theorem 6.15.1] and [22, Theorem 6.4].) The following is an analogue of this result for fibrewise algebraic holomorphic maps to algebraically special elliptic manifold.

Theorem 4.1. *Assume that Ω is as in Theorem 3.1 and Y is an algebraically special elliptic manifold. Given a map $f : \Omega \to Y$ of class $\mathscr{A}(\Omega, Y)$ (i.e., f is holomorphic and fibrewise algebraic), a* *compact $\mathscr{O}(\Omega)$-convex set $K \subset \Omega$, and a homotopy of holomorphic maps $f_t : U \to Y$ ($t \in [0,1)$ on an open neighbourhood U of K with $f_0 = f|_U$, there are holomorphic maps $F : \Omega \times \mathbb{C} \to Y$ such that $F(\cdot, 0) = f$ and $F(\cdot, t) \in \mathscr{A}(\Omega, Y)$ approximates f_t as closely as desired uniformly on K and uniformly in $t \in [0,1]$. In particular, a holomorphic map $\Omega \to Y$ that is homotopic to a map in $\mathscr{A}(\Omega, Y)$ is a limit of maps in $\mathscr{A}(\Omega, Y)$ uniformly on compacts. This holds in particular for the Teichmüller family $\Omega = V(g, n)$ for any $g \geqslant 0$ and $n \geqslant 1$.*

Proof. It suffices to inspect the proof of [19, Theorem 3.1] and apply Theorem 3.2. We recall the main steps. Let $p : E \to Y$ be an algebraic vector bundle and $s : E \to Y$ a dominating algebraic spray. Consider the commuting diagram



where the map $\tilde{p}$ in the first column is the f-pullback of the vector bundle $p : E \to Y$, the map ι in the top row is the natural inclusion, and $\tilde{s} = s \circ \iota$ is a dominating holomorphic spray over f (the pullback of s by f). Note that $\tilde{s} = f$ holds on the zero section of f^*E. Since f is fibrewise algebraic and s is algebraic, the maps ι and $\tilde{s}$ are algebraic on $f^*E|\Omega_x$ for every $x \in X$. (Here, $\pi : \Omega \to X$ and $\Omega_x = \pi^{-1}(x)$ are as in Theorem 3.1.) Replacing the spray bundle (E, p, s) with a sufficiently high iterate of itself (see [20, Definition 6.3.5]) and shrinking U around K, [20, Proposition 6.5.1] provides a lift of the homotopy of holomorphic maps $f_t : U \to Y$ in the statement of the theorem to a homotopy of holomorphic sections $\xi_t : U \to f^*E|_U$, with ξ_0 the zero section, such that $\tilde{s} \circ \xi_t = f_t$ for $t \in [0,1]$.

Assume now that the vector bundle $p : E \to Y$ is algebraically trivial. Then, every iterate of (E, p, s) is also algebraically trivial. Since the map $f : \Omega \to Y$ is fibrewise algebraic, the pullback bundle $f^*E \to \Omega$ is fibrewise algebraically trivial. Hence, sections of $f^*E \to \Omega$ can be identified with maps from Ω to the fibre, which is a Euclidean space. Thus, Theorem 3.2 lets us approximate every section ξ_t uniformly on K by a holomorphic fibrewise algebraic section $\tilde{\xi}_t : \Omega \to f^*E$. The map $F_t = \tilde{s} \circ \tilde{\xi}_t : \Omega \to Y$ is then of class $\mathscr{A}(\Omega, Y)$ and it approximates f_t on K for every $t \in [0,1]$. The approximation can be made uniform in $t \in [0,1]$ by including [0,1] in a standard way into $\mathbb{C}$ and applying Theorem 3.2 with the compact holomorphically convex set $K \times [0,1]$ in $\Omega \times \mathbb{C}$. (The homotopy $\{\xi_t\}_{t \in [0,1]}$ is first approximated on $K \times [0,1]$ by a holomorphic map in a neighbourhood of $K \times [0,1]$ in $\Omega \times \mathbb{C}$ using Mergelyan's theorem.) This gives a holomorphic map $F : \Omega \times \mathbb{C} \to Y$ satisfying the conclusion of the theorem which is algebraic on $\Omega_x \times \mathbb{C}$ for every $x \in X$.

The last statement follows from the fact that a homotopy $f_t : \Omega \to Y$ ($t \in [0,1]$) between holomorphic maps f_0, f_1 can be deformed, with fixed ends at $t = 0, 1$, to a homotopy consisting of holomorphic maps when Ω is Stein and Y is an Oka manifold (which holds in our situation). □

Remark 4.2. The relative algebraic Oka principle in [19, Theorem 3.1] holds for all algebraically elliptic target manifolds Y. (It was stated for the ostensibly bigger class of algebraically subelliptic manifolds, but it has recently been shown by Kaliman and Zaidenberg [39] that these two classes of manifolds coincide. See also [22, Theorem 6.2] for a more complete list of equivalent algebraic ellipticity conditions.) We do not know whether Theorem 4.1 holds for this larger class of manifolds.

An example of a flexible manifold, which is of particular interest in the theory of minimal surfaces in Euclidean spaces $\mathbb{R}^n$, $n \geqslant 3$, is the punctured null quadric

$$\mathbf{A} = \{z = (z_1, \dots, z_n) \in \mathbb{C}^n \setminus \{0\} : z_1^2 + z_2^2 + \cdots + z_n^2 = 0\}.$$

(See [4, Proposition 1.15.3] and [8].) Algebraic 1-forms with values in $\mathbf{A}$ on finitely punctured compact Riemann surfaces, which satisfy the real period vanishing conditions on closed curves in M, give rise to conformal minimal surfaces $M \to \mathbb{R}^n$ of finite total curvature via the Enneper–Weierstrass representation formula, and vice versa. (See [44] and [4, Chapter 4] for a discussion of this topic.) Hence, the fact that the Teichmüller family $V(g,n)$ for $n \geqslant 1$ admits nontrivial fibrewise algebraic maps $V(g,n) \to \mathbf{A}$ (see Theorem 4.1) raises the following question.

Problem 4.3. Is it possible to realise the fibres of $V(g,n)$ as a family of immersed minimal surfaces of finite total curvature in some $\mathbb{R}^k$, $k \geqslant 3$, depending holomorphically on the parameter $t \in T(g,n)$?

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