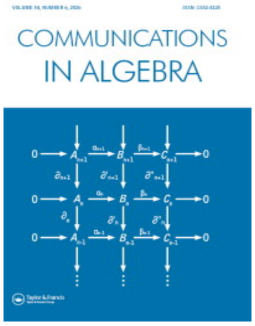

Taylor & Francis
Taylor & Francis Group

Communications in Algebra

ISSN: 0092-7872 (Print) 1532-4125 (Online) Journal homepage: www.tandfonline.com/journals/lagb20

Primes and absolutely or non-absolutely irreducible elements in atomic domains

Victor Fadinger, Sophie Frisch, Sarah Nakato, Daniel Smertnig & Daniel Windisch

To cite this article: Victor Fadinger, Sophie Frisch, Sarah Nakato, Daniel Smertnig & Daniel Windisch (2026) Primes and absolutely or non-absolutely irreducible elements in atomic domains, Communications in Algebra, 54:6, 2631-2648, DOI: 10.1080/00927872.2025.2586234

To link to this article: https://doi.org/10.1080/00927872.2025.2586234

Published online: 08 Jan 2026.

COMMUNICATIONS IN ALGEBRA®
2026, VOL. 54, NO. 6, 2631–2648
https://doi.org/10.1080/00927872.2025.2586234

Primes and absolutely or non-absolutely irreducible elements in atomic domains

Victor Fadinger[a], Sophie Frisch[b], Sarah Nakato[c], Daniel Smertnig[d], and Daniel Windisch[e]

[a]Institute for Mathematics and Scientific Computing, University of Graz, Graz, Austria; [b]Institut für Analysis und Zahlentheorie, Graz University of Technology, Graz, Austria; [c]Department of Mathematics, Kabale University, Kabale, Uganda; [d]Faculty of Mathematics and Physics (FMF), University of Ljubljana and Institute of Mathematics, Physics and Mechanics (IMFM), Ljubljana, Slovenia; [e]KU Leuven, Leuven (Heverlee), Belgium

ABSTRACT
We give examples of atomic integral domains satisfying each of the eight logically possible combinations of existence or nonexistence of the following kinds of elements: (1) primes, (2) absolutely irreducible elements that are not prime, and (3) irreducible elements that are not absolutely irreducible. A nonzero non-unit is called absolutely irreducible (or, a strong atom) if every one of its powers factors uniquely into irreducibles.

ARTICLE HISTORY
Received 10 January 2025
Revised 23 August 2025
Communicated by Scott Chapman

KEYWORDS
Absolutely irreducible elements; non-absolutely irreducible elements; non-unique factorization; integer-valued polynomials; irreducible elements; transfer homomorphisms; zero-sum sequences

2020 MATHEMATICS SUBJECT CLASSIFICATION
13A05; 11S05; 11R09; 13B25; 13F20; 11C08

1. Introduction

Among atomic domains, that is, domains in which every nonzero non-unit is a product of irreducibles, unique factorization domains are characterized by the fact that all irreducibles are prime.

Chapman and Krause [10] showed for rings of integers in number fields that $\mathcal{O}_K$ is a UFD if and only if every irreducible element is absolutely irreducible—meaning that each of its powers factors uniquely into irreducibles—a weaker property than prime.

Their result prompts the question whether this characterization of unique factorization domains holds in greater generality: among Dedekind domains, for instance. The answer is no.

We have the following implications (whose converses do not hold):

$$\text{prime} \quad \Longrightarrow \quad \text{absolutely irreducible} \quad \Longrightarrow \quad \text{irreducible}$$

This gives us three kinds of elements that may or may not exist in a given domain:

(i) Primes
(ii) Absolutely irreducibles that are not prime
(iii) Irreducibles that are not absolutely irreducible

CONTACT Sophie Frisch frisch@math.tugraz.at Institut für Analysis und Zahlentheorie, Graz University of Technology, Kopernikusgasse 24, 8010 Graz, Austria.

— and, therefore, eight logically possible combinations of existence or nonexistence of each kind of element.

Monoid examples for all eight scenarios are easy to find, compared to examples in integral domains. For instance, Baginski and Kravitz [7] provide examples of non-factorial monoids whose irreducibles are all absolutely irreducible, but not prime.

We will show that all eight scenarios occur in atomic domains.

Remark 1.1. Some cases are trivial: Atomic domains without any irreducible elements, let alone absolutely irreducible or prime elements, are just fields. (There are also non-atomic domains without irreducible elements, the so-called antimatter domains [2].)

Atomic domains containing primes but no other irreducible elements are precisely those UFDs that are not fields, as mentioned.

For the six non-trivial combinations we now proceed to give examples.

In the following table, plus indicates existence, and minus, nonexistence. R_1 and R_2 are certain Dedekind domains with class group $\mathbb{Z}^n$, see Propositions 8.1 and 9.1, respectively.

Result	Example	Irreducible but not absolutely irreducible	Absolutely irreducible but not prime	Prime
6.2,6.4	$\mathbb{Z}[\sqrt{-14}]$	+	+	+
4.9	$\mathrm{Int}(\mathcal{O}_K)$	+	+	-
5.1,5.2	$\mathbb{R}+X\mathbb{C}[X]$	+	-	+
3.2	$\mathbb{R}+X\mathbb{C}[[X]]$	+	-	-
9.1	R_2	—	+	+
8.1	R_1	—	+	—
1.1	UFDs	—	—	+
1.1	Fields	—	—	—

2. Preliminaries

2.1. Factorization terms

We recall some concepts and terminology related to factorization. For a comprehensive introduction to non-unique factorizations, we refer to the textbook by Geroldinger and Halter-Koch [15].

The terms that we here define for a monoid H we will use mostly (but not only) in the context of an integral domain R. In that case, the monoid in question is understood to be $(R \setminus \{0\}, \cdot)$.

Definition 2.1. Let $(H, \cdot)$ be a commutative monoid.

(i) We denote the group of units of the monoid H by $H^\times$.

(ii) $r \in H$ is said to be *irreducible* in H (or, an *atom* of H) if it is a non-unit that is not a product of two non-units of H.

(iii) A *factorization* of a non-unit $r \in H$ is an expression

$$r = a_1 \cdot \ldots \cdot a_n,$$

where $n \geq 1$ and a_i is irreducible in H for $1 \leq i \leq n$.

The number n of irreducible factors is called the *length* of the factorization.

(iv) $r, s \in H$ are *associated* in H if there exists a unit $u \in H$ such that $r = us$. We denote this by $r \sim s$.

(v) Two factorizations into irreducibles of the same element,

$$r = a_1 \cdot \ldots \cdot a_n = b_1 \cdot \ldots \cdot b_m, \quad (1)$$

are called *essentially the same* if $n = m$ and, after re-indexing, $a_j \sim b_j$ for $1 \le j \le m$. Otherwise, the factorizations in (1) are called *essentially different*.

(vi) $(H, \cdot)$ is called *atomic* if every non-unit has a factorization.
(vii) $(H, \cdot)$ is *factorial* if H is atomic and any two factorizations of an element are essentially the same.
(ix) $(H, \cdot)$ is *half-factorial* if H is atomic and any two factorizations of an element have the same length, i.e., the same number of irreducible factors.
(x) In a half-factorial monoid, the *length* of a nonzero element h, denoted $\ell(h)$, is defined as the length of a factorization of h into irreducibles.

Definition 2.2. A commutative monoid $(H, \cdot)$ is called cancellative if $ab = ac$ implies $b = c$, for any $a, b, c \in H$.

The quotient group $\mathbf{q}(H)$ of a cancellative monoid is the group defined on the set of equivalence classes of pairs $(a, b) \in H \times H$ with respect to the equivalence relation

$$(a, b) \simeq (c, d) \iff ad = bc,$$

endowed with the multiplication

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ab}{cd},$$

where $\frac{a}{b}$ denotes the equivalence class of (a, b).

Definition 2.3. Let H be a cancellative commutative monoid.

(i) An irreducible element $r \in H$ is called *absolutely irreducible* (or, a *strong atom*), if for all natural numbers n, every factorization of r^n is essentially the same as $r^n = r \cdot \ldots \cdot r$.
(ii) If $r \in H$ is irreducible, but not absolutely irreducible, it is called *non-absolutely irreducible*.

2.2. Transfer homomorphisms

Transfer homomorphisms are a key tool in factorization theory. They are used to study non-unique factorization in a domain (or monoid) using a simpler "model" monoid. In this section, we show that transfer homomorphisms preserve absolute irreducibility (in the forward direction).

Definition 2.4. [15, Definition 3.2.1] Let H and M be cancellative commutative monoids. A monoid homomorphism $\theta\colon H \longrightarrow M$ is called a transfer homomorphism if it has the following properties:

(i) $M = \theta(H)M^\times$ and $\theta^{-1}(M^\times) = H^\times$
(ii) If $h \in H$ and $b, c \in M$ such that $\theta(h) = bc$, then there exist $v, w \in H$ such that $h = vw$ and $\theta(v) \sim b$ and $\theta(w) \sim c$.

Fact 2.5. *[15, Proposition 3.2.3] Let $\theta\colon H \longrightarrow M$ be a transfer homomorphism.*

(i) An element $u \in H$ is irreducible in H if and only if $\theta(u)$ is irreducible in M.
(ii) H is atomic if and only if M is atomic.

Lemma 2.6. *Let $\theta\colon H \longrightarrow M$ be a transfer homomorphism, and $c \in H$ an irreducible element. If c is absolutely irreducible in H then $\theta(c)$ is absolutely irreducible in M.*

Proof. If $\theta(c)$ is not absolutely irreducible in M, then there exists an irreducible element $a \in M$, not associated to $\theta(c)$, that divides $\theta(c)^m$ for some $m \in \mathbb{N}$. By the first of the defining properties of a transfer homomorphism, we may assume $a = \theta(b)$ for some $b \in H$. Since $\theta(b)$ is irreducible, it follows by the second of the defining properties of a transfer homomorphism that this b is an irreducible element of H, dividing c^m. Also, b cannot be associated to c, because otherwise $\theta(b)$ would be associated to $\theta(c)$. □

Remark 2.7. The converse of Lemma 2.6 does not hold. A non-absolutely irreducible element of H may be mapped to an absolutely irreducible element of M by a transfer homomorphism $\theta\colon H \longrightarrow M$. We illustrate this by an example: If H is a half-factorial commutative monoid and $M = (\mathbb{N}_0, +)$, then the function

$$\begin{aligned} \theta\colon H &\longrightarrow M \\ a &\longmapsto \ell(a), \end{aligned}$$

where $\ell(a)$ denotes the length of a (that is, the number of irreducibles in a factorization of a), is a transfer homomorphism.

If $c \in H$ is irreducible in H, then $\theta(c)$ is irreducible in M, by Fact 2.5. The unique irreducible of M is 1 (which is prime). Hence $\theta(c) = 1$ for all irreducible $c \in H$, and in particular, every irreducible in M is absolutely irreducible. However, there exist half-factorial monoids H that contain non-absolutely irreducibles, for instance $H = \mathcal{O}_K \setminus \{0\}$ with $\mathcal{O}_K$ a ring of algebraic integers whose class group is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ (see Fact 6.3).

2.3. Krull monoids

Recall that an integral domain is Krull if and only if it is completely integrally closed and Mori. The concepts involved in this characterization depend only on the multiplicative monoid and can thus be used to define Krull monoids.

A cancellative commutative monoid H is called *completely integrally closed* if it contains all elements of $\mathbf{q}(H)$ that are almost integral over H, that is, those $c \in \mathbf{q}(H)$ for which there exists some $d \in H$ such that for all $n \in \mathbb{N}$, $dc^n \in H$.

Likewise, a cancellative commutative monoid $(H, \cdot)$ is *Mori* if it satisfies the ascending chain condition for divisorial ideals. Here, a divisorial ideal is defined just like a divisorial ideal of an integral domain, with the quotient group $\mathbf{q}(H)$ of the cancellative monoid taking the place of the quotient field of an integral domain.

A cancellative commutative monoid is a *Krull monoid* if it is completely integrally closed and Mori.

With this definition of Krull monoid, an integral domain D is Krull if and only if $D \setminus \{0\}$ is a Krull monoid [15, Thm. 2.10.2.3].

We refer to [15, Chapter 2] for the algebraic theory of Krull monoids and for more details on the terminology we just introduced.

Let H be a Krull monoid and let $\mathfrak{X}(H)$ denote the set of nonempty divisorial prime (semigroup) ideals. The nonempty divisorial ideals of H form a free abelian monoid with basis $\mathfrak{X}(H)$ with respect to the divisorial product; the nonempty divisorial fractional ideals form a free abelian group on the same basis. Explicitly, every nonempty divisorial ideal $\mathfrak{a}$ of H is uniquely expressible as a divisorial product of prime ideals

$$\mathfrak{a} = \mathfrak{p}_1 \cdot \cdots \cdot_v \mathfrak{p}_r = \left(\prod_{\mathfrak{p} \in \mathfrak{X}(H)} \mathfrak{p}^{\mathsf{v}_\mathfrak{p}(\mathfrak{a})} \right)_v .$$

The set $\{\, \mathfrak{p} \in \mathfrak{X}(H) : \mathsf{v}_\mathfrak{p}(\mathfrak{a}) > 0 \,\} = \{\, \mathfrak{p} \in \mathfrak{X}(H) : \mathfrak{p} \supseteq \mathfrak{a} \,\}$ is the *support* of $\mathfrak{a}$. The class group $G = \mathcal{C}(H)$ of H is the quotient of the group of nonempty divisorial fractional ideals of H by the subgroup of principal fractional ideals. Let $[\mathfrak{a}] \in G$ denote the class of $\mathfrak{a}$. In factorization theory, the set $G_0 = \{\, [\mathfrak{p}] : \mathfrak{p} \in \mathfrak{X}(H) \,\}$ of classes containing prime divisors is of central importance.

We have two main examples of Krull monoids in mind: the first are the Dedekind domains, which are Krull domains. In fact, Dedekind domains (apart from fields, which usually count as Dedekind domains, too) are precisely the one-dimensional Krull domains [15, Thm. 2.10.6]. In a Dedekind domain D, the map $\mathfrak{a} \mapsto \mathfrak{a} \setminus \{0\}$ is a bijection between ring ideals of D and divisorial semigroup ideals of $D \setminus \{0\}$.

The divisorial product is the usual product of ideals, and the class group is the usual one (the group of fractional ideals by principal fractional ideals).

The following is another class of Krull monoids.

Definition 2.8. Let G be an additively written abelian group and $G_0 \subseteq G$ a nonempty subset. Let $\mathcal{F}(G_0)$ be the free abelian monoid with basis G_0.

(i) The elements of $\mathcal{F}(G_0)$ are called *sequences* over G_0 and are of the form

$$S = \prod_{g \in G_0} g^{n_g},$$

where $n_g = \mathsf{v}_g(S) \in \mathbb{N} \cup \{0\}$ with $n_g = 0$ for almost all $g \in G_0$.

(ii) The *length* of a sequence S is

$$|S| = \sum_{g \in G_0} \mathsf{v}_g(S) \quad \in \mathbb{N} \cup \{0\}$$

and the *sum* of S is

$$\sigma(S) = \sum_{g \in G_0} \mathsf{v}_g(S) g \quad \in G.$$

The *support* of S is

$$\text{supp}(S) = \{ g \in G_0 : \mathsf{v}_g(S) > 0 \}.$$

(iii) The monoid

$$\mathcal{B}(G_0) = \left\{ S \in \mathcal{F}(G_0) \ : \ \sigma(S) = 0 \right\}$$

is called the *monoid of zero-sum sequences* over G_0 or the *block monoid*.

The irreducibles of $\mathcal{B}(G_0)$ are the *minimal zero-sum sequences*: nonempty sequences whose sum is 0, but which do not contain a nonempty proper subsequence whose sum is 0.

The following key theorem links the factorization theory of Krull monoids to that of monoids of zero-sum sequences, showing that the latter serve as a combinatorial model for the factorization in Krull monoids and, in particular, Dedekind domains. A proof can be found in [15, Theorem 3.4.10.1]. For more expository accounts of this theory see [14, Theorem 1.3.4.2], the survey [16], or the expository article [6].

Theorem 2.9. *Let H be a Krull monoid with class group G, and let $G_0 \subseteq G$ be the set of classes containing prime divisors. Then there exists a transfer homomorphism*

$$\theta \colon H \to \mathcal{B}(G_0), \quad a \mapsto [\mathfrak{p}_1] \cdots [\mathfrak{p}_r],$$

where $aH = \mathfrak{p}_1 \cdot \dots \cdot_v \mathfrak{p}_r$ with $\mathfrak{p}_i \in \mathfrak{X}(H)$.

The homomorphism θ is called the *block homomorphism* of H.

3. Rings whose irreducible elements are all non-absolutely irreducible

Examples of atomic domains all of whose irreducibles are non-absolutely irreducible occur among generalized power series rings.

Remark 3.1. Let $K_1 \subseteq K_2$ be fields, $n \in \mathbb{N}$, and $R = K_1 + X^n K_2[[X]]$. Let H be the multiplicative monoid $R \setminus \{0\}$ and M_n be the numerical monoid $\{0\} \cup (n + \mathbb{N}_0)$. Then the monoid homomorphism

$$\theta \colon H \longrightarrow M_n$$
$$uX^\ell \longmapsto \ell$$

is a transfer homomorphism, where u is a unit of $K_2[[X]]$ and $\ell \in M_n$.

Proposition 3.2. *Let $K_1 \subseteq K_2$ be fields and $n \in \mathbb{N}$, and set $R = K_1 + X^n K_2[[X]]$. Then the following are equivalent.*

(i) $K_1 = K_2$ *and* $n = 1$.
(ii) *R is a UFD.*
(iii) *Every irreducible of R is prime.*
(iv) *R has a prime element.*
(v) *R has an absolutely irreducible element.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) follow immediately.

For (v) $\Longrightarrow$ (i), suppose $K_1 = K_2$ and $n > 1$. Then the units of R are the elements of R with constant term in $K_2 \setminus \{0\}$. It follows from Remark 3.1 and Fact 2.5 that the irreducible elements of R are the polynomials of the form $r = uX^m$, where u is a unit of $K_2[[X]]$ and $n \le m \le 2n - 1$. Every irreducible of the form r is not absolutely irreducible since for any $n \le t \le 2n - 1$, with $t \ne m$,

$$r^t = u^t X^t \cdot \underbrace{X^t \cdots X^t}_{m-1 \text{ copies}}$$

is a factorization of r essentially different from

$$\underbrace{r \cdots r}_{t \text{ copies}}.$$

Suppose $K_1 \ne K_2$. Then the units of R are the elements of R with constant term in $K_1 \setminus \{0\}$. Similarly, the irreducible elements of R are the polynomials of the form uX^m, where u is a unit of $K_2[[X]]$ and $n \le m \le 2n - 1$. Moreover, if u_1, u_2 are units of $K_2[[X]]$, then $u_1 X^m \not\simeq u_2 X^m$ if $u_1 u_2^{-1}$ has a constant term in $K_2 \setminus K_1$. It follows that each irreducible of the form uX^m is not absolutely irreducible since

$$(uX^m)^2 = ucX^m \cdot uc^{-1}X^m$$

is a factorization of $(uX^m)^2$ essentially different from $uX^m \cdot uX^m$, where $c \in K_2 \setminus K_1$. □

4. Rings with both absolutely and non-absolutely irreducible elements, but no primes

As an example of an atomic domain that has no prime element and contains both absolutely and non-absolutely irreducible elements, we propose the ring of integer-valued polynomials $\mathrm{Int}(\mathbb{Z})$, or, more generally, $\mathrm{Int}(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the ring of integers in a number field K. For a domain D with quotient field K, the ring of integer-valued polynomials on D is

$$\mathrm{Int}(D) = \{f \in K[x] \mid f(D) \subseteq D\}.$$

Definition 4.1. The *fixed divisor* of $f \in \mathrm{Int}(D)$, abbreviated $\mathrm{fd}(f)$, is the ideal of D generated by $f(D)$. If the fixed divisor is a principal ideal, we say $\mathrm{fd}(f) = c$, by abuse of notation, for $\mathrm{fd}(f) = (c)$. A polynomial $f \in \mathrm{Int}(D)$ with $\mathrm{fd}(f) = 1$ is called *image-primitive.*

It is clear that $\mathrm{fd}(f) \cdot \mathrm{fd}(g) \supseteq \mathrm{fd}(fg)$, but note that the inclusion may be strict. In any case, all divisors in $\mathrm{Int}(D)$ of an image-primitive polynomial $f \in \mathrm{Int}(D)$ are image-primitive.

4.1. No prime elements

Anderson, Cahen, Chapman, and Smith [3] showed that $\mathrm{Int}(\mathbb{Z})$ has no prime element by using the fact that $\mathrm{Int}(\mathbb{Z})$ is a Prüfer domain whose maximal ideals are known and are not principal. They argue that a Prüfer domain never has any principal prime ideals other than (0) and, possibly, maximal ideals. At the same time, no maximal ideal of $\mathrm{Int}(\mathbb{Z})$ is principal (or even finitely generated). Their argument readily generalizes to $\mathrm{Int}(\mathcal{O}_K)$.

We will here give an elementary, more explicit, proof that $\mathrm{Int}(\mathbb{Z})$ (and, more generally, $\mathrm{Int}(\mathcal{O}_K)$) has no prime element, by exhibiting, for every potential prime element p, a product ab such that p divides ab, but p divides neither a nor b.

The only nontrivial fact needed is that every nonconstant polynomial in $\mathbb{Z}[x]$ has zeros modulo infinitely many primes, and, more generally, for every number field K, every nonconstant polynomial in $\mathcal{O}_K[x]$ has zeros modulo infinitely many maximal ideals of $\mathcal{O}_K$.

To see this we refer to a few facts about d-rings, a notion introduced independently by Brizolis [8], and Gunji and McQuillan [19].

Definition 4.2. A domain D is a *d-ring* if for every nonconstant polynomial $f \in D[x]$ there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.

So, D being a d-ring just means that a polynomial $f \in D[x]$ cannot map D into the set of units of D unless f is a constant. It is easy to see that $\mathbb{Z}$ is a d-ring. Indeed, any $f \in \mathbb{Z}[x]$ such that $f(\mathbb{Z}) \subseteq \{1, -1\}$ must be constant.

Alternatively, d-rings can be characterized as those domains for which every integer-valued rational function is an integer-valued polynomial. We summarize what we need to know about d-rings (cf. [9], §VII.2).

Fact 4.3. *[8, Lemma 1.3], [19, Prop. 1]. The following are equivalent:*

(i) *D is a d-ring.*
(ii) *For every nonconstant $f \in D[x]$, the intersection of the maximal ideals M of D for which f has a zero modulo M is (0).*
(iii) *For every nonconstant $f \in \mathrm{Int}(D)$, there exists a maximal ideal M of D and an element $d \in D$ such that $f(d) \in M$.*

We conclude from Fact 4.3 that every nonconstant polynomial in $\mathbb{Z}[x]$ has zeros modulo infinitely many primes.

Lemma 4.4 (Anderson, Cahen, Chapman, and Smith [3]). $\mathrm{Int}(\mathbb{Z})$ *has no prime element.*

Proof. First, no constant can be prime, because, if $p \in \mathbb{Z}$ is a nonzero non-unit, then p divides $(x - r_1)\cdots(x - r_p)$, where $r_1, \ldots, r_p$ is a complete set of residues modulo $p\mathbb{Z}$, but, since $(x - r_i)/p$ is not integer-valued, p does not divide any individual linear factor.

Now consider $G \in \mathrm{Int}(\mathbb{Z})$ nonconstant, $G = \frac{g}{d}$ with $g(x) \in \mathbb{Z}[x]$ and $d \in \mathbb{Z}$. Let $p \in \mathbb{Z}$ be prime such that g has a zero modulo p but the polynomial function induced by g is not constant zero modulo p. (Such a prime p exists because g has a zero modulo infinitely many primes, but only finitely many primes divide the fixed divisor of g.)

Since $g \in \mathbb{Z}[x]$, the residue class of $g(r)$ modulo p depends only on the residue class of r modulo p. Let $r_1, \ldots, r_k$ be a complete set of representatives of those residue classes modulo p on which g takes a nonzero value modulo p. Let $h(x) = \prod_{i=1}^{k}(x - r_i)$. Note that $p \nmid h(c)$ when c is a zero modulo p of g (such as exist by assumption), so that $\frac{h(x)}{p}$ is not integer-valued.

Then, both $\frac{h(x)G(x)}{p}$ and $\frac{(G(x)+p)h(x)}{p}$ are integer-valued. This means that $G(x)$ divides $\frac{(G(x)+p)h(x)G(x)}{p}$, but $G(x)$ divides neither $G(x)+p$ (not even in $\mathbb{Q}[x]$) nor $\frac{h(x)G(x)}{p}$ (because $\frac{h(x)}{p}$ is not integer-valued). □

To generalize to $\mathrm{Int}(\mathcal{O}_K)$, we note that $\mathcal{O}_K$ is a d-ring for every number field K:

Fact 4.5. *[19, Prop. 3, Corollary 2]. Let $D \subseteq R$ be domains, and D a d-ring.*

(i) If R is integral over D, then R is a d-ring.
(ii) If R is finitely generated as a ring over D, then R is a d-ring.

Since $\mathbb{Z}$ is a d-ring, it follows by Fact 4.5 that $\mathcal{O}_K$, too, is a d-ring.

Lemma 4.6. *Let $\mathcal{O}_K$ be the ring of integers in a number field K. Then* $\text{Int}(\mathcal{O}_K)$ *has no prime element.*

Proof. First, no constant can be prime, because, if $p \in \mathcal{O}_K$ is a non-zero non-unit, then p divides $(x - r_1)\cdots(x - r_k)$, where $r_1, \ldots, r_k$ is a complete set of residues modulo $p\mathcal{O}_K$, but p does not divide any individual linear factor.

Now consider $G \in \text{Int}(\mathcal{O}_K)$ nonconstant, $G(x) = \frac{g}{d}$ with $g(x) \in \mathcal{O}_K[x]$ and $d \in \mathcal{O}_K$. Because there are only finitely many ramified primes and only finitely many primes dividing the fixed divisor of g, there exists an unramified prime $p \in \mathbb{Z}$, say $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$, such that

(i) g has a zero modulo P_1,
(ii) no P_i divides the fixed divisor of g.

Let k be the maximal number of different residue classes modulo any one P_j on which g assumes a non-zero value modulo P_j. Choose $R = \{r_1, \ldots, r_k\} \subseteq \mathcal{O}_K$ such that

(i) for each P_j, R contains a complete set of representatives of the residue classes modulo P_j on which g assumes a nonzero value modulo P_j,
(ii) $g(r_i)$ is not zero modulo P_1 for any $r_i \in R$.

Let $h = \prod_{i=1}^{k}(x - r_i)$, then

$$G \mid \frac{g(x)h(x)(g(x) + dp)}{dp} = G(x)\frac{h(x)(g(x) + dp)}{p}, \text{ but}$$

$$G \nmid \frac{g(x)h(x)}{dp} \text{ and } G \nmid (g(x) + dp).$$

□

4.2. Absolutely irreducible elements

Examples of absolutely irreducible elements of $\text{Int}(\mathbb{Z})$ include the binomial polynomials

$$\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!}$$

for $n > 1$. If n is prime, this is elementary, as already McClain [23] remarked in her honor's thesis. For general n, it is non-trivial and was shown by Rissner and the fifth author [26]. Their result has been generalized to function fields by Tichy and the fifth author [27], but we are here concerned with rings of integer-valued polynomials on number fields, where we can provide quite elementary examples of absolutely irreducible elements as follows:

Lemma 4.7. *For any number field K, there exist absolutely irreducible elements in* $\text{Int}(\mathcal{O}_K)$.

Proof. Let $p \in \mathcal{O}_K$ be an irreducible element with square-free factorization into prime ideals; $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$. (Such an element exists because there are unramified primes and $\mathcal{O}_K$ is atomic.) Then let $q = \max_{1\leq j\leq r}[\mathcal{O}_K : P_j]$. W.l.o.g., $[\mathcal{O}_K : P_1] = q$. Let $r_1, \ldots, r_q$ be a complete system of residues modulo P_1, containing one (but not more than one) complete system of residues modulo P_i for each i and not containing a complete system of residues modulo any other primes. Set

$$f(x) = (x - r_1) \cdots (x - r_q) \quad \text{and} \quad F(x) = \frac{f(x)}{p}.$$

We will show that F is absolutely irreducible.

Suppose F^m factors as $F^m = G_1 \cdots G_s$, with each G_i irreducible in $\mathrm{Int}(\mathcal{O}_K)$, and $G_i = c_i g_i$ with $c_i \in K$, g_i monic in $K[x]$. Since F^m is image-primitive, so is G_i for each i. This means that $c_i = (\mathrm{fd}(g_i))^{-1}$.

Let v be the essential valuation corresponding to P_1, normalized to have value group $\mathbb{Z}$. Then, in particular, $v(c_i) = -v(\mathrm{fd}(g_i))$, therefore,

$$\sum_{i=1}^{s} v(c_i) = v(p^{-m}) = -m \quad \text{and} \quad \sum_{i=1}^{s} v(\mathrm{fd}(g_i)) = m.$$

This can only happen if each g_i is a power of f; $g_i = f^{m_i}$ with $\sum_{i=1}^{s} m_i = m$.

To see this, consider that each g_i is a product of monic linear polynomials $(x - r_j)$, $1 \le j \le q$, where $r_1, \ldots, r_q$ form a complete system of residues modulo P_1. For such a polynomial $g = \prod_{j=1}^{q} (x - r_j)^{k_j}$, clearly $v(\mathrm{fd}(g)) = \min_{1 \le j \le q} k_j$.

Returning to $F^m = c_1 g_1 \cdots c_s g_s$, where $g_i = \prod_{j=1}^{q} (x - r_j)^{k_{ij}}$ then, of course, $\sum_{i=1}^{s} k_{ij} = m$ for each j, while, on the other hand,

$$m = \sum_{i=1}^{s} v(\mathrm{fd}(g_i)) = \sum_{i=1}^{s} \min_{1 \le j \le q} k_{ij}.$$

This implies that for each $1 \le i \le s$ and $1 \le h \le q$, necessarily $k_{ih} = \min_{1 \le j \le q} k_{ij}$, so that each linear factor occurs in g_i to the same exponent. If $m_i = \min_{1 \le j \le q} k_{ij}$ then $g_i = f^{m_i}$.

Now $c_i = (\mathrm{fd}(g_i))^{-1} = p^{-m_i}$, and, therefore, $G_i = c_i g_i = f^{m_i} p^{-m_i} = F^{m_i}$. As $c_i g_i = G_i$ was assumed irreducible, $m_i = 1$ follows. □

There are many other examples of absolutely irreducible elements in $\mathrm{Int}(\mathcal{O}_K)$, or more generally in $\mathrm{Int}(D)$, where D is a Dedekind domain with at least one finite residue field and torsion class group [12, Corollary 8.9.].

4.3. *Non-absolutely irreducible elements*

As an example of a non-absolutely irreducible element of $\mathrm{Int}(\mathbb{Z})$, consider

$$f = \frac{x(x^2+3)}{2}, \text{ noting that } f^2 = \frac{x^2(x^2+3)}{4} \cdot (x^2+3).$$

This example, taken from the third author's paper [24] on non-absolutely irreducible integer-valued polynomials, generalizes to $\mathrm{Int}(\mathcal{O}_K)$ as follows.

Lemma 4.8. *For any number field K, there exist non-absolutely irreducible elements in* $\mathrm{Int}(\mathcal{O}_K)$.

Proof. Let $p \in \mathcal{O}_K$ be an irreducible element with square-free factorization into prime ideals; $p\mathcal{O}_K = P_1 \cdot \ldots \cdot P_r$. (Such an element exists because there are unramified primes and $\mathcal{O}_K$ is atomic.) Then let $q = \max_{1 \le j \le r} [\mathcal{O}_K : P_j]$. W.l.o.g., $[\mathcal{O}_K : P_1] = q$. Let $r_1, \ldots, r_q$ be a complete system of residues modulo P_1, containing one (but not more than one) complete system of residues modulo P_i for each i and not containing a complete system of residues modulo any other primes. Set $g(x) = (x - r_1)^2$ and $h(x) = (x - r_2) \cdots (x - r_q)$.

Let H, G of the same degree as g and h, respectively, be irreducible in $K[x]$ and non-associated in $K[x]$, such that for any product of copies of g and h, the fixed divisor is the same as that of any modified product in which some copies of g have been replaced by G and some copies of h by H. (That such G and H exist has been shown by some of the present authors together with R. Rissner [11, Lemma 3.3].)

Let

$$F(x) = \frac{G(x)H(x)}{p}.$$

Then F is irreducible in $\mathrm{Int}(\mathcal{O}_K)$, but not absolutely irreducible, because

$$F^2 = \frac{G(x)H(x)^2}{p^2} \cdot G(x).$$

□

Regarding non-absolutely irreducible elements of $\mathrm{Int}(\mathcal{O}_K)$, we can likewise generalize examples where the n-th power of an irreducible element has factorizations of length other than n (for instance, [24, Example 4.4]) from $\mathrm{Int}(\mathbb{Z})$ to $\mathrm{Int}(\mathcal{O}_K)$ by using [11, Lemma 3.3] as in the above proof. To summarize:

Theorem 4.9. *For any number field K, the ring of integer-valued polynomials on algebraic integers,*

$$\mathrm{Int}(\mathcal{O}_K) = \{f \in K[x] \mid f(\mathcal{O}_K) \subseteq \mathcal{O}_K\},$$

is a ring without prime elements containing both absolutely irreducible and non-absolutely irreducible elements.

Proof. The non-existence of primes is shown in Lemma 4.6; the existence of absolutely irreducible elements in Lemma 4.7, and the existence of non-absolutely irreducible elements in Lemma 4.8. □

5. Rings with non-absolutely irreducible elements and primes, but no other absolutely irreducible elements

Examples of atomic domains that have no absolutely irreducible elements, but contain both prime elements and non-absolutely irreducibles arise for instance from the D+M construction [1].

Example 5.1. Let $R = K_1 + xK_2[x]$, where $K_1 \subsetneq K_2$ are fields. Then R is atomic and its irreducible elements are of the form

(i) ax, where $a \in K_2$ or
(ii) $a(1 + xf(x))$, where $a \in K_1, f(x) \in K_2[x]$, and $1 + xf(x)$ is irreducible in $K_2[x]$, see [1].

Furthermore, every irreducible of the form $a(1 + xf(x))$ is prime [1]. The irreducible elements of the form ax are not absolutely irreducible, because

$$(ax)^2 = acx \cdot ac^{-1}x$$

with $c \in K_2 \setminus K_1$.

With a bit more effort, we can even construct Noetherian semilocal integral domains of this type.

Proposition 5.2. *Let D and V be one-dimensional Noetherian local integral domains that have a common quotient field L, and set $R = D \cap V$. Suppose further that the following hold:*

(i) V is a discrete rank one valuation ring of L.
(ii) There exists a prime element π of V that is also a prime element of R. [1]
(iii) All the valuation rings that appear as localizations of the integral closure of D are distinct from V.

[1] Of course, the prime element π of V is unique up to associativity *in* V. In the course of the proof we will see $\pi V \cap R = \pi R$, so all associates of π in V that are contained in R are also associated to π *in* R.

(iv) D is not a unique factorization domain.
(v) There are at least two non-associated irreducible elements of D that are also irreducible in R.

Then R is a one-dimensional Noetherian integral domain (so, in particular, an atomic domain) that has exactly two maximal ideals and precisely one prime element up to associativity, namely π. In addition, it has irreducible elements distinct from π and all of these are not absolutely irreducible.

Proof. First, for the sake of completeness, we want to recall the full argument why D cannot be contained in V (which will be needed later in the proof). Assume to the contrary that $D \subseteq V$. Since V is integrally closed, the integral closure $\overline{D}$ of D is also contained in V.

Now there are two cases. In case $\pi V \cap \overline{D} = (0)$, all nonzero elements of $\overline{D} \setminus \{0\}$ are invertible in V and hence $L = (\overline{D} \setminus \{0\})^{-1}\overline{D} \subseteq V$, which is a contradiction. Otherwise, πV lies over a maximal ideal of $\overline{D}$ and hence $\overline{D}_{\pi V \cap \overline{D}} = V$. This is in contradiction to the assumption in Proposition 5.2. So, in total, we infer that D is not contained in V.

Denote by M the maximal ideal of D. We are now in the situation of [25, Theorem 3], which implies that $D = R_{M \cap R}$ and $V = R_{\pi V \cap R}$. Note that the prime ideals $M \cap R$ and $\pi V \cap R$ are not comparable by set-theoretic inclusion because, otherwise, D or V could not be a one-dimensional ring. So, we can apply [21, Theorem 105] and get that $M \cap R$ and $\pi V \cap R$ are exactly the maximal ideals of R. In particular, using the prime ideal correspondence under localization, R is a one-dimensional domain.

Because R is one-dimensional and π is a prime element of R by assumption, we must have $\pi R = \pi V \cap R$. In particular, $\pi \notin M$, and so π is a unit of D but a non-unit of V. Thus we are able to use [20, Corollary 1.20] and infer that R is Noetherian (and therefore atomic).

It is now left to show that the irreducible elements of R not associated to π are not absolutely irreducible. Since π is prime in R, all these irreducibles lie in $(M \cap R) \setminus (\pi V \cap R)$. Now, 5.2 says that there are at least two of them and hence, by [10, Lemma 2.1] they cannot be absolutely irreducible. □

We now give concrete examples of integral domains D and V as in 5.2. In particular, we thus find another atomic integral domain that has prime elements and irreducible elements that are not absolutely irreducible, but has no absolutely irreducible elements that are not prime.

Example 5.3. Let K be any field in which -1 is not a square, for instance $K = \mathbb{Q}$, and let X be an indeterminate over K. Define D and V as the localizations

$$D = K[X^2, X^3]_{(X^2,X^3)}, \ V = K[X]_{(X^2+1)},$$

and set $R = D \cap V$.

Note that $(X^2 + 1)$ is indeed a prime ideal of $K[X]$ since -1 is not a square in K. Both, D and V, are one-dimensional Noetherian local integral domains and their quotient field is just the function field $K(X)$. In order to see this for D, just note that $K[X^2, X^3]$ is a numerical semigroup algebra and hence one-dimensional and Noetherian.

Moreover, V clearly is a discrete rank one valuation ring with prime element $\pi = X^2+1$. The integral closure of D is the valuation ring $K[X]_{(X)}$ that is indeed distinct from V.

Next we show that $X^2 + 1$ is also a prime element of R. For this, we can just argue that it generates the prime ideal $P = \mathfrak{m}_V \cap R$, where $\mathfrak{m}_V$ denotes the unique maximal ideal of V.

Let $g \in P$. As g is, a fortiori, an element of D, we can write it in the form $g = \frac{h}{s}$, where $h, s \in K[X^2, X^3]$ and $s \notin (X^2, X^3)K[X^2, X^3]$. On the other hand, g is also in $\mathfrak{m}_V$ and we can therefore write it as $g = \frac{a}{b}(X^2 + 1)$ with $a, b \in K[X]$ and $b \notin (X^2 + 1)K[X]$.

It is our goal to show that $X^2 + 1$ divides g in R. Since $\frac{a}{b} \in V$ by its choice, it suffices to prove that $\frac{a}{b} \in D$. In order to do this, we clear denominators in the equation $\frac{h}{s} = \frac{a}{b}(X^2 + 1)$ and arrive at

$$bh = sa(X^2 + 1)$$

that we can view as an identity in $K[X]$. Since X^2+1 does not divide b in $K[X]$, it has to divide h and the unique cofactor is $\frac{sa}{b} \in K[X]$. Since $h \in K[X^2,X^3]$ has no linear term, it follows from $h = \frac{sa}{b}(X^2+1)$ that the polynomial $\frac{sa}{b}$ also has no linear term, that is, $\frac{sa}{b} \in K[X^2,X^3]$. By choice, $s \in K[X^2,X^3] \setminus (X^2,X^3)K[X^2,X^3]$ and hence $\frac{a}{b} = \frac{1}{s} \cdot \frac{sa}{b} \in D$.

As a last step, we argue that the elements X^2 and X^3 are still irreducible in R. They are both elements of $R \cap (X^2,X^3)K[X^2,X^3]$ and, therefore, non-units. In the following, for an irreducible polynomial p of $K[X]$, we denote by v_p the p-adic valuation on $K(X)$.

We carry out the argument for X^2; it is then analogous for X^3. So, decompose $X^2 = f \cdot g$ where $f, g \in R$. We want to show that either f or g is a unit of R. The elements of $D = K[X^2,X^3]_{(X^2,X^3)}$ (and therefore those of R) either have X-adic valuation 0 or ≥ 2. Since $\mathsf{v}_X(f) + \mathsf{v}_X(g) = \mathsf{v}_X(X^2) = 2$, we can assume without loss of generality that $\mathsf{v}_X(f) = 2$ and $\mathsf{v}_X(g) = 0$.

Furthermore, as elements of $V = K[X]_{(X^2+1)}$, the (X^2+1)-adic valuation of f and g is ≥ 0. Hence, the equality $0 = \mathsf{v}_{X^2+1}(X^2) = \mathsf{v}_{X^2+1}(f) + \mathsf{v}_{X^2+1}(g)$ implies that $\mathsf{v}_{X^2+1}(g) = 0$. To conclude, g is an element of R that is in neither of the two maximal ideals of R and therefore a unit. This finishes the example.

6. Rings with prime elements, absolutely irreducible elements that are not prime, and non-absolutely irreducible elements

For an atomic domain that has prime elements, absolutely irreducible elements that are not prime, and non-absolutely irreducible elements, we consider certain Krull domains and subrings of the ring of integer-valued polynomials.

Example 6.1. Let $R = \mathrm{Int}(\mathbb{Z}) \subseteq \mathbb{Q}[x]$ be the ring of integer-valued polynomials as in Section 4 (or any atomic domain satisfying ACCP and having both absolutely irreducible and non-absolutely irreducible elements).

We show that $T = R[y]$, the polynomial ring in one indeterminate over R, is an atomic domain that has primes, as well as both absolutely and non-absolutely irreducible elements.

Since R is an integral domain, y is a prime element of T.

Again because R is an integral domain, by degree considerations, the set of polynomials of degree zero in $R[y]$ (which we may identify with $R \setminus \{0\}$) forms a saturated (i.e., divisor closed) multiplicatively closed subset of $R[y]$ containing all the units of $R[y]$.

A constant in $R[y]$, therefore, is a unit, or absolutely irreducible, or non-absolutely irreducible, in $R[y]$ if and only if it has the respective property in R.

Since R contains both absolutely and non-absolutely irreducible elements, $R[y]$ contains both absolutely and non-absolutely irreducible elements (namely, the constants with the respective property in R), as well as a prime element, y.

Also, since R satisfies ACCP, so does $R[y]$, and $R[y]$ is, therefore, atomic.

Proposition 6.2. *For each prime number p, the ring*

$$R(p) = \left\{ \frac{g}{p^n} \in \mathrm{Int}(\mathbb{Z}) \text{ with } g \in \mathbb{Z}[X] \text{ and } n \in \mathbb{N}_0 \right\}$$

has infinitely many prime elements. Furthermore, $R(p)$ has both absolutely irreducible elements that are not prime and non-absolutely irreducible elements.

Proof. First, each prime number $q \neq p$ is prime in $R(p)$: If q divides a product $(g_1/p^{n_1}) \cdots (g_r/p^{n_r})$ in $R(p)$, then q divides $g_1 \cdots g_r$ in $\mathbb{Z}[X]$. Because $\mathbb{Z}[X]$ is a UFD with q prime, without restriction $g_1 = qf_1$ with $f_1 \in \mathbb{Z}[X]$. For all $a \in \mathbb{Z}$, also $b := (qf_1(a))/p^{n_1} \in \mathbb{Z}$, and since $q \neq p$, it follows that $f_1(a)/p^{n_1} \in \mathbb{Z}$, showing $f_1/p^{n_1} \in R(p)$. We have therefore shown that q divides g_1/p^{n_1} in $R(p)$, so q is prime in $R(p)$.

Now, let $r_1, \dots, r_p$ be a complete system of residues modulo p and set

$$f(x) = \frac{(x - r_1) \cdots (x - r_p)}{p}.$$

Then f is absolutely irreducible in $R(p)$. The polynomial f is not prime because f divides $(x - r_1) \cdots (x - r_p)$ but it does not divide any individual linear factor.

Furthermore, let $a_1, \dots, a_p, b_{p+1}, \dots, b_{p^2}$ be a complete system of residues modulo p^2 with $b_i \not\equiv 0 \pmod{p}$ for $p + 1 \leq i \leq p^2$. Let $c_1, c_2 \in \mathbb{Z}$ such that $c_1 \equiv c_2 \equiv 0 \pmod{p^2}$ and $c_1 \neq c_2$. Set $g(x) = \prod_{k=p+1}^{p^2} (x - b_k)$ and

$$f(x) = \frac{g(x)(x - c_1)(x - c_2)^{e-1}}{p^e},$$

where $e = \mathsf{v}_p(p^2!) = p + 1$. Then f is irreducible in $R(p)$, but not absolutely irreducible, because

$$f^2 = \frac{g(x)(x - c_1)^2(x - c_2)^{e-2}}{p^e} \cdot \frac{g(x)(x - c_2)^e}{p^e}$$

is a factorization of f^2 essentially different from $f \cdot f$. □

The non-absolutely irreducible elements of $R(p)$ whose n-th power has factorizations of different lengths can be constructed by adapting known examples in $\mathrm{Int}(\mathbb{Z})$ [24, Examples 4.1 and 4.4].

Fact 6.3. *Let $\mathcal{O}_K$ be the ring of integers of a number field K. Then the following hold.*

(i) $\mathcal{O}_K$ has infinitely many prime elements.
(ii) If $\mathcal{O}_K$ is not a unique factorization domain, then $\mathcal{O}_K$ has absolutely irreducible elements that are not prime, and non-absolutely irreducible elements, see [10, Theorem 3.1] or Corollary 7.4.

Example 6.4. Let $d = -ec$, where $e, c \in \mathbb{Z}$, $e, c \geq 2$, d is square-free and $d \not\equiv 1 \pmod 4$. Set $R = \mathbb{Z}[\sqrt{d}]$. Firstly, the element $\sqrt{d} = \sqrt{-ec}$ is irreducible in R, but not absolutely irreducible, because $(\sqrt{d})^2 = -e \cdot c$ is a nontrivial factorization of $(\sqrt{d})^2$ (not necessarily into irreducibles).

Secondly, let $r = a + b\sqrt{d} \in R$ and $N(r) = a^2 - db^2$, the norm of r. Then 2 is irreducible in R because $N(2) = 4$ and $N(r) \neq 2$ for all $r \in R$ (because $d \leq -6$). The element 2 is not prime in R because

(i) if $d \equiv 2 \pmod 4$, then $2 \mid (\sqrt{d})^2$, but $2 \nmid \sqrt{d}$, and
(ii) if $d \equiv 3 \pmod 4$, then $2 \mid (1 + \sqrt{d})(1 - \sqrt{d})$, but 2 does not divide the individual factors.

More generally, by [22, Theorem 25], the prime decomposition of $2R$ is

(i) $2R = (2, \sqrt{d})^2$ if $d \equiv 2 \pmod 4$, and
(ii) $2R = (2, 1 + \sqrt{d})^2$ if $d \equiv 3 \pmod 4$.

It follows by [10, Theorem 3.1] that 2 is absolutely irreducible in R.

Lastly, it follows by [22, Theorem 25] that every odd prime p such that $p \nmid d$ and $d^{\frac{p-1}{2}} \equiv -1 \pmod p$, is prime in R.

7. Absolutely irreducible elements in Krull monoids

The examples in Sections 8 and 9 will be constructed using Dedekind domains. The multiplicative monoid of nonzero elements of a Dedekind domain is a Krull monoid [15, Chapter 2.10]. To lay the groundwork for Sections 8 and 9, we therefore first prove some results on absolutely irreducible elements in Krull monoids, culminating in Theorem 7.2, which forms the basis of Propositions 8.1 and 9.1.

As corollaries we also recover a generalization of a theorem of Chapman and Krause (Corollary 7.4) and a theorem of Angermüller (Corollary 7.5). Corollary 7.5 in particular rules out the existence of Krull monoids having non-prime irreducibles, but having every absolutely irreducible prime. This shows that no example as in Section 5 is possible among Krull monoids. Corollary 7.4 shows that, if every class of the class group of a Krull monoid H contains a prime divisor and there exist no non-absolutely irreducible irreducibles, then H is already factorial. This explains why in Sections 8 and 9 we have to construct domains in which the set of classes containing prime divisors is a proper subset of the entire class group.

In this section, let H be a Krull monoid with class group G. Let $G_0 \subseteq G$ denote the set of classes containing prime divisors, and let $G_1 \subseteq G_0$ denote the set of classes containing *exactly* one prime divisor.

Absolutely irreducible elements in Krull monoids have been characterized in various ways. The following extends [15, Proposition 7.1.4] (where $G = G_0$ is assumed) and parts of [18, Proposition 4.7] (where $H = \mathcal{B}(G_0)$ with G torsion-free). A similar characterization is given in [4, Lemma 5].

Proposition 7.1. *Let $\emptyset \neq S \subseteq \mathfrak{X}(H)$ be a finite set. The following statements are equivalent.*

(i) There exists an absolutely irreducible $a \in H$ with $\operatorname{supp}(aH) = S$.
(ii) The set S is minimal in $\{ \operatorname{supp}(bH) : b \in H \setminus H^\times \}$
(iii) The set S is minimal in $\{ \operatorname{supp}(bH) : b \in H$ is irreducible$\}$.
(iv) The family $([\mathfrak{p}])_{\mathfrak{p}\in S}$ in G is $\mathbb{Z}_{\geq 0}$-linearly dependent, and every proper subfamily is $\mathbb{Z}_{\geq 0}$-linearly independent.
(v) The family $([\mathfrak{p}])_{\mathfrak{p}\in S}$ in G is $\mathbb{Z}_{\geq 0}$-linearly dependent, and every proper subfamily is $\mathbb{Z}$-linearly independent.

In case the equivalent conditions hold, the absolutely irreducible element with support S is uniquely determined up to associativity.

Proof. (i) ⇒ (ii) If $\operatorname{supp}(bH) \subsetneq S$, then $b \mid a^n$ for some $n \geq 1$ and b is not associated to a.

(ii) ⇒ (iii) Trivial.

(iii) ⇒ (ii) Suppose there exists $b \in H \setminus H^\times$ with $\operatorname{supp}(bH) \subsetneq S$. Let $u \in H$ be an irreducible element dividing b. Then $\operatorname{supp}(uH) \subsetneq S$.

(ii) ⇔ (iv) Statement (iv) is just a explicit way of stating (ii).

(iv) ⇒ (v) Fix a nonzero vector $(\alpha_\mathfrak{p})_{\mathfrak{p}\in S} \in \mathbb{Z}_{\geq 0}^S$ such that $\sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}] = 0$. Suppose, for the sake of contradiction, that there exists $S' \subsetneq S$ and a nonzero vector $(\beta_\mathfrak{p})_{\mathfrak{p}\in S'} \in \mathbb{Z}^{S'}$ such that $\sum_{\mathfrak{p}\in S'} \beta_\mathfrak{p}[\mathfrak{p}] = 0$. By the minimality of S, there must exist $\mathfrak{p} \in S'$ with $\beta_\mathfrak{p} < 0$. Let $\mathfrak{q} \in S'$ be such that $\beta_\mathfrak{q} < 0$ and so that $\alpha_\mathfrak{q}/\left|\beta_\mathfrak{q}\right|$ is minimal among all $\alpha_\mathfrak{p}/\left|\beta_\mathfrak{p}\right|$ with $\beta_\mathfrak{p} < 0$. Then

$$\left|\beta_\mathfrak{q}\right| \alpha_\mathfrak{p} + \alpha_\mathfrak{q}\beta_\mathfrak{p} \geq 0$$

for all $\mathfrak{p} \in S$, and equality holds for $\mathfrak{p} = \mathfrak{q}$. Since

$$\left|\beta_\mathfrak{q}\right| \sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}] + \alpha_\mathfrak{q} \sum_{\mathfrak{p}\in S'} \beta_\mathfrak{p}[\mathfrak{p}] = 0,$$

this contradicts the $\mathbb{Z}_{\geq 0}$-linear independence of $([\mathfrak{p}])_{\mathfrak{p}\in S\setminus\{\mathfrak{q}\}}$.

(v) ⇒ (i) Consider the group homomorphism $\sigma : \mathbb{Z}^S \to G$ given by $\sigma((\alpha_\mathfrak{p})_{\mathfrak{p}\in S}) = \sum_{\mathfrak{p}\in S} \alpha_\mathfrak{p}[\mathfrak{p}]$. Our assumptions ensure that there exists some $(\alpha_\mathfrak{p})_{\mathfrak{p}\in S} \in \mathbb{Z}_{>0}^S \cap \ker(\sigma)$. Moreover, the image of σ contains a torsion-free subgroup of rank $|S| - 1$. Thus $\ker(\sigma)$ is free of rank one. Since we know that $\ker(\sigma)$ contains a vector with all positive coordinates, we can also choose a generator (as an abelian group) $(\beta_\mathfrak{p})_{\mathfrak{p}\in S}$ with all positive coordinates. The zero-sum sequences with support S correspond to elements of $\ker(\sigma)$ with positive entries, and they are all positive multiples of $(\beta_\mathfrak{p})_{\mathfrak{p}\in S}$.

Hence there is, up to associativity, a unique irreducible with support contained in S. This irreducible is then necessarily absolutely irreducible. □

While the absolute irreducibility of elements does not lift along the transfer homomorphism $H \to \mathcal{B}(G_0)$ in general (Remark 2.7), additional knowledge of the set G_1 nevertheless allows us to characterize when *every* irreducible is absolutely irreducible using $\mathcal{B}(G_0)$.

Theorem 7.2. *Let H be a Krull monoid with class group G, let G_0 be the set of classes containing prime divisors, and let G_1 be the set of classes containing precisely one prime divisor. The following are equivalent.*

(i) Every irreducible of H is absolutely irreducible.
(ii) Every irreducible of $\mathcal{B}(G_0)$ is absolutely irreducible, and for every irreducible $U \in \mathcal{B}(G_0)$ and every $g \in G_0 \setminus G_1$ it holds that $\mathsf{v}_g(U) \le 1$.

Proof. Let $\theta\colon H \to \mathcal{B}(G_0)$ denote the block homomorphism.

(i) ⇒ (ii) Since θ is a transfer homomorphism, the monoid $\mathcal{B}(G_0)$ also has the property that every irreducible is absolutely irreducible (by Lemma 2.6).

For the second property, suppose that there exists an irreducible $U \in \mathcal{B}(G_0)$ that is of the form $U = g^2 T$ with $T \in \mathcal{F}(G_0)$, and that there exist $\mathfrak{p} \ne \mathfrak{q} \in \mathfrak{X}(H)$ such that $[\mathfrak{p}] = [\mathfrak{q}] = g$. We may assume $T = [\mathfrak{r}_1] \cdots [\mathfrak{r}_k]$ with $\mathfrak{r}_i \in \mathfrak{X}(H)$. Let $\mathfrak{a} = \mathfrak{r}_1 \cdots_v \mathfrak{r}_k$. Then $\mathfrak{p}^2 \cdot_v \mathfrak{a}$ and $\mathfrak{p} \cdot_v \mathfrak{q} \cdot_v \mathfrak{a}$ are principal ideals, say $aH = \mathfrak{p}^2 \cdot_v \mathfrak{a}$ and $bH = \mathfrak{p} \cdot_v \mathfrak{q} \cdot_v \mathfrak{a}$ with $a, b \in H$. Since θ is a transfer homomorphism, the irreducibility of U implies that of a and b. However $a \mid b^2$ and so b is not absolutely irreducible, contradicting our assumption.

(ii) ⇒ (i) Let $a \in H$ be irreducible. Then aH has a unique factorization

$$aH = \mathfrak{p}_1^{e_1} \cdots_v \mathfrak{p}_k^{e_k} \cdot_v \mathfrak{q}_1 \cdots_v \mathfrak{q}_l,$$

with pairwise distinct prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_k$ with $[\mathfrak{p}_i] \in G_1$ and exponents $e_i \ge 1$, and prime ideals $\mathfrak{q}_1, \ldots, \mathfrak{q}_l$ with $[\mathfrak{q}_i] \in G_0 \setminus G_1$. (Assumption (ii) applied to $\theta(a)$ implies $[\mathfrak{q}_i] \ne [\mathfrak{q}_j]$ for $i \ne j$.)

Suppose that $b \in H$ is an irreducible with $b \mid a^n$ for some $n \ge 1$. Then $\theta(b) \mid \theta(a)^n$. Since $\theta(a)$ is absolutely irreducible, we get $\theta(a) = \theta(b)$. Since $[\mathfrak{p}_i] \in G_1$ for all $1 \le i \le k$, the image $\theta(b)$ fully determines the multiplicity of each $\mathfrak{p}_i$ in bH. Now

$$bH = \mathfrak{p}_1^{e_1} \cdots_v \mathfrak{p}_k^{e_k} \cdot_v \mathfrak{q}_{i_1} \cdots_v \mathfrak{q}_{i_m},$$

for some $1 \le i_1 \le \cdots \le i_m \le l$. Since $[\mathfrak{q}_{i_r}] \in G_0 \setminus G_1$ for $1 \le r \le m$, assumption (ii) applied to $\theta(b)$ implies $[\mathfrak{q}_{i_r}] \ne [\mathfrak{q}_{i_s}]$ for $r \ne s$. Thus $1 \le i_1 < \cdots < i_m \le l$. Now $|\theta(a)| = |\theta(b)|$ shows $\{i_1, \ldots, i_m\} = \{1, \ldots, l\}$, and so $aH = bH$. □

7.1. Consequences for Krull monoids

As stated at the beginning of the section, we now note some easy consequences of Theorem 7.2 that connect to the existing literature and are of interest in their own right. Aside from the fact that Corollary 7.4 can be used to obtain (ii) of Fact 6.3, these results are however not used in this paper.

Given Theorem 7.2, and the fact that we are often interested in cases when $G_0 = G$ and $G_1 = \emptyset$, it is useful to determine when every irreducible of $\mathcal{B}(G)$ is absolutely irreducible. The equivalence of the first two statements of the following corollary is well-known.

Corollary 7.3. *Let G be an abelian group. Then the following are equivalent for the monoid of zero-sum sequences $\mathcal{B}(G)$.*

(i) $|G| \le 2$.
(ii) $\mathcal{B}(G)$ is factorial.
(iii) Every irreducible of $\mathcal{B}(G)$ is absolutely irreducible.

Proof. (i) ⇒ (ii): We recall the (well-known) argument. If G is trivial, then the sequence 0 (that is, the sequence of length 1 consisting of the single element $0 \in G$) is the only irreducible of $\mathcal{B}(G)$, and $\mathcal{B}(G) \cong$

$\mathbb{N}_0$ is factorial. If $G = \{0, g\} \cong \mathbb{Z}/2\mathbb{Z}$, then 0 and g^2 are the only irreducibles of $\mathcal{B}(G)$, and they are both prime, so that $\mathcal{B}(G) \cong \mathbb{N}_0^2$.

(ii) ⇒ (iii): Trivial.

(iii) ⇒ (i): We prove the contrapositive. Assume $|G| \geq 3$. Then one of the following three cases must occur.

- *G contains an element g of finite order $n \geq 3$.* Consider the irreducibles $S = g^n$, $S' = (-g)^n$ and $T = g(-g) \in \mathcal{B}(G)$. Then $SS' = T^n$, and so T is not absolutely irreducible (note $g \neq -g$).
- *G contains two independent elements g, h both of order 2.* Consider the irreducibles $S = g^2$, $S' = h^2$, $S'' = (g+h)^2$, and $T = gh(g+h) \in \mathcal{B}(G)$. Then $SS'S'' = T^2$.
- *G contains an element g of infinite order.* Consider $S = (3g)(-g)^3$, $S' = (3g)^2(-2g)^3$ and $T = (-g)(-2g)(3g) \in \mathcal{B}(G)$. Then S, S', and T are irreducible and $SS' = T^3$. □

The statements of the previous lemma are further equivalent to $\mathcal{B}(G)$ being half-factorial [15, Theorem 3.4.11.5].

The following (straightforwardly) generalizes the theorem of Chapman and Krause [10, Corollary 3.2], who proved the statement for $H = \mathcal{O}_K^\bullet$ with $\mathcal{O}_K$ a ring of algebraic integers in a number field.

Corollary 7.4. *Let H be a Krull monoid such that every class in the class group G contains a prime divisor (that is, $G = G_0$). Then every irreducible element of H is absolutely irreducible if and only if H is factorial.*

Proof. If H is factorial, then every irreducible is prime and hence absolutely irreducible.

For the converse, suppose that every irreducible of H is absolutely irreducible. By Theorem 7.2, every irreducible of $\mathcal{B}(G)$ must be absolutely irreducible. Thus $|G| \leq 2$ by Corollary 7.3. To show that G is trivial it therefore suffices to show $G \ncong \mathbb{Z}/2\mathbb{Z}$.

Suppose that $G = \{0, g\} \cong \mathbb{Z}/2\mathbb{Z}$. By (b) of [15, Theorem 2.5.4.1] there must exist at least two distinct nonzero divisorial prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of nontrivial class $[\mathfrak{p}] = [\mathfrak{q}] = g$. Therefore $\mathfrak{p}^2$, $\mathfrak{q}^2$ and $\mathfrak{p} \cdot_v \mathfrak{q}$ are all principal and generated by irreducibles u, v, w, say, $\mathfrak{p}^2 = uH$, $\mathfrak{q}^2 = vH$ and $\mathfrak{p} \cdot_v \mathfrak{q} = wH$. Now $w^2 \sim uv$ shows that w is not absolutely irreducible. □

We will see below, in Proposition 9.1, that some assumption on G_0 is necessary for the previous proposition. We also recover a theorem of Angermüller; see [5, Theorem 1(e)] for a generalization to monadically Krull monoids.

Corollary 7.5. [4, Corollary 1(c)] *A Krull monoid H is factorial if and only if every absolutely irreducible element is prime.*

Proof. If H is factorial, then every irreducible is prime.

For the converse, suppose that every absolutely irreducible element is prime. Suppose that H is not factorial. Then there exist non-prime irreducibles. Let $u \in H$ be a non-prime irreducible with $\mathrm{supp}(uH)$ minimal among all non-prime irreducibles. Then $\mathrm{supp}(uH)$ is in fact minimal among all irreducibles: if $p \in H$ is a prime, then pH is a divisorial prime ideal and hence $\mathrm{supp}(pH) = \{pH\} \subseteq \mathrm{supp}(uH)$ would imply that pH appears in the factorization of uH into divisorial prime ideals, meaning $p \mid u$, which is impossible since u is a non-prime irreducible. Then u is absolutely irreducible by Proposition 7.1, and therefore prime, a contradiction. □

Remark 7.6. Absolutely irreducible elements in Krull monoids have been studied in different settings and under different names. For instance, in [18] they play a very central role in the setting of $H = \mathcal{B}(G_0)$ with G torsion-free, and are called *elementary atoms*. If H is a normal affine monoid, then the absolutely irreducible elements correspond precisely to the extremal rays of the polyhedral convex cone, whereas the irreducible elements form the Hilbert basis of the monoid. We refer to [13], in particular to Section 4 and therein to Remarks 13 and 16 for a discussion of the terminologies.

8. Rings with irreducible elements that are all absolutely irreducible, but none of them prime

Proposition 8.1. *There exists a Dedekind domain that is not half-factorial and such that all of its irreducible elements are absolutely irreducible but none of them prime.*

Proof. Let $n \geq 2$ be an integer, let $G = \mathbb{Z}^n$, and let $\{e_1, e_2, \ldots, e_n\}$ the standard $\mathbb{Z}$-basis for G. Define $f = \sum_{i=1}^{n} e_i$ and set

$$G_0 = \{\pm e_i, \pm f\}.$$

Consider the monoid of zero-sum sequences $\mathcal{B}(G_0)$ over G_0. The irreducible elements of $\mathcal{B}(G_0)$ are

$$\{e_i(-e_i), f(-f), e_1e_2\cdots e_n(-f), (-e_1)(-e_2)\cdots(-e_n)f\}.$$

Because the supports of these sequences are pairwise incomparable, all of them are absolutely irreducible.

Let

$$U = e_1e_2\cdots e_n(-f) \quad \text{and} \quad V = (-e_1)(-e_2)\cdots(-e_n)f.$$

Then in $\mathcal{B}(G_0)$ we have the non-unique factorization

$$UV = (e_1(-e_1))\cdot(e_2(-e_2))\cdots(e_n(-e_n))\cdot f(-f).$$

It follows from [17, Theorem 8] that there exists a Dedekind domain D with class group $G = \mathbb{Z}^n$ and G_0 precisely the set of classes containing prime ideals. There exists a transfer homomorphism $\varphi\colon D \setminus \{0\} \to \mathcal{B}(G_0)$ (see Theorem 2.9). In particular, the domain D is not half-factorial and an element $a \in D$ is irreducible if and only if the corresponding zero-sum sequence $\varphi(a) \in \mathcal{B}(G_0)$ is irreducible.

Since $0 \notin G_0$, the trivial ideal class of D contains no prime ideals. Hence D contains no prime element. Finally, every irreducible element of $\mathcal{B}(G_0)$ is square-free, and so Theorem 7.2 implies that every irreducible element of D is absolutely irreducible. □

9. Rings with all irreducible elements absolutely irreducible but not all prime

In contrast to Proposition 8.1, the following result gives an analogous example of a Dedekind domain, but this time it contains a prime element.

Proposition 9.1. *There exists a Dedekind domain D that is not half-factorial, contains a prime element, and such that all of its irreducible elements are absolutely irreducible.*

Proof. Repeat the proof of Proposition 8.1 with the set

$$G_0 = \left\{0, \pm e_i, \sum_{i=1}^{n} e_i, \sum_{i=1}^{n} -e_i\right\}.$$

Note that $0 \in G_0$ and hence there exists a nonzero principal prime ideal in D and therefore a prime element. □

Acknowledgments

The authors wish to thank the second of two anonymous referees for contributing an additional example of a ring containing primes and both absolutely and non-absolutely irreducible elements to Section 6, namely, Example 6.1.

Funding

V. Fadinger was supported by the Austrian Science Fund (FWF) [10.55776/W1230]. S. Frisch and D. Windisch acknowledge support by the Austrian Science Fund (FWF) [10.55776/P30934]. S. Nakato was supported in part by Africa-UniNet under the project P142_Uganda: Uganda-Austria Collaboration in Algebra and Geometry D. Smertnig was funded in whole or in part by the Slovenian Research and Innovation Agency (ARIS): Grants P1-0288, J1-60025 and the Austrian Science Fund (FWF) [10.55776/P36742].

References

[1] Anderson, D. D., Anderson, D. F., Zafrullah, M. (1991). Rings between $D[X]$ and $K[X]$. *Houston J. Math.* 17(1): 109–129.
[2] Anderson, D. D., Coykendall, J., Hill, L., Zafrullah, M. (2007). Monoid domain constructions of antimatter domains. *Commun. Algebra* 35(10):3236–3241.
[3] Anderson, D. F., Cahen, P.-J., Chapman, S. T., and Smith, W. W. (1995). Some factorization properties of the ring of integer-valued polynomials. In: *Zero-Dimensional Commutative Rings (Knoxville, TN, 1994)*, volume 171 of Lecture Notes in Pure and Applied Mathematics, pp. 125–142. New York: Dekker.
[4] Angermüller, G. (2020). Strong atoms in Krull monoids. *Semigroup Forum* 101(1):11–18.
[5] Angermüller, G. (2022). Strong atoms in monadically Krull monoids. *Semigroup Forum* 104(1):10–17.
[6] Baginski, P., Chapman, S. T. (2011). Factorizations of algebraic integers, block monoids, and additive number theory. *Amer. Math. Monthly* 118(10):901–920.
[7] Baginski, P., Kravitz, R. (2010). A new characterization of half-factorial Krull monoids. *J. Algebra Appl.* 9(5): 825–837.
[8] Brizolis, D. (1975). Hilbert rings of integral-valued polynomials. *Commun. Algebra* 3(12):1051–1081.
[9] Cahen, P.-J., Chabert, J.-L. (1997). *Integer-Valued Polynomials*, volume 48 of Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society.
[10] Chapman, S. T., Krause, U. (2012). A closer look at non-unique factorization via atomic decay and strong atoms. In: *Progress in Commutative Algebra 2*. Berlin: Walter de Gruyter, pp. 301–315.
[11] Frisch, S., Nakato, S., Rissner, R. (2019). Sets of lengths of factorizations of integer-valued polynomials on Dedekind domains with finite residue fields. *J. Algebra* 528:231–249.
[12] Frisch, S., Nakato, S., and Rissner, R. (2022). Split absolutely irreducible integer-valued polynomials over discrete valuation domains. *J. Algebra* 602:247–277.
[13] García-Sánchez, P. A., Krause, U., Llena, D. (2023). Factorization in monoids by stratification of atoms and the Elliott problem. *Quaest. Math.* 46(6):1195–1219.
[14] Geroldinger, A. (2009). Additive group theory and non-unique factorizations. In: *Combinatorial Number Theory and Additive Group Theory*, Adv. Courses Math. CRM Barcelona, pp. 1–86. Basel: Birkhäuser Verlag.
[15] Geroldinger, A., Halter-Koch, F. (2006). *Non-Unique Factorizations*, volume 278 of Pure and Applied Mathematics (Boca Raton). Boca Raton, FL: Chapman & Hall/CRC. Algebraic, combinatorial and analytic theory.
[16] Geroldinger, A., Zhong, Q. (2020). Factorization theory in commutative monoids. *Semigroup Forum* 100(1):22–51.
[17] Gilmer, R., Heinzer, W., and Smith, W. W. (1996). On the distribution of prime ideals within the ideal class group. *Houston J. Math.* 22(1):51–59.
[18] Grynkiewicz, D. J. (2022). *The Characterization of Finite Elasticities—Factorization Theory in Krull Monoids via Convex Geometry*, volume 2316 of Lecture notes in mathematics. Cham: Springer.
[19] Gunji, H., McQuillan, D. L. (1975). On rings with a certain divisibility property. *Michigan Math. J.* 22(4):289–299.
[20] Heinzer, W., Ohm, J. (1972). Noetherian intersections of integral domains. *Trans. Amer. Math. Soc.* 167:291–308.
[21] Kaplansky, I. (1970). *Commutative Rings*. Jeffery-Williams lecture. Boston: Allyn and Bacon.
[22] Marcus, D. A. (1977). *Number Fields*. New York-Heidelberg: Springer-Verlag.
[23] McClain, B. A. (2004). Factorisation properties of integer-valued polynomials. Bachelor's thesis, Trinity University, Texas, U.S.
[24] Nakato, S. (2020). Non-absolutely irreducible elements in the ring of integer-valued polynomials. *Commun. Algebra* 48(4):1789–1802.
[25] Prekowitz, B. (1973). Intersections of quasi-local domains. *Trans. Amer. Math. Soc.* 181:329–339.
[26] Rissner, R., Windisch, D. (2021). Absolute irreducibility of the binomial polynomials. *J. Algebra* 578:92–114.
[27] Tichy, R., Windisch, D. (2024). Irreducibility properties of Carlitz' binomial coefficients for algebraic function fields. *Finite Fields Appl.* 96:Paper No. 102413, 10.