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GAFA Geometric And Functional Analysis

DIRECT AND INVERSE SPECTRAL CONTINUITY FOR DIRAC OPERATORS

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Abstract. The half-line Dirac operators with L^2-potentials can be characterized by their spectral data. It is known that the spectral correspondence is a homeomorphism: close potentials give rise to close spectral data and vice versa. We prove the first explicit two-sided uniform estimate related to this continuity in the general L^2-case. The proof is based on an exact solution of the inverse spectral problem for Dirac operators with δ-interactions on a half-lattice in terms of the Schur's algorithm for analytic functions.

Contents

1 Introduction . . . 352
  1.1 Dirac operators . . . 352
  1.2 Sylvester-Winebrenner theorem . . . 353
  1.3 The main result . . . 354
  1.4 Historical remarks . . . 355
  1.5 Schur's algorithm . . . 356
  1.6 Kronig-Penney model . . . 356
  1.7 Plan of the paper . . . 358
2 Dirac operators with periodic Weyl functions . . . 359
  2.1 Canonical Hamiltonian systems . . . 359
  2.2 Dirac operators with measures . . . 362
  2.3 Auxiliary lemmas . . . 370
  2.4 Proof of Theorem 1.4 . . . 371
  2.5 Proof of Corollaries 1.5-1.7 . . . 373
  2.6 Piecewise constant Hamiltonians . . . 378
3 Spectral continuity estimates for potentials in $L^2(\mathbb{R}_+)$ . . . 380
  3.1 Estimates for Schur's algorithm . . . 380
  3.2 Approximation of $L^2(\mathbb{R}_+)$-potentials by discrete measures . . . 386
  3.3 Approximation of $S_2(\mathbb{C}_+)$-functions by periodic Schur functions . . . 389
  3.4 Proof of Theorem 1.3 . . . 394
4 Sylvester-Winebrenner theorem . . . 395
  4.1 Properties of the Sylvester-Winebrenner metric . . . 395
  4.2 Spectral map is a homeomorphism . . . 398

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4.3 Direct map is not uniformly continuous . . . 399
4.4 Inverse map is not uniformly continuous . . . 403
References . . . 407

1 Introduction

The problem of stable reconstruction of differential operators from their spectral data is a classical subject of spectral theory. Here, *stability* means that the nonlinear map relating coefficients of the operators and their spectral data is a *homeomorphism* between some topological spaces. An overview of stability results for operators with discrete spectrum can be found in Hryniv [Hry11, Hry11], Horváth and Kiss [HK10]; see also Savchuk and Shkalikov [SS10], Chelkak, Kargaev, and Korotyaev [KC09, CKK04]. Spectral stability for operators with nonempty continuous spectrum is much less studied. We give some historical remarks below in Sect. 1.4. This paper is devoted to the stability of the solutions of direct and inverse spectral problems for Dirac operators on the half-line $\mathbb{R}_+ = [0, +\infty)$.

1.1 Dirac operators. The Dirac operator $\mathcal{D}_q$ on $\mathbb{R}_+$ with a square summable potential $q \in L^2(\mathbb{R}_+)$ is defined by the differential expression

$$\mathcal{D}_q : X \mapsto JX' + QX, \qquad Q = \begin{pmatrix} \operatorname{Im} q & \operatorname{Re} q \\ \operatorname{Re} q & -\operatorname{Im} q \end{pmatrix}, \tag{1.1}$$

where $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and X belongs to the set of all locally absolutely continuous vector-valued functions in the Hilbert space

$$L^2(\mathbb{R}_+, \mathbb{C}^2) = \left\{ Y : \mathbb{R}_+ \to \mathbb{C}^2 : \|Y\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} = \int_{\mathbb{R}_+} \|Y(x)\|^2_{\mathbb{C}^2}\, dx < \infty \right\}$$

satisfying a self-adjoint boundary condition $\langle X(0), \left(\begin{smallmatrix} \sin\alpha \\ \cos\alpha \end{smallmatrix}\right)\rangle_{\mathbb{C}^2} = 0$, and such that $\mathcal{D}_q X \in L^2(\mathbb{R}_+, \mathbb{C}^2)$. With this domain, $\mathcal{D}_q$ is the densely defined self-adjoint operator on $L^2(\mathbb{R}_+, \mathbb{C}^2)$, see [Lev91], [Den06]. Without loss of generality, we will work with the boundary condition corresponding to $\alpha = 0$. To describe appropriate spectral data for $\mathcal{D}_q$, consider the matrix-valued solution $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right)$ of the Dirac system $JN'(x,z) + Q(x)N(x,z) = zN(x,z)$, $N(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $z \in \mathbb{C}$. The Weyl function of $\mathcal{D}_q$ is defined by

$$m_q(z) = \lim_{x \to +\infty} \frac{n_{22}(x,z)}{n_{21}(x,z)}, \qquad \operatorname{Im} z > 0.$$

This function belongs to the Herglotz class in the upper half-plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$, i.e., it is analytic and takes $\mathbb{C}_+$ into itself. See [Den06] for the Weyl theory of Dirac operators from the perspective of Krein systems. It is well-known that the Weyl function m_q determines $\mathcal{D}_q$ uniquely. From the point of view of spectral correspondence it is more convenient to work with the Cayley transform of m_q, i.e., with the Schur function f_q of $\mathcal{D}_q$. Recall that an analytic function f on $\mathbb{C}_+$ is said to belong to the Schur class $S(\mathbb{C}_+)$ if $|f(z)| \leqslant 1$ for all $z \in \mathbb{C}_+$. The Schur function f_q of $\mathcal{D}_q$ is determined by

$$m_q = i\frac{1+f_q}{1-f_q}. \tag{1.2}$$

Each function $f \in S(\mathbb{C}_+)$ has the nontangential boundary values almost everywhere on $\mathbb{R}$ [Gar81]. As usual, we use the same letter f for the function in the unit ball of $L^\infty(\mathbb{R})$ defined by these boundary values. In a moment we will see that Schur functions of $\mathcal{D}_q$ with $q \in L^2(\mathbb{R}_+)$ belong to the set

$$S_2(\mathbb{C}_+) = \Big\{ f \in S(\mathbb{C}_+) : \, \log(1-|f|^2) \in L^1(\mathbb{R}) \Big\}. \tag{1.3}$$

The set $S_2(\mathbb{C}_+)$ is a complete metric space with respect to the metric

$$\rho_{S_2}(f,g) = \sqrt{\int_{\mathbb{R}} -\log\Big(1-\Big|\frac{f-g}{1-\overline{f}g}\Big|^2\Big)\, dx}. \tag{1.4}$$

It can be shown that $f_n \to g$ in $S_2(\mathbb{C}_+)$ if and only if $\|\log(1-|f_n|^2)\|_{L^1(\mathbb{R}_+)} \to \|\log(1-|g|^2)\|_{L^1(\mathbb{R}_+)}$ and $f_n \to g$ in Lebesgue measure on $\mathbb{R}$, see Lemma 4.4.

1.2 Sylvester-Winebrenner theorem. Our starting point is the following fundamental result that stems from the paper [SW99] by Sylvester and Winebrenner.

Theorem 1.1 (Sylvester-Winebrenner theorem). *The correspondence $\mathcal{F} : q \mapsto f_q$ is a homeomorphism from $L^2(\mathbb{R}_+)$ onto $S_2(\mathbb{C}_+)$. Moreover, we have*

$$\int_{\mathbb{R}_+} |q(x)|^2\, dx = \frac{1}{\pi}\int_{\mathbb{R}} -\log(1-|f_q(x)|^2)\, dx. \tag{1.5}$$

The fact that $\mathcal{F}$ is a bijection from $L^2(\mathbb{R}_+)$ onto $S_2(\mathbb{C}_+)$, as well as identity (1.5), was established by Denisov [Den06] following original ideas of Sylvester and Winebrenner [SW99]. We use arguments from the same paper [SW99] to prove continuity of $\mathcal{F}$ and $\mathcal{F}^{-1}$, see details in Sect. 4.2.

Theorem 1.1 belongs to a general direction in spectral theory that relies on the usage of trace formulae (or *sum rules*, in the terminology of B. Simon [Sim11]). This direction often leads to the most general results when one is interested in complete characterization theorems (*"spectral gems"* [Sim11]). See, e.g., [KS03, KS09, DKS10, Yud18, BD20, BD21, DEY21]. The proofs of such theorems, however, do not involve reconstruction procedures for potentials from the spectral data, and in particular, they do not imply any *continuity estimates* related to the spectral correspondence. To illustrate the situation, let us rewrite the sum rule (1.5) in the form

$$\|q-0\|^2_{L^2(\mathbb{R}_+)} = \rho_{S_2}(f_q, 0)^2/\pi.$$

Having this identity, it is natural to expect that quantities $\|q-\tilde{q}\|_{L^2(\mathbb{R}_+)}$, $\rho_{S_2}(f_q, f_{\tilde{q}})$ control each other for q, $\tilde{q} \in L^2(\mathbb{R})$. Moreover, Theorem 1.1 says that $\|q_n - \tilde{q}\|_{L^2(\mathbb{R}_+)} \to 0$ if and only if $\rho_{S_2}(f_{q_n}, f_{\tilde{q}}) \to 0$, making this expectation even more plausible. It turns out, however, that it is false. In fact, we have the following theorem.

Theorem 1.2. *There are potentials u_n, $\tilde{u}_n$, q_n, $\tilde{q}_n$ in the unit ball of $L^2(\mathbb{R}_+)$ such that*

$$\lim_{n\to\infty} \|u_n - \tilde{u}_n\|_{L^2(\mathbb{R}_+)} = 0, \ \ \textit{but} \ \lim_{n\to\infty} \rho_{S_2}(f_{u_n}, f_{\tilde{u}_n}) > 0, \tag{1.6}$$

$$\lim_{n\to\infty} \|q_n - \tilde{q}_n\|_{L^2(\mathbb{R}_+)} > 0, \ \ \textit{but} \ \lim_{n\to\infty} \rho_{S_2}(f_{q_n}, f_{\tilde{q}_n}) = 0. \tag{1.7}$$

In other words, the homeomorphisms $\mathcal{F}$, $\mathcal{F}^{-1}$ in Theorem 1.1 are not uniformly continuous on bounded subsets of $L^2(\mathbb{R}_+)$, $S_2(\mathbb{C}_+)$.

As an “explanation” for (1.6), (1.7), let us mention that the continuous operators $\mathcal{F}$, $\mathcal{F}^{-1}$ are not linear and the closed unit balls in $L^2(\mathbb{R}_+)$, $S_2(\mathbb{C}_+)$ are not compact. In particular, the standard general arguments are not applicable here and the lack of uniform continuity of $\mathcal{F}$, $\mathcal{F}^{-1}$ is possible. To prove (1.6), we construct some explicit sequences of potentials u_n, $\tilde{u}_n$. The proof of (1.7) is more delicate. Here we use a very important observation of Volberg and Yuditskii [VY02] on the non-injectivity of the scattering map for Jacobi matrices. This observation was transferred to the setting of the nonlinear Fourier transform by Tao and Thiele [TT03] and to Dirac operators by the first author and Denisov [BD24].

1.3 The main result. Let us now turn to positive results. For every $f \in S_2(\mathbb{C}_+)$, the function $|f|^2$ is comparable to $|\log(1-|f|^2)|$ on the set E where $|f| \leqslant 1/2$, and the complement $\mathbb{R}\setminus E$ has a finite measure as $\log(1-|f|^2) \in L^1(\mathbb{R})$. It follows that $f \in L^2(\mathbb{R})$. Thus, the Fourier transform of any element $f \in S_2(\mathbb{C}_+)$ is well defined and belongs to $L^2(\mathbb{R})$. We will denote it by $\hat{f}$, so that

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-i\xi x}\, dx \tag{1.8}$$

if f is integrable. We will need the following Wiener-type norm and the weighted L^1-norm:

$$\|f\|_{W^1_A(\mathbb{R}_+)} = \int_{\mathbb{R}_+} |\hat{f}(\xi)| e^{-A\xi}\, d\xi, \qquad \|q\|_{L^1_A(\mathbb{R}_+)} = \int_{\mathbb{R}_+} |q(\xi)| e^{-A\xi}\, d\xi.$$

Our main result is the following theorem.

Theorem 1.3. *Let q, $\tilde{q} \in L^2(\mathbb{R}_+)$, and let $f_q, f_{\tilde{q}}$ be the Schur functions (1.2) of the corresponding Dirac operators $\mathcal{D}_q$, $\mathcal{D}_{\tilde{q}}$ (1.1). Then we have*

$$c_1 \|q-\tilde{q}\|_{L^1_{2A}(\mathbb{R}_+)} \leqslant \|f_q - f_{\tilde{q}}\|_{W^1_A(\mathbb{R}_+)} \leqslant c_2 \|q-\tilde{q}\|_{L^1_{2A}(\mathbb{R}_+)} \tag{1.9}$$

for $c_1 = \sqrt{\pi/2}$, $c_2 = 2\sqrt{2\pi}$ and any $A \geqslant 12 \max\left(\|q\|^2_{L^2(\mathbb{R}_+)}, \|\tilde{q}\|^2_{L^2(\mathbb{R}_+)}\right)$.

We would like to stress that (1.9) is a uniform estimate. This makes it much stronger than the continuity property in Theorem 1.1, cf. (1.6), (1.7). Note also that (1.9) is nontrivial and new even in the case $\tilde{q} = 0$.

1.4 Historical remarks. Before proceed with further results, let us give a few historical remarks. Perhaps, the most general stability result in the one-dimensional spectral theory is the Krein-de Branges spectral theorem (see Sect. 5.2 of [Rem18]). It gives stability of the solution of direct and inverse spectral problems for canonical Hamiltonian systems and implies spectral stability for various other classical operators (Schrödinger and Dirac operators, Krein strings, Jacobi matrices). However, the proof of Krein-de Branges theorem, at least in its current form, cannot give explicit stability estimates, because it uses the following general topological argument to prove the fact that the solution map is a homeomorphism:

a continuous bijection between two Hausdorff compacts
is a homeomorphism. (1.10)

Moreover, the usage of compactness arguments similar to (1.10) forces to deal with very weak variants of stability, because for this approach closed bounded subsets in the metric spaces under consideration need to be compact. Even implicit stability control with respect to the norms like $\|\cdot\|_{L^p(\mathbb{R}_+)}$ or $\|\cdot\|_{L^1_A(\mathbb{R}_+)}$ via general Krein-de Branges theory is not possible because these norms define topologies that are not locally compact.

For classical operators, there are well-known constructive methods to solve inverse spectral problems developed by Gelfand-Levitan and Krein (see Marchenko [Mar06] for an excellent historical overview) and more recent methods by Belishev-Mikhaylov [BM12] and Makarov-Poltoratski [MP23]. In principle, one can use these methods to prove spectral continuity by accurate estimation of all quantities appearing in the proofs. See, e.g., historically first stability results by Marchenko [Mar68], Lundina and Marchenko [LM69], Marchenko and Maslov [MM70], as well as their recent extensions by Xu [Xu21], Xu and Bondarenko [XB22]. Section 6 in Denisov [Den06] is devoted to spectral continuity for $L^2_{\mathrm{loc}}(\mathbb{R}_+)$-potentials.

On the other hand, it seems difficult to get *optimal* estimates from a direct analysis of concrete classical methods because they are rather involved and even "just" characterization theorems (isomorphisms between potentials and spectral data without proving continuity) are already very nontrivial. In particular, the control of stability in classical methods comes from consideration of integral equations on intervals $[0, R]$, $R > 0$, it becomes weaker and weaker with R growing to infinity if we do not impose strong integrability assumptions on the potential.

This explains why we choose another road and first treat the case of the so-called *exactly solvable models*. In this case a detailed analysis is possible and gives explicit two-sided estimates. We then approximate a general Dirac operator with potential $q \in L^2(\mathbb{R}_+)$ by a sequence of exactly solvable models and arrive at Theorem 1.3. Our main instrument is the classical Schur's algorithm whose definition we now recall.

1.5 Schur's algorithm. The Schur class $S(\mathbb{D})$ in the open unit disk $\mathbb{D}=\{z\in\mathbb{C}: |z|<1\}$ consists of analytic functions F on $\mathbb{D}$ such that $|F(z)|\leqslant 1$ for all $z\in\mathbb{D}$. We will deal with the subset of $S(\mathbb{D})$ defined by

$$S_*(\mathbb{D})=\{F\in S(\mathbb{D}):\ F \text{ is not a finite Blaschke product}\}.$$

The Schur's algorithm for $F\in S_*(\mathbb{D})$ is the following iterative procedure:

$$F_0=F, \qquad zF_{k+1}=\frac{F_k-F_k(0)}{1-\overline{F_k(0)}F_k}, \qquad k\geqslant 0, \qquad z\in\mathbb{D}. \tag{1.11}$$

All functions produced by Schur's algorithm belong to $S_*(\mathbb{D})$, moreover, each step $F_k\mapsto F_{k+1}$ is the bijection from $S_*(\mathbb{D})$ onto itself. The numbers $\{F_k(0)\}_{k\geqslant 0}$ in (1.11) are called the recurrence coefficients of F. We also will need a version of Schur's algorithm for periodic functions in the upper half-plane $\mathbb{C}_+$. Fix $\ell>0$ and define

$$S_{\ell,*}(\mathbb{C}_+)=\Big\{f:\mathbb{C}_+\to\mathbb{D}:\ f(z)=F(e^{2i\ell z}),\ F\in S_*(\mathbb{D})\Big\}. \tag{1.12}$$

A function $f\in S(\mathbb{C}_+)$ belongs to $S_{\ell,*}(\mathbb{C}_+)$ if and only if $f(z+\pi/\ell)=f(z)$ for every $z\in\mathbb{C}_+$, and there is no finite Blaschke product B in $\mathbb{D}$ such that $f=B(e^{2i\ell z})$. Relation (1.11) for $f(z)=F(e^{2i\ell z})$, $F\in S_*(\mathbb{D})$, takes the form

$$f_0=f, \qquad e^{2i\ell z}f_{k+1}=\frac{f_k-f_k(\infty)}{1-\overline{f_k(\infty)}f_k}, \qquad k\geqslant 0, \qquad z\in\mathbb{C}_+, \tag{1.13}$$

where $f_k(\infty)=\lim_{y\to\infty}f_k(iy)=F_k(0)$, see Lemma 2.9. It is natural to call the numbers $\{f_k(\infty)\}_{k\in\mathbb{Z}_+}$ the recurrence coefficients of $f\in S_{\ell,*}(\mathbb{C}_+)$. They determine functions f, F uniquely. In fact, the knowledge of first n recurrence coefficients of f allows to approximate it with accuracy $2e^{-2n\ell y}$ in the half-plane $\operatorname{Im} z\geqslant y>0$, cf. (1.3.43) in [Sim05].

1.6 Kronig-Penney model. Let us consider the half-line Dirac operators $\mathcal{D}_q$ (1.1) on $\mathbb{R}_+$ whose potentials

$$q=\sum_{k\in\mathbb{Z}_+}c_k\delta_{\ell k}, \qquad c_k\in\mathbb{C}, \tag{1.14}$$

now are linear combinations of point masses (usually, they are called δ-interactions) supported on the half-lattice $\ell\mathbb{Z}_+=\{\ell k:\ k\in\mathbb{Z},\ k\geqslant 0\}$ of sparseness $\ell>0$. The spectral theory of this class of Dirac operators can be considered in the framework of the relativistic Kronig-Penney model for massless particles (for comparison, in the original Kronig-Penney model [D+31] for a non-relativistic electron in a one-dimensional crystal, Schrödinger operators with δ-interactions on the lattice $\mathbb{Z}$ were used, and $c_k\equiv c$ in the classical case). The Kronig-Penney model and its various generalizations are called exactly solvable models, meaning that the resolvents of the operators under consideration (Schrödinger or Dirac) as well as associated quantities (spectra, resonances, eigenfunctions, etc) can be often found explicitly in terms of the potential q. Solvable models attracted an enormous attention in theoretical physics and mathematics. We refer the reader to the classical monograph [A+88] (1988) by Albeverio, Gesztesy, Høegh-Krohn, and Holden, to the review chapter by Exner in the second edition [A+05] (2005) of this monograph, and to later survey by Kostenko and Malamud [KM13] (2013). The most close mathematical references to our work are [CMP13, LS14, Hug98, GS87]. The main distinction of our setting from the previous considerations comes from the fact that exactly solvable relativistic models are usually studied for radial massive Dirac operators

$$\mathcal{D}_{\mathbf{m},q}: X \mapsto c\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X'(x) + \mathbf{m}c^2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} X(x) + Q(x)X(x),$$

with some positive parameters c, $\mathbf{m}$ (corresponding to the velocity of light and the mass of the particle, see Sect. 4.6.6 in [Tha92]). We consider the case where $\mathbf{m} = 0$ and choose physical units so that $c = 1$. We also assume that Q is real and written in the second canonical form, i.e., $Q = Q^*$, $\operatorname{trace} Q = 0$, see (1.1). These assumptions are standard for the spectral theory of Dirac operators [LS91] and for its applications to the nonlinear Schrödinger equation [FT07] (the massless Dirac operator is the auxiliary operator for the inverse scattering transform method for NLSE) but less common in the area of exactly solvable models.

Essential part of the literature devoted to exactly solvable models deals with direct problems: knowing potential q (a measure on some discrete subset of $\mathbb{R}$), one determines some spectral characteristics of the corresponding Schrödinger or Dirac operator. On the other hand, the full spectral characterization (the Weyl function or the spectral measure and the corresponding Fourier transform) is not known even for potentials q supported on a lattice in the simplest massless case $\mathbf{m} = 0$. Moreover, it is not immediate if it is possible to describe spectral measures in closed form in terms of q. Indeed, for general q of the form (1.14) the corresponding spectral measures could have a complex structure and arbitrary spectral type (e.g., singular continuous component is not excluded). Below we show that such a description indeed exists, and, moreover, it turns out to be very simple and explicit (modulo nonlinearity and generality of the problem). To state our second main result, we need the following bijection $\varkappa: \mathbb{D} \to \mathbb{C}$:

$$\varkappa(w) = \frac{\overline{w}}{2|w|} \log \frac{1 + |w|}{1 - |w|}, \qquad w \in \mathbb{C}. \tag{1.15}$$

Theorem 1.4. *Let q be a discrete complex-valued measure on $\mathbb{R}_+$ such that* $\operatorname{supp} q \subset \ell\mathbb{Z}_+$, *$\ell > 0$. Then the Schur function f_q of $\mathcal{D}_q$ is a nondegenerate π/ℓ-periodic function, i.e., $f_q \in S_{\ell,*}(\mathbb{C}_+)$, where the set $S_{\ell,*}(\mathbb{C}_+)$ is defined in (1.12). Any element of $S_{\ell,*}(\mathbb{C}_+)$ arises uniquely in this way. Moreover, q and f_q determine each other via*

$$q(\{\ell k\}) = \varkappa(f_{q,k}(\infty)), \qquad k \in \mathbb{Z}_+, \tag{1.16}$$

where $\{f_{q,k}(\infty)\}_{k \in \mathbb{Z}_+}$ is the sequence of recurrence coefficients of f_q, see (1.13).

It is astonishing that the Schur's algorithm [Sch17] (1917) and the Kronig-Penney model [D+31] (1931) met in Theorem 1.4 almost a century after their independent development and generalizations in various directions. In fact, Theorem 1.4 completely reduces the spectral theory of the relativistic Kronig-Penney model for massless particles to the theory of orthogonal polynomials on the unit circle [Sze75, Sim05]. To illustrate this, we formulate some immediate spectral consequences. Below we assume that q is an arbitrary discrete complex-valued measure on $\mathbb{R}_+$ such that $\operatorname{supp} q \subset \mathbb{Z}_+$, and $\mathcal{D}_q$ is the corresponding Dirac operator. We will use the standard notation $\sigma_{ac}(\mathcal{D}_q)$, $\sigma_p(\mathcal{D}_q)$, $\sigma_{sc}(\mathcal{D}_q)$ for the absolutely continuous, pure point, and singular continuous parts of its spectrum.

COROLLARY 1.5 (Rakhmanov). *If* $\sigma_{ac}(\mathcal{D}_q) = \mathbb{R}$, *then* $q(\{k\}) \to 0$ *as* $k \to +\infty$.

COROLLARY 1.6 (Baxter). *If* $\sum_{k \geqslant 0} |q(\{k\})| < \infty$, *then* $\sigma_p(\mathcal{D}_q) = \sigma_{sc}(\mathcal{D}_q) = \emptyset$, *and the main spectral measure* μ_q *of* $\mathcal{D}_q$ *has the form* $\mu_q = w_q\,dx$, *where* $w_q = \sum_{k\in\mathbb{Z}} c_k e^{2ikx}$ *is such that* $\sum_{k\in\mathbb{Z}} |c_k| < \infty$, $\min_{\mathbb{R}} w_q > 0$. *The converse is also true.*

COROLLARY 1.7 (Szegő-Golinskii-Ibragimov). *If* $\sum_{k\geqslant 0} k|q(\{k\})|^2 < \infty$, *then* $\sigma_p(\mathcal{D}_q) = \sigma_{sc}(\mathcal{D}_q) = \emptyset$, *and the main spectral measure* μ_q *of* $\mathcal{D}_q$ *has the form* $\mu_q = w_q\,dx$, *where* $\log w_q = \sum_{k\in\mathbb{Z}} c_k e^{2ikx}$ *is such that* $\sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty$. *The converse is also true.*

Generalized versions of Schur's algorithm appeared in spectral theory previously. For example, in Sect. 10 of [Den06] Denisov discusses an analogue of Schur's algorithm in the form of the differential equation

$$\frac{df_\ell(z)}{d\ell} = -izf_\ell(z) + A(\ell) - \overline{A(\ell)}f_\ell^2(z), \qquad \ell \in \mathbb{R}_+, \quad z \in \mathbb{C}_+, \quad A(\ell) = q(\ell/2)/2,$$

for Schur functions f_ℓ of potentials $q_\ell = q(\ell + \cdot)$ generated by $q \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, see (10.3) in [Den06]. See also Poltoratski [Pol24] for a similar Ricatti equation for certain meromorphic inner functions arising on the spectral side of Dirac system. These equations are more difficult to analyze in the perturbation regime than the recursive relation (1.13) in the classical Schur's algorithm.

1.7 Plan of the paper.

We start the next section with the introduction of canonical Hamiltonian systems and Dirac operators with measures. We prove Theorem 1.4 in Sect. 2.4. Theorem 1.3 is proved in Sect. 3. We derive it from continuity estimates for Schur's algorithm (Sect. 3.1) and Theorem 1.4. In Sects. 4.2-4.4 we show that the spectral correspondence $\mathcal{F} : q \mapsto f_q$ is a homeomorphism that is not uniformly continuous in both directions, thus proving Theorem 1.1 and Theorem 1.2.

2 Dirac operators with periodic Weyl functions

2.1 Canonical Hamiltonian systems. A canonical Hamiltonian system on $\mathbb{R}_+ = [0, +\infty)$ is the differential equation of the form

$$JX'(x,z) = z\mathcal{H}(x)X(x,z), \qquad x \in \mathbb{R}_+, \quad z \in \mathbb{C}, \tag{2.1}$$

where, as before, $J = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, the derivative of the vector-valued function $X : \mathbb{R}_+ \times \mathbb{C} \to \mathbb{C}^2$ is taken with respect to $x \in \mathbb{R}_+$, and $\mathcal{H}$ is a Hamiltonian. Here and below by a Hamiltonian we mean a matrix-valued mapping on $\mathbb{R}_+$,

$$\mathcal{H} : x \mapsto \begin{pmatrix} h_1 & h \\ h & h_2 \end{pmatrix}, \qquad \mathcal{H}(x) \geqslant 0,$$

whose entries h_1, h_2, h are real-valued functions in $L^1_{\mathrm{loc}}(\mathbb{R}_+) = \{f : f \in L^1[0,\ell] \text{ for all } \ell > 0\}$. We also assume that $\mathcal{H}$ is not identically zero on any subset of $\mathbb{R}_+$ of positive Lebesgue measure. A Hamiltonian $\mathcal{H}$ is called singular if $\mathcal{H} \notin L^1(\mathbb{R}_+)$, or, equivalently,

$$\int_{\mathbb{R}_+} \operatorname{trace} \mathcal{H}(x)\, dx = +\infty.$$

The set of singular Hamiltonians on $\mathbb{R}_+$ will be denoted by $\mathbb{H}_{sing}$. Let Θ, Φ denote the solutions of (2.1) satisfying $\Theta(0,z) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\Phi(0,z) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Take $\omega \in \mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\}$ (here and below ∞ is regarded as the element of the Riemann sphere, and the linear fractional transformations involving ∞ are understood accordingly). To each Hamiltonian $\mathcal{H} \in \mathbb{H}_{sing}$ one can associate the Weyl function,

$$m_{\mathcal{H}}(z) = \lim_{x \to +\infty} \frac{\Phi^+(x,z) + \Phi^-(x,z)\omega}{\Theta^+(x,z) + \Theta^-(x,z)\omega}, \qquad \Theta = \begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix}, \quad \Phi = \begin{pmatrix} \Phi^+ \\ \Phi^- \end{pmatrix}, \quad z \in \mathbb{C}_+. \tag{2.2}$$

It is known that the limit above exists for every $z \in \mathbb{C}_+$ and does not depend on the choice of ω. Moreover, the Weyl function, $m_{\mathcal{H}}$, is analytic in $\mathbb{C}_+$ and takes $\mathbb{C}_+$ into $\mathbb{C}_+$ unless it coincides with a constant $c \in \mathbb{R} \cup \{\infty\}$. Weyl's theory for canonical Hamiltonian systems can be found in [HSW00], [Rem18], [Rom14]. We also define the Schur's function $f_{\mathcal{H}}$ by

$$f_{\mathcal{H}} = \frac{m_{\mathcal{H}} - i}{m_{\mathcal{H}} + i}, \qquad m_{\mathcal{H}} = i\frac{1 + f_{\mathcal{H}}}{1 - f_{\mathcal{H}}}. \tag{2.3}$$

Analytic functions taking $\mathbb{C}_+$ into $\mathbb{C}_+$ form the Herglotz-Nevanlinna class $N(\mathbb{C}_+)$. The set

$$\overline{N}(\mathbb{C}_+) = N(\mathbb{C}_+) \cup \mathbb{R} \cup \{\infty\}$$

is the compactification of $N(\mathbb{C}_+)$ when the latter is equipped with the topology of convergence on compact subsets in $\mathbb{C}_+$. This topology (we extend it to $\overline{N}(\mathbb{C}_+)$) is metrizable with the metric, e.g.,

$$\rho_c(m,\tilde{m}) = \max_{|z-i|\leqslant 1/2} \frac{2|m(z)-\tilde{m}(z)|}{\sqrt{1+|m(z)|^2}\sqrt{1+|\tilde{m}(z)|^2}},$$
$$\rho_c(m,\infty) = \max_{|z-i|\leqslant 1/2} \frac{2}{\sqrt{1+|m(z)|^2}}.$$

One can choose other metrics on $\overline{N}(\mathbb{C}_+)$ determining the same compact topological space, see discussion on page 109 in [Rem18].

Different singular Hamiltonians $\mathcal{H}, \tilde{\mathcal{H}}$ can have equal Weyl functions. For instance, it is not difficult to check that if $\tilde{\mathcal{H}}(x) = \xi'(x)\mathcal{H}(\xi(x))$ almost everywhere on $\mathbb{R}_+$ for some locally absolutely continuous increasing bijection $\xi : \mathbb{R}_+ \to \mathbb{R}_+$, then $m_{\mathcal{H}} = m_{\tilde{\mathcal{H}}}$. It will be convenient to call such $\mathcal{H}$, $\tilde{\mathcal{H}}$ equivalent, so that $\mathbb{H}_{sing}$ becomes the set of classes of equivalent Hamiltonians:

$$\mathbb{H}_{sing} = \{\mathcal{H} \text{ is a singular Hamiltonian on } \mathbb{R}_+\}/\sim.$$

One can check that each class of equivalence in $\mathbb{H}_{sing}$ contains the unique (up to values on a set of measure zero) element $\mathcal{H}^{tr}$ such that $\operatorname{trace}\mathcal{H}^{tr} = 1$ on $\mathbb{R}_+$. One can turn $\mathbb{H}_{sing}$ into a compact Hausdorff space by defining the topology via the metric, e.g.,

$$\begin{aligned} d(\mathcal{H},\tilde{\mathcal{H}}) &= \sum_{n\geqslant 0} 2^{-n} \frac{d_n(\mathcal{H}^{tr},\tilde{\mathcal{H}}^{tr})}{1+d_n(\mathcal{H}^{tr},\tilde{\mathcal{H}}^{tr})}, \\ d_n(\mathcal{H}^{tr},\tilde{\mathcal{H}}^{tr}) &= \sup_{0\leqslant t\leqslant n} \left\| \int_0^t (\mathcal{H}^{tr}(s) - \tilde{\mathcal{H}}^{tr}(s))\, ds \right\|. \end{aligned} \tag{2.4}$$

Compactness of $\mathbb{H}_{sing}$ follows from Riesz representation theorem, see the proof of Theorem 5.4 in [Rem18].

The following theorem is a key result of Krein – de Branges spectral theory of canonical Hamiltonian systems [KK68], [Bra68], [DM76], [Rem18], [Rom14].

Theorem 2.1 (Krein – de Branges theorem). *The correspondence $\mathcal{H} \mapsto m_{\mathcal{H}}$ is the homeomorphism of compact metric spaces $\mathbb{H}_{sing}$, $\overline{N}(\mathbb{C}_+)$.*

For a discussion (and some surprisingly deep applications) of the continuity part in Theorem 2.1, see Sect. 5 in [Rem18], Sect. 2 in [EKT18], or Sect. 3 in [ELS211]. Currently, explicit estimates related to the continuity properties of the homeomorphism in Theorem 2.1 are not known. However, in contrast to the Sylvester-Winebrenner theorem (Theorem 1.1), the homeomorphism in Krein-de Branges theorem are uniformly continuous in both directions by compactness and the Heine-Cantor theorem.

Let $\mathcal{H} \in \mathbb{H}_{sing}$ be a singular Hamiltonian on $\mathbb{R}_+ = [0,+\infty)$. The fundamental matrix solution corresponding to $\mathcal{H}$ is the locally absolutely-continuous (with respect to $x \in \mathbb{R}_+$) matrix-valued mapping $M : \mathbb{R}_+ \times \mathbb{C} \to \mathbf{SL}(2,\mathbb{C})$ satisfying the differential equation

$$JM'(x,z) = z\mathcal{H}(x)M(x,z), \qquad M(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \tag{2.5}$$

almost everywhere on $\mathbb{R}_+$, where the differentiation is taken with respect to $x \in \mathbb{R}_+$ and $z \in \mathbb{C}$ is a spectral parameter. Note that we have $M = \big(\Theta, \Phi\big)$ in terms of the solutions Θ, Φ of (2.1) satisfying $\Theta(0,z) = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$, $\Phi(0,z) = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Let us write $w_1 \doteq Aw_2$ for two complex numbers w_1, w_2 and a matrix A with $\det A \neq 0$ if

$$w_1 = \frac{a_{11}w_2 + a_{12}}{a_{21}w_2 + a_{22}}, \qquad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

It is straightforward to generalize this definition to the case where w_1, w_2 can admit the value ∞. For a matrix $A \in \mathbf{SL}(2,\mathbb{C})$, we have $w_1 \doteq Aw_2$ if and only if $w_2 \doteq A^{-1}w_1$, $w_{1,2} \in \mathbb{C} \cup \{\infty\}$. Note also that the definition (2.2) for $m_{\mathcal{H}}$ can be rewritten in the following form:

$$m_{\mathcal{H}} \doteq \lim_{x\to\infty} \sigma_1 M(x,z)^T \sigma_1 \omega, \tag{2.6}$$

where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and M^T stands for the transposed matrix M. Relation (2.3) between the Weyl and Schur functions reads as

$$m_q \doteq Lf_q, \quad f_q \doteq Rm_q, \qquad L = \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}. \tag{2.7}$$

LEMMA 2.2. *For $\mathcal{H} \in \mathbb{H}_{sing}$ and $A \in \mathbf{SL}(2,\mathbb{R})$ define $\mathcal{H}_A = A^*\mathcal{H}A$. Then we have*

(a) *$M_A = A^{-1}MA$, where M_A and M are solutions of (2.5) with Hamiltonians $\mathcal{H}_A$ and $\mathcal{H}$;*

(b) *$m_{\mathcal{H}_A} \doteq \sigma_1 A^* \sigma_1 m_{\mathcal{H}}$, where $m_{\mathcal{H}_A}$ and $m_{\mathcal{H}}$ are the Weyl functions corresponding to $\mathcal{H}_A$ and $\mathcal{H}$.*

Proof. The proof is a calculation. We have $J^{-1} = -J$ and

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad A^{-1} = \begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix}, \quad JA^{-1}J = \begin{pmatrix} -a_{11} & -a_{21} \\ -a_{12} & -a_{22} \end{pmatrix} = -A^*.$$

Therefore we can write

$$\begin{aligned} J(A^{-1}MA)' &= JA^{-1}M'A = (JA^{-1}J)(J^{-1}M'A) = A^*(JM')A = zA^*HMA \\ &= z(A^*HA)A^{-1}MA, \end{aligned}$$

which proves assertion (a) of the lemma. To prove (b), take $\omega \in \mathbb{R}$. Formula (2.6) for $m_{\mathcal{H}_A}$ takes the form

$$m_{\mathcal{H}_A} \doteq \lim_{x\to\infty} \sigma_1 M_A(x,z)^T \sigma_1 \omega \doteq \lim_{x\to\infty} \sigma_1 (A^{-1}M(x,z)A)^T \sigma_1 \omega$$

$$\doteq \lim_{x\to\infty} \sigma_1 A^* M(x,z)^T (A^{-1})^* \sigma_1 \omega.$$

Note that $\sigma_1(A^{-1})^*\sigma_1 \in \mathbf{SL}(2,\mathbb{R})$, hence the number $\tilde{\omega} \doteq \sigma_1(A^{-1})^*\sigma_1\omega$ is in $\mathbb{R}$. Then, we have

$$m_{\mathcal{H}_A} \doteq \lim_{x\to\infty} \sigma_1 A^* M(x,z)^T \sigma_1 \tilde{\omega} \doteq \sigma_1 A^* \sigma_1 \lim_{x\to\infty} \sigma_1 M(x,z)^T \sigma_1 \tilde{\omega} \doteq \sigma_1 A^* \sigma_1 m_{\mathcal{H}}.$$

This proves the lemma. □

Given a Hamiltonian $\mathcal{H} \in \mathbb{H}_{sing}$ and $\ell > 0$, define the Hamiltonian $\mathcal{H}_\ell = \mathcal{H}(\ell + \cdot)$ on $\mathbb{R}_+$. Let $m_{\mathcal{H}_\ell}$ be the Weyl function of $\mathcal{H}_\ell$. For the following lemma see, e.g., the proof of formula (2.13) in [BD20].

Lemma 2.3. *We have* $m_{\mathcal{H}}(z) \doteq \sigma_1 M(\ell,z)^T \sigma_1 m_{\mathcal{H}_\ell}(z)$ *for all* $z \in \mathbb{C}_+$, *where* M *is the solution of* (2.5).

Lemma 2.4. *Let* H *be a singular Hamiltonian on* $\mathbb{R}_+$. *Define*

$$\mathcal{H}(x) = \begin{cases} \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), & x \in [0,\ell), \\ H(x-\ell), & x \geqslant \ell, \end{cases} \qquad E_{\ell z} = \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix}.$$

Then we have $m_{\mathcal{H}} \doteq E_{\ell z} m_H$ *and* $f_{\mathcal{H}} = e^{2i\ell z} f_H$.

Proof. Notice that $M(t,x) = E_{xz}$ solves (2.1) with the Hamiltonian $\mathcal{H}$ for $x \leqslant \ell$. Lemma 2.3 then gives

$$m_{\mathcal{H}} \doteq \sigma_1 \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix}^T \sigma_1 m_H \doteq \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix} m_H \tag{2.8}$$

as claimed. Applying relation (2.7) we get

$$\begin{aligned} f_{\mathcal{H}} &\doteq R m_{\mathcal{H}} \doteq R E_{\ell z} m_H = R E_{\ell z} L f_H \\ &\doteq \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} f_H \doteq \begin{pmatrix} e^{2i\ell z} & 0 \\ 0 & e^{2i\ell z} \end{pmatrix} f_H. \end{aligned}$$

The proof of the lemma is concluded. □

2.2 Dirac operators with measures. Let us denote by $\mathcal{M}$ the set of all signed complex Borel measures on $\mathbb{R}$ with $\operatorname{supp}\mu \subset \mathbb{R}_+$ such that the total variation of μ is finite on all intervals $[0,L]$, $L \geqslant 0$. Each element $q \in \mathcal{M}$ generates the matrix potential

$$Q = \begin{pmatrix} \mu_2 & \mu_1 \\ \mu_1 & -\mu_2 \end{pmatrix}, \qquad \begin{cases} \mu_1 = \operatorname{Re} q, \\ \mu_2 = \operatorname{Im} q. \end{cases} \tag{2.9}$$

Conversely, each matrix potential Q of the form (2.9) with real entries $\mu_{1,2} \in \mathcal{M}$ generates a complex-valued measure $q = \mu_1 + i\mu_2$ in $\mathcal{M}$ that we also refer to as the potential. It is possible to associate a self-adjoint Dirac operator $\mathcal{D}_q$ to each $q \in \mathcal{M}$. That was done by Zeng in paper [Zen23] devoted to Dirac operators with measures. In particular, Zeng [Zen23] proved relation of these operators to canonical Hamiltonian systems, that we will use in the proof of Theorem 1.4. For the reader's convenience, we give a summary of some results from [Zen23]. Since our measures $q \in \mathcal{M}$ might have a nontrivial point mass at $x = 0$ (the case formally excluded by Zeng, though he mentioned that it can be covered as well), we make necessary modifications of definitions.

Take $q \in \mathcal{M}$, construct the corresponding matrix potential Q as in (2.9), and define the solution of the differential equation

$$JN_q'(x) + Q(x)N_q(x) = 0, \qquad \lim_{x\to 0,\, x<0} N_q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.10}$$

as the unique 2×2 continuous from the right matrix function N_q such that

$$JN_q(x) + \int_{(-\infty,x]} \mathbf{g}(Q(\{x_1\})J^*)Q(x_1)N_q(x_1) = J, \qquad x \in \mathbb{R}, \tag{2.11}$$

where

$$\mathbf{g}(T) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{n=2}^{\infty} \frac{T^{n-1}}{n!},$$

for a 2×2 matrix T. This definition of the solution N_q of (2.10) via (2.11), as well as the existence and uniqueness of such a solution are due to Persson [Per88] (see Theorem 3.1 in [Per88]). The reader might note appearance of J^* in the expression $\mathbf{g}(Q(\{x_1\})J^*)$ in (2.11). This multiplicative factor J^* does not appear in the work [Per88] of Persson. The explanation is simple: (2.10) is in fact the differential equation for JN_q (namely, $JN_q' + QJ^*\cdot JN_q = 0$), and the coefficient in front of JN_q in (2.10) is QJ^*, not Q. Then, we need to use same coefficient in (2.11). Finally, $\mathbf{g}(Q(\{x_1\})J^*)Q(x_1)N_q(x_1)$ is just the short way to write $\mathbf{g}(Q(\{x_1\})J^*)Q(x_1)J^*\cdot JN_q(x_1)$.

For regular potentials, the matrix-valued function N_q solving (2.11) coincides with the classical solution of (2.10). More precisely, if $q = s\,dx$ for some $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$, and we define S by $S\,dx = JQ$, then $g(Q(\{x_1\})J^*) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ for every $x_1 \in \mathbb{R}_+$, and (2.10), (2.11) are both equivalent to the integral equation

$$N_q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^x S(x_1)N_q(x_1)\,dx_1,\ \ x \geqslant 0, \qquad N_q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ on } (-\infty, 0),$$

that can be solved by iterations:

$$\begin{aligned} N_q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_0^x S(x_1)\,dx_1 + \int_0^x S(x_1)\int_0^{x_1} S(x_2)\,dx_2\,dx_1 \\ + \int_0^x S(x_1)\int_0^{x_1} S(x_2)\int_0^{x_2} S(x_3)\,dx_3\,dx_2\,dx_1 + \cdots . \end{aligned} \tag{2.12}$$

Here, the series converges in the matrix norm and the n-th term can be bounded by

$$\begin{aligned}\left\| \int_0^x \dots \int_0^{x_{n-1}} S(x_1)S(x_2)\cdots S(x_n)\,dx_n \dots dx_1 \right\| &\leqslant \frac{1}{n!}\left(\int_0^x \|S(x_1)\|\,dx_1\right)^n \\ &\leqslant \frac{(4\|s\|_{L^1[0,x]})^n}{n!}.\end{aligned} \tag{2.13}$$

For a general $q \in \mathcal{M}$, the solution N_q can be approximated by solutions of regular equations. More precisely, if φ is a nonnegative continuous function supported on $[-1,1]$, $\|\varphi\|_{L^1(\mathbb{R})} = 1$, and $\varphi_\varepsilon = \varepsilon^{-1}\varphi(x/\varepsilon)$, $\varepsilon > 0$, is the corresponding approximate unity, then the solutions of regularized equations

$$JN'_{q^{(\varepsilon)}}(x) + Q^{(\varepsilon)}(x)N_{q^{(\varepsilon)}}(x) = 0, \quad N_{q^{(\varepsilon)}}(-1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad q^{(\varepsilon)}(x) = \int_{\mathbb{R}} \varphi_\varepsilon(y-x)q(y), \tag{2.14}$$

converge pointwise on $\mathbb{R}$ to N_q, see [Per88] (we use initial condition at $x=-1$ instead of $x=0$ because $\operatorname{supp} q^{(\varepsilon)} \subset [-1,\infty)$ for small $\varepsilon > 0$, one can also use any other point in $(-\infty, 0)$ for the initial condition). In particular, for every $q \in \mathcal{M}$ we have $\det N_q = 1$ on $\mathbb{R}$ and the multiplicative property

$$N_q(w_2) = N_q(w_2, w_1)N_q(w_1), \qquad -\infty < w_1 < w_2 < \infty, \tag{2.15}$$

holds, where $N_q(w_2, w_1)$ is the value at $w_2 - w_1$ of the solution of (2.10) for the potential $x \mapsto \chi_{\mathbb{R}_+}(x)q(x+w_1)$, $\chi_{\mathbb{R}_+}$ being the indicator function of $\mathbb{R}_+$. Indeed, it is enough to use the multiplicative property for regularized solutions $N_{q^{(\varepsilon)}}$ and take the limit.

For regular potentials q (i.e., for $q = s\,dx$ with some $s \in L^1_{\mathrm{loc}}(\mathbb{R}_+)$) the corresponding Dirac operators are related to canonical systems as follows. One need to take the solution of (2.10) and define the Hamiltonian

$$\mathcal{H}_q(x) = N_q^*(x)N_q(x), \qquad x \geqslant 0. \tag{2.16}$$

Then, the Dirac operator $\mathcal{D}_q$ and the operator of canonical system $\mathcal{D}_{\mathcal{H}_q}$ are unitary equivalent, see, e.g., [Rom14] or Sect. 2.4 in [Bes20]. As we will see in a moment, the same relation holds for $q \in \mathcal{M}$.

At first, we need some notation. By $\mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$ we will denote the set of all continuous from the right functions $X : \mathbb{R} \to \mathbb{C}^2$ such that their coordinate functions have a finite variation on any interval $[0, L]$, $L > 0$, and the restriction of X to the set $(-\infty, 0)$ is a constant vector in $\mathbb{C}^2$. This constant vector will be denoted by $X(0-)$. This agrees with the standard notation

$$X(0-) = \lim_{x \to 0,\, x<0} X(x),$$

because much stronger property $X(x)=X(0-)$ is assumed for all $x<0$ if $X\in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$. Recall that for $\alpha\in[0,2\pi)$ we denoted by e_α the vector

$$e_\alpha=\begin{pmatrix}\sin\alpha\\ \cos\alpha\end{pmatrix}.$$

For $q\in\mathcal{M}$ and $X\in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$, we will say that $\mathcal{D}_qX\in L^2(\mathbb{R}_+,\mathbb{C}^2)$ if there exists a function $Y\in L^2(\mathbb{R}_+,\mathbb{C}^2)$ solving equation

$$JX'(x)+QX(x)=Y(x),\qquad x\geqslant 0,$$

in the sense of Persson, i.e., such that for all $x\in\mathbb{R}$ we have

$$JX(x)+\int_{(-\infty,x]}g(Q(\{x_1\})J^*)Q(x_1)X(x_1)=JX(0-)+\int_{(-\infty,x]}Y(x_1)\,dx_1,\quad (2.17)$$

where we extended Y by zero to $(-\infty,0)$. Here, Q is defined by (2.9), in particular, QX is a vector-valued measure. Let us define the domain of the Dirac $\mathcal{D}_q$ on $\mathbb{R}_+$ corresponding to the boundary condition $\alpha\in[0,2\pi)$ by

$$\begin{gathered}\operatorname{dom}\mathcal{D}_q=\Big\{X\in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)\cap L^2(\mathbb{R}_+,\mathbb{C}^2)\colon \langle X(0-),e_\alpha\rangle_{\mathbb{C}^2}=0,\\ \mathcal{D}_qX\in L^2(\mathbb{R}_+,\mathbb{C}^2)\text{ in the sense (2.17)}\Big\}.\end{gathered}$$

For $X\in\operatorname{dom}\mathcal{D}_q$, we define

$$\mathcal{D}_q: X\mapsto Y$$

for the unique (up to values on sets of Lebesgue measure zero) function $Y\in L^2(\mathbb{R}_+,\mathbb{C}^2)$ in (2.17). With this definition, $\mathcal{D}_q$ sends functions on $\mathbb{R}$ into functions in $L^2(\mathbb{R}_+,\mathbb{C}^2)$. Since we are interested in D_q as a densely defined self-adjoint operator on $L^2(\mathbb{R}_+,\mathbb{C}^2)$, an additional step is needed to place $\operatorname{dom}\mathcal{D}_q$ into $L^2(\mathbb{R}_+,\mathbb{C}^2)$. For this, we note that the values of $X\in\operatorname{dom}\mathcal{D}_q$ on $\mathbb{R}$ as well as the function $Y=\mathcal{D}_qX$ in (2.17) depend solely on the restriction of X to $\mathbb{R}_+$. Indeed, we only need to check that for $X\in\operatorname{dom}D_q$ the value $X(0-)$ is determined by the restriction of X to $\mathbb{R}_+$. For this, we substitute $x=0$ into (2.17) and get

$$\begin{aligned}JX(0-)-JX(0)&=g(Q(\{0\})J^*)Q(\{0\})X(0)\\ &=\left[\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{n=2}^{\infty}\frac{(Q(\{0\})J^*)^{n-1}}{n!}\right]Q(\{0\})X(0)\\ &=J\left[\sum_{n=1}^{\infty}\frac{(J^*Q(\{0\}))^n}{n!}\right]X(0)=Je^{J^*Q(\{0\})}X(0)-JX(0).\end{aligned}$$

Thus, the value

$$X(0-)=e^{J^*Q(\{0\})}X(0)\qquad (2.18)$$

is determined by the restriction of X onto $\mathbb{R}_+$ and we can consider $\operatorname{dom}\mathcal{D}_q$ as the subset of $L^2(\mathbb{R}_+,\mathbb{C}^2)$. A similar argument applies to any point $x\in\mathbb{R}$. In fact, we have

$$X(x-)=e^{J^*Q(\{x\})}X(x),\qquad x\in\mathbb{R}, \tag{2.19}$$

while $Q(\{x\})\neq 0$ for at most countable set of points $x\in\mathbb{R}$. Similar argument applied to (2.11) gives

$$N_q(x-)=e^{J^*Q(\{x\})}N_q(x),\qquad x\in\mathbb{R}. \tag{2.20}$$

Let us now recall the definition of the operator of canonical system. We will deal with Hamiltonians $\mathcal{H}$ of *rank two* almost everywhere on $\mathbb{R}_+$, for the general case see [Rom14] or [Rem18] (the latter book considers linear relations instead of operators to cover the most general situation). At first, we denote by $\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ the set of all functions $X:\mathbb{R}_+\to\mathbb{C}^2$ such that their coordinate functions are absolutely continuous on any compact subset of $\mathbb{R}_+$. The Hilbert space $L^2(\mathcal{H})$ is defined by

$$L^2(\mathcal{H})=\Big\{X:\mathbb{R}_+\to\mathbb{C}^2:\ \|X\|^2_{L^2(\mathcal{H})}=\int_{\mathbb{R}_+}\langle\mathcal{H}(x)X(x),X(x)\rangle_{\mathbb{C}^2}<\infty\Big\}.$$

For $\tilde X\in\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$, we will say that $\mathcal{D}_{\mathcal{H}}\tilde X\in L^2(\mathcal{H})$ if there exists a function $\tilde Y\in L^2(\mathcal{H})$ solving the equation

$$J\tilde X'(x)=\mathcal{H}(x)\tilde Y(x),\qquad x\geqslant 0,$$

or, in other words, such that

$$J\tilde X(x)=J\tilde X(0)+\int_0^x\mathcal{H}\tilde Y(x_1)\,dx_1,\qquad x\in\mathbb{R}_+. \tag{2.21}$$

The domain of the canonical system operator $\mathcal{D}_{\mathcal{H}}$ on $\mathbb{R}_+$ corresponding to the boundary condition $\alpha\in[0,2\pi)$ is defined by

$$\operatorname{dom}\mathcal{D}_{\mathcal{H}}=\Big\{\tilde X\in\mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)\cap L^2(\mathcal{H}):\Big\langle\tilde X(0),e_\alpha\Big\rangle_{\mathbb{C}^2}=0,$$
$$\mathcal{D}_{\mathcal{H}}\tilde X\in L^2(\mathcal{H})\text{ in the sense of (2.21)}\Big\}.$$

For $\tilde X\in\operatorname{dom}\mathcal{D}_{\mathcal{H}}$, we define

$$\mathcal{D}_{\mathcal{H}}:\tilde X\mapsto\tilde Y$$

for the unique (up to values on sets of Lebesgue measure zero) function $\tilde Y\in L^2(\mathcal{H})$ in (2.21).

Proposition 2.5. *Let $q\in\mathcal{M}$, define $\mathcal{H}_q$ by (2.16). Then the Dirac operator $\mathcal{D}_q$ in $L^2(\mathbb{R}_+,\mathbb{C}^2)$ defined on the domain $\operatorname{dom}\mathcal{D}_q$ for the boundary condition $\alpha\in[0,2\pi)$ is unitarily equivalent to the operator $\mathcal{D}_{\mathcal{H}_q}$ in $L^2(\mathcal{H}_q)$ defined on the domain $\operatorname{dom}\mathcal{D}_{\mathcal{H}_q}$ for the same boundary condition α. In particular, $\mathcal{D}_q$ is a densely defined self-adjoint* *operator on* $L^2(\mathbb{R}_+,\mathbb{C}^2)$*. The unitary equivalence is given by the multiplication operator* $U : X \mapsto N_q^{-1}X$ *from* $L^2(\mathbb{R}_+,\mathbb{C}^2)$ *to* $L^2(\mathcal{H}_q)$*. We have* $U(\operatorname{dom}\mathcal{D}_q) = \operatorname{dom}\mathcal{D}_{\mathcal{H}_q}$*.*

Proof. Recall that $\mathcal{H}_q = N_q^* N_q$. In particular, we have $\det\mathcal{H}_q = \det(N_q)^2 = 1$ everywhere on $\mathbb{R}_+$. The same relation $\mathcal{H}_q = N_q^* N_q$ tells us that $X \in L^2(\mathbb{R}_+,\mathbb{C}^2)$ if and only if $UX \in L^2(\mathcal{H})$, and, moreover, $\|X\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} = \|UX\|_{L^2(\mathcal{H})}$. Thus, the multiplication operator $U : X \mapsto N_q^{-1}X$ is unitary from $L^2(\mathbb{R}_+,\mathbb{C}^2)$ to $L^2(\mathcal{H}_q)$. For functions $X \in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$, $\tilde{X} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$, related by $\tilde{X} = N_q^{-1}X$ we have $\left\langle \tilde{X}(0), e_\alpha \right\rangle_{\mathbb{C}^2} = 0$ if and only if $\left\langle N_q^{-1}(0)X(0), e_\alpha \right\rangle_{\mathbb{C}^2} = 0$, i.e., $\left\langle N_q^{-1}(0)e^{-J^*Q(\{0\})}X(0-), e_\alpha \right\rangle_{\mathbb{C}^2} = 0$, where $X(0-)$ is defined by (2.18). However, (2.20) gives $N_q(0) = e^{-J^*Q(\{0\})}N_q(0-) = e^{-J^*Q(\{0\})}$. Thus, $N_q^{-1}(0)e^{-J^*Q(\{0\})} = I$ and we have $\left\langle \tilde{X}(0), e_\alpha \right\rangle_{\mathbb{C}^2} = 0$ if and only if $\left\langle X(0-), e_\alpha \right\rangle_{\mathbb{C}^2} = 0$. Now, the result is a consequence of the following two lemmas. ☐

Lemma 2.6. *If a function* $X \in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$ *solves* (2.17) *for some* $Y \in L^2(\mathbb{R}_+,\mathbb{C}^2)$*, then* $\tilde{X} = UX$ *solves* (2.21) *with* $\tilde{Y} = UY$*. In particular,* $\tilde{Y} \in L^2(\mathcal{H})$ *and* $\tilde{X} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$.

Lemma 2.7. *If a function* $\tilde{X} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ *solves* (2.21) *for some* $\tilde{Y} \in L^2(\mathcal{H})$*, then* $X = U^{-1}\tilde{X}$ *solves* (2.17) *with* $Y = U^{-1}\tilde{Y}$*. In particular,* $Y \in L^2(\mathbb{R}_+,\mathbb{C}^2)$ *and* $X \in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$.

Proof of Lemma 2.6. The result is Claim 3.3 in [Zen23]. We repeat the proof for the reader's convenience. Integral equation (2.11) shows that N_q is of locally bounded variation, i.e., its columns belong to the space $\mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$. Hence $\tilde{X} \in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$ as a product of $\mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$ functions. Moreover, (2.19) and (2.20) imply

$$\begin{aligned}\tilde{X}(x-) &= N_q(x-)^{-1}X(x-) = N_q(x)^{-1}e^{-J^*Q(\{x\})}e^{J^*Q(\{x\})}X(x)\\ &= N_q(x)^{-1}X(x) = \tilde{X}(x),\end{aligned}$$

i.e., $\tilde{X}$ is continuous. For simplicity, we write below $\mathbf{G}(x_1)$ instead of $\mathbf{g}(Q(\{x_1\})J^*)$. Equality (2.17) rewrites as

$$JN_q(x)\tilde{X}(x) + \int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)N_q(x_1)\tilde{X}(x_1) = J\tilde{X}(0) + \int_{(-\infty,x]} Y(x_1)\,dx_1. \tag{2.22}$$

Denote $M(x) = \int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)N_q(x_1)$. For 2×2 matrix-valued functions Θ_1, Θ_2 whose columns belong to $\mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$, the formula of integration by parts (see assertion (iv) in Theorem (21.67) in [HS69]) reads as

$$\begin{aligned}\int_{(-\infty,x]} (d\Theta_1(x_1))\Theta_2(x_1-) &= \Theta_1(x)\Theta_2(x) - \Theta_1(-\infty)\Theta_2(-\infty)\\ &\quad - \int_{(-\infty,x]} \Theta_1(x_1)\,d\Theta_2(x_1),\end{aligned}$$

where $d\Theta_1$, $d\Theta_2$ denote the representing measures of Θ_1, Θ_2 as functions of bounded variations. Clearly, if Θ_2 is continuous, we can replace $\Theta_2(x_1-)$ by $\Theta_2(x_1)$. Applying this formula to $\Theta_1 = M$ and to the continuous matrix-valued function $\Theta_2 = (\tilde{X}, 0)$ with columns $\tilde{X}$, $\binom{0}{0}$, and considering the first column in the resulting expressions, we get

$$\int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)N_q(x_1)\tilde{X}(x_1) = M(x)\tilde{X}(x) - \int_{(-\infty,x]} M(x_1)\, d\tilde{X}(x_1).$$

Equality (2.11) gives $M(x) = J - JN_q(x)$, $x \in \mathbb{R}$, hence

$$\begin{aligned}\int_{(-\infty,x]} &\mathbf{G}(x_1)Q(x_1)N_q(x_1)\tilde{X}(x_1)\\ &= (J - JN_q(x))\tilde{X}(x) - \int_{(-\infty,x]} (J - JN_q(x_1))\, d\tilde{X}(x_1)\\ &= \int_{(-\infty,x]} JN_q(x_1)\, d\tilde{X}(x_1) - JN_q(x)\tilde{X}(x) + J\tilde{X}(-\infty).\end{aligned}$$

Combining this with (2.22), we get (recall that X is constant and $N_q = I$ on $(-\infty, 0)$, so $\tilde{X}(-\infty) = \tilde{X}(0-) = \tilde{X}(0)$ by continuity)

$$\int_{(-\infty,x]} JN_q(x_1)\, d\tilde{X}(x_1) = \int_{(-\infty,x]} Y(x_1)\, dx_1.$$

We see that the measures $JN_q(x_1)\, d\tilde{X}(x_1)$ and $Y(x_1)\, dx_1$ coincide, hence $\tilde{X}$ is absolutely continuous and

$$\begin{aligned}J\tilde{X}(x) - J\tilde{X}(0) &= \int_0^x J\, d\tilde{X}(x_1) = \int_0^x JN_q(x_1)^{-1}J^{-1}JN_q(x_1)\, d\tilde{X}(x_1)\\ &= \int_0^x JN_q(x_1)^{-1}J^{-1}Y(x_1)\, dx_1 = \int_0^x JN_q(x_1)^{-1}J^{-1}N_q(x_1)\tilde{Y}(x_1)\, dx_1\\ &= \int_0^x \mathcal{H}(x_1)\tilde{Y}(x_1)\, dx_1,\end{aligned}$$

because $JA^{-1}J^{-1} = A^*$ for every real 2×2 matrix A with unit determinant. The lemma follows. □

Proof of Lemma 2.7. The proof is similar to the proof of Lemma 2.6. Assume that $\tilde{X} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ solves (2.21) for some $\tilde{Y} \in L^2(\mathcal{H})$. Set $X = U^{-1}\tilde{X}$, $Y = U^{-1}\tilde{Y}$. Using $\mathcal{H}_q = N_q^* N_q$ we see that $Y \in L^2(\mathbb{R}_+, \mathbb{C}^2)$ with $\|Y\|_{L^2(\mathbb{R}_+,\mathbb{C}^2)} = \|\tilde{Y}\|_{L^2(\mathcal{H}_q)}$. Next, since columns of N_q belong to $\mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$ by (2.11) and $\tilde{X} \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}_+)$ by the assumption of the lemma, we have $X \in \mathrm{BV}^r_{\mathrm{loc}}(\mathbb{R}_+)$. Thus, it remains to show that X, Y satisfy (2.17). Let us proceed as in the proof of Lemma 2.6. Namely, we denote $\mathbf{G}(x) = \mathbf{g}(Q(\{x\})J^*)$, $M(x) = \int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)N_q(x_1)$, and use integration by parts to get the formula

$$\begin{aligned}\int_{(-\infty,x]} &\mathbf{G}(x_1)Q(x_1)N_q(x_1)\tilde{X}(x_1)\\ &= \int_{(-\infty,x]} JN_q(x_1)\, d\tilde{X}(x_1) - JN_q(x)\tilde{X}(x) + J\tilde{X}(-\infty),\end{aligned}$$

that we rewrite next in the form

$$\int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)X(x_1) = \int_{(-\infty,x]} JN_q(x_1)\,d\tilde{X}(x_1) - JX(x) + JX(0-). \tag{2.23}$$

Since $\tilde{X}$ satisfies (2.21), we have

$$\begin{aligned} JN_q(x_1)\,d\tilde{X}(x_1) &= JN_q(x_1)J^*\mathcal{H}_q\tilde{Y}(x_1)\,dx_1 = JN_q(x_1)J^*N_q(x_1)^*Y(x_1)\,dx_1 \\ &= Y(x_1)\,dx_1, \end{aligned}$$

where we used the fact that $JAJ^* = (A^*)^{-1}$ for real matrices with unit determinant. Now equation (2.23) takes the form

$$\int_{(-\infty,x]} \mathbf{G}(x_1)Q(x_1)X(x_1) = \int_{(-\infty,x]} Y(x_1)\,dx_1 - JX(x) + JX(0-),$$

which is equivalent to (2.17). □

Knowing that operators $\mathcal{D}_q$ and $\mathcal{D}_{\mathcal{H}_q}$ are unitary equivalent, it is of no surprise that they have the same spectral measures and Weyl functions. Recall that the Weyl function of $\mathcal{D}_q$ for $q \in L^2(\mathbb{R}_+)$ is defined by

$$m_q(z) = \lim_{x\to+\infty} \frac{n_{22}(x,z)}{n_{21}(x,z)}, \qquad \operatorname{Im} z > 0,$$

where $N = \left(\begin{smallmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{smallmatrix}\right)$ is the solution of the Dirac system $JN'(x,z) + Q(x)N(x,z) = zN(x,z)$, $N(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $z \in \mathbb{C}$. Let us use the same definition for potentials $q \in \mathcal{M}$, with the interpretation of solution N in the sense of Persson:

$$JN(x,z) + \int_{(-\infty,x]} \mathbf{g}((Q(\{x_1\}) - z)J^*)(Q(x_1) - z)N(x_1,z) = J, \qquad x \in \mathbb{R}. \tag{2.24}$$

Note that $N_q(x) = N(x,0)$, $x \in \mathbb{R}_+$, for the function N_q solving (2.11). A variant of Lemma 2.6 for N in place of X implies that $N(x,z) = N_q(x)M(x,z)$, where M is the fundamental matrix solution for the Hamiltonian $\mathcal{H}_q = N_q^*N_q$. In particular, for $\omega = \infty$, $\tilde{\omega}(x) \doteq \sigma_1 N_q(x)^T\sigma_1\omega$ and each $\operatorname{Im} z > 0$ we have

$$m_q(z) = \lim_{x\to+\infty} \sigma_1 N(x,z)^T\sigma_1\omega = \lim_{x\to+\infty} \sigma_1 M(x,z)^T\sigma_1\tilde{\omega}(x) = m_{\mathcal{H}_q}(z),$$

because $\tilde{\omega}(x) \in \mathbb{R}\cup\infty$ for every $x \in \mathbb{R}_+$ (according to the Weyl circles analysis, for every $\mathcal{H} \in \mathbb{H}_{sing}$, $z \in \mathbb{C}_+$, the sets $\sigma_1 M(x,z)^T\sigma_1(\mathbb{C}_+\cup\mathbb{R}\cup\infty)$ shrink to the singleton set $\{m_{\mathcal{H}}(z)\}$ as $x \to \infty$, see, e.g., Sect. 8 in [Rom14]). Thus, for every $q \in \mathcal{M}$ we have

$$m_q = m_{\mathcal{H}_q}, \qquad \mathcal{H}_q = N_q^*N_q, \tag{2.25}$$

as in the standard theory for $q \in L^1_{\rm loc}(\mathbb{R}_+)$. The Schur function f_q is then defined via (1.2). Clearly, we have $f_q = f_{\mathcal{H}_q}$. Our final remark concerning the general theory of Dirac operators with measures is the spectral theorem for these operators. For potentials $q \in L^1_{\rm loc}(\mathbb{R}_+)$, it can be found in Sects. 7, 14 of [Den06].

PROPOSITION 2.8. *Let $q \in \mathcal{M}$. Define the main spectral measure μ_q of the Dirac operator $\mathcal{D}_q$ on $\mathbb{R}_+$ to be the representing measure of the harmonic function* $\operatorname{Im} m_q = \operatorname{Re}(\frac{1+f_q}{1-f_q})$, *i.e., by*

$$\frac{1-|f_q(z)|^2}{|1-f_q(z)|^2} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|x-z|^2}\, d\mu_q(x), \qquad z \in \mathbb{C}_+, \tag{2.26}$$

where f_q is the Schur function of $\mathcal{D}_q$. Then the operator $\mathcal{D}_q$ on $L^2(\mathbb{R}_+,\mathbb{C}^2)$ is unitarily equivalent to the multiplication operator by the independent variable in $L^2(\mu_q)$. The operator $V : L^2(\mathbb{R}_+,\mathbb{C}^2) \to L^2(\mu_q)$ of unitary equivalence is densely defined by

$$V : X \mapsto \frac{1}{\sqrt{\pi}}\int_{\mathbb{R}_+} \left\langle X(x), N_1(x,\overline{z})\right\rangle_{\mathbb{C}^2} dx, \qquad N_1(x,z) = N(x,z)\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right),$$

on smooth functions $X : \mathbb{R}_+ \to \mathbb{C}$ with compact support.

Proof. In view of (2.25), the result is a direct consequence of Proposition 2.5 and the spectral theorem for operators of canonical systems (see Sect. 8 in [Rom14], or Chap. 3 in [Rem18]). □

In what follows we will deal with a subclass of $\mathcal{M}$ – potentials q supported on lattices. To simplify notation, we define

$$\mathcal{M}_\ell = \{q \in \mathcal{M} :\ \operatorname{supp} q \subset \ell\mathbb{Z}_+\}.$$

Take $q \in \mathcal{M}_\ell$ and let Q be the matrix potential associated with q via (2.9). Define also the constant 2×2 matrices $Q[\ell k] = Q(\{\ell k\})$. We have

$$N_q(x) = e^{JQ[\ell n]} e^{JQ[\ell(n-1)]} \cdot \ldots \cdot e^{JQ[0]}, \qquad n = \lfloor x/\ell \rfloor, \qquad x \geqslant 0. \tag{2.27}$$

Here $\lfloor y \rfloor$ is the integer part of y (the maximal integer k such that $k \leqslant y$).

2.3 Auxiliary lemmas. Recall that $\mathbb{D}$ denotes the open unit disk $\{z \in \mathbb{C} :\ |z| < 1\}$. Take a number $\ell > 0$ and consider the conformal map $e^{2i\ell z} : \Omega_{\mathbb{C}_+} \to \Omega_{\mathbb{D}}$ from $\Omega_{\mathbb{C}_+} = \{z \in \mathbb{C}_+ : |\operatorname{Re} z| < \pi/2\ell\}$ to $\Omega_{\mathbb{D}} = \{z \in \mathbb{D} : z \notin (-1,0]\}$. We will denote by $\omega_\ell : \Omega_{\mathbb{D}} \to \Omega_{\mathbb{C}_+}$ the inverse conformal map.

LEMMA 2.9. *Suppose that the Schur function $g \in S(\mathbb{C}_+)$ satisfies $g(z+\pi/\ell) = g(z)$ for some $\ell > 0$ and all $z \in \mathbb{C}_+$, i.e., g is π/ℓ–periodic. Then $G = g(\omega_\ell(\cdot)) \colon \Omega_{\mathbb{D}} \to \mathbb{C}$ extends analytically to the whole open unit disk $\mathbb{D}$. In particular,*

(a) *there exists the finite limit $g(\infty) = \lim_{y\to+\infty} g(iy)$;*
(b) *if $g(\infty) = 0$, then $g = e^{2i\ell z} f$, where f is also π/ℓ–periodic and $f \in S(\mathbb{C}_+)$.*

Thus, one step of the Schur's algorithm $e^{2i\ell z} g_1 = \frac{g-g(\infty)}{1-\overline{g(\infty)}g}$ for a π/ℓ–periodic Schur function g in $\mathbb{C}_+$ corresponds to the one step of the Schur's algorithm $zG_1 = \frac{G-G(0)}{1-\overline{G(0)}G}$ for its counterpart G in $\mathbb{D}$.

Proof. Since $g(z+\pi/\ell)=g(z)$ for all $z\in\mathbb{C}_+$, the function $G=g(\omega_\ell(\cdot))$ extends to a continuous function on $\mathbb{D}\setminus\{0\}$. Therefore, G is analytic in $\mathbb{D}\setminus\{0\}$. The point 0 is a removable singularity of G because G is bounded. From here we see that there exists the limit

$$G(0)=\lim_{\varepsilon\to0}G(\varepsilon)=\lim_{\varepsilon\to0}g(\omega_\ell(\varepsilon))=\lim_{y\to+\infty}g(iy)=g(\infty).$$

To prove (b), notice that $g(\infty)=0$ implies $G(0)=0$. Then by Schwarz lemma we have $G(\lambda)=\lambda F(\lambda)$, $\lambda\in\mathbb{D}$, for some $F\in S(\mathbb{D})$. In particular, $g(z)=\omega_\ell^{-1}(z)F(\omega_\ell^{-1}(z))=e^{2i\ell z}f(z)$ for $f\in S(\mathbb{C}_+)$. □

LEMMA 2.10. *Let* $A=\left(\begin{smallmatrix}a&b\\b&-a\end{smallmatrix}\right)$ *be a zero-trace symmetric matrix. Then we have*

$$\exp(A)=\cosh(\lambda)\begin{pmatrix}1&0\\0&1\end{pmatrix}+\frac{\sinh(\lambda)}{\lambda}A,\qquad\lambda=\sqrt{a^2+b^2}.$$

Proof. We have $A^2=\left(\begin{smallmatrix}a&b\\b&-a\end{smallmatrix}\right)\left(\begin{smallmatrix}a&b\\b&-a\end{smallmatrix}\right)=\lambda^2\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. Therefore

$$\exp(A)=\sum_{k=0}^{\infty}\frac{A^k}{k!}=\sum_{k=0}^{\infty}\frac{A^{2k}}{(2k)!}+\sum_{k=0}^{\infty}\frac{A^{2k+1}}{(2k+1)!}=\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k)!}\begin{pmatrix}1&0\\0&1\end{pmatrix}+\sum_{k=0}^{\infty}\frac{\lambda^{2k}}{(2k+1)!}A$$

The series in the right hand-side are Taylor series of $\cosh(\lambda)$ and $\sinh(\lambda)/\lambda$. The lemma follows. □

2.4 Proof of Theorem 1.4.

Given a Hamiltonian $\mathcal{H}$ on $\mathbb{R}_+$, and $0\leqslant u<v\leqslant\infty$, let us denote by $M(v,u,z)$ the value at $v-u$ of the fundamental matrix solution for the Hamiltonian $\mathcal{H}(u+\cdot)$ on $\mathbb{R}_+$.

LEMMA 2.11. *If* $q\in\mathcal{M}_\ell$ *and* f_q *is the Schur function of* $\mathcal{D}_q$, *then* $f_q(z+\pi/\ell)=f_q(z)$ *for all* $z\in\mathbb{C}_+$.

Proof. Let $\mathcal{H}_q=N_q^*N_q$ for the matrix-valued function N_q in (2.27), and let M_q be the fundamental solution of the corresponding canonical system (2.5). The Hamiltonian $\mathcal{H}_q$ is a constant rank two Hamiltonian on $[\ell k,\ell(k+1))$ for every $k\in\mathbb{Z}_+$. By Lemma 2.2, the function $M_k(z)=M_q(\ell(k+1),\ell k,z)$ has the form $M_k(z)=A_k^{-1}E_{\ell z}A_k$ for $A_k=\sqrt{\mathcal{H}_q(\ell k)}$ and the function $E_{\ell z}$ defined in Lemma 2.4. In particular, M_k is π/ℓ–antiperiodic for every $k\in\mathbb{Z}_+$, i.e., it satisfies $M_k(z+\pi/\ell)=-M_q(z)$ for all $z\in\mathbb{C}_+$. For each n the chain rule for solutions of ODE gives

$$M_q(\ell n,z)=M_{n-1}(z)M_{n-2}(z)\cdots M_0(z).$$

Therefore, $M_q(\ell n,z)$ is π/ℓ-periodic or π/ℓ-antiperiodic depending on the parity of $n\in\mathbb{Z}_+$. Since $M_q(\ell n,z)$ and $-M_q(\ell n,z)$ generate the same fractional linear transform, we obtain π/ℓ-periodicity of $m_{\mathcal{H}_q}$ from its definition:

$$m_{\mathcal{H}_q}(z)\doteq\lim_{x\to\infty}\sigma_1M_q(x,z)^T\sigma_1\omega\doteq\lim_{n\to\infty,\,n\in\mathbb{Z}}\sigma_1M_q(\ell n,z)^T\sigma_1\omega,\qquad z\in\mathbb{C}_+.$$

To conclude, recall that f_q is related to $m_q=m_{\mathcal{H}_q}$ via (1.2). □

LEMMA 2.12. *Let $q \in \mathcal{M}_\ell$ for some $\ell > 0$, and let f_q be the Schur function of $\mathcal{D}_q$. Then $f_q(\infty)$ exists and satisfies*

$$f_q(\infty) = \varkappa^{-1}(q(\{0\})),$$

where $\varkappa$ is defined in (1.15). Moreover, the Schur function of $\mathcal{D}_{q_\ell}$, $q_\ell = q(\ell + \cdot)$, satisfies

$$e^{2i\ell z} f_{q_\ell}(z) = \frac{f_q(z) - f_q(\infty)}{1 - \overline{f_q(\infty)} f_q(z)}, \qquad z \in \mathbb{C}_+. \tag{2.28}$$

Proof. Recall that f_q, f_{q_ℓ} coincide with Schur functions of Hamiltonians $\mathcal{H}_q = N_q^* N_q$, $\mathcal{H}_{q_\ell} = N_{q_\ell}^* N_{q_\ell}$, where

$$\begin{aligned} N_q(x) &= e^{JQ[\ell n]} e^{JQ[\ell(n-1)]} \cdots e^{JQ[0]}, && n = \lfloor x/\ell \rfloor, && x \geqslant 0, \\ N_{q_\ell}(x) &= e^{JQ[\ell(n+1)]} e^{JQ[\ell(n-1)]} \cdots e^{JQ[1]}, && n = \lfloor x/\ell \rfloor, && x \geqslant 0. \end{aligned}$$

In particular, we have

$$\mathcal{H}_q(x) = \begin{cases} N_q(0)^* N_q(0), & x \in [0, \ell), \\ N_q(0)^* \mathcal{H}_{q_\ell}(x - \ell) N_q(0), & x \geqslant \ell. \end{cases}$$

Lemma 2.2, Lemma 2.4 and relation (2.7) give

$$m_q \doteq \sigma_1 N_q(0)^* \sigma_1 E_{\ell z} m_{q_\ell}, \qquad f_q \doteq R \sigma_1 N_q(0)^* \sigma_1 E_{\ell z} L f_{q_\ell}.$$

Denote $\zeta = q(\{0\})$. We have $JQ[0] = \left(\begin{smallmatrix} -\operatorname{Re}\zeta & \operatorname{Im}\zeta \\ \operatorname{Im}\zeta & \operatorname{Re}\zeta \end{smallmatrix}\right)$. Lemma 2.10 gives

$$N_q(0)^* = N_q(0) = e^{JQ[0]} = \cosh(|\zeta|) \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{\sinh(|\zeta|)}{|\zeta|} \cdot JQ[0].$$

Furthermore, the straightforward calculation shows

$$\begin{aligned} RE_{\ell z}L &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix} = 2i \begin{pmatrix} e^{i\ell z} & 0 \\ 0 & e^{-i\ell z} \end{pmatrix}, \\ R\sigma_1 JQ[0] \sigma_1 E_{\ell z} L &= \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} \operatorname{Re}\zeta & \operatorname{Im}\zeta \\ \operatorname{Im}\zeta & -\operatorname{Re}\zeta \end{pmatrix} \begin{pmatrix} \cos \ell z & \sin \ell z \\ -\sin \ell z & \cos \ell z \end{pmatrix} \begin{pmatrix} i & i \\ -1 & 1 \end{pmatrix}, \\ &= 2i \begin{pmatrix} 0 & \overline{\zeta} e^{-i\ell z} \\ \zeta e^{i\ell z} & 0 \end{pmatrix}. \end{aligned}$$

When working with the relation "$\doteq$", the constant factor $2i$ can be cancelled, hence

$$f_q \doteq \begin{pmatrix} \cosh(|\zeta|) e^{i\ell z} & \frac{\sinh(|\zeta|)}{|\zeta|} \overline{\zeta} e^{-i\ell z} \\ \frac{\sinh(|\zeta|)}{|\zeta|} \zeta e^{i\ell z} & \cosh(|\zeta|) e^{-i\ell z} \end{pmatrix} f_{q_\ell} \doteq \begin{pmatrix} e^{2i\ell z} & \frac{\tanh(|\zeta|)}{|\zeta|} \overline{\zeta} \\ \frac{\tanh(|\zeta|)}{|\zeta|} \zeta e^{2i\ell z} & 1 \end{pmatrix} f_{q_\ell}.$$

Note that the function $\varkappa^{-1} : \zeta \mapsto \frac{\tanh|\zeta|}{|\zeta|}\overline{\zeta}$ from $\mathbb{C}$ to $\mathbb{D}$ is inverse to the function $\varkappa : s \mapsto \frac{\bar{s}}{2|s|}\log\frac{1+|s|}{1-|s|}$ from $\mathbb{D}$ to $\mathbb{C}$. This follows from the fact that $\tanh^{-1}|s| = \frac{1}{2}\log\frac{1+|s|}{1-|s|}$. Therefore the obtained formula for f_q rewrites as

$$f_q = \begin{pmatrix} e^{2i\ell z} & \varkappa^{-1}(\zeta) \\ \overline{\varkappa^{-1}(\zeta)}e^{2i\ell z} & 1 \end{pmatrix} f_{q_\ell}, \qquad f_q = \frac{e^{2i\ell z} f_{q_\ell} + \varkappa^{-1}(\zeta)}{\overline{\varkappa^{-1}(\zeta)}e^{2i\ell z} f_{q_\ell} + 1},$$

$$e^{2i\ell z} f_{q_\ell} = \frac{f_q - \varkappa^{-1}(\zeta)}{1 - \overline{\varkappa^{-1}(\zeta)} f_q}.$$

It remains to prove that $f_q(\infty) = \varkappa^{-1}(\zeta)$. For this we simply write

$$\lim_{y\to\infty}(f_q(iy) - \varkappa^{-1}(\zeta)) = \lim_{y\to\infty} e^{2i\ell\cdot iy} f_{q_\ell}(iy)(1 - \overline{\varkappa^{-1}(\zeta)} f_q(iy)) = 0,$$

where we used the fact that both f_q and f_{q_ℓ} are bounded in $\mathbb{C}_+$. ☐

Proof of Theorem 1.4. Let q be a discrete measure on $\mathbb{R}_+$ such that $\operatorname{supp} q \subset \ell\mathbb{Z}_+$ for some $\ell > 0$. The Schur function $f_q : \mathbb{C}_+ \to \mathbb{D}$ of $\mathcal{D}_q$ on $\mathbb{R}_+$ coincides with the Schur function of the Hamiltonian $\mathcal{H}_q = N_q^* N_q$ on $\mathbb{R}_+$, where N_q is defined in (2.27). By Lemma 2.11, f_q is π/ℓ-periodic in $\mathbb{C}_+$. For $\varkappa$ defined in (1.15), Lemma 2.12 states that $q(\{0\}) = \varkappa(f_q(\infty))$ and f_{q_ℓ} is the first Schur's iterate of f_q. Then (1.16) follows by the induction principle. Since there exists infinitely many Schur iterates of f_q (equivalently, $|f_{q,k}(\infty)| < 1$ for each $k \in \mathbb{Z}_+$), we have $f_q \in S_{\ell,*}(\mathbb{C}_+)$. It remains to show that for every $f \in S_{\ell,*}(\mathbb{C}_+)$ there exists $q \in \mathcal{M}_\ell$ such that $f = f_q$. Take $f \in S_{\ell,*}(\mathbb{C}_+)$, find its recurrence coefficients $\{f_k(\infty)\}_{k\in\mathbb{Z}_+}$, and define q by

$$q(\{k\ell\}) = \varkappa(f_k(\infty)), \qquad k \in \mathbb{Z}_+.$$

The first part of the proof shows that recurrence coefficients of f and f_q coincide. The same is true for the corresponding Schur functions $F = f \circ \omega_\ell$, $F_q = f_q \circ \omega_\ell$ in $\mathbb{D}$ in Lemma 2.9 and the standard recurrence coefficients in Schur's algorithm in $\mathbb{D}$ for these functions. It follows that $F = F_q$, see formula (1.3.43) in [Sim05]. Thus, $f = f_q$ and the proof is completed. ☐

2.5 Proof of Corollaries 1.5-1.7. Consider a Schur function $F \in S_*(\mathbb{D})$. Note that $\frac{1+zF}{1-zF}$, $\frac{1+F}{1-F}$ are analytic functions in $\mathbb{D}$ with positive real part. It follows that there are finite nonnegative measures σ, ν on the unit circle $\mathbb{T}$ being the boundary values of the corresponding positive harmonic functions in $\mathbb{D}$, i.e.,

$$\frac{1 - |\lambda F(\lambda)|^2}{|1 - \lambda F(\lambda)|^2} = \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}\xi|^2}\, d\sigma(\xi), \tag{2.29}$$

$$\frac{1 - |F(\lambda)|^2}{|1 - F(\lambda)|^2} = \int_{\mathbb{T}} \frac{1 - |\lambda|^2}{|1 - \bar{\lambda}\xi|^2}\, d\nu(\xi), \tag{2.30}$$

for all $\lambda \in \mathbb{D}$, see Section I.3 in [Gar81]. Taking $\lambda = 0$, one can see that σ is a probability measure on $\mathbb{T}$. In general, the measure ν is not probabilistic and there is no simple way of expressing $\nu/\nu(\mathbb{T})$ in terms of σ. However, $\nu/\nu(\mathbb{T})$ coincides with the measure $\sigma_{\alpha,1}$ generated via (2.29) by the first Schur iterate $F_{\alpha,1}(\lambda) = \frac{1}{\lambda}\frac{F_\alpha(\lambda)-F_\alpha(0)}{1-\overline{F_\alpha(0)}F_\alpha(\lambda)}$ of the Schur function $F_\alpha = \alpha F$ for a specific value $\alpha \in \mathbb{T}$. More precisely, we have the following relation.

Lemma 2.13. *Suppose that $F \in S(\mathbb{D})$ and $\alpha \in \mathbb{T}$ are such that $\alpha = \frac{1-\overline{F(0)}}{1-F(0)}$, $|F(0)| < 1$. Then*

$$\frac{1-|\lambda F_{1,\alpha}|^2}{|1-\lambda F_{\alpha,1}|^2} = \frac{1-|F(0)|^2}{|1+\alpha F(0)|^2}\frac{1-|F|^2}{|1-F|^2}, \qquad \lambda \in \mathbb{D}. \tag{2.31}$$

In particular, we have $\nu/\nu(\mathbb{T}) = \sigma_{\alpha,1}$, i.e.,

$$\frac{1-|\lambda F_{\alpha,1}(\lambda)|^2}{|1-\lambda F_{\alpha,1}(\lambda)|^2} = \frac{1}{\nu(\mathbb{T})}\int_{\mathbb{T}}\frac{1-|\lambda|^2}{|1-\bar\lambda\xi|^2}\,d\nu(\xi), \qquad \lambda \in \mathbb{D}. \tag{2.32}$$

Proof. We have $\lambda F_{\alpha,1} = \frac{F_\alpha - F_\alpha(0)}{1-\overline{F_\alpha(0)}F_\alpha} = \frac{\alpha F - \alpha F(0)}{1-\overline{F(0)}F}$. It follows that

$$1-|\lambda F_{\alpha,1}|^2 = \frac{(1-|F(0)|^2)(1-|F|^2)}{|1-\overline{F(0)}F|^2},$$

and, taking into account $\overline{F(0)} + \alpha = 1 + \alpha F(0)$, we get

$$1-\lambda F_{\alpha,1} = \frac{1-\overline{F(0)}F - \alpha F + \alpha F(0)}{1-\overline{F(0)}F} = \frac{(1+\alpha F(0))(1-F)}{1-\overline{F(0)}F}.$$

These two relations yield (2.31). Integrating (2.31) over $\mathbb{T}$, we see that $\frac{1-|F(0)|^2}{|1+\alpha F(0)|^2} = \frac{1}{\nu(\mathbb{T})}$. Now, (2.32) follows from (2.30). □

Let a Schur function $F \in S_*(\mathbb{D})$ and a probability measure σ on $\mathbb{T}$ be related as in (2.29). There is a strong connection between properties of orthogonal polynomials in $L^2(\sigma)$ and the properties of F. Let $\{\Phi_k\}_{k\geqslant 0}$ be the family of polynomials such that $\Phi_k = z^k + \dots + \Phi_k(0)$ for each $k \geqslant 0$ and

$$(\Phi_{k_1}, \Phi_{k_2})_{L^2(\sigma)} = 0 \ \text{ for all } \ 0 \leqslant k_1 < k_2 < \infty.$$

Let also $\Phi_k^*(z) = z^k\overline{\Phi_k(1/\bar z)}$ denote the so-called reflected orthogonal polynomials. It is well-known (see, e.g., Theorem 1.5.2 in [Sim05]) that

$$\Phi_{k+1}(z) = z\Phi_k(z) - \overline{\alpha_k}\Phi_k^*(z), \qquad k \geqslant 0, \tag{2.33}$$

for all $z \in \mathbb{C}$ and some sequence $\{\alpha_k\}_{k\geqslant 0} \subset \mathbb{D}$, which is called the sequence of recurrence coefficients of σ. Moreover, any $\{\alpha_k\}_{k\geqslant 0} \subset \mathbb{D}$ arises uniquely as the sequence of recurrence coefficients of some probability measure σ supported on an infinite subset of $\mathbb{T}$, see Sect. 1.7 in [Sim05]. The following theorem is due to Geronimus [Ger44], its modern exposition can be found in Sect. 3.1 of [Sim05].

Theorem 2.14 (Geronimus' theorem). *Assume that a Schur function* $F \in S_*(\mathbb{D})$ *and a probability measure* σ *on* $\mathbb{T}$ *are related by* (2.29). *Let* $\{F_k(0)\}_{k\geqslant 0}$ *be the recurrence coefficients of* F, *see* (1.11), *and let* $\{\alpha_k\}_{k\geqslant 0} \subset \mathbb{D}$ *be the recurrence coefficients of* σ *in* (2.33). *Then* $F_k(0) = \alpha_k$ *for all* $k \geqslant 0$.

Geronimus' theorem allows to formulate many results of the theory of orthogonal polynomials in the language of Schur functions. Let us state several landmark theorems of the theory in this form. In all of them, $\sigma = w_\sigma\, dm_{\mathbb{T}} + \sigma_s$ (respectively, $\nu = w_\nu\, dm_{\mathbb{T}} + \nu_s$) is the Radon-Nikodym decomposition of σ (respectively, ν) into the absolutely continuous and singular parts. Here and below $m_{\mathbb{T}}$ denotes the Lebesgue measure on the unit circle $\mathbb{T}$ normalized so that $m_{\mathbb{T}}(\mathbb{T}) = 1$. For F, ν related by (2.30) it is known that

$$w_\nu(\xi) = \frac{1 - |F(\xi)|^2}{|1 - F(\xi)|^2} \text{ for Lebesgue almost all } \xi \in \mathbb{T}, \tag{2.34}$$

and

$$\nu = *\lim_{r\to 1} \frac{1 - |F(r\xi)|^2}{|1 - F(r\xi)|^2}\, dm_{\mathbb{T}}(\xi), \tag{2.35}$$

where $*\lim$ stands for the *-weak convergence of measures. For the proof of relations (2.34), (2.35), see Theorem I.5.3, Theorem I.3.1 in [Gar81]. Similar relations hold for the measure σ (for σ, one need to replace F by ξF in (2.34), (2.35)). In the next three results we assume that a Schur function $F \in S_*(\mathbb{D})$ and measures σ, ν on $\mathbb{T}$ are related by (2.29), (2.30) and $\{F_k(0)\}_{k\geqslant 0} = \{\alpha_k\}_{k\geqslant 0}$ are the recurrence coefficients of F, σ.

Theorem 2.15 (Rakhmanov's theorem). *If* $|F| < 1$ *almost everywhere on* $\mathbb{T}$, *then* $\lim_{k\to\infty} F_k(0) = 0$.

Proof. By (2.34) if $|F| < 1$ almost everywhere on $\mathbb{T}$, then $w_\sigma > 0$ almost everywhere on $\mathbb{T}$. By the classical Rakhmanov's theorem [Rak77], see also Sect. 9.1 in [Sim05], this implies $\lim_{k\geqslant 0} \alpha_k = 0$. It remains to use Geronimus theorem. ☐

Below, $W^1(\mathbb{T})$ stands for the Wiener algebra of all absolutely convergent Fourier series on $\mathbb{T}$.

Theorem 2.16 (Baxter's theorem). *The following assertions are equivalent*:

1. $F \in S_*(\mathbb{D}) \cap W^1(\mathbb{T})$, $\max_{\xi\in\mathbb{T}} |F(\xi)| < 1$;
2. $\sum_{k\geqslant 0} |F_k(0)| < \infty$;
3. $\sigma = w_\sigma\, dm_{\mathbb{T}}$, *where* $w_\sigma \in W^1(\mathbb{T})$, $\inf_{\mathbb{T}} w_\sigma > 0$;
4. $\nu = w_\nu\, dm_{\mathbb{T}}$, *where* $w_\nu \in W^1(\mathbb{T})$, $\inf_{\mathbb{T}} w_\nu > 0$.

Proof. Equivalence (2) $\Leftrightarrow$ (3) is the original Baxter's theorem [Bax63] (modulo Geronimus' theorem 2.14). Equivalence (1) $\Leftrightarrow$ (2)&(3) was proved by Golinskii in [Gol97]. So, it suffices to prove the equivalence (2) $\Leftrightarrow$ (4). For this we take

$\alpha \in \mathbb{T}$, set $F_\alpha = \alpha F$ and note that (2) is equivalent to $\sum_{k\geqslant 0}|F_{\alpha,k}(0)| < \infty$ because $F_{\alpha,k}(0) = \alpha F_k(0)$ for all $k \geqslant 0$. Assumption $\sum_{k\geqslant 1}|F_{\alpha,k}(0)| < \infty$ is equivalent to the assumption (2) for $F_{\alpha,1}$, the first Schur iterate of F_α. In turn, the latter is equivalent to the fact that the probability measure $\sigma_{\alpha,1}$ corresponding to $F_{\alpha,1}$ satisfies assumption (3) with σ replaced by $\sigma_{\alpha,1}$. Now, choosing $\alpha = \frac{1-\overline{F(0)}}{1-F(0)}$ we see from Lemma 2.13 that $\sigma_{\alpha,1} = \nu/\nu(\mathbb{T})$, and the result follows. □

Next theorem deals with the Sobolev space $H_{1/2}(\mathbb{T})$ – the set of functions

$$H_{1/2}(\mathbb{T}) = \Big\{g \in L^1(\mathbb{T}) : \sum_{k\in\mathbb{Z}} |k||\hat{g}(k)|^2 < \infty\Big\},$$

where $\hat{g}(k)$ is the k-th Fourier coefficient of a function on the unit circle $\mathbb{T}$, see (3.3).

Theorem 2.17 (Szegő-Golinskii-Ibragimov theorem). *The following assertions are equivalent*:

1. $\sum_{k\geqslant 0} k|F_k(0)|^2 < \infty$;
2. $\sigma = w_\sigma\, dm_{\mathbb{T}}$, *where* $w_\sigma = e^{\varphi}$ *for some* $\varphi \in H_{1/2}(\mathbb{T})$;
3. $\nu = w_\nu\, dm_{\mathbb{T}}$, *where* $w_\nu = e^{\psi}$ *for some* $\psi \in H_{1/2}(\mathbb{T})$.

Proof. The proof is similar to the proof of Theorem 2.16. The equivalence (1) ⇔ (2) is the original Szegő-Golinskii-Ibragimov theorem [GI71] (modulo Geronimus theorem 2.14). Let us show that (3) is equivalent to (1). For this we take $\alpha \in \mathbb{T}$, set $F_\alpha = \alpha F$ and note that (1) is equivalent to $\sum_{k\geqslant 0} k|F_{\alpha,k}(0)|^2 < \infty$ because $F_{\alpha,k}(0) = \alpha F_k(0)$ for all $k \geqslant 0$. Assumption $\sum_{k\geqslant 0} k|F_{\alpha,k}(0)|^2 < \infty$ is equivalent to the assumption (1) for $F_{\alpha,1}$, the first Schur iterate of F_α. In turn, the latter is equivalent to the fact that the probability measure $\sigma_{\alpha,1}$ corresponding to $F_{\alpha,1}$ satisfies assumption (2) with σ replaced by $\sigma_{\alpha,1}$. Now, choosing $\alpha = \frac{1-\overline{F(0)}}{1-F(0)}$ we see from Lemma 2.13 that $\sigma_{\alpha,1} = \nu/\nu(\mathbb{T})$, and the result follows. □

Let $q \in \mathcal{M}$, and let f_q is the Schur function of $\mathcal{D}_q$. Recall that the main spectral measure μ_q of the Dirac operator $\mathcal{D}_q$ on $\mathbb{R}_+$ is defined by

$$\frac{1-|f_q(z)|^2}{|1-f_q(z)|^2} = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\operatorname{Im} z}{|x-z|^2}\, d\mu_q(x), \qquad z \in \mathbb{C}_+. \tag{2.36}$$

Writing $\mu_q = w_q\, dx + \mu_{q,s}$ for $\mu_{q,s} \perp dx$, we obtain

$$w_\sigma(x) = \frac{1-|f_q(x)|^2}{|1-f_q(x)|^2} \text{ for Lebesgue almost all } x \in \mathbb{R}, \tag{2.37}$$

and

$$\mu_q = *\lim_{\varepsilon\downarrow 0} \frac{1-|f_q(x+i\varepsilon)|^2}{|1-f_q(x+i\varepsilon)|^2}\, dx, \tag{2.38}$$

where the limit is understood in the $*$-weak sense. These relations are analogous to (2.34), (2.35) and have essentially the same proofs.

Given $q \in \mathcal{M}_\ell$, we know that $f_q \in S_{\ell,*}(\mathbb{C}_+)$, see Theorem 1.4. By Lemma 2.9, there exists $F_q \in S_*(\mathbb{D})$ such that $F_q(e^{2i\ell z}) = f_q(z)$, $z \in \mathbb{C}_+$. Below we prove Corollaries 1.5-1.7 based on Theorem 1.4 and the following proposition.

Proposition 2.18. *Let $q \in \mathcal{M}_\ell$ and $f_q \in S_{\ell,*}(\mathbb{C}_+)$ be the measure-valued potential and its Schur function. Also let $F_q \in S_*(\mathbb{D})$ be such that $F_q(e^{2i\ell z}) = f_q(z)$, $z \in \mathbb{C}_+$. Then the spectral measure μ_q of $\mathcal{D}_q$ is periodic with period π/ℓ and satisfies*

$$\frac{1-|F_q(\lambda)|^2}{|1-F_q(\lambda)|^2} = \frac{\ell}{\pi}\int_{[-\frac{\pi}{2\ell},\frac{\pi}{2\ell})} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{2i\ell x}|^2}\, d\mu_q(x), \qquad \lambda \in \mathbb{D}. \tag{2.39}$$

In particular, if ν_q is the measure on $\mathbb{T}$ generated by F_q via (2.30), then $\frac{\ell}{\pi}\mu_q(E) = \nu_q(E^*)$, where $E^* = \{e^{2i\ell x},\, x \in E\}$ for every Borel set $E \subset [-\frac{\pi}{2\ell},\frac{\pi}{2\ell})$.

Proof. Set $w_\varepsilon(x) = \frac{1-|f_q(x+i\varepsilon)|^2}{|1-f_q(x+i\varepsilon)|^2}$, $x \in \mathbb{R}$, and take arbitrary $x_0 \in \mathbb{R}$. By (2.38), for the proof of (2.39) it suffices to show that for every $\lambda \in \mathbb{D}$ we have

$$\frac{1-|F_q(\lambda)|^2}{|1-F_q(\lambda)|^2} = \lim_{\varepsilon\to 0}\frac{\ell}{\pi}\int_{[x_0-\frac{\pi}{2\ell},x_0+\frac{\pi}{2\ell})} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{2i\ell x}|^2} w_\varepsilon(x)\, dx. \tag{2.40}$$

For $\varepsilon > 0$, denote $r_\varepsilon = e^{-2\ell\varepsilon}$. In view of $F_q(e^{2i\ell z}) = f_q(z)$, $z \in \mathbb{C}_+$, relation (2.40) can be rewritten in the form

$$\begin{aligned}
\frac{1-|F_q(\lambda)|^2}{|1-F_q(\lambda)|^2} &= \lim_{\varepsilon\to 0}\frac{\ell}{\pi}\int_{[x_0-\frac{\pi}{2\ell},x_0+\frac{\pi}{2\ell})} \frac{1-|\lambda|^2}{|1-\bar{\lambda}e^{2i\ell x}|^2}\frac{1-|F_q(e^{2i\ell(x+i\varepsilon)})|^2}{|1-F_q(e^{2i\ell(x+i\varepsilon)})|^2}\, dx\\
&= \lim_{\varepsilon\to 0}\int_{\mathbb{T}} \frac{1-|\lambda|^2}{|1-\bar{\lambda}\xi|^2}\frac{1-|F_q(r_\varepsilon\xi)|^2}{|1-F_q(r_\varepsilon\xi)|^2}\, dm_{\mathbb{T}}(\xi)\\
&= \lim_{\varepsilon\to 0}\frac{1-|F_q(r_\varepsilon\lambda)|^2}{|1-F_q(r_\varepsilon\lambda)|^2},
\end{aligned}$$

which holds by continuity of F_q in $\mathbb{D}$. Periodicity of μ_q follows from the fact that $x_0 \in \mathbb{R}$ in (2.40) is arbitrary, while the left hand side does not depend on x_0. Relation $\frac{\ell}{\pi}\mu_q(E) = \nu_q(E^*)$ is just the change of variables in the Lebesgue integral. ☐

In the proof of corollaries below we assume that f_q and F_q are related as in Proposition 2.18. In particular, their Schur's iterations satisfy $f_{q,k}(\infty) = F_{q,k}(0)$ for all $k \geqslant 0$.

Proof of Corollary 1.5. Suppose that $\sigma_{ac}(\mathcal{D}_q) = \mathbb{R}$. It follows that $|f_q| < 1$ almost everywhere on $\mathbb{R}_+$, see (2.37). Then, the corresponding Schur function F_q in $\mathbb{D}$ is such that $|F_q| < 1$ almost everywhere on $\mathbb{T}$. By Rakhmanov's theorem 2.15, it follows that $F_{q,k}(0) \to 0$ as $k \to +\infty$. Then, Theorem 1.4 and the fact that $f_{q,k}(\infty) = F_{q,k}(0)$ imply $q(\{k\}) \to 0$ as $k \to +\infty$. ☐

Proof of Corollary 1.6. By Theorem 1.4, $\sum_{k\geqslant 0}|q(\{k\})| < \infty$ if and only if $\sum_{k\geqslant 0}|F_{q,k}(0)| < \infty$. By Baxter's theorem 2.16, this is equivalent to $\nu = w_\nu\, dm_{\mathbb{T}}$,

$w_\nu \in W^1(\mathbb{T})$, $\inf_{\mathbb{T}} w_\nu > 0$, for the measure ν constructed from F_q by (2.30). By Proposition 2.18, we have $\mu_q = w_q\, dx$, where $w_q(x) = w_\nu(e^{2ix})$ is a continuous π-periodic function on $\mathbb{R}$ with $\inf_{x\in\mathbb{R}} |w_q(x)| = \inf_{\xi\in\mathbb{T}} |w_\nu(\xi)| > 0$. Moreover, for every $k \in \mathbb{Z}$ we have

$$\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-2ikx} w_q(x)\, dx = \int_{\mathbb{T}} \overline{\xi}^k w_\nu(\xi)\, dm_{\mathbb{T}}.$$

It follows that assumption $\sum_{k\geqslant 0} |q(\{k\})| < \infty$ implies $\mu_q = w_q\, dx$ for a π-periodic function w_q such that $\inf_{x\in\mathbb{R}} |w_q(x)| > 0$ and

$$w_q = \sum_{k\in\mathbb{Z}} c_k e^{2ikx}, \qquad \sum_{k\in\mathbb{Z}} |c_k| = \sum_{k\in\mathbb{Z}} |\hat{w}_\nu(k)| = \|w_\nu\|_{W^1(\mathbb{T})} < \infty.$$

The argument is reversible: any measure μ_q with these properties gives rise to the function F_q such that $\sum_{k\geqslant 0} |F_{q,k}(0)| < \infty$, i.e., $\sum_{k\geqslant 0} |q(\{k\})| < \infty$ by Theorem 1.4. □

Proof of Corollary 1.7. By Theorem 1.4, assumption $\sum_{k\geqslant 0} k|q(\{k\})|^2 < \infty$ is equivalent to the assumption $\sum_{k\geqslant 0} k|F_{q,k}(0)|^2 < \infty$. Then, Szegő-Golinskii-Ibragimov theorem 2.17 says that the last condition is equivalent to $\nu = e^{\psi}\, dm_{\mathbb{T}}$, $\psi \in H_{1/2}(\mathbb{T})$ for the measure ν related to F_q via (2.30). By Proposition 2.18, this can be further equivalently reformulated in the following way: $\mu_q = w_q\, dx$ for a positive π-periodic function w_q that satisfies $\log w_q(x) = \psi(e^{2ix})$ on $\mathbb{R}$. For these objects, we have

$$\log w_q(x) = \sum_{k\in\mathbb{Z}} c_k e^{2ikx}, \qquad c_k = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} e^{-2ikx} \log w_q\, dx = \int_{\mathbb{T}} \overline{\xi}^k \psi(\xi)\, dm_{\mathbb{T}}.$$

In particular, the series $\sum_{k\in\mathbb{Z}} |k||c_k|^2 = \sum_{k\in\mathbb{Z}} |k||\hat{\psi}(k)|^2$ converge or not simultaneously. From here we see that $\sum_{k\geqslant 0} k|q(\{k\})|^2 < \infty$ if and only if $\mu_q = w_q\, dx$, where $\log w_q = \sum_{k\in\mathbb{Z}} c_k e^{2ikx}$ with $\sum_{k\in\mathbb{Z}} |k||c_k|^2 < \infty$. □

2.6 Piecewise constant Hamiltonians. Since Dirac systems generated by purely atomic singular measures are closely related to piecewise constant Hamiltonians (see (2.25) and (2.27)), some results of Sect. 2.3 can be rewritten in terms of canonical Hamiltonian systems. Below we formulate two theorems of such kind and discuss the main difference that appear in the analysis of corresponding Schur functions.

Given $\ell > 0$, we set $\overline{N_\ell}(\mathbb{C}_+) = \{m \in \overline{N}(\mathbb{C}_+) :\ m(z) = m(z + \pi/\ell) \text{ for all } z \in \mathbb{C}_+\}$, where, as before, $\overline{N}(\mathbb{C}_+) = N(\mathbb{C}_+) \cup \mathbb{R} \cup \{\infty\}$ stands for the extended Herglotz-Nevanlinna class. Note that π/ℓ is not necessary the minimal period of a function $m \in \overline{N_\ell}(\mathbb{C}_+)$. In particular, we have $\mathbb{C}_+ \cup \mathbb{R} \cup \{\infty\} \subset \overline{N_\ell}(\mathbb{C}_+)$ for each $\ell > 0$. For $n \in \mathbb{Z}_+$, we set $\Delta_{\ell,n} = [\ell n, \ell n + \ell)$.

Theorem 2.19. *For every function $m \in \overline{N_\ell}(\mathbb{C}_+)$, there exists $\mathcal{H} \in \mathbb{H}_{sing}$, $n_0 \in \mathbb{Z}_+ \cup \{\infty\}$, such that $m = m_{\mathcal{H}}$ and*

(a) *$\mathcal{H}$ is constant on each interval $\Delta_{\ell,n}$, $0 \leqslant n < n_0$;*

(b) $\det \mathcal{H} = 1$ *on* $\cup_{0 \leqslant n < n_0} \Delta_{\ell,n}$;
(c) $\mathcal{H}$ *is a constant nonnegative rank-one matrix on* $[\ell n_0, +\infty)$.

Conversely, the Weyl function of any Hamiltonian $\mathcal{H} \in \mathbb{H}_{sing}$ *satisfying* (*a*)*–*(*c*) *belongs to* $\overline{N_\ell}(\mathbb{C}_+)$.

Let us note that assertions (a) and (c) are non-void only if $n_0 > 0$ and $n_0 < \infty$, correspondingly.

Piecewise constant Hamiltonians and periodic Weyl functions were previously studied by Eichinger, Lukić, Simanek [ELS212] and Makarov, Poltoratski [MP23]. In these papers the authors found a relation of such Hamiltonians to orthogonal polynomials on the unit circle. For a subsequent development of the area, see Poltoratski and Zhang [PR23]. It is possible to prove Theorem 2.19 based on the relation from [ELS212], [MP23] or derive it directly from Theorem 1.4. For Theorem 2.19, the choice of model (Dirac systems or canonical Hamiltonian systems) is not important. However, on the level of Schur functions we will see an important difference. Let us fix $\ell > 0$, take $\mathcal{H}$ satisfying (a)–(c), and define the functions f_n° by

$$f_0^\circ = \frac{m_{\mathcal{H}} - i}{m_{\mathcal{H}} + i}, \qquad e^{2i\ell z} f_{n+1}^\circ = \frac{m_{\mathcal{H}_{\ell n}} - m_{\mathcal{H}_{\ell n}}(\infty)}{m_{\mathcal{H}_{\ell n}} - \overline{m_{\mathcal{H}_{\ell n}}(\infty)}}, \qquad 0 \leqslant n < n_0. \tag{2.41}$$

Symbol $^\circ$ is used here to distinguish f_n° from the Schur functions f_n of the Dirac operators $\mathcal{D}_{q_{\ell n}}$ generated by the potential q such that $\mathcal{H} = \mathcal{H}_q$, see Proposition 2.5. In fact, we have $f_0^\circ = f_0$, while in general $f_n^\circ \neq f_n$ for $n \geqslant 1$.

Theorem 2.20. *Let* ℓ, $\mathcal{H}$, n_0 *be as in Theorem* 2.19, *and let* f_n°, $0 \leqslant n \leqslant n_0$, *be the functions defined by* (2.41). *Then* f_n° *belong to* $S_\ell(\mathbb{C}_+)$ *and satisfy the generalized Schur's recursion*:

$$e^{2i\ell z} f_{n+1}^\circ = \frac{1 - \overline{f_n^\circ(\infty)}}{1 - f_n^\circ(\infty)} \frac{f_n^\circ - f_n^\circ(\infty)}{1 - \overline{f_n^\circ(\infty)} f_n^\circ}, \qquad 0 \leqslant n < n_0. \tag{2.42}$$

Moreover, if we set $a_n = f_n^\circ(\infty)$, *then the Hamiltonian* $\mathcal{H}$ *can be recovered from its Schur function* $f_{\mathcal{H}} = f_0^\circ$ *by*

$$\begin{aligned} \mathcal{H}(\ell n) &= \prod_{k=0}^{n} \frac{1 - |a_k|^2}{|1 - a_k|^2} \cdot U_n^* U_n, \qquad U_n = A_n A_{n-1} \cdot \ldots \cdot A_0, \\ A_n &= \begin{pmatrix} \frac{|1-a_n|^2}{1-|a_n|^2} & \frac{-2\operatorname{Im} a_n}{1-|a_n|^2} \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{2.43}$$

for all $0 \leqslant n < n_0$.

This theorem can be also derived from results of [ELS212], [MP23] or directly from Theorem 1.4. We skip this derivation because our main objective is the usage of Schur's algorithm for proving spectral stability results. From this perspective, the classical Schur's algorithm (that appears in Theorem 1.4) turns out to be much more accessible than its generalized version (2.42). The difference comes from a purely "algebraic" obstacle – the presence in (2.42) of the multiplicative factor $\frac{1-\overline{f_n^\circ(\infty)}}{1-f_n^\circ(\infty)}$ makes perturbative analysis on the side of Schur functions much more complicated. It would be interesting to derive continuity estimates in Krein-de Branges theorem based on (2.42). This direction remains open.

3 Spectral continuity estimates for potentials in $L^2(\mathbb{R}_+)$

Our aim in this section is to prove Theorem 1.3. The strategy of the proof is to approximate potentials $q, \tilde{q} \in L^2(\mathbb{R}_+)$ by discrete measures supported on $\ell\mathbb{Z}_+$, $\ell \to 0$. By Theorem 1.4, solution of the direct and inverse spectral problems for measures supported on $\ell\mathbb{Z}_+$ relies on a Schur algorithm. This and the continuity estimates for the Schur algorithm that we prove below will be the principal ingredients of the proof of Theorem 1.3.

3.1 Estimates for Schur's algorithm. Recall that the Schur's algorithm for functions $F \in S_*(\mathbb{D})$ is defined by

$$F_0 = F, \qquad zF_{k+1} = \frac{F_k - F_k(0)}{1-\overline{F_k(0)}F_k}, \qquad k \geqslant 0. \tag{3.1}$$

It is known that the mapping $F \mapsto \{F_k(0)\}_{k\geqslant 0}$ is a bijection from $S_*(\mathbb{D})$ onto the space $\ell^0(\mathbb{Z}_+, \mathbb{D})$ consisting of all sequences $\{q(k)\}_{k\in\mathbb{Z}_+}$ indexed by nonnegative integers $\mathbb{Z}_+ = \{k \in \mathbb{Z}: \ k \geqslant 0\}$ such that $|q(k)| < 1$ for every $k \in \mathbb{Z}_+$. Moreover, one can introduce topologies on $S_*(\mathbb{D})$, $\ell^0(\mathbb{Z}_+, \mathbb{D})$ so that this mapping became a homeomorphism, see Sect. 1.3.6 in [Sim05]. One version of the Szegő theorem (see Sect. 2.7.8 in [Sim05]) states that for every function $F \in S_*(\mathbb{D})$ we have the following relation between recurrence coefficients $\{F_k(0)\}_{k\geqslant 0}$ and the logarithmic integral of F:

$$\eta(F) := \prod_{k=0}^{\infty}(1-|F_k(0)|^2) = \exp\left(\int_{\mathbb{T}} \log(1-|F|^2)\, dm_{\mathbb{T}}\right). \tag{3.2}$$

Since $|F_k(0)| < 1$ for each k and $|F| \leqslant 1$ almost everywhere on $\mathbb{T}$, the quantities $\prod_{k=0}^{\infty}(1-|F_k(0)|^2)$ and $\exp\left(\int_{\mathbb{T}} \log(1-|F|^2)\, dm_{\mathbb{T}}\right)$ are defined for every $F \in S_*(\mathbb{D})$ but could be, in general, equal to zero. We will denote by $\mathrm{Sz}(\mathbb{T})$ the class of all functions $F \in S_*(\mathbb{D})$ such that $\eta(F) > 0$. Equivalently, $F \in \mathrm{Sz}(\mathbb{T})$ if $F \in S_*(\mathbb{D})$ and any of the following two equivalent assertions holds:

- $\sum_{k\geqslant 0} |F_k(0)|^2 < \infty$, or
- $\log(1-|F|^2) \in L^1(\mathbb{T})$.

For a function $h \in L^1(\mathbb{T})$ and $k \in \mathbb{Z}$, let us denote by $\hat{h}(k)$ its k-th Fourier coefficient,

$$\hat{h}(k) = \int_{\mathbb{T}} h(\xi)\bar{\xi}^k\, dm_{\mathbb{T}}(\xi). \tag{3.3}$$

Take $r > 0$. Denote by $W^1(r\mathbb{T})$ the linear space of functions $h \in L^1(\mathbb{T})$ such that $\sum_{k\in\mathbb{Z}} r^k|\hat{h}(k)| < \infty$ equipped with the norm $\|\sum_{k\in\mathbb{Z}} c_k z^k\|_{W^1(r\mathbb{T})} = \sum_{k\in\mathbb{Z}} r^k|c_k|$. If $h_{1,2} \in L^2(\mathbb{T})$, then $h_1h_2 \in L^1(\mathbb{T})$ and the following multiplicative inequality holds:

$$\begin{aligned}\|h_1h_2\|_{W^1(r\mathbb{T})} &= \sum_{k\in\mathbb{Z}} r^k|(\hat{h}_1 * \hat{h}_2)(k)| \\ &\leqslant \sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}} r^k|\hat{h}_1(j)\hat{h}_2(k-j)| \\ &= \sum_{j\in\mathbb{Z}} r^j|\hat{h}_1(j)| \sum_{k\in\mathbb{Z}} r^{k-j}|\hat{h}_2(k-j)| \\ &= \|h_1\|_{W^1(r\mathbb{T})}\cdot\|h_2\|_{W^1(r\mathbb{T})}.\end{aligned}$$

Consider the space

$$B^1(r\mathbb{T}) = \Big\{F \in S_*(\mathbb{D}) : \sum_{k\geqslant 0} r^k|F_k(0)| < \infty\Big\}.$$

For Schur functions $F, G \in \mathrm{Sz}(\mathbb{T})$, and $r \in (0,1)$, we are going to estimate

$$\|F-G\|_{W^1(r\mathbb{T})} = \sum_{k=0}^{\infty} r^k|\hat{F}(k) - \hat{G}(k)| \tag{3.4}$$

in terms of

$$\rho_{B^1(r\mathbb{T})}(F,G) := \sum_{k=0}^{\infty} r^k|F_k(0) - G_k(0)|. \tag{3.5}$$

We set $\eta(F,G) = \min(\eta(F), \eta(G))$, where $\eta(\cdot)$ is defined in (3.2).

Theorem 3.1. *For all $F, G \in \mathrm{Sz}(\mathbb{T})$, $r \in (0,1)$ such that $1 - r \geqslant 12\log(\eta(F,G)^{-1})$ we have*

$$\frac{1}{2}\rho_{B^1(r\mathbb{T})}(F,G) \leqslant \|F-G\|_{W^1(r\mathbb{T})} \leqslant 2\rho_{B^1(r\mathbb{T})}(F,G). \tag{3.6}$$

By definition, we have $B^1(r\mathbb{T}) \subset S_*(\mathbb{D})$ for all $r \in (0,1)$. The situation changes for $r = 1$ and $r > 1$. Schur functions F lying in $B^1(\mathbb{T})$ were considered by Baxter in his famous work [Bax63], see Theorem 2.16 above. For $r > 1$, Schur functions have exponentially decaying recurrence coefficients. This case was studied by P. Nevai and V. Totik in [NT89], see also Sect. 7 in [Sim05], [Sim07], [GM06] and references in these works. However, we are not aware of inequalities analogous to (3.6) for $r \geqslant 1$. This remains an interesting open direction. We also do not know if (3.6) holds (possibly with some different constants) if we change 12 in the statement of Theorem 3.1 by arbitrary $\varepsilon > 0$. See Lemma 3.3 below for the case where $g = 0$.

For the proof of Theorem 3.1 we need some auxiliary results. Take a function $F \in \mathrm{Sz}(\mathbb{T})$. The inequality $x \leqslant -\log(1-x)$ holds for all $x \in [0,1)$, let us apply it to

(3.2). We have

$$\sum_{k=0}^{\infty}|F_k(0)|^2 \leqslant \sum_{k=0}^{\infty} -\log(1-|F_k(0)|^2) = \log\eta(F)^{-1}, \tag{3.7}$$

$$\sum_{k=0}^{\infty}|\hat{F}(k)|^2 = \int_{\mathbb{T}}|F|^2\,dm_{\mathbb{T}} \leqslant \int_{\mathbb{T}} -\log(1-|F|^2)\,dm_{\mathbb{T}} = \log\eta(F)^{-1}. \tag{3.8}$$

Lemma 3.2. *Assume that* $F \in \mathrm{Sz}(\mathbb{T})$, $r \in (0,1)$ *are such that* $1-r \geqslant T\log\eta(F)^{-1}$ *for some* $T>0$. *Then* $\|F_n\|^2_{W^1(r\mathbb{T})} \leqslant 1/T$ *for every* $n \geqslant 0$.

Proof. From (3.8) we get

$$\|F_n\|_{W^1(r\mathbb{T})} = \sum_{k=0}^{\infty} r^k|\hat{F}_n(k)| \leqslant \left(\sum_{k=0}^{\infty} r^{2k}\cdot\sum_{k=0}^{\infty}|\hat{F}_n(k)|^2\right)^{1/2} \leqslant \sqrt{\frac{\log\eta(F_n)^{-1}}{1-r^2}}.$$

The definition (3.2) of $\eta(F)$ yields $\eta(F) \leqslant \eta(F_n)$ for each $n \geqslant 0$, hence

$$\|F_n\|^2_{W^1(r\mathbb{T})} \leqslant \frac{\log\eta(F_n)^{-1}}{1-r^2} \leqslant \frac{\log\eta(F)^{-1}}{1-r} \leqslant \frac{1}{T}. \qquad \square$$

Lemma 3.3. *Assume that* $F \in \mathrm{Sz}(\mathbb{T})$, $r \in (0,1)$ *are such that* $1-r \geqslant T\log\eta(F)^{-1}$ *for some* $T>0$. *Then*

$$\frac{T-1}{T}\sum_{k=0}^{\infty}|F_k(0)|r^k \leqslant \|F\|_{W^1(r\mathbb{T})} \leqslant \frac{T+1}{T}\sum_{k=0}^{\infty}|F_k(0)|r^k.$$

Proof. One step $F_n \mapsto F_{n+1}$ of the Schur's algorithm (3.1) can be rewritten in the following form:

$$zF_{n+1} = F_n - F_n(0) + z\overline{F_n(0)}F_nF_{n+1}. \tag{3.9}$$

Then,

$$\begin{aligned} z^2F_{n+1} &= zF_n - zF_n(0) + z^2\overline{F_n(0)}F_nF_{n+1} \\ &= F_{n-1} - F_{n-1}(0) - zF_n(0) + z\overline{F_{n-1}(0)}F_{n-1}F_n + z^2\overline{F_n(0)}F_nF_{n+1}. \end{aligned}$$

Iterating (3.9) further, we obtain

$$z^{n+1}F_{n+1} = F - \sum_{k=0}^{n} z^kF_k(0) + \sum_{k=0}^{n} z^{k+1}\overline{F_k(0)}F_kF_{k+1}.$$

If we send $n \to \infty$, this becomes

$$F = \sum_{k=0}^{\infty} z^kF_k(0) - \sum_{k=0}^{\infty} z^{k+1}\overline{F_k(0)}F_kF_{k+1}, \qquad z \in \mathbb{D}. \tag{3.10}$$

Lemma 3.2 applied for F_k and F_{k+1} gives

$$\begin{aligned}\Big\|\sum_{k=0}^{\infty} z^{k+1}\overline{F_k(0)}F_kF_{k+1}\Big\|_{W^1(r\mathbb{T})} &\leqslant \sum_{k=0}^{\infty} r^{k+1}|F_k(0)|\cdot\|F_k\|_{W^1(r\mathbb{T})}\cdot\|F_{k+1}\|_{W^1(r\mathbb{T})}\\ &\leqslant \frac{1}{T}\sum_{k=0}^{\infty} r^k|F_k(0)|,\end{aligned}$$

where we used the multiplicative property of the norm $\|\cdot\|_{W^1(r\mathbb{T})}$. The lemma follows from (3.10) and the triangle inequality. □

Lemma 3.4. *Assume that* $F, G \in \mathrm{Sz}(\mathbb{T})$, $r \in (0,1)$ *are such that* $1-r \geqslant T\log\eta(F,G)^{-1}$ *for some* $T>0$. *Then we have*

$$\sum_{k=0}^{\infty} |G_k(0)|\cdot\|F_k\|_{W^1(r\mathbb{T})} \leqslant \frac{T+1}{T^2}.$$

Proof. From the previous lemma we get

$$\|F_k\|_{W^1(r\mathbb{T})} \leqslant \frac{T+1}{T}\sum_{l=0}^{\infty} r^l|F_{k+l}(0)|.$$

We use this, the Cauchy-Schwarz inequality, and (3.7) to obtain

$$\begin{aligned}\sum_{k=0}^{\infty}|G_k(0)|\cdot\|F_k\|_{W^1(r\mathbb{T})} &\leqslant \frac{T+1}{T}\sum_{k=0}^{\infty}|G_k(0)|\sum_{l=0}^{\infty} r^l|F_{k+l}(0)|\\ &= \frac{T+1}{T}\sum_{l=0}^{\infty} r^l\sum_{k=0}^{\infty}|G_k(0)|\cdot|F_{k+l}(0)|\\ &\leqslant \frac{T+1}{T}\sum_{l=0}^{\infty} r^l\cdot\sqrt{\log\eta(F_l)^{-1}\log\eta(G)^{-1}}\\ &\leqslant \frac{(T+1)\log\eta(F,G)^{-1}}{T(1-r)}.\end{aligned}$$

Since $1-r \geqslant T\log\eta(F,G)^{-1}$, the lemma is proved. □

Proof of Theorem 3.1. We first prove Theorem 3.1 under the assumption $F_{n+1} = G_{n+1} = 0$ for some $n \geqslant 0$. For simplicity, in some places we write $\|\cdot\|$ instead of $\|\cdot\|_{W^1(r\mathbb{T})}$. Let us argue by induction. Fix $k \in \mathbb{Z}_+$ and assume that

$$\frac{1}{2}\sum_{l=0}^{\infty} r^l|F_{j+l}(0)-G_{j+l}(0)| \leqslant \|F_j-G_j\|_{W^1(r\mathbb{T})} \leqslant 2\sum_{l=0}^{\infty} r^l|F_{j+l}(0)-G_{j+l}(0)| \qquad (3.11)$$

for every $j>k$. We want to prove (3.11) for $j=k$. Note that (3.11) clearly holds for $j \geqslant n+1$, therefore, there is no problem with the initial step of induction. Relation

(3.10) for F_k, G_k gives

$$\begin{aligned}F_k - G_k =& \Big(\sum_{l=0}^{\infty} z^l F_{k+l}(0) - \sum_{l=0}^{\infty} z^{l+1}\overline{F_{k+l}(0)}F_{k+l}F_{k+l+1}\Big) - \\ &- \Big(\sum_{l=0}^{\infty} z^l G_{k+l}(0) - \sum_{l=0}^{\infty} z^{l+1}\overline{G_{k+l}(0)}G_{k+l}G_{k+l+1}\Big) \\ =& \sum_{l=0}^{\infty} z^l (F_{k+l}(0) - G_{k+l}(0)) \\ &- \sum_{l=0}^{\infty} z^{l+1}\Big(\overline{F_{k+l}(0)}F_{k+l}F_{k+l+1} - \overline{G_{k+l}(0)}G_{k+l}G_{k+l+1}\Big).\end{aligned}$$

For every $s = k+l$, the expression in the last brackets can be rewritten in the form

$$\begin{aligned}\overline{F_s(0)}F_sF_{s+1} - \overline{G_s(0)}G_sG_{s+1} = &(\overline{F_s(0)} - \overline{G_s(0)})F_sF_{s+1} + \\ &+ \overline{G_s(0)}F_s(F_{s+1} - G_{s+1}) + \overline{G_s(0)}G_{s+1}(F_s - G_s).\end{aligned}$$

Therefore, we have

$$\begin{aligned}F_k - G_k =& \sum_{l=0}^{\infty} z^l \Big(F_{k+l}(0) - G_{k+l}(0)\Big) - \sum_{l=0}^{\infty} z^{l+1} F_{k+l}F_{k+l+1}\Big(\overline{F_{k+l}(0)} - \overline{G_{k+l}(0)}\Big) - \\ &- \sum_{l=0}^{\infty} z^{l+1}\overline{G_{k+l}(0)}F_{k+l}\Big(F_{k+l+1} - G_{k+l+1}\Big) \\ &- \sum_{l=0}^{\infty} z^{l+1}\overline{G_{k+l}(0)}G_{k+l+1}\Big(F_{k+l} - G_{k+l}\Big).\end{aligned} \tag{3.12}$$

For the second term in the latter sum we use the triangle inequality, the multiplicative property of the norm $\|\cdot\|_{W^1(r\mathbb{T})}$, and Lemma 3.2 to obtain

$$\begin{aligned}&\Big\|\sum_{l=0}^{\infty} z^{l+1}F_{k+l}F_{k+l+1}\Big(\overline{F_{k+l}(0)} - \overline{G_{k+l}(0)}\Big)\Big\| \\ &\leqslant \sum_{l=0}^{\infty} r^{l+1}\|F_{k+l}\| \cdot \|F_{k+l+1}\| \cdot \Big|F_{k+l}(0) - G_{k+l}(0)\Big| \\ &\leqslant \frac{1}{T}\sum_{l=0}^{\infty} r^l |F_{k+l}(0) - G_{k+l}(0)|.\end{aligned} \tag{3.13}$$

Let us rewrite the third and the fourth term in (3.12) in the form

$$\sum_{l=0}^{\infty} z^{l+1}\overline{G_{k+l}(0)}F_{k+l}\Big(F_{k+l+1} - G_{k+l+1}\Big) = \sum_{l=1}^{\infty} z^l\overline{G_{k+l-1}(0)}F_{k+l-1}\Big(F_{k+l} - G_{k+l}\Big),$$

$$\sum_{l=0}^{\infty} z^{l+1}\overline{G_{k+l}(0)}G_{k+l+1}\Big(F_{k+l}-G_{k+l}\Big)=\sum_{l=1}^{\infty} z^{l} z\overline{G_{k+l}(0)}G_{k+l+1}\Big(F_{k+l}-G_{k+l}\Big)$$
$$+ z\overline{G_k(0)}G_{k+1}\Big(F_k-G_k\Big).$$

We see that the sum of these two expressions equals

$$z\overline{G_k(0)}G_{k+1}(F_k-G_k)+\sum_{l=1}^{\infty} z^{l}\Big(\overline{G_{k+l-1}(0)}F_{k+l-1}+z\overline{G_{k+l}(0)}G_{k+l+1}\Big)\Big(F_{k+l}-G_{k+l}\Big).$$

For every $l \geqslant 1$ we have $k+l>k$, hence $F_{k+l}-G_{k+l}$ can be estimated by the induction assumption (3.11). Together with Lemma 3.4 this gives (we use again the multiplicative property of the norm $\|\cdot\|_{W^1(r\mathbb{T})}$)

$$\Big\|\sum_{l=1}^{\infty} z^{l}(z\overline{G_{k+l}(0)}G_{k+l+1}+\overline{G_{k+l-1}(0)}F_{k+l-1})\Big(F_{k+l}-G_{k+l}\Big)\Big\| \leqslant$$
$$\leqslant 2\sum_{l=1}^{\infty} r^{l}\Big(r|G_{k+l}(0)|\cdot\|G_{k+l+1}\|+|G_{k+l-1}(0)|\cdot\|F_{k+l-1}\|\Big)$$
$$\times\sum_{t=0}^{\infty} r^{t}|F_{k+l+t}(0)-G_{k+l+t}(0)|$$
$$\leqslant 2\sum_{l=1}^{\infty}\Big(|G_{k+l}(0)|\cdot\|G_{k+l+1}\|+|G_{k+l-1}(0)|\cdot\|F_{k+l-1}\|\Big)$$
$$\cdot\sup_{\ell\geqslant 1}\sum_{t=0}^{\infty} r^{t+\ell}|F_{k+\ell+t}(0)-G_{k+\ell+t}(0)|$$
$$\leqslant \frac{4(T+1)}{T^2}\sum_{s=1}^{\infty} r^{s}|F_{k+s}(0)-G_{k+s}(0)|. \tag{3.14}$$

Denote $C(T)=\frac{4(T+1)}{T^2}+\frac{1}{T}=\frac{5T+4}{T^2}$. The substitution of (3.13) and (3.14) into (3.12) implies

$$(1-C(T))\sum_{s=0}^{\infty} r^{s}|F_{k+s}(0)-G_{k+s}(0)| \leqslant$$
$$\leqslant \|(F_k-G_k)-z\overline{G_k(0)}G_{k+1}(F_k-G_k)\| \leqslant$$
$$\leqslant (1+C(T))\sum_{s=0}^{\infty} r^{s}|F_{k+s}(0)-G_{k+s}(0)|. \tag{3.15}$$

By Lemma 3.2, we have $\|\overline{G_k(0)}G_{k+1}\| \leqslant \|G_k\|\cdot\|G_{k+1}\| \leqslant 1/T$, hence

$$\frac{T-1}{T}\|F_k-G_k\| \leqslant \|(F_k-G_k)-z\overline{G_k(0)}G_{k+1}(F_k-G_k)\| \leqslant \frac{T+1}{T}\|F_k-G_k\|. \tag{3.16}$$

Therefore, for $T > 1$ from (3.15) and (3.16) we see that

$$\frac{T(1-C(T))}{T+1}\sum_{l=0}^{\infty} r^l |F_{k+l}(0) - G_{k+l}(0)| \leqslant \|F_k - G_k\|_{W^1(r\mathbb{T})},$$

$$\frac{T(1+C(T))}{T-1}\sum_{l=0}^{\infty} r^l |F_{k+l}(0) - G_{k+l}(0)| \geqslant \|F_k - G_k\|_{W^1(r\mathbb{T})}.$$

To complete the induction step, it remains to check that for $T = 12$ we have

$$\frac{1}{2} \leqslant \frac{T(1-C(T))}{T+1}, \qquad \frac{T(1+C(T))}{T-1} \leqslant 2,$$

where $C(T) = \frac{5T+4}{T^2}$. This is indeed the case.

To finish the proof, we need to get rid of the assumption $F_{n+1} = G_{n+1} = 0$. For this, we take arbitrary Schur functions $F, G \in \mathrm{Sz}(\mathbb{T})$, fix $n \in \mathbb{Z}_+$, and consider the functions $F^{(n)}, G^{(n)} \in \mathrm{Sz}(\mathbb{T})$ such that

$$F_k^{(n)}(0) = \begin{cases} F_k(0), & k \leqslant n, \\ 0, & k > n, \end{cases} \qquad G_k^{(n)}(0) = \begin{cases} G_k(0), & k \leqslant n, \\ 0, & k > n. \end{cases}$$

From (3.2) we know that $\eta(F^{(n)}, G^{(n)}) \geqslant \eta(F, G)$, hence the previous part of the proof works for $F^{(n)}$ and $G^{(n)}$. It gives

$$\frac{1}{2}\|F^{(n)} - G^{(n)}\|_{W^1(r\mathbb{T})} \leqslant \rho_{B^1(r\mathbb{T})}(F^{(n)}, G^{(n)}) \leqslant 2\|F^{(n)} - G^{(n)}\|_{W^1(r\mathbb{T})}.$$

As $n \to \infty$, we have

$$\rho_{B^1(r\mathbb{T})}(F^{(n)}, G^{(n)}) \to \rho_{B^1(r\mathbb{T})}(F, G), \qquad \|F^{(n)} - G^{(n)}\|_{W^1(r\mathbb{T})} \to \|F - G\|_{W^1(r\mathbb{T})}.$$

Indeed, the first convergence immediately follows from the definition (3.5) of the metric $\rho_{B^1(r\mathbb{T})}$; the second convergence holds because $\hat{F}(k) = \widehat{F^{(n)}}(k)$, $\hat{G}(k) = \widehat{G^{(n)}}(k)$ for all $n \geqslant k$, see Theorem 1.5.5 in [Sim05]. □

3.2 Approximation of $L^2(\mathbb{R}_+)$-potentials by discrete measures.

In this short section we prove some auxiliary lemmas related to the approximation of functions in $L^2(\mathbb{R}_+)$ by measures supported on $\ell\mathbb{Z}_+$ as $\ell \to 0$. These lemmas will be used in the proof of Theorem 1.3. Let $q \in L^2(\mathbb{R}_+)$, $\ell > 0$. From now on and till the end of the paper, we denote

$$q_\ell = \sum_{k=0}^{\infty} q[\ell k]\delta_{\ell k}, \qquad q[\ell k] = \int_{\ell k}^{\ell(k+1)} q(x)\,dx, \qquad Q[\ell k] = \begin{pmatrix} \operatorname{Im} q[\ell k] & \operatorname{Re} q[\ell k] \\ \operatorname{Re} q[\ell k] & -\operatorname{Im} q[\ell k] \end{pmatrix}, \tag{3.17}$$

where $\delta_{\ell k}$ is the point mass measure concentrated at the point ℓk. This notation should not be confused with the notation $q_\ell = q(\ell + \cdot)$ used in previous sections.

Lemma 3.5. *For $q \in L^2(\mathbb{R}_+)$, denote by $\mathcal{H}_q$, $\mathcal{H}_{q_\ell}$ the Hamiltonians on $\mathbb{R}_+$ corresponding to q, q_ℓ, respectively, see (2.16). Then $\mathcal{H}_{q_\ell} \to \mathcal{H}_q$ as $\ell \to 0$ uniformly on compact subsets of $\mathbb{R}_+$. In particular, we have $f_{q_\ell} \to f_q$ on compact subsets of $\mathbb{C}_+$ for the corresponding Schur functions by Krein-de Branges Theorem 2.1.*

Proof. Fix an arbitrary $L > 0$. Since $\mathcal{H}_q = N_q^* N_q$ and $\mathcal{H}_{q_\ell} = N_{q_\ell}^* N_{q_\ell}$, it suffices to show that $\|N_{q_\ell} - N_q\|$ tends to 0 uniformly on $[0, L]$ as $\ell \to 0$. Take $x \in [0, L]$ and set $k_* = \lfloor x/\ell \rfloor$. We have $N_q(x) = N_q(x, \ell k_*) N_q(\ell k_*, \ell(k_* - 1)) \cdot \ldots \cdot N_q(2\ell, \ell) N_q(\ell)$, see (2.15). For every $\ell > 0$, $k \in \mathbb{Z}_+$, let $\Delta_{\ell,k} = [\ell k, \ell k + \ell)$. Then,

$$\int_{\Delta_{\ell,k}} |q(x)|\, dx \leqslant \sqrt{\ell} \left(\int_{\Delta_{\ell,k}} |q(x)|^2\, dx \right)^{1/2} = \sqrt{\ell} \|q\|_{L^2(\Delta_{\ell,k})}.$$

Series representation (2.12) and inequality (2.13) give

$$N_q(\ell(k+1), \ell k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_{\Delta_{\ell,k}} JQ(x_1)\, dx_1 + O\left(\ell \|q\|^2_{L^2(\Delta_{\ell,k})}\right), \tag{3.18}$$

$$\|N_q(\ell(k+1), \ell k)\| = 1 + O\left(\|q\|_{L^1(\Delta_{\ell,k})}\right), \tag{3.19}$$

where $O(\cdot)$ is uniform with respect to $\ell \in [0, 1]$ and $k \in \mathbb{Z}_+$. By the definition (2.27) of N_{q_ℓ}, we have $N_{q_\ell}(x) = e^{JQ[\ell k_*]} e^{JQ[\ell(k_*-1)]} \cdot \ldots \cdot e^{JQ[\ell]} e^{JQ[0]}$. Relation (3.18) gives

$$e^{JQ[\ell k]} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + JQ[\ell k] + O(\|Q[\ell k]\|^2) = N_q(\ell(k+1), \ell k) + O\left(\ell \|q\|^2_{L^2(\Delta_{\ell,k})}\right). \tag{3.20}$$

As before, we have

$$\|e^{JQ[\ell k]}\| = 1 + O\left(\|q\|_{L^1(\Delta_{\ell,k})}\right). \tag{3.21}$$

Let us use the following telescopic sum relation

$$a_1 a_2 \cdots a_{k_*} - b_1 b_2 \cdots b_{k_*} = \sum_{k=1}^{k_*} a_1 \cdots a_{k-1} (a_k - b_k) b_{k+1} \cdots b_{k_*},$$

where the empty products (appearing for $k = 1$ and $k = k_*$ in the r.h.s.) are understood as 1. We obtain

$$\|N_q(\ell k_*) - N_{q_\ell}(x)\| \leqslant C_{q,x} C_{q_\ell,x} \sum_{k=1}^{k_*} \|N_q(\ell k, \ell(k-1)) - e^{JQ[\ell k]}\|,$$

for

$$C_{q_\ell,x} = \max_{1 \leqslant k \leqslant k_*} \prod_{j=1}^{k-1} \|N_q(\ell j, \ell(j-1))\|, \qquad C_{q,x} = \max_{1 \leqslant k \leqslant k_*} \prod_{j=k+1}^{k_*} \|e^{JQ[\ell j]}\|.$$

The estimates (3.19), (3.21) tell us that $C_{q_\ell,x}$, $C_{q,x}$ are uniformly bounded in $\ell \in [0,1]$, $x \in [0,L]$. Relation (3.20) implies

$$\sum_{k=0}^{k_*} \|N_q(\ell(k+1),\ell k) - e^{JQ[\ell k]}\| = O\left(\ell \|q\|^2_{L^2([0,x])}\right),$$

which tends to zero as $\ell \to 0$ (the constant in $O(\cdot)$ is uniform in $\ell \in [0,1]$, $x \in [0,L]$). Finally, (3.19) implies that $\|N_q(x) - N_q(\ell k_*)\| \to 0$ as $\ell \to 0$ uniformly in $x \in [0,L]$, and the proof is completed. □

Lemma 3.6. *For every $q \in L^2(\mathbb{R}_+)$ we have*

$$\lim_{\ell\to 0} \frac{1}{\ell} \sum_{k\geqslant 0} |q[\ell k]|^2 = \|q\|^2_{L^2(\mathbb{R}_+)}. \tag{3.22}$$

Proof. Recall that $\Delta_{\ell,k} = [\ell k, \ell k + \ell)$. The Cauchy-Schwarz inequality gives

$$\frac{1}{\ell}\sum_{k\geqslant 0} |q[\ell k]|^2 = \frac{1}{\ell}\sum_{k\geqslant 0}\left|\int_{\Delta_{\ell,k}} q(x)\,dx\right|^2 \leqslant \frac{1}{\ell}\sum_{k\geqslant 0} \ell \cdot \int_{\Delta_{\ell,k}} |q(x)|^2\,dx = \|q\|^2_{L^2(\mathbb{R}_+)}. \tag{3.23}$$

In particular, both sides of (3.22) with lim replaced by lim sup or lim inf depend continuously on q in $L^2(\mathbb{R}_+)$-norm. Hence, it suffices to prove (3.22) only on a dense subset of $L^2(\mathbb{R}_+)$. Let $C_0^\infty(\mathbb{R}_+)$ be the set of infinitely smooth functions on $\mathbb{R}_+$ with compact support. Take some $q \in C_0^\infty(\mathbb{R}_+)$ and let $R > 0$ be such that $\operatorname{supp} q \subset [0,R]$. Then

$$\frac{1}{\ell}|q[\ell k]|^2 - \int_{\Delta_{\ell,k}} |q(x)|^2\,dx = \int_{\Delta_{\ell,k}} q(x)\left(\frac{1}{\ell}\int_{\Delta_{\ell,k}} \overline{q(y)}\,dy - \overline{q(x)}\right)dx = O(\ell^2), \quad \ell \to 0.$$

We also have $q[\ell k] = 0$ for $k > R/\ell$, hence

$$\begin{aligned}\frac{1}{\ell}\sum_{k\geqslant 0}|q[\ell k]|^2 = \sum_{k=0}^{[R/\ell]} \frac{1}{\ell}|q[\ell k]|^2 &= \sum_{k=0}^{[R/\ell]}\left(\int_{\Delta_{\ell,k}} |q(x)|^2\,dx + O(\ell^2)\right)\\ &= \int_0^\infty |q(x)|^2\,dx + O(\ell^2 \cdot R/\ell) = \|q\|^2_{L^2} + O(R\ell),\end{aligned}$$

which tends to zero with $\ell \to 0$ because R is fixed. □

Lemma 3.7. *For $q \in L^2(\mathbb{R}_+)$ and $A > 0$ we have*

$$\lim_{\ell\to 0}\sum_{k\geqslant 0} e^{-A\ell k}|q[\ell k]| = \int_{\mathbb{R}_+} e^{-Ax}|q(x)|dx. \tag{3.24}$$

Proof. The argument is similar to the proof of the previous lemma. It is easy to check using Cauchy-Schwarz inequality and (3.23) that both sides of (3.24) with lim replaced by lim sup or lim inf depend on q continuously in $L^2(\mathbb{R}_+)$-norm. Therefore, we need to prove (3.24) only for $q \in C_0^\infty(\mathbb{R}_+)$. As $\ell \to 0$, we have the following estimates

$$|q[\ell k]| = \ell|q(\ell k)| + O(\ell^2), \qquad \int_{\Delta_{\ell,k}} |q(x)|\,dx = \ell|q(\ell k)| + O(\ell^2),$$
$$\int_{\Delta_{\ell,k}} (e^{-A\ell k} - e^{-Ax})\,dx = O(e^{-A\ell k}\ell^2).$$

Therefore,

$$\begin{aligned} &e^{-A\ell k}|q[\ell k]| - \int_{\Delta_{\ell,k}} e^{-Ax}|q(x)|dx \\ &\quad = e^{-A\ell k}\Big(|q[\ell k]| - \int_{\Delta_{\ell,k}} |q(x)|\,dx\Big) \\ &\qquad + \int_{\Delta_{\ell,k}} (e^{-A\ell k} - e^{-Ax})|q(x)|dx = O(e^{-A\ell k}\ell^2), \quad \ell \to 0. \end{aligned}$$

Summing up over all $k \geqslant 0$, we get

$$\Big|\sum_{k\geqslant 0} e^{-A\ell k}|q[\ell k]| - \int_{\mathbb{R}_+} e^{-Ax}|q(x)|dx\Big| = O\Big(\sum_{k\geqslant 0} e^{-A\ell k}\ell^2\Big) = O\Big(\frac{\ell^2}{1-e^{-A\ell}}\Big) = O(\ell),$$

which completes the proof. □

3.3 Approximation of $S_2(\mathbb{C}_+)$-functions by periodic Schur functions.

Take $q \in L^2(\mathbb{R}_+)$, $\ell > 0$, and define $q_\ell \in \mathcal{M}_\ell$ by (3.17). Let f_{q_ℓ} be the Schur function of $\mathcal{D}_{q_\ell}$. By Lemma 2.11, this function is periodic with period π/ℓ. From Lemma 2.9 we know that there exists a Schur function F_{q_ℓ} in $\mathbb{D}$ that satisfies

$$f_{q_\ell}(z) = F_{q_\ell}(e^{2i\ell z}), \qquad f_{q_\ell}(\infty) = F_{q_\ell}(0), \qquad z \in \mathbb{C}_+. \tag{3.25}$$

It is natural to expect that f_{q_ℓ} in some sense approximates f_q as $\ell \to 0$. This is indeed the case, we study this approximation below. For $\ell > 0$, we let g_{q_ℓ} be such that $g_{q_\ell}(x) = f_{q_\ell}(x)$ for $|x| \leqslant \pi/(2\ell)$ and $g_{q_\ell}(x) = 0$ for $|x| > \pi/(2\ell)$. Note that g_{q_ℓ} is not analytic and defined only on $\mathbb{R}$. We are going to prove the convergence of g_{q_ℓ} to f_q in the following metric space X of measurable functions on $\mathbb{R}$,

$$X = \{r \in L^\infty(\mathbb{R}) \colon \|r\|_{L^\infty(\mathbb{R})} \leqslant 1,\ \log(1-|r|^2) \in L^1(\mathbb{R})\}. \tag{3.26}$$

For every $r, r_1, r_2 \in X$ define

$$\begin{aligned} \rho_X^2(r_1,r_2) &= \int_{\mathbb{R}} -\log\Big(1 - \Big|\frac{r_1 - r_2}{1-\overline{r_1}r_2}\Big|^2\Big)\,dx \\ &= \int_{\mathbb{R}} -\log\Big(\frac{(1-|r_1|^2)(1-|r_2|^2)}{|1-\overline{r_1}r_2|^2}\Big)\,dx, \end{aligned} \tag{3.27}$$

$$E(r_1,r_2) = \int_{\mathbb{R}} -\log|1-\overline{r_1}r_2|\,dx, \qquad E(r) = E(r,r). \tag{3.28}$$

Formula (3.27) can be written in the following form:

$$\rho_X^2(r_1,r_2) = E(r_1) + E(r_2) - 2E(r_1,r_2). \tag{3.29}$$

Theorem 1.6 in [SW99] states that ρ_X is a metric on X. In particular, it satisfies the triangle inequality

$$\rho_X(r_1,r_3) \leqslant \rho_X(r_1,r_2) + \rho_X(r_2,r_3), \qquad r_1,r_2,r_3 \in X. \tag{3.30}$$

This follows from the fact that the function $d\colon \mathbb{D}\times\mathbb{D}\to\mathbb{R}$,

$$d^2(z,w) = -\log\left(1 - \left|\frac{z-w}{1-\overline{z}w}\right|^2\right)$$

is a metric on $\mathbb{D}$. Indeed, by Cauchy-Schwarz inequality, we have

$$\begin{aligned}\rho_X(r_1,r_3) &= \int_{\mathbb{R}} d^2(r_1,r_3)\,dx = \int_{\mathbb{R}} \Big(d(r_1,r_2)+d(r_2,r_3)\Big)^2\,dx \\ &\leqslant \int_{\mathbb{R}} d^2(r_1,r_2)\,dx + \int_{\mathbb{R}} d^2(r_2,r_3)\,dx + 2\sqrt{\int_{\mathbb{R}} d^2(r_1,r_2)\,dx \int_{\mathbb{R}} d^2(r_2,r_3)\,dx} \\ &= \rho_X^2(r_1,r_2) + \rho_X^2(r_2,r_3) + 2\rho_X(r_1,r_2)\rho_X(r_2,r_3) \\ &= \Big(\rho_X(r_1,r_2)+\rho_X(r_2,r_3)\Big)^2.\end{aligned}$$

One can see that the space $S_2(\mathbb{C}_+)$ in (1.4) is a closed subspace of X with the induced norm. To prove convergence of g_{q_ℓ} to f_q in X, we will need the following lemma.

LEMMA 3.8. *For every $q \in L^2(\mathbb{R}_+)$ we have*

$$\frac{1}{\pi}\lim_{\ell\to 0} E(g_{q_\ell}) = \lim_{\ell\to 0}\frac{1}{\ell}\log(\eta(F_{q_\ell})) = -\|q\|^2_{L^2(\mathbb{R}_+)}. \tag{3.31}$$

Proof. We have $f_{q_\ell}(x) = F_{q_\ell}(e^{2i\ell x})$ a. e. on $[-\pi/2\ell, \pi/2\ell]$, hence

$$\begin{aligned}E(g_{q_\ell}) &= \int_{-\pi/2\ell}^{\pi/2\ell} \log(1-|f_{q_\ell}(x)|^2)\,dx = \int_{-\pi/2\ell}^{\pi/2\ell} \log(1-|F_{q_\ell}(e^{2i\ell x})|^2)\,dx \\ &= \frac{1}{2\ell}\int_{-\pi}^{\pi} \log(1-|F_{q_\ell}(e^{iy})|^2)\,dy = \frac{\pi}{\ell}\log\eta(F_{q_\ell}),\end{aligned}$$

where the factor 2π appears in the last inequality because of the normalization of the measure $m_{\mathbb{T}}$ used in (3.2). Therefore it suffices to prove only the second equality in (3.31). From the Szegő theorem, formula (3.2), and (3.25) we know that

$$\log(\eta(F_{q_\ell})) = \sum_{k\geqslant 0}\log(1-|F_{q_\ell,k}(0)|^2) = \sum_{k\geqslant 0}\log(1-|f_{q_\ell,k}(\infty)|^2). \tag{3.32}$$

Theorem 1.4 states that $\varkappa(f_{q_\ell,k}(\infty))=q[\ell k]$, where $\varkappa$ is defined in (1.15). As $w\to 0$, we have

$$\varkappa(w)-\overline{w}=o(|w|), \qquad |\varkappa(w)|^2-|w|^2=o\left(|w|^2\right).$$

Therefore from (3.23) and Lemma 3.6 we get

$$\sum_{k\geqslant 0}\left||f_{q_\ell,k}(\infty)|^2-|q[\ell k]|^2\right|=\sum_{k\geqslant 0}o\left(|q[\ell k]|^2\right)=o\left(\ell\right), \qquad \ell\to 0,$$

$$\frac{1}{\ell}\sum_{k\geqslant 0}|f_{q_\ell,k}(\infty)|^2=\frac{1}{\ell}\sum_{k\geqslant 0}|q[\ell k]|^2+o(1)=\|q\|^2_{L^2(\mathbb{R}_+)}+o(1), \qquad \ell\to 0. \tag{3.33}$$

In particular, we get $\sup_k|f_{q_\ell,k}(\infty)|^2=O(\ell)$ uniformly for small ℓ. It follows that

$$\left||f_{q_\ell,k}(\infty)|^2+\log(1-|f_{q_\ell,k}(\infty)|^2)\right|=O\left(|f_{q_\ell,k}(\infty)|^4\right)=O\left(\ell\,|f_{q_\ell,k}(\infty)|^2\right), \qquad \ell\to 0,$$

$$\sum_{k\geqslant 0}\left||f_{q_\ell,k}(\infty)|^2+\log(1-|f_{q_\ell,k}(\infty)|^2)\right|=O\left(\ell\sum_{k\geqslant 0}|f_{q_\ell,k}(\infty)|^2\right)=O(\ell^2), \qquad \ell\to 0,$$

$$\frac{1}{\ell}\sum_{k\geqslant 0}|f_{q_\ell,k}(\infty)|^2=-\frac{1}{\ell}\sum_{k\geqslant 0}\log(1-|f_{q_\ell,k}(\infty)|^2)+O(\ell), \qquad \ell\to 0.$$

The latter combined with (3.32) and (3.33) concludes the proof. □

Lemma 3.9. *The functions g_{q_ℓ} converge to f_q in X as $\ell\to 0$.*

The proof of this lemma uses two function-theoretic results whose proofs we postpone until the next section. Firstly, if $r_n,r\in S_2(\mathbb{C}_+)$ are such that $r_n\to r$ uniformly on compact subsets in $\mathbb{C}_+$, then

$$\lim_{n\to\infty}E(r_n,\varphi)=E(r,\varphi)\ \text{ for every }\ \varphi\in L^1(\mathbb{R}):\ \|\varphi\|_{L^\infty(\mathbb{R})}<1. \tag{3.34}$$

This is Lemma 4.5. Secondly, we have $\rho_X(r_n,r)\to 0$ for $r_n,r\in X$ if and only if

$$\lim_{n\to\infty}E(r_n)=E(r)\ \text{ and (3.34) holds.} \tag{3.35}$$

This is assertion (d) of Lemma 4.4.

Proof of Lemma 3.9. By Lemma 3.8 and the isometric relation (1.5) in Theorem 1.1, we have

$$\begin{aligned}\lim_{\ell\to 0}E(g_{q_\ell})&=\lim_{\ell\to 0}\int_{-\pi/2\ell}^{\pi/2\ell}-\log(1-|f_{q_\ell}(x)|^2)\,dx\\&=\pi\|q\|^2_{L^2(\mathbb{R}_+)}=\int_{\mathbb{R}}-\log(1-|f_q|^2)\,dx=E(f_q).\end{aligned}$$

So, we only need to prove that $E(g_{q_\ell},\varphi)\to E(f_q,\varphi)$ for every function $\varphi\in L^1(\mathbb{R})$ satisfying $\|\varphi\|_{L^\infty(\mathbb{R})}<1$. Fix such a function φ. Lemma 3.5, Krein – de Branges

Theorem 2.1 and (3.34) imply

$$\lim_{\ell\to 0}\int_{\mathbb{R}}\log|1-\overline{f_{q_\ell}}\varphi|\,dx=\int_{\mathbb{R}}\log|1-\overline{f_q}\varphi|\,dx.$$

Furthermore, we have

$$\begin{aligned}\left|\int_{\mathbb{R}}\log|1-\overline{f_{q_\ell}}\varphi|\,dx-\int_{\mathbb{R}}\log|1-\overline{g_{q_\ell}}\varphi|\,dx\right|&=\left|\int_{|x|>\pi/2\ell}\log|1-\overline{f_{q_\ell}}\varphi|\,dx\right|\\&\leqslant C\int_{|x|>\pi/2\ell}|\varphi(x)|\,dx,\end{aligned}$$

for a constant C depending only on $\|\varphi\|_{L^\infty(\mathbb{R})}$. The r.h.s. in the last formula tends to 0 as $\ell\to 0$. Thus, we get $\lim_{\ell\to 0}E(g_{q_\ell},\varphi)=\lim_{\ell\to 0}E(f_{q_\ell},\varphi)=\lim_{\ell\to 0}E(f_q,\varphi)$, and the proof is concluded by the application of (3.35). $\square$

Lemma 3.10. *For $q,\tilde q\in L^2(\mathbb{R}_+)$ and $A>0$, we have*

$$\lim_{\ell\to 0}\sum_{k\geqslant 0}|\hat F_{q_\ell}(k)-\hat F_{\tilde q_\ell}(k)|e^{-A\ell k}=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+}e^{-A\xi/2}|\hat f_q-\hat f_{\tilde q}|\,d\xi.$$

Proof. First of all, let us relate $\hat F_{q_\ell}$ and $\hat g_{q_\ell}$. For every $\ell>0$ and $k\in\mathbb{Z}_+$, we have

$$\begin{aligned}\hat g_{q_\ell}(2\ell k)&=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}g_{q_\ell}(x)e^{-i2\ell kx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2\ell}^{\pi/2\ell}f_{q_\ell}(x)e^{-i2\ell kx}\,dx\\&=\frac{1}{\sqrt{2\pi}}\int_{-\pi/2\ell}^{\pi/2\ell}F_{q_\ell}(e^{2i\ell x})e^{-i2\ell kx}\,dx\\&=\frac{\sqrt{2\pi}}{2\ell}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}F_{q_\ell}(e^{iy})e^{-iky}\,dy=\frac{\sqrt{2\pi}}{2\ell}\hat F_{q_\ell}(k).\end{aligned}$$

It follows that

$$\sum_{k\geqslant 0}|\hat F_{q_\ell}(k)-\hat F_{\tilde q_\ell}(k)|e^{-A\ell k}=\frac{1}{\sqrt{2\pi}}\sum_{k\geqslant 0}2\ell|\hat g_{q_\ell}(2\ell k)-\hat g_{\tilde q_\ell}(2\ell k)|e^{-A\ell k}. \tag{3.36}$$

Take arbitrary $r_1,r_2\in X$ and observe that the inequality $-\log(1-x)\geqslant x$ for $x\in[0,1)$ implies

$$\rho_X^2(r_1,r_2)=\int_{\mathbb{R}}-\log\left(1-\left|\frac{r_1-r_2}{1-\overline{r_1}r_2}\right|^2\right)dx\geqslant\int_{\mathbb{R}}\left|\frac{r_1-r_2}{1-\overline{r_1}r_2}\right|^2dx\geqslant\frac14\int_{\mathbb{R}}|r_1-r_2|^2\,dx.$$

It follows that for every $r_1,r_2\in X$ we have $2\rho_X(r_1,r_2)\geqslant\|r_1-r_2\|_{L^2(\mathbb{R})}$. Then Lemma 3.9 gives

$$\lim_{\ell\to 0}\|g_{q_\ell}-f_q\|_{L^2(\mathbb{R})}=0,\qquad\lim_{\ell\to 0}\|g_{\tilde q_\ell}-f_{\tilde q}\|_{L^2(\mathbb{R})}=0. \tag{3.37}$$

The latter, the Cauchy-Schwarz inequality and the isometric property of the Fourier transform imply

$$\lim_{\ell\to 0}\int_{\mathbb{R}_+}|\hat{g}_{q_\ell}(\xi)-\hat{g}_{\tilde{q}_\ell}(\xi)|e^{-A\xi/2}\,d\xi=\int_{\mathbb{R}_+}|\hat{f}_q(\xi)-\hat{f}_{\tilde{q}}(\xi)|e^{-A\xi/2}\,d\xi.$$

Thus, the claim of the lemma will follow from

$$\lim_{\ell\to 0}\left|\int_{\mathbb{R}_+}|\hat{g}_{q_\ell}(\xi)-\hat{g}_{\tilde{q}_\ell}(\xi)|e^{-A\xi/2}\,d\xi-\sum_{k\geqslant 0}2\ell|\hat{g}_{q_\ell}(2\ell k)-\hat{g}_{\tilde{q}_\ell}(2\ell k)|e^{-A\ell k}\right|=0. \tag{3.38}$$

Denote $h=\hat{g}_{q_\ell}-\hat{g}_{\tilde{q}_\ell}$. Then the value under the limit in (3.38) equals

$$\begin{aligned}&\left|\int_{\mathbb{R}_+}|h(\xi)|e^{-A\xi/2}\,d\xi-\sum_{k\geqslant 0}2r|h(2\ell k)|e^{-A\ell k}\right|\\&\qquad\leqslant\sum_{k\geqslant 0}\int_{2\ell k}^{2\ell(k+1)}\left|h(\xi)e^{-A\xi/2}-h(2\ell k)e^{-A\ell k}\right|\,d\xi.\end{aligned} \tag{3.39}$$

For every $\xi\in\mathbb{R}$ we have $|h(\xi)|\leqslant\|g_{q_\ell}-g_{\tilde{q}_\ell}\|_{L^1(\mathbb{R})}$. Moreover, there is an absolute constant $C>0$ such that uniformly for all small $\ell>0$, all $k\in\mathbb{Z}_+$ and $\xi\in[2\ell k, 2\ell(k+1)]$ we have

$$\begin{aligned}|h(\xi)-h(2\ell k)|&\leqslant C|\xi-2\ell k|\cdot\|g_{q_\ell}-g_{\tilde{q}_\ell}\|_{L^1(\mathbb{R})},\\|e^{-A\xi/2}-e^{-A\ell k}|&\leqslant C|\xi-2\ell k|e^{-A\ell k}.\end{aligned}$$

It follows that

$$\left|h(\xi)e^{-A\xi/2}-h(2\ell k)e^{-A\ell k}\right|\leqslant 2Ce^{-A\ell k}|\xi-2\ell k|\cdot\|g_{q_\ell}-g_{\tilde{q}_\ell}\|_{L^1(\mathbb{R})}.$$

Recall that $g_{q_\ell},g_{\tilde{q}_\ell}$ are supported on the interval of length π/ℓ and that $\|g_{q_\ell}\|_{L^2(\mathbb{R})}$, $\|g_{\tilde{q}_\ell}\|_{L^2(\mathbb{R})}$ are uniformly bounded in ℓ by (3.37). We get

$$\begin{aligned}\|g_{q_\ell}-g_{\tilde{q}_\ell}\|_{L^1(\mathbb{R})}&\leqslant\sqrt{\pi/\ell}\|g_{q_\ell}-g_{\tilde{q}_\ell}\|_{L^2(\mathbb{R})}\leqslant\sqrt{\pi/\ell}\Big(\|g_{q_\ell}\|_{L^2(\mathbb{R})}+\|g_{\tilde{q}_\ell}\|_{L^2(\mathbb{R})}\Big)\\&=O(1/\sqrt{\ell}),\qquad \ell\to 0.\end{aligned}$$

Therefore, the k-th term in the right-hand side of (3.39) is $O(\ell^{3/2}e^{-A\ell k})$ as $\ell\to 0$ and the total sum can be estimated by

$$O\Big(\sum_{k\geqslant 0}\ell^{3/2}e^{-A\ell k}\Big)=O\Big(\frac{\ell^{3/2}}{1-e^{-A\ell}}\Big)=O(\ell^{1/2}),\qquad \ell\to 0.$$

In particular, it tends to 0 as $\ell\to 0$, as was required in (3.38). The proof is concluded.

□

3.4 Proof of Theorem 1.3.

Recall that we need to prove

$$\sqrt{\frac{\pi}{2}}\int_{\mathbb{R}_+}|q-\tilde{q}|e^{-2Ax}\,dx \leqslant \int_{\mathbb{R}_+}|\hat{f}_q-\hat{f}_{\tilde{q}}|e^{-A\xi}\,d\xi \leqslant 2\sqrt{2\pi}\int_{\mathbb{R}_+}|q-\tilde{q}|e^{-2Ax}\,dx, \tag{3.40}$$

where $q,\tilde{q}\in L^2(\mathbb{R}_+)$ and $A\in\mathbb{R}$ satisfies $A\geqslant 12\max\left(\|q\|^2_{L^2(\mathbb{R}_+)},\|\tilde{q}\|^2_{L^2(\mathbb{R}_+)}\right)$. Without loss of generality, we can assume that

$$A>12\max\left(\|q\|^2_{L^2(\mathbb{R}_+)},\|\tilde{q}\|^2_{L^2(\mathbb{R}_+)}\right), \tag{3.41}$$

the claim for $A=12\max\left(\|q\|^2_{L^2(\mathbb{R}_+)},\|\tilde{q}\|^2_{L^2(\mathbb{R}_+)}\right)$ will follow by a limiting argument (it worth be mentioned that the constant 12 is not optimal). Fix $\varepsilon>0$. Let us show that the inequality

$$\begin{aligned}\frac{1}{2(1+\varepsilon)}\sum_{k\geqslant 0}e^{-A\ell k}|q[\ell k]-\tilde{q}[\ell k]| &\leqslant \sum_{k\geqslant 0}e^{-A\ell k}|\hat{F}_{q_\ell}(k)-\hat{F}_{\tilde{q}_\ell}(k)|\\ &\leqslant \frac{2}{1-\varepsilon}\sum_{k\geqslant 0}e^{-A\ell k}|q[\ell k]-\tilde{q}[\ell k]|\end{aligned} \tag{3.42}$$

holds for all sufficiently small ℓ. Then (3.40) will follow if we take $\ell\to 0$ in (3.42), apply Lemma 3.7 and Lemma 3.10, and send $\varepsilon\to 0$. Thus, we can now focus on (3.42). Let $r=e^{-A\ell}$ and recall the definition (3.4) of the metric in $W^1(r\mathbb{T})$. We have

$$\sum_{k\geqslant 0}e^{-A\ell k}|\hat{F}_{q_\ell}(k)-\hat{F}_{\tilde{q}_\ell}(k)| = \sum_{k\geqslant 0}r^k|\hat{F}_{q_\ell}(k)-\hat{F}_{\tilde{q}_\ell}(k)| = \|F_{q_\ell}-F_{\tilde{q}_\ell}\|_{W^1(r\mathbb{T})}. \tag{3.43}$$

Note that $1-r=1-e^{-A\ell}=A\ell+o(\ell)$ as $\ell\to 0$. Lemma 3.8 shows that

$$\max\left(\|q\|^2_{L^2},\|\tilde{q}\|^2_{L^2}\right)\ell = \max\left(\log\eta(F_{q_\ell})^{-1},\log\eta(F_{\tilde{q}_\ell})^{-1}\right)+o(\ell), \quad \ell\to 0.$$

Then assumption (3.41) for small ℓ implies

$$1-r\geqslant 12\max\left(\log\eta(F_{q_\ell})^{-1},\log\eta(F_{\tilde{q}_\ell})^{-1}\right).$$

Therefore, Theorem 3.1 applies to Schur functions $F_{q_\ell},F_{\tilde{q}_\ell}$ on the circle of radius $r=e^{-A\ell}$ if ℓ is small enough. It gives

$$\frac{1}{2}\rho_{B^1(r\mathbb{T})}(F_{q_\ell},F_{\tilde{q}_\ell})\leqslant\|F_{q_\ell}-F_{\tilde{q}_\ell}\|_{W^1(r\mathbb{T})}\leqslant 2\rho_{B^1(r\mathbb{T})}(F_{q_\ell},F_{\tilde{q}_\ell}). \tag{3.44}$$

According to the definition (3.5) of metric $\rho_{B^1(r\mathbb{T})}$ and (3.25), we have

$$\rho_{B^1(r\mathbb{T})}(F_{q_\ell},F_{\tilde{q}_\ell})=\sum_{k\geqslant 0}r^k|F_{q_\ell,k}(0)-F_{\tilde{q}_\ell,k}(0)|. \tag{3.45}$$

Equality (3.25) and Theorem 1.4 state

$$q[\ell k]=\varkappa(F_{q_\ell,k}(0)), \qquad \tilde{q}[\ell k]=\varkappa(F_{\tilde{q}_\ell,k}(0)).$$

The straightforward calculation shows that for $u,v\in\mathbb{C}$ small enough we have

$$|(\varkappa(u)-\varkappa(v))+(\overline{u}-\overline{v})|\leqslant\varepsilon|u-v|. \tag{3.46}$$

From (3.33) and (3.25) we know that $\lim\limits_{\ell\to 0}F_{q_\ell,k}(0)=\lim\limits_{\ell\to 0}F_{\tilde{q}_\ell,k}(0)=0$ hence (3.46) applies. It gives

$$\Big|(q[\ell k]-\tilde{q}[\ell k])+(\overline{F_{q_\ell,k}(0)}-\overline{F_{\tilde{q}_\ell,k}(0)})\Big|\leqslant\varepsilon\Big|F_{q_\ell,k}(0)-F_{\tilde{q}_\ell,k}(0)\Big|,$$

$$(1-\varepsilon)\Big|F_{q_\ell,k}(0)-F_{\tilde{q}_\ell,k}(0)\Big|\leqslant\Big|q[\ell k]-\tilde{q}[\ell k]\Big|\leqslant(1+\varepsilon)\Big|F_{q_\ell,k}(0)-F_{\tilde{q}_\ell,k}(0)\Big|.$$

It follows from (3.45) that

$$\frac{1}{1+\varepsilon}\sum_{k\geqslant 0}e^{-A\ell k}|q[\ell k]-\tilde{q}[\ell k]|\leqslant\rho_{B^1(r\mathbb{T})}(F_{q_\ell},F_{\tilde{q}_\ell})\leqslant\frac{1}{1-\varepsilon}\sum_{k\geqslant 0}e^{-A\ell k}|q[\ell k]-\tilde{q}[\ell k]|. \tag{3.47}$$

To establish (3.42), we substitute (3.47) and (3.43) into (3.44). □

4 Sylvester-Winebrenner theorem

In the first part of this section we study the metric space X and its subspace $S_2(\mathbb{C}_+)$ defined in (3.26) and (1.3), respectively. Then we use this analysis to prove that the mapping $\mathcal{F}: q\mapsto f_q$ is a homeomorphism from $L^2(\mathbb{R}_+)$ onto $S_2(\mathbb{C}_+)$. The fact that $\mathcal{F}$ is a bijection from $L^2(\mathbb{R}_+)$ to $S_2(\mathbb{C}_+)$ was proved in Theorem 11.4 in [Den06]. Therefore, only continuity properties of this mapping need to be investigated. We check continuity of $\mathcal{F}$, $\mathcal{F}^{-1}$ in Sect. 4.2 and prove the lack of uniform continuity of $\mathcal{F}$, $\mathcal{F}^{-1}$ in Sects. 4.3, 4.4, correspondingly. This last part implies Theorem 1.2.

4.1 Properties of the Sylvester-Winebrenner metric. Recall that the quantities $E(r_1,r_2)$, $E(r)$, for $r_1,r_2,r\in X$ are defined in (3.28). Our aim here is to prove the following result.

Theorem 4.1. *Let $r_n,r\in S_2(\mathbb{C}_+)$. Then $r_n\to r$ in $S_2(\mathbb{C}_+)$ if and only if $r_n\to r$ uniformly on compact subsets of $\mathbb{C}_+$ and $E(r_n)\to E(r)$.*

The observation below was already used in the proof of Lemma 3.10, but we repeat it here for convenience.

LEMMA 4.2. *For every $r_1,r_2\in X$ we have $2\rho_X(r_1,r_2)\geqslant\|r_1-r_2\|_{L^2(\mathbb{R})}$.*

Proof. Let us apply the inequality $-\log(1-x)\geqslant x$ in formula (3.27) defining the metric ρ_X. We obtain

$$\rho_X^2(r_1,r_2)=\int_{\mathbb{R}}-\log\left(1-\left|\frac{r_1-r_2}{1-\overline{r_1}r_2}\right|^2\right)dx\geqslant\int_{\mathbb{R}}\left|\frac{r_1-r_2}{1-\overline{r_1}r_2}\right|^2dx\geqslant\frac{1}{4}\int_{\mathbb{R}}|r_1-r_2|^2\,dx. \quad\square$$

Lemma 4.3. *Let $r_n, r \in X$ be such that r_n are bounded in X and $r_n \to r$ as $n \to \infty$ in the Lebesgue measure on $\mathbb{R}$. Then $\lim\limits_{n\to\infty} E(r_n,\varphi) = E(r,\varphi)$ for every $\varphi \in X$.*

Proof. Considering $r_2 = 0$ in Lemma 4.2, we see that $\{r_n\}_{n\geqslant 0}$ is a uniformly bounded sequence in $L^2(\mathbb{R})$. Since $r_n \to r$ in the Lebesgue measure on $\mathbb{R}$, we then have $r_n \to r$ weakly in $L^2(\mathbb{R})$. Moreover, for each $k \geqslant 1$, $\{r_n^k\}_{n\geqslant 0}$ is also a uniformly bounded sequence in $L^2(\mathbb{R})$ and $r_n^k \to r^k$ in the Lebesgue measure on $\mathbb{R}$. Thus, we have $r_n^k \to r^k$ weakly in $L^2(\mathbb{R})$ for every $k \geqslant 1$. Let us take $\varphi \in X$ and represent $E(r_n,\varphi)$ in the form

$$E(r_n,\varphi) = \operatorname{Re}\int_{\mathbb{R}} -\log(1-\overline{r_n}\varphi)\,dx = \operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}} \frac{(\overline{r_n}\varphi)^k}{k}\,dx + \operatorname{Re}\int_{\mathbb{R}}\sum_{k=N+1}^{\infty}\frac{(\overline{r_n}\varphi)^k}{k}\,dx.$$

Here and below $\log z$ denotes the analytic branch of the logarithm in $\mathbb{C}\setminus(-\infty,0]$ such that $\log 1 = 0$. We are going to show that

$$\lim_{N\to\infty}\sup_n\left|\operatorname{Re}\int_{\mathbb{R}}\sum_{k=N+1}^{\infty}\frac{(\overline{r_n}\varphi)^k}{k}\,dx\right| = 0.$$

For this we write

$$\begin{aligned}\left|\int_{\mathbb{R}}\sum_{k=N+1}^{\infty}\frac{(\overline{r_n}\varphi)^k}{k}\,dx\right|^2 &\leqslant \left(\int_{\mathbb{R}}|r_n\varphi|^N\sum_{k=1}^{\infty}\frac{|r_n\varphi|^k}{k}\,dx\right)^2\\ &\leqslant \left(\int_{\mathbb{R}}|r_n\varphi|^N\sqrt{\log(1-|r_n|^2)\log(1-|\varphi|^2)}\,dx\right)^2\\ &\leqslant \int_{\mathbb{R}}\log(1-|r_n|^2)\,dx\int_{\mathbb{R}}|\varphi|^{2N}\log(1-|\varphi|^2)\,dx,\end{aligned}$$

where we used Cauchy-Schwarz inequality (for sums and for integrals) and the fact that $|r_n| \leqslant 1$ on $\mathbb{R}$. Since $\{r_n\}_{n\geqslant 0}$ is bounded in X, we have $\sup_n \int_{\mathbb{R}}|\log(1-|r_n|^2)|\,dx = \sup_n \rho_X^2(0,r_n) < \infty$. Moreover, $\int_{\mathbb{R}}|\varphi|^{2N}\log(1-|\varphi|^2)\,dx \to 0$ by the Lebesgue dominated convergence theorem with the majorant $|\log(1-|\varphi|^2)|$. Thus, we only need to check that

$$\lim_{N\to\infty}\lim_{n\to\infty}\operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}}\frac{(\overline{r_n}\varphi)^k}{k}\,dx = E(r,\varphi).$$

We have $\varphi \in L^2(\mathbb{R})$ by Lemma 4.2 for $r_1 = \varphi$, $r_2 = 0$. In view of $\|\varphi\|_{L^\infty(\mathbb{R})} \leqslant 1$, this implies $\varphi^k \in L^2(\mathbb{R})$ for every $k \geqslant 1$. Then for each fixed $N \geqslant 1$ we obtain

$$\lim_{n\to\infty}\operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}}\frac{(\overline{r_n}\varphi)^k}{k}\,dx = \operatorname{Re}\int_{\mathbb{R}}\sum_{k=1}^{N}\frac{(\overline{r}\varphi)^k}{k}\,dx$$

from the weak convergence $r_n^k \to r^k$ in $L^2(\mathbb{R})$. Moreover,

$$\left|\sum_{k=1}^{N} \frac{(\overline{r}\varphi)^k}{k}\right| \leqslant \sum_{k=1}^{N} \frac{|r\varphi|^k}{k} \leqslant \sqrt{\log(1-|r|^2)\log(1-|\varphi|^2)},$$

where the r.h.s. belongs to $L^1(\mathbb{R})$ because $r,\varphi \in X$. Thus, by the Lebesgue dominated convergence theorem we have

$$\lim_{N\to\infty} \operatorname{Re}\int_{\mathbb{R}} \sum_{k=1}^{N} \frac{(\overline{r}\varphi)^k}{k}\,dx = \operatorname{Re}\int_{\mathbb{R}} \sum_{k=1}^{\infty} \frac{(\overline{r}\varphi)^k}{k} = \int_{\mathbb{R}} -\log|1-\overline{r}\varphi|\,dx = E(r,\varphi). \qquad \square$$

Lemma 4.4. *Let $r_n, r \in X$. The following assertions are equivalent*:

(a) r_n *converges to* r *in* X;
(b) $\lim_{n\to\infty} E(r_n) = E(r)$ *and* r_n *converges to* r *in Lebesgue measure on* $\mathbb{R}$;
(c) $\lim_{n\to\infty} E(r_n) = E(r)$ *and* $\lim_{n\to\infty} E(r_n,\varphi) = E(r,\varphi)$ *for every* $\varphi \in X$.
(d) $\lim_{n\to\infty} E(r_n) = E(r)$ *and* $\lim_{n\to\infty} E(r_n,\varphi) = E(r,\varphi)$ *for every* $\varphi \in L^1(\mathbb{R})$ *with* $\|\varphi\|_{L^\infty(\mathbb{R})} < 1$;
(e) $\lim_{n\to\infty} E(r_n) = E(r)$ *and* $\lim_{n\to\infty} E(r_n,r) = E(r)$;

Proof. Since $E(r) = \rho_X^2(0,r)$ for every $r \in X$, we have (a) ⇒ (b) by Lemma 4.2. Then, (b) ⇒ (c) by Lemma 4.3. Clearly, (c) ⇒ (d). To show that (d) ⇒ (a), we fix $\varepsilon > 0$, take $r \in X$ and find $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that $\rho_X(\varphi,r) < \varepsilon$ (for instance, one can take $\varphi = \chi_F r$, where χ_F is the characteristic function of the set $F = \{x : |r(x)| \leqslant 1 - 1/N\} \cap [-N,N]$ for a sufficiently large N). Then, as $n \to \infty$, we have

$$\begin{aligned}\rho_X(r_n,r) &\leqslant \rho_X(r_n,\varphi) + \rho_X(\varphi,r) = \sqrt{E(r_n) + E(\varphi) - 2E(r_n,\varphi)} + \rho_X(\varphi,r)\\ &\to 2\rho_X(\varphi,r)\end{aligned}$$

by assumption (d) and formula (3.29). Thus, $\rho_X(r_n,r) < 3\varepsilon$ for all n large enough. Since $\varepsilon > 0$ is arbitrary, this yields (a). Equivalence of (a) and (e) is a simple consequence of (3.29). □

Lemma 4.5. *If $r_n, r \in S_2(\mathbb{C}_+)$ are such that $r_n \to r$ uniformly on compact subsets in $\mathbb{C}_+$, then*

$$\lim_{n\to\infty} E(r_n,\varphi) = E(r,\varphi) \tag{4.1}$$

for every $\varphi \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ with $\|\varphi\|_{L^\infty(\mathbb{R})} < 1$.

Proof. As in the proof of Lemma 4.3, we have

$$E(r_n,\varphi) = \operatorname{Re}\int_{\mathbb{R}} -\log(1-\overline{r_n}\varphi)\,dx = \operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}} \frac{(\overline{r_n}\varphi)^k}{k}\,dx + \operatorname{Re}\int_{\mathbb{R}} \sum_{k=N+1}^{\infty} \frac{(\overline{r_n}\varphi)^k}{k}\,dx,$$

and

$$\lim_{N\to\infty}\sup_n\left|\operatorname{Re}\int_{\mathbb{R}}\sum_{k=N+1}^{\infty}\frac{(\overline{r_n}\varphi)^k}{k}\,dx\right| \leqslant \lim_{N\to\infty}\int_{\mathbb{R}}|\varphi|^N\sum_{k=1}^{\infty}\frac{|\varphi|^k}{k}\,dx$$
$$= \lim_{N\to\infty}\int_{\mathbb{R}}|\varphi|^N\log(1-|\varphi|)\,dx = 0,$$

because $\log(1-|\varphi|)\in L^1(\mathbb{R})$. Thus, to prove that $E(r_n,\varphi)\to E(r,\varphi)$ it suffices to check that

$$\lim_{n\to\infty}\operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}}\frac{(\overline{r_n}\varphi)^k}{k}\,dx = \operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}}\frac{(\overline{r}\varphi)^k}{k}\,dx, \tag{4.2}$$

$$\lim_{N\to\infty}\operatorname{Re}\sum_{k=1}^{N}\int_{\mathbb{R}}\frac{(\overline{r}\varphi)^k}{k}\,dx = \int_{\mathbb{R}}\operatorname{Re}\sum_{k=1}^{\infty}\frac{(\overline{r}\varphi)^k}{k}\,dx = E(r,\varphi). \tag{4.3}$$

As in the proof of Lemma 4.3, relation (4.3) is a consequence of the Lebesgue dominated convergence theorem (this time – with the majorant $|\log(1-|\varphi|)|$). So, we can focus on the proof of (4.2). Since $|\bar{r}_n^k|\leqslant 1$ and $\varphi^k\in L^1(\mathbb{R})$ for all n, k, it suffices to prove that

$$\lim_{n\to\infty}\int_{\mathbb{R}}\overline{r_n}^k\psi\,dx = \int_{\mathbb{R}}\overline{r}^k\psi\,dx \tag{4.4}$$

for functions ψ from a dense subset of $L^1(\mathbb{R})$. By our assumption, $\lim_{n\to\infty} r_n^k(z) = r^k(z)$ for every $k\geqslant 1$ and $z\in\mathbb{C}_+$. Thus, (4.4) holds for ψ from the set of all finite linear combinations of Poisson kernels $P_z : x\mapsto \frac{1}{\pi}\operatorname{Im}\frac{1}{x-z}$, $z\in\mathbb{C}_+$. This set is dense in $L^1(\mathbb{R})$, which completes the proof. □

Proof of Theorem 4.1. Let $r_n, r\in S_2(\mathbb{C}_+)$ be such that $r_n\to r$ in $S_2(\mathbb{C}_+)$. Then $E(r_n)\to E(r)$ by Lemma 4.4. Moreover, we have $r_n\to r$ in the Hardy space $H^2(\mathbb{C}_+)$ by Lemma 4.2. In particular, $r_n\to r$ uniformly on compact subsets of $\mathbb{C}_+$. Conversely, assume that $r_n\to r$ uniformly on compact subsets of $\mathbb{C}_+$ and $E(r_n)\to E(r)$. Then Lemma 4.5 and Lemma 4.4 imply $r_n\to r$ in $S_2(\mathbb{C}_+)$. □

4.2 Spectral map is a homeomorphism. As we mentioned at the beginning of Sect. 4, the fact that the mapping $\mathcal{F}: q\mapsto f_q$ is a bijection from $L^2(\mathbb{R}_+)$ to $S_2(\mathbb{C}_+)$ is proved in Theorem 11.4 in [Den06]; formula (1.5) is contained in Corollary 11.2 in the same paper [Den06]. The proof of continuity in Theorem 1.1 relies on the following known lemma.

Lemma 4.6. *Suppose that $\{q_n\}$ is a bounded sequence in $L^2(\mathbb{R}_+)$, and let $\{f_{q_n}\}$ be the corresponding sequence of Schur functions. Then, $\{q_n\}$ converges weakly in $L^2(\mathbb{R}_+)$ if and only if $\{f_{q_n}\}$ converges uniformly on compact subsets of $\mathbb{C}_+$.*

Proof. Suppose that a sequence $\{q_n\}$ converges weakly to $q \in L^2(\mathbb{R}_+)$. Then we have

$$\lim_{n\to\infty} \max_{0\leqslant y\leqslant L} \left| \int_0^y q_n(x)\,dx - \int_0^y q(x)\,dx \right| = 0$$

for every $L \geqslant 0$. It follows that the corresponding Weyl functions, $\{m_{q_n}\}$, converge on compact subsets on $\mathbb{C}_+$ to the Weyl function m_q of q. To prove this fact it suffices to note that the solutions of equations

$$JN'_{q_n}(x) + Q_n(x)N_{q_n}(x) = 0, \qquad N_{q_n}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad Q_n = \begin{pmatrix} \operatorname{Im} q_n & \operatorname{Re} q_n \\ \operatorname{Re} q_n & -\operatorname{Im} q_n \end{pmatrix}$$

converge to the solution N_q of the corresponding equation for q uniformly on $[0, L]$ for each $L \geqslant 0$ (see (2.12), (2.13)) and use Theorem 2.1 for Hamiltonians $\mathcal{H}_{q_n} = N^*_{q_n} N_{q_n}$, $\mathcal{H}_q = N_q^* N_q$. The locally uniform convergence $m_{q_n} \to m_q$ on $\mathbb{C}_+$ implies the locally uniform convergence $f_{q_n} \to f_q$.

Conversely, assume that $\{q_n\}$ is a bounded sequence in $L^2(\mathbb{R}_+)$ such that Schur functions f_{q_n} converge on compact subsets of $\mathbb{C}_+$ to some function f. Using the weak compactness of closed bounded subsets of $L^2(\mathbb{R}_+)$, we see that from each subsequence $\{q_{n_k}\}$ one can extract another subsequence $\{q_{n_{k_j}}\}$ weakly converging to a function $q \in L^2(\mathbb{R}_+)$. From the first part of the proof we get $f_q = f$. In particular, q is determined uniquely by f (the spectral correspondence $q \mapsto f_q$ is a bijection, see Theorem 11.4 in [Den06]) and the weak limit of $\{q_{n_k}\}$ equals q for all subsequences of the sequence $\{q_n\}$. It follows that $\{q_n\}$ weakly converges to q. □

Proof of continuity in Theorem 1.1. Let us prove that the map $\mathcal{F} : L^2(\mathbb{R}_+) \to S_2(\mathbb{C}_+)$ is a homeomorphism assuming that this map is a bijection and the sum rule (1.5) holds for all $q \in L^2(\mathbb{R}_+)$. Consider some potentials $q_n, q \in L^2(\mathbb{R}_+)$ and let $f_n, f \in S_2(\mathbb{C}_+)$ be the corresponding Schur functions. It is well-known that $q_n \to q$ in $L^2(\mathbb{R}_+)$ if and only if $\|q_n\|_{L^2(\mathbb{R}_+)} \to \|q\|_{L^2(\mathbb{R}_+)}$ and $q_n \to q$ weakly in $L^2(\mathbb{R}_+)$. In view of (1.5), we have $\|q_n\|_{L^2(\mathbb{R}_+)} \to \|q\|_{L^2(\mathbb{R}_+)}$ if and only if $E(f_n) \to E(f)$. In particular, the sequence $\{q_n\}$ is bounded in $L^2(\mathbb{R}_+)$ if and only if the sequence $\{f_n\}$ is bounded in $S_2(\mathbb{C}_+)$. If one of these sequences is bounded, we know from Lemma 4.6 that $q_n \to q$ weakly in $L^2(\mathbb{R}_+)$ if and only if $f_n \to f$ on compact subsets of $\mathbb{C}_+$. Thus, $q_n \to q$ in $L^2(\mathbb{R}_+)$ if and only if $E(f_n) \to E(f)$ and $f_n \to f$ on compact subsets of $\mathbb{C}_+$. By Theorem 4.1 this is equivalent to the convergence $f_n \to f$ in $S_2(\mathbb{C}_+)$. This proves that the map $\mathcal{F} : L^2(\mathbb{R}_+) \to S_2(\mathbb{C}_+)$ is a homeomorphism. □

4.3 Direct map is not uniformly continuous.

Our next aim is to prove the following proposition.

Proposition 4.7. *There are potentials u_n, $\tilde{u}_n$ in the unit ball of $L^2(\mathbb{R}_+)$ such that (1.6) holds. In other words, the direct homeomorphism $\mathcal{F} : S_2(\mathbb{C}_+) \to L^2(\mathbb{R}_+)$ in Theorem 1.1 is not uniformly continuous on bounded subsets of $S_2(\mathbb{C}_+)$.*

Proof. We construct an explicit example. For $T \geqslant 1$, $v \geqslant 0$, denote by $q_{T,v}$ the constant imaginary-valued potential on $[0,T]$ with positive imaginary part that satisfies $\|q_{T,v}\|_{L^2[0,T]} = v$. Extend it by zero to (T,∞). Also let $\varepsilon(T)$ be some positive number that satisfy $100e^{-\sqrt{T}/6} \leqslant \varepsilon(T) \leqslant T^{-1}$. We are going to show that

$$\|q_{T,1} - q_{T,1-\varepsilon(T)}\|_{L^2(\mathbb{R}_+)} = \varepsilon(T), \qquad \liminf_{T\to\infty} \rho_{S_2}(f_{q_{T,1}}, f_{T,1-\varepsilon(T)}) > 0. \tag{4.5}$$

The claim will then follow. Note that the first relation in (4.5) is a simple consequence of the definition of $q_{T,1}$, $q_{T,1-\varepsilon(T)}$, and we only need to check the second relation. To fix notation, take $p > 0$ and let $q \in L^2(\mathbb{R}_+)$ be the piece-wise constant potential such that $q = ip$ on $[0,T]$, $q = 0$ on (T,∞). Then $Q(t) = \left(\begin{smallmatrix} p & 0 \\ 0 & -p \end{smallmatrix}\right)$ on $[0,T]$ and the fundamental solution $N(t,z)$ of the corresponding Dirac system $JN' + QN = zN$, $N(0,z) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, for $0 \leqslant t \leqslant T$, $z \in \mathbb{C}$, is given by

$$N(t,z) = \exp(tJ(Q - z)) = \exp\left(t\left(\begin{smallmatrix} 0 & p+z \\ p-z & 0 \end{smallmatrix}\right)\right),$$

that can be checked by the differentiation with respect to t. Then, recall Lemma 2.10,

$$N(T,z) = \begin{pmatrix} \cosh(T\lambda) & \frac{p+z}{\lambda}\sinh(T\lambda) \\ \frac{\lambda}{p+z}\sinh(T\lambda) & \cosh(T\lambda) \end{pmatrix}, \qquad \lambda = \sqrt{p^2 - z^2}.$$

Notice that $N(T,z)$ is an entire function in z and its does not depend on the choice of the root $\lambda = \sqrt{p^2 - z^2}$. Indeed, this follows from the fact that functions $\cosh(T\lambda)$, $\sinh(T\lambda)/\lambda$ depend only on the value λ^2. Further, for $t > T$ we have $Q(t) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, hence

$$N(t,z) = \exp(-(t-T)zJ)N(T,z), \qquad t \geqslant T.$$

The fundamental matrix solution of the canonical system with the Hamiltonian $\mathcal{H}_q = N(t,0)^* N(t,0)$ that corresponds to the Dirac operator with the potential q is given by $M(t,z) = N(t,0)^{-1}N(t,z)$, see, e.g., Sect. 2.4 in [Bes20]. Therefore, the Weyl function of this canonical system equals

$$\begin{aligned} m_{\mathcal{H}_q} &\doteq \lim_{t\to\infty} \sigma_1 M(t,z)^T \sigma_1 \omega \doteq \lim_{t\to\infty} \sigma_1 N(t,z)^T (N(t,0)^{-1})^T \sigma_1 \omega \\ &\doteq \lim_{t\to\infty} \sigma_1 N(t,z)^T \sigma_1 \tilde{\omega}(t), \end{aligned}$$

where $\tilde{\omega}(t) = \sigma_1 (N(t,0)^{-1})^T \sigma_1 \omega$ belongs to $\mathbb{C}_+$ because $\omega \in \mathbb{C}_+$ and $\sigma_1(N(t,0)^{-1})^T \sigma_1 \in \mathbf{SL}(2,\mathbb{R})$. So, we have

$$\begin{aligned} \lim_{t\to\infty} \sigma_1 N(t,z)^T \sigma_1 \tilde{\omega}(t) &\doteq \sigma_1 N(T,z)^T \sigma_1 \lim_{t\to\infty} \sigma_1 \exp(-(t-T)zJ)^T \sigma_1 \tilde{\omega}(t) \\ &\doteq \sigma_1 N(T,z)^T \sigma_1 i \doteq N(T,z)i, \end{aligned}$$

where we used the fact that the fractional-linear transformation with the matrix $\sigma_1 \exp(-szJ)^T\sigma_1$ for large $s>0$ maps $\mathbb{C}_+$ into a small disk with center at i. Then,

$$\begin{aligned} f_q &= \frac{m_q - i}{m_q + i} = \frac{m_{\mathcal{H}_q} - i}{m_{\mathcal{H}_q} + i} \doteq \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} N(T,z)i \\ &\doteq \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix} \begin{pmatrix} i\cosh(T\lambda) + \frac{p+z}{\lambda}\sinh(T\lambda) \\ i\frac{\lambda}{p+z}\sinh(T\lambda) + \cosh(T\lambda) \end{pmatrix} \\ &= \frac{\left(\frac{p+z}{\lambda} + \frac{\lambda}{p+z}\right)\sinh(T\lambda)}{2i\cosh(T\lambda) + \left(\frac{p+z}{\lambda} - \frac{\lambda}{p+z}\right)\sinh(T\lambda)} = \frac{\frac{2p}{\lambda}\sinh(T\lambda)}{2i\cosh(T\lambda) + \frac{2z}{\lambda}\sinh(T\lambda)} \\ &= \frac{-ip\sinh(T\lambda)}{\lambda\cosh(T\lambda) - iz\sinh(T\lambda)}. \end{aligned}$$

From now on we will assume that $z = x$ with real $x \in [c_1p, c_2p]$, where $c_1 = \sqrt{3/4}$, $c_2 = \sqrt{8/9}$. In this case $p^2 - x^2 > 0$, hence λ can be chosen positive, $\lambda \in [p/3, p/2]$. For brevity, we will write $f = f_q$ and $\varepsilon = \varepsilon(T)$ in the remaining part of the proof. We have

$$\begin{aligned} f(x) &= \frac{-ip(e^{T\lambda} - e^{-T\lambda})}{\lambda(e^{T\lambda} + e^{-T\lambda}) - ix(e^{T\lambda} - e^{-T\lambda})} = \frac{-ip + ipe^{-2T\lambda}}{(\lambda - ix) + (\lambda + ix)e^{-2T\lambda}} \\ &= g(x) + e^{-2T\lambda}h(x), \\ g(x) &= \frac{-ip}{\lambda - ix}, \qquad h(x) = e^{2T\lambda}(f_q(x) - g(x)) = \frac{-2ip\lambda}{(x+i\lambda)^2 - p^2e^{-2T\lambda}}. \end{aligned}$$

Let us denote $q = q_{T,1}$ and $q_\varepsilon = q_{T,1-\varepsilon}$, and let $f_\varepsilon, g_\varepsilon, h_\varepsilon$ be the functions corresponding to q_ε. We have

$$\begin{aligned} |g(x) - g_\varepsilon(x)| &= \left|\frac{p}{\lambda - ix} - \frac{p_\varepsilon}{\lambda_\varepsilon - ix}\right| = \frac{|p\lambda_\varepsilon - p_\varepsilon\lambda + ix(p_\varepsilon - p)|}{|(\lambda - ix)(\lambda_\varepsilon - ix)|} \geqslant \frac{|p\lambda_\varepsilon - p_\varepsilon\lambda|}{pp_\varepsilon} \\ &= |\lambda_\varepsilon/p_\varepsilon - \lambda/p| = \left|\sqrt{1 - x^2/p_\varepsilon^2} - \sqrt{1 - x^2/p^2}\right| \\ &= \frac{x^2/p^2 - x^2/p_\varepsilon^2}{\sqrt{1 - x^2/p_\varepsilon^2} + \sqrt{1 - x^2/p^2}} \\ &\geqslant \frac{x^2/p_\varepsilon^2 - x^2/p^2}{2} \geqslant \frac{x^2}{p^2}(1 - p_\varepsilon^2/p^2) \geqslant c_1^2(1 - p_\varepsilon^2/p^2). \end{aligned}$$

Recall that $p = 1/\sqrt{T}$, $p_\varepsilon = (1-\varepsilon)/\sqrt{T}$. Hence $1 - p_\varepsilon^2/p^2 = 1 - (1-\varepsilon)^2 \geqslant \varepsilon$ for small ε (equivalently, for large T). It follows that

$$|g(x) - g_\varepsilon(x)| \geqslant \frac{c_1^2\varepsilon}{2} = \frac{3\varepsilon}{8}, \qquad x \in [c_1p, c_2p]. \tag{4.6}$$

Furthermore, we have $\lambda \geqslant \lambda_\varepsilon \geqslant p/4$ for small ε, hence $e^{-2T\lambda} \leqslant e^{-2T\lambda_\varepsilon} \leqslant e^{-2Tp/4} = e^{-\sqrt{T}/2}$ and for T large enough we have

$$|f(x)-g(x)| = e^{-2T\lambda}|h(x)| = \frac{2e^{-2T\lambda}p\lambda}{|(x+i\lambda)^2 - p^2e^{-2T\lambda}|} \leqslant \frac{2e^{-2T\lambda}p\lambda}{p^2(1-e^{-2T\lambda})} \leqslant e^{-\sqrt{T}/2}, \tag{4.7}$$

$$|f_\varepsilon(x)-g_\varepsilon(x)| = e^{-2T\lambda_\varepsilon}|h_\varepsilon(x)| = \frac{2e^{-2T\lambda_\varepsilon}p_\varepsilon\lambda_\varepsilon}{|(x+i\lambda_\varepsilon)^2 - p_\varepsilon^2e^{-2T\lambda_\varepsilon}|} \leqslant \frac{2e^{-2T\lambda_\varepsilon}p_\varepsilon\lambda_\varepsilon}{p_\varepsilon^2(1-e^{-2T\lambda_\varepsilon})} \leqslant e^{-\sqrt{T}/2}. \tag{4.8}$$

The triangle inequality implies

$$\begin{aligned} |f(x)-f_\varepsilon(x)| &\geqslant |g(x)-g_\varepsilon(x)| - |f_q(x)-g(x)| - |g_\varepsilon(x)-f_\varepsilon(x)| \\ &\geqslant \frac{3}{8}\varepsilon - e^{-2T\lambda}|h(x)| - e^{-2T\lambda_\varepsilon}|h_\varepsilon(x)| \geqslant \frac{3}{8}\varepsilon - 2e^{-\sqrt{T}/2} \geqslant \frac{\varepsilon}{4}. \end{aligned} \tag{4.9}$$

By definition of λ, we have

$$|g(x)|^2 = \left|\frac{-ip}{\lambda - ix}\right|^2 = \frac{p^2}{\lambda^2+x^2} = 1, \qquad \left|\frac{g-g_\varepsilon}{1-\overline{g}g_\varepsilon}\right| = \left|\frac{g-g_\varepsilon}{\overline{g}g - \overline{g}g_\varepsilon}\right| = 1.$$

We are ready to estimate $\rho_{S_2}(f, f_\varepsilon)$. We have

$$\left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon} - \frac{g-g_\varepsilon}{1-\overline{g}g_\varepsilon}\right| = \left|\frac{(f-f_\varepsilon)(1-\overline{g}g_\varepsilon) - (g-g_\varepsilon)(1-\overline{f}f_\varepsilon)}{(1-\overline{g}g_\varepsilon)(1-\overline{f}f_\varepsilon)}\right| \leqslant \frac{4(|f-g|+|f_\varepsilon-g_\varepsilon|)}{\left|(1-\overline{g}g_\varepsilon)(1-\overline{f}f_\varepsilon)\right|}.$$

For the denominator we write $|1-\overline{g}g_\varepsilon| = |g-g_\varepsilon|$, $|1-\overline{f}f_\varepsilon| \geqslant |f-f_\varepsilon|$ and apply (4.6), (4.9); for the numerator we use (4.7) and (4.8). This gives

$$1 - \left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon}\right| \leqslant \left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon} - \frac{g-g_\varepsilon}{1-\overline{g}g_\varepsilon}\right| \leqslant \frac{4\cdot 2\cdot e^{-\sqrt{T}/2}}{3\varepsilon/8\cdot\varepsilon/4} < \frac{2^8e^{-\sqrt{T}/2}}{3\varepsilon^2} \leqslant \frac{e^{-\sqrt{T}/6}}{2}.$$

It follows that

$$1 - \left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon}\right|^2 \leqslant e^{-\sqrt{T}/6}, \qquad -\log\left(1 - \left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon}\right|^2\right) \geqslant \frac{\sqrt{T}}{6}.$$

Finally, we write

$$\rho_{S_2}(f, f_\varepsilon)^2 \geqslant \int_{c_1p}^{c_2p} -\log\left(1 - \left|\frac{f-f_\varepsilon}{1-\overline{f}f_\varepsilon}\right|^2\right)dx \geqslant \frac{(c_2-c_1)p\sqrt{T}}{6} = \frac{c_2-c_1}{6} > 0.$$

The proof is concluded. □

REMARK 4.8. The argument can be adapted to prove that $\mathcal{F}$ is not uniformly continuous on any ball of radius $\delta > 0$ with center at 0.

4.4 Inverse map is not uniformly continuous. In this section we prove that the mapping $\mathcal{F}^{-1}\colon S_2(\mathbb{C}_+) \to L^2(\mathbb{R}_+)$ from Theorem 1.1 is not uniformly continuous. The proof is based on the non-injectivity of the direct scattering transform of the Dirac operator on $\mathbb{R}$. Let us introduce the required objects, following notation from [BD24]. The results of the present section are adaptations of the results from Sect. 6 in [BG24], where the discrete scattering transform was considered.

We start with the function $q \in L^2(\mathbb{R})$ and introduce $q^+, q^- \in L^2(\mathbb{R}_+)$,

$$q^+(x) = -\overline{q(x)}, \qquad q^-(x) = q(-x), \qquad x \geqslant 0.$$

The functions f^+ and f^- are the Schur functions corresponding to q^+ and q^- respectively as described in the introduction of the present paper, see (1.2). From Theorem 1.1 we know that $\log(1-|f^\pm|^2) \in L^1(\mathbb{R})$ hence there exist outer in $\mathbb{C}_+$ functions $\mathfrak{a}^\pm$ satisfying $|\mathfrak{a}^\pm|^{-2} = 1 - |f^\pm|^2$, see Theorem 4.4 in [Gar81]. Outer function is uniquely defined by its boundary values up to a constant unimodular factor. As in [BD24], we normalize $\mathfrak{a}^\pm$ by $\mathfrak{a}^\pm(i\infty) = 1$, i.e., we let

$$\mathfrak{a}^\pm(z) = \exp\left(\frac{1}{\pi i}\int_{\mathbb{R}} \frac{\log|\mathfrak{a}^\pm|}{x-z}\,dx\right) = \exp\left(\frac{-1}{2\pi i}\int_{\mathbb{R}} \frac{\log(1-|f^\pm|^2)}{x-z}\,dx\right), \quad z \in \mathbb{C}_+. \tag{4.10}$$

The analytic functions $\mathfrak{b}^\pm$ are defined so that $f^\pm = \mathfrak{b}^\pm/\mathfrak{a}^\pm$ in $\mathbb{C}_+$. Finally, we let

$$a = \mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^-, \qquad b = \mathfrak{a}^-\overline{\mathfrak{b}^+} - \mathfrak{b}^-\overline{\mathfrak{a}^+}, \qquad \mathbf{r}_q = b/a. \tag{4.11}$$

The function a is outer in $\mathbb{C}_+$, and

$$|a|^2 = 1 + |b|^2, \qquad 1 - |\mathbf{r}_q|^2 = 1 - \frac{|b|^2}{|a|^2} = \frac{|a|^2-|b|^2}{|a|^2} = |a|^{-2}. \tag{4.12}$$

The function $\mathbf{r}_q$ is called the reflection coefficient of q and the mapping $q \mapsto \mathbf{r}_q$ acting from $L^2(\mathbb{R})$ to the space X defined in (3.26) is called the direct scattering transform. It has some symmetries, see Lemma 2.1 in [BD24]. We will need a part of this lemma that concerns translation of the potential.

Lemma 4.9 (Lemma 2.1 in [BD24]). *Let $q \in L^2(\mathbb{R})$. We have $\mathbf{r}_{q_s}(\lambda) = e^{-i\lambda s}\mathbf{r}_q(\lambda)$ for almost every $\lambda \in \mathbb{R}$, where $s \in \mathbb{R}$ and $q_s\colon x \mapsto q(x-s) \in L^2(\mathbb{R})$ is the s-shift of q.*

The following proposition is also well-known.

Proposition 4.10. *For every $q \in L^2(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} |q|^2\,dx = \frac{1}{\pi}\int_{\mathbb{R}} -\log(1-|\mathbf{r}_q|^2)\,dx.$$

Proof. In the upper half-plane $\mathbb{C}_+$ we can write

$$a = \mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^- = \mathfrak{a}^+\mathfrak{a}^-\left(1 - \frac{\mathfrak{b}^+}{\mathfrak{a}^+}\cdot\frac{\mathfrak{b}^-}{\mathfrak{a}^-}\right) = \mathfrak{a}^+\mathfrak{a}^-\left(1 - f^+f^-\right).$$

Therefore, (4.12) implies

$$\int_{\mathbb{R}} \log(1-|\mathbf{r}_q|^2)\,dx = -\int_{\mathbb{R}} \log|\mathfrak{a}^+|^2\,dx - \int_{\mathbb{R}} \log|\mathfrak{a}^-|^2\,dx - \int_{\mathbb{R}} \log|1-f^+f^-|^2\,dx.$$

Recall that $\mathfrak{a}^\pm$ are defined so that $|\mathfrak{a}^\pm|^{-2} = 1-|f^\pm|^2$ a. e. on $\mathbb{R}$. Theorem 1.1 gives

$$\begin{aligned} -\int_{\mathbb{R}} \log|\mathfrak{a}^+|^2\,dx &= \int_{\mathbb{R}} \log(1-|f^+|^2)\,dx = -\pi\int_{\mathbb{R}_+} |q^+|^2\,dx, \\ -\int_{\mathbb{R}} \log|\mathfrak{a}^-|^2\,dx &= \int_{\mathbb{R}} \log(1-|f^-|^2)\,dx = -\pi\int_{\mathbb{R}_+} |q^-|^2\,dx, \\ \int_{\mathbb{R}} \log|\mathfrak{a}^+|^2\,dx + \int_{\mathbb{R}} \log|\mathfrak{a}^-|^2\,dx &= \pi\int_{\mathbb{R}_+} |q^+|^2\,dx + \pi\int_{\mathbb{R}_+} |q^-|^2\,dx = \pi\int_{\mathbb{R}} |q|^2\,dx. \end{aligned}$$

Thus, to finish the proof of the lemma we need to show that

$$\int_{\mathbb{R}} \log|1-f^+f^-|^2\,dx = 0. \tag{4.13}$$

We have $f^\pm \in L^2(\mathbb{R}) \cap H^\infty(\mathbb{C}_+)$. It follows that $f^\pm \in H^2(\mathbb{C}_+)$, see Corollary 4.3 in [Gar81]. We also know that $|f^+f^-| < 1$ in $\mathbb{C}_+$ by the maximum modulus principle, hence $\log(1-f^+f^-)$ is a well-defined analytic function in $\mathbb{C}_+$, it satisfies

$$\begin{aligned} |\log(1-f^+f^-)|^2 &= \left|\sum_{k\geqslant 1} \frac{(f^+f^-)^k}{k}\right|^2 \leqslant \sum_{k\geqslant 1} \frac{|f^+|^{2k}}{k} \cdot \sum_{k\geqslant 1} \frac{|f^-|^{2k}}{k} \\ &= |\log(1-|f^+|^2)| \cdot |\log(1-|f^-|^2)|. \end{aligned}$$

Therefore, for every $y > 0$ we can write

$$\begin{aligned} &\int_{\mathbb{R}} |\log(1+(f^+f^-)(x+iy))|\,dx \\ &\leqslant \int_{\mathbb{R}} |\log(1-|f^+(x+iy)|^2)|^{1/2} \cdot |\log(1-|f^-(x+iy)|^2)|^{1/2}\,dx \\ &\leqslant \left(\int_{\mathbb{R}} |\log(1-|f^+(x+iy)|^2)|\,dx\right)^{1/2} \left(\int_{\mathbb{R}} |\log(1-|f^-(x+iy)|^2)|\,dx\right)^{1/2}. \end{aligned} \tag{4.14}$$

Inclusion $(f^\pm)^k \in H^2(\mathbb{C}_+)$ holds for all $k \geqslant 1$, hence

$$\begin{aligned} \int_{\mathbb{R}} |\log(1-|f^\pm(x+iy)|^2)|\,dx &= \int_{\mathbb{R}} \sum_{k\geqslant 1} \frac{|f^\pm(x+iy)|^{2k}}{k}\,dx = \sum_{k\geqslant 1} \int_{\mathbb{R}} \frac{|f^\pm(x+iy)|^{2k}}{k}\,dx \\ &\leqslant \sum_{k\geqslant 1} \int_{\mathbb{R}} \frac{|f^\pm(x)|^{2k}}{k}\,dx = \int_{\mathbb{R}} |\log(1-|f^\pm(x)|^2)|\,dx. \end{aligned}$$

The latter shows that the value of the l.h.s. in (4.14) is uniformly bounded in y, which means $\log(1+f^+f^-) \in H^1(\mathbb{C}_+)$. The integral of any function in $H^1(\mathbb{C}_+)$ over the real line $\mathbb{R}$ is 0, see Lemma 3.7 in [Gar81], hence

$$\int_{\mathbb{R}} \log|1-f^+f^-|^2\,dx = 2\,\mathrm{Re}\int_{\mathbb{R}} \log(1-f^+f^-)\,dx = 0.$$

This gives (4.13) and ends the proof. ☐

PROPOSITION 4.11. *If $q_n \to q$ in $L^2(\mathbb{R})$ as $n\to\infty$ then $\mathbf{r}_{q_n}\to\mathbf{r}_q$ in X as $n\to\infty$.*

Proof. We start by rewriting (4.11) in the following form:

$$\mathbf{r}_q = \frac{\mathfrak{a}^-\overline{\mathfrak{b}^+}-\mathfrak{b}^-\overline{\mathfrak{a}^+}}{\mathfrak{a}^+\mathfrak{a}^- - \mathfrak{b}^+\mathfrak{b}^-} = \frac{\overline{\mathfrak{a}^+}}{\mathfrak{a}^+}\,\frac{\overline{f^+}-f^-}{1-f^+f^-}. \tag{4.15}$$

We have $\|q_n^+-q^+\|_{L^2(\mathbb{R}_+)}\to 0$ and $\|q_n^- - q^-\|_{L^2(\mathbb{R}_+)}\to 0$ as $n\to\infty$. Then by Theorem 1.1 we get $f_n^+\to f^+$ and $f_n^-\to f^-$ in $S_2(\mathbb{C}_+)$ as $n\to\infty$. In particular, $f_n^\pm \to f^\pm$ in Lebesgue measure on $\mathbb{R}$ by Lemma 4.4. Therefore, $\log(1-|f_n^+|^2)$ converge in measure to $\log(1-|f^+|^2)$. Theorem 1.1 also gives

$$\begin{aligned}\lim_{n\to\infty}\|\log(1-|f_n^+|^2)\|_{L^1(\mathbb{R})} &= \lim_{n\to\infty}\pi\|q_n^+\|^2_{L^2(\mathbb{R}_+)}\\ &= \pi\|q^+\|^2_{L^2(\mathbb{R}_+)} = \|\log(1-|f^+|^2)\|_{L^1(\mathbb{R})}.\end{aligned}$$

Then Scheffé's lemma, see Lemma 5.10 in [Wil91], implies that $2\log|\mathfrak{a}_n^+| = -\log(1-|f_n^+|^2)$ converge to $2\log|\mathfrak{a}^+| = -\log(1-|f^+|^2)$ in $L^1(\mathbb{R})$. Integral representation (4.10) of $\mathfrak{a}^+$ gives

$$\frac{\overline{\mathfrak{a}^+(z)}}{\mathfrak{a}^+(z)} = \exp\left(-\frac{2}{\pi i}\int_{\mathbb{R}}\frac{\mathrm{Re}(x-z)}{|x-z|^2}\log|\mathfrak{a}^\pm|\,dx\right) = \exp\left(2i\cdot Q_y * \log|\mathfrak{a}^\pm|\right),$$

where $y=\mathrm{Im}\,z$ and $Q_y = \frac{1}{\pi}\frac{t}{t^2+y^2}$ is the conjugate kernel for the upper half-plane, see Section III.1 in [Gar81]. Harmonic conjugation is L^1- weakly continuous, see Section III.2 in [Gar81], hence the functions $\overline{\mathfrak{a}_n^+}/\mathfrak{a}_n^+$ converge to $\overline{\mathfrak{a}^+}/\mathfrak{a}^+$ in Lebesgue measure on $\mathbb{R}$ as $n\to\infty$. From here and (4.15) we see that $\mathbf{r}_{q_n}\to\mathbf{r}_q$ in measure on $\mathbb{R}$. By Proposition 4.10 we also have

$$\lim_{n\to\infty}E(\mathbf{r}_{q_n}) = \lim_{n\to\infty}\pi\int_{\mathbb{R}}|q_n|^2\,dx = \pi\int_{\mathbb{R}}|q|^2\,dx = E(\mathbf{r}_q).$$

The convergence in X now follows from Lemma 4.4. ☐

Next proposition is analogous to the noninjectivity of the scattering transform for Jacobi matrices discovered by Volberg and Yuditskii in [VY02].

PROPOSITION 4.12 (Example 6.1 in [BD24]). *There exist potentials $q,\tilde{q}\in L^2(\mathbb{R})$ such that $q\neq\tilde{q}$ in $L^2(\mathbb{R})$ with $\mathbf{r}_q=\mathbf{r}_{\tilde{q}}$ a.e. in $\mathbb{R}$.*

LEMMA 4.13. *Let $q\in L^2(\mathbb{R}_+)$ be supported on $(-\infty,0]$. Then $\mathbf{r}_q=-f^-$. Moreover, $\mathbf{r}_q=f_{q^\#}$, where $q^\#\in L^2(\mathbb{R}_+)$ is such that $q^\#(x)=-q^-(x)=q(-x)$ for $x\geqslant 0$.*

Proof. The potential q is supported on $(-\infty,0]$ hence $q^+=0$ and

$$f^+=0,\quad \mathfrak{a}^+=1,\quad \mathfrak{b}^+=0,\qquad a=\mathfrak{a}^-,\quad b=-\mathfrak{b}^-,\qquad \mathbf{r}_q=\frac{b}{a}=\frac{-\mathfrak{b}^-}{\mathfrak{a}^-}=-f^-.$$

We also have $f_{q^\#}=-f_{q^-}$, see Sect. 7 in [Den06]. The equality $\mathbf{r}_q=f_{q^\#}$ follows. □

PROPOSITION 4.14. *There are potentials q_n, $\tilde{q}_n$ in the unit ball of $L^2(\mathbb{R}_+)$ such that (1.7) holds. In other words, the inverse homeomorphism $\mathcal{F}^{-1}: S_2(\mathbb{C}_+)\to L^2(\mathbb{R}_+)$ in Theorem 1.1 is not uniformly continuous on bounded subsets of $S_2(\mathbb{C}_+)$.*

Proof. Let q and $\tilde{q}$ be the potentials from Proposition 4.12, denote $\mathbf{r}=\mathbf{r}_q=\mathbf{r}_{\tilde{q}}$. For any $\delta>0$, considerations similar to those in Example 6.1 in [BD24] allow to assume without loss of generality that $E(\mathbf{r})\leqslant\delta$. In particular, we can assume without loss of generality that potentials q, $\tilde{q}$ lie in the unit ball of $L^2(\mathbb{R}_+)$. For $N\in\mathbb{R}$, let

$$q_N(x)=\begin{cases}q(x), & x\leqslant N,\\ 0, & x>N,\end{cases}\qquad q_{N,s}(x)=q_N(x+N),$$

$$\tilde{q}_N(x)=\begin{cases}\tilde{q}(x), & x\leqslant N,\\ 0, & x>N,\end{cases}\qquad \tilde{q}_{N,s}(x)=\tilde{q}_N(x+N).$$

We have $q_N\to q$ and $\tilde{q}_N\to\tilde{q}$ in $L^2(\mathbb{R})$ hence for large N

$$\|q_{N,s}-\tilde{q}_{N,s}\|_{L^2(\mathbb{R})}=\|q_N-\tilde{q}_N\|_{L^2(\mathbb{R})}\geqslant\|q-\tilde{q}\|_{L^2(\mathbb{R})}/2. \tag{4.16}$$

By Proposition 4.11, we have $\mathbf{r}_{q_N}\to\mathbf{r}$ and $\mathbf{r}_{\tilde{q}_N}\to\mathbf{r}$ in X as $n\to\infty$. Therefore, for every $\varepsilon>0$ we can choose large N such that

$$\rho_X(\mathbf{r}_{q_N},\mathbf{r})\leqslant\varepsilon/2,\quad \rho_X(\mathbf{r},\mathbf{r}_{\tilde{q}_N})\leqslant\varepsilon/2,\qquad \rho_X(\mathbf{r}_{q_N},\mathbf{r}_{\tilde{q}_N})\leqslant\rho_X(\mathbf{r}_{q_N},\mathbf{r})+\rho_X(\mathbf{r},\mathbf{r}_{\tilde{q}_N})\leqslant\varepsilon.$$

Potentials $q_{N,s}$ and $\tilde{q}_{N,s}$ are the N-shifts of q_N and $\tilde{q}_N$ respectively, hence Lemma 4.9 applies. It gives

$$\mathbf{r}_{q_{N,s}}(x)=e^{iNx}\mathbf{r}_{q_N}(x),\qquad \mathbf{r}_{\tilde{q}_{N,s}}(x)=e^{iNx}\mathbf{r}_{\tilde{q}_N}(x),$$
$$\rho_X(\mathbf{r}_{q_{N,s}},\mathbf{r}_{\tilde{q}_{N,s}})=\rho_X(\mathbf{r}_{q_N},\mathbf{r}_{\tilde{q}_N})\leqslant\varepsilon.$$

Both $q_{N,s}$ and $\tilde{q}_{N,s}$ are supported on $(-\infty,0]$ hence $\mathbf{r}_{q_{N,s}}=f_{q^\#_{N,s}}$ and $\mathbf{r}_{\tilde{q}_{N,s}}=f_{\tilde{q}^\#_{N,s}}$ by Lemma 4.13. Since $\varepsilon>0$ is arbitrary, and

$$\|q^\#_{N,s}-\tilde{q}^\#_{N,s}\|_{L^2(\mathbb{R}_+)}=\|q_N-\tilde{q}_N\|_{L^2(\mathbb{R})}\geqslant\|q-\tilde{q}\|_{L^2(\mathbb{R})}/2,$$
$$\rho_{S_2}(f_{q^\#_{N,s}},f_{\tilde{q}^\#_{N,s}})=\rho_X(\mathbf{r}_{q_{N,s}},\mathbf{r}_{\tilde{q}_{N,s}})\leqslant\varepsilon,$$

this concludes the proof. □

Proof of Theorem 1.2. The theorem is a combination of Proposition 4.7 and Proposition 4.14. □

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References

[A+88] Albeverio, S., Gesztesy, F., Høegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics. Texts and Monographs in Physics. Springer, New York (1988).

[A+05] Albeverio, S., Gesztesy, F., Høegh-Krohn, R., Holden, H.: Solvable Models in Quantum Mechanics, 2nd edn. AMS Chelsea Publishing, Providence (2005). With an appendix by P. Exner.

[Bax63] Baxter, G.: A norm inequality for a "finite-section" Wiener-Hopf equation. *Ill. J. Math.* **7**(1), 97–103 (1963).

[BM12] Belishev, M.I., Mikhaylov, V.S.: Unified approach to classical equations of inverse problem theory. *J. Inverse Ill-Posed Probl.* **20**(4), 461–488 (2012).

[BD20] Bessonov, R., Denisov, S.: A spectral Szegő theorem on the real line. *Adv. Math.* **359**, 106851 (2020).

[BD21] Bessonov, R., Denisov, S.: De Branges canonical systems with finite logarithmic integral. *Anal. PDE* **14**(5), 1509–1556 (2021).

[BG24] Bessonov, R., Gubkin, P.: Stability of Schur's iterates and fast solution of the discrete integrable NLS. J. Spectr. Theory (2024). Published online first.

[Bes20] Bessonov, R.V.: Szegő condition and scattering for one-dimensional Dirac operators. *Constr. Approx.* **51**(2), 273–302 (2020).

[BD24] Bessonov, R.V., Denisov, S.A.: Sobolev norms of L^2-solutions to the nonlinear Schrödinger equation. *Pac. J. Math.* **331**(2), 217–258 (2024).

[CMP13] Carlone, R., Malamud, M., Posilicano, A.: On the spectral theory of Gesztesy–Seba realizations of 1-D Dirac operators with point interactions on a discrete set. *J. Differ. Equ.* **254**(9), 3835–3902 (2013).

[CKK04] Chelkak, D., Kargaev, P., Korotyaev, E.: Inverse problem for harmonic oscillator perturbed by potential, characterization. *Commun. Math. Phys.* **249**(1), 133–196 (2004).

[DEY21] Damanik, D., Eichinger, B., Yuditskii, P.: Szegő's theorem for canonical systems: the Arov gauge and a sum rule. *J. Spectr. Theory* **11**(3), 1255–1277 (2021).

[DKS10] Damanik, D., Killip, R., Simon, B.: Perturbations of orthogonal polynomials with periodic recursion coefficients. *Ann. Math. (2)* **171**(3), 1931–2010 (2010).

[Bra68] de Branges, L.: Hilbert Spaces of Entire Functions. Prentice-Hall, Englewood Cliffs (1968).

[Den06] Denisov, S.A.: Continuous analogs of polynomials orthogonal on the unit circle and Kreĭn systems. *Int. Math. Res. Surv.* **2006**, 54517 (2006).

[DM76] Dym, H., McKean, H.P.: Gaussian Processes, Function Theory, and the Inverse Spectral Problem. Probability and Mathematical Statistics, vol. 31. Academic Press, New York (1976).

[EKT18] Eckhardt, J., Kostenko, A., Teschl, G.: Spectral asymptotics for canonical systems. *J. Reine Angew. Math.* **736**, 285–315 (2018).

[ELS211] Eichinger, B., Lukic, M., Simanek, B.: An approach to universality using Weyl m-functions (2021a). arXiv preprint. arXiv:2108.01629.

[ELS212] Eichinger, B., Lukić, M., Simanek, B.: An approach to universality using Weyl m-functions. Ann. Math. (2021b). arXiv preprint, to appear in. arXiv:2108.01629.

[FT07] Faddeev, L., Takhtajan, L.: Hamiltonian Methods in the Theory of Solitons. Classics in Mathematics. Springer, Berlin (2007). English edition. Translated from the 1986 Russian original by Alexey G. Reyman.

[Gar81] Garnett, J.B.: Bounded Analytic Functions. Pure and Applied Mathematics, vol. 96. Academic Press, New York (1981).

[GM06] Geronimo, J.S., Martínez-Finkelshtein, A.: On extensions of a theorem of Baxter. *J. Approx. Theory* **139**(1–2), 214–222 (2006).

[Ger44] Geronimus, J.: On polynomials orthogonal on the circle, on trigonometric moment-problem and on allied Carathéodory and Schur functions. *Rec. Math. [Mat. Sb.] N.S.* **15**(57), 99–130 (1944).

[GS87] Gesztesy, F., Šeba, P.: New analytically solvable models of relativistic point interactions, 13, 345–358 (1987).

[GI71] Golinskii, B.L., Ibragimov, I.A.: A limit theorem of G. Szegő. *Izv. Akad. Nauk SSSR, Ser. Mat.* **35**, 408–427 (1971).

[Gol97] Golinskii, L.: On Schur functions and Szegő orthogonal polynomials. In: Topics in Interpolation Theory, Leipzig, 1994. Oper. Theory Adv. Appl., vol. 95, pp. 195–203. Birkhäuser, Basel (1997).

[HSW00] Hassi, S., De Snoo, H., Winkler, H.: Boundary-value problems for two-dimensional canonical systems. *Integral Equ. Oper. Theory* **36**(4), 445–479 (2000).

[HS69] Hewitt, E., Stromberg, K.: Real and Abstract Analysis. A Modern Treatment of the Theory of Functions of a Real Variable. Springer, New York-Berlin (1969).

[HK10] Horváth, M., Kiss, M.: Stability of direct and inverse eigenvalue problems for Schrödinger operators on finite intervals. *Int. Math. Res. Not.* **11**, 2022–2063 (2010).

[Hry11] Hryniv, R.O.: Analyticity and uniform stability in the inverse singular Sturm-Liouville spectral problem. *Inverse Probl.* **27**(6), 065011 (2011).

[Hry11] Hryniv, R.O.: Analyticity and uniform stability in the inverse spectral problem for Dirac operators. *J. Math. Phys.* **52**(6), 063513 (2011).

[Hug98] Hughes, R.J.: Relativistic Kronig-Penney-type Hamiltonians. *Integral Equ. Oper. Theory* **31**(4), 436–448 (1998).

[KK68] Kac, I.S., Krein, M.G.: On the spectral functions of the string. *Supplement II to the Russian edition of F.V. Atkinson, Discrete and continuous boundary problems*, 1968. English translation: *Am. Math. Soc. Transl. (2)* **103**, 19–102 (1974).

[KS03] Killip, R., Simon, B.: Sum rules for Jacobi matrices and their applications to spectral theory. *Ann. Math. (2)* **158**(1), 253–321 (2003).

[KS09] Killip, R., Simon, B.: Sum rules and spectral measures of Schrödinger operators with L^2 potentials. *Ann. Math. (2)* **170**(2), 739–782 (2009).

[KC09] Korotyaev, E.L., Chelkak, D.S.: The inverse Sturm-Liouville problem with mixed boundary conditions. *Algebra Anal.* **21**(5), 114–137 (2009).

[KM13] Kostenko, A., Malamud, M.: 1-D Schrödinger operators with local point interactions: a review. In: Spectral Analysis, Differential Equations and Mathematical Physics: A Festschrift in Honor of Fritz Gesztesy's 60th Birthday. Proc. Sympos. Pure Math., vol. 87, pp. 235–262. Am. Math. Soc., Providence (2013).

[D+31] Kronig, R. De L., Penney, W.G., Fowler, R.H.: Quantum mechanics of electrons in crystal lattices. *Proc. R. Soc. Lond. Ser. A, Contain. Pap. Math. Phys. Character* **130**(814), 499–513 (1931).

[Lev91] Levitan, B.M.: Sturm-Liouville and Dirac Operators (1991).

[LS91] Levitan, B.M., Sargsjan, I.S.: Sturm-Liouville and Dirac Operators. Mathematics and Its Applications (Soviet Series), vol. 59. Kluwer Academic, Dordrecht (1991). Translated from the Russian.

[LS14] Lotoreichik, V., Simonov, S.: Spectral analysis of the half-line Kronig-Penney model with Wigner–Von Neumann perturbations. *Rep. Math. Phys.* **74**(1), 45–72 (2014).

[LM69] Lundina, D.S., Marchenko, V.A.: A refinement of the inequalities which characterize the stability of the inverse problem of scattering theory. *Mat. Sb. (N.S.)* **78**(120), 475–484 (1969).

[MP23] Makarov, N., Poltoratski, A.: Etudes for the inverse spectral problem. *J. Lond. Math. Soc. (2)* **108**(3), 916–977 (2023).

[Mar68] Marchenko, V.: Stability of the inverse problem in scattering theory. *Math. USSR Sb.* **6**(2), 125 (1968).

[Mar06] Marchenko, V.: The generalized shift, transformation operators, and inverse problems. In: Mathematical Events of the Twentieth Century, pp. 145–162. Springer, Berlin (2006).

[MM70] Marchenko, V.A., Maslov, K.V.: Stability of the problem of the reconstruction of the Sturm-Liouville operator in terms of the spectral function. *Mat. Sb. (N.S.)* **81**(123), 525–551 (1970).

[NT89] Nevai, P., Totik, V.: Orthogonal polynomials and their zeros. *Acta Sci. Math.* **53**(1–2), 99–104 (1989).

[Per88] Persson, J.: Fundamental theorems for linear measure differential equations. Mathematica scandinavica, 19–43 (1988).

[Pol24] Poltoratski, A.: Pointwise convergence of the non-linear Fourier transform. *Ann. Math. (2)* **199**(2), 741–793 (2024).

[PR23] Poltoratski, A., Zhang, A.R.: Periodic approximations in inverse spectral problems for canonical Hamiltonian systems. *J. Funct. Anal.* **284**(11), 109883 (2023).

[Rak77] Rakhmanov, E.A.: The asymptotic behavior of the ratio of orthogonal polynomials. *Mat. Sb. (N.S.)*, 103 (145)(2):237–252, 319 (1977).

[Rem18] Remling, Ch.: Spectral Theory of Canonical Systems. De Gruyter Studies in Mathematics, vol. 70. de Gruyter, Berlin (2018).

[Rom14] Romanov, R.: Canonical systems and de Branges spaces (2014). arXiv:1408.6022. Preprint.

[SS10] Savchuk, A.M., Shkalikov, A.A.: Inverse problems for the Sturm-Liouville operator with potentials in Sobolev spaces: uniform stability. *Funkc. Anal. Prilozh.* **44**(4), 34–53 (2010).

[Sch17] Schur, J.: Über potenzreihen, die im innern des einheitskreises beschränkt sind. *J. Reine Angew. Math.* **147**, 205–232 (1917).

[Sim05] Simon, B.: Orthogonal Polynomials on the Unit Circle. Part 1: Classical Theory. American Mathematical Society Colloquium Publications., vol. 1. Am. Math. Soc., Providence (2005).

[Sim07] Simon, B.: Orthogonal polynomials with exponentially decaying recursion coefficients. CRM Proceedings and lecture notes **42** (2007).

[Sim11] Simon, B.: Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials. M. B. Porter Lectures. Princeton University Press, Princeton (2011).

[SW99] Sylvester, J., Winebrenner, D.P.: Linear and nonlinear inverse scattering. *SIAM J. Appl. Math.* **59**(2), 669–699 (1999).

[Sze75] Szegő, G.: Orthogonal Polynomials, 4th edn. Amer. Math. Soc. Colloquium Publications., vol. XXIII. Am. Math. Soc., Providence (1975).

[TT03] Tao, T., Thiele, Ch.: Nonlinear Fourier analysis (2003, 2012). https://arxiv.org/abs/1201.5129. Preprint arXiv, 2003, 2012.

[Tha92] Thaller, B.: The Dirac Equation. Texts and Monographs in Physics. Springer, Berlin (1992).

[VY02] Volberg, A., Yuditskii, P.: On the inverse scattering problem for Jacobi matrices with the spectrum on an interval, a finite system of intervals or a Cantor set of positive length. *Commun. Math. Phys.* **226**(3), 567–605 (2002).

[Wil91] Williams, D.: Probability with Martingales. Cambridge Mathematical Textbooks. Cambridge University Press, Cambridge (1991).

[Xu21] Xu, X.-Ch.: Stability of direct and inverse scattering problems for the self-adjoint Schrödinger operators on the half-line. *J. Math. Anal. Appl.* **501**(2), Paper No. 125217 (2021).

[XB22] Xu, X.-Ch., Bondarenko, N.P.: Stability of the inverse scattering problem for the self-adjoint matrix Schrödinger operator on the half line. *Stud. Appl. Math.* **149**(3), 815–838 (2022).

[Yud18] Yuditskii, P.: Killip-Simon problem and Jacobi flow on GMP matrices. *Adv. Math.* **323**, 811–865 (2018).

[Zen23] Zeng, J.: Spectral theory of Dirac operators with measures. PhD thesis, University of Oklahoma, Norman (2023).

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