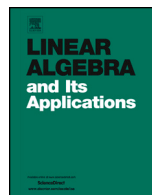




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Positive commutators of positive square-zero operators

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ABSTRACT

In this paper we first consider the question which nonnegative matrices are commutators of nonnegative square-zero matrices. Then, we treat infinite-dimensional analogues of these results for operators on the Banach lattices $L^p[0, 1]$ and ℓ^p ($1 \leq p < \infty$). In the last setting we need to extend the notion of the nonnegative rank of a nonnegative matrix.

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1. Introduction

Positive commutators of positive operators on Banach lattices have been the subject of extensive research; see e.g. [3,9,7,13–15,8,10–12]. A systematic investigation of their properties began with [3], where the authors studied the spectral properties of the posi-

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tive commutator $[A, B] := AB - BA$ formed by positive compact operators A and B . In particular, they showed that if A and B are nonnegative matrices such that the commutator $C = [A, B]$ is nonnegative as well, then, up to similarity with a permutation matrix, C is a strictly upper triangular matrix, and so it is nilpotent. Recently, the authors of [17] have studied commutators of square-zero operators on Hilbert spaces. In particular, they completely characterized commutators of square-zero matrices by proving that commutators are, up to similarity, precisely direct sums $A \oplus (-A) \oplus N^2$, where A is an invertible matrix and N is a nilpotent matrix.

In this paper we first study the question which nonnegative nilpotent matrices are commutators of nonnegative square-zero matrices. This is contained in Section 2. Infinite-dimensional analogues of these results for operators on the Banach lattices $L^p[0, 1]$ and ℓ^p ($1 \leq p < \infty$) are considered in Section 3, where the nonnegative rank of positive operators on vector lattices is also introduced.

Since some of our results hold in general setting of vector lattices, we recall some basic definitions and properties of vector lattices and operators on them. For the terminology and details not explained here we refer the reader to [16] or [2] or [1]. Let X be a vector lattice with the positive cone X^+ . A subspace A of X is called an *ideal* whenever $|x| \leq |y|$ and $y \in A$ imply $x \in A$. In the finite-dimensional case $X = \mathbb{R}^n$ ideals are also called *standard subspaces*, since they are precisely linear spans of some standard unit vectors. An order dense ideal of X is said to be a *band*. In the case $X = L^p(\mu)$ ($1 \leq p < \infty$) bands are precisely (norm) closed ideals. The band

$$S^d := \{x \in X : |x| \wedge |y| = 0 \text{ for all } y \in S\}$$

is called the *disjoint complement* of a set S of X .

A linear operator T on X is *positive* if T leaves the positive cone X^+ invariant, that is, $T(X^+) \subseteq X^+$. In the finite-dimensional case $X = \mathbb{R}^n$ these are precisely nonnegative $n \times n$ matrices. Let T be a positive operator on a vector lattice X . The *null ideal* $\mathcal{N}(T)$ is the ideal in X defined by

$$\mathcal{N}(T) = \{x \in X : T|x| = 0\}.$$

The *nonnegative rank* of a nonnegative $m \times n$ matrix A is equal to the smallest nonnegative integer k for which there exist a nonnegative $m \times k$ matrix L and a nonnegative $k \times n$ matrix R such that $A = LR$. We denote it by $\text{rank}^+(A)$. To obtain the usual rank (denoted by $\text{rank}(A)$), we drop the condition that L and R must be nonnegative. It is easy to see that $\text{rank}^+(A)$ is the smallest nonnegative integer k such that there exist nonnegative vectors u_1, \dots, u_k and v_1, \dots, v_k such that $A = \sum_{i=1}^k u_i v_i^T$.

Finally, we recall Bohnenblust’s result. By [4, Theorem 7.1], every separable infinite-dimensional Banach lattice $L^p(\mu)$ ($1 \leq p < \infty$) is isometric and order isomorphic to one of the following Banach lattices: ℓ^p , $L^p[0, 1]$, $\ell^p \oplus L^p[0, 1]$ or $\ell_n^p \oplus L^p[0, 1]$.

2. The finite-dimensional case

In this section we study the question which nonnegative matrices are commutators of nonnegative square-zero matrices. We begin with the special case.

Theorem 2.1. *For a nonnegative $n \times n$ matrix T the following assertions are equivalent.*

- (i) *There exist nonnegative $n \times n$ matrices M and N such that $T = MN$ and $M^2 = N^2 = NM = 0$.*
- (ii) *There exists a nonnegative $n \times n$ matrix U such that $T = U^2$ and $U^3 = 0$.*
- (iii) *There exists a decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$ on standard subspaces with respect to which the operator T is of the form*

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive operator $T_{13}: L_3 \rightarrow L_1$ with $\text{rank}^+(T_{13}) \leq \dim(L_2)$.

Proof. (i) \Rightarrow (ii) Define $U := M+N$. Since $M^2 = N^2 = NM = 0$, we have $U^2 = MN = T$ and $U^3 = MN(M+N) = 0$.

(ii) \Rightarrow (iii) Define standard subspaces $L_1 = \mathcal{N}(U)$, $L_2 = \mathcal{N}(U)^d \cap \mathcal{N}(U^2)$ and $L_3 = \mathcal{N}(U^2)^d$. With respect to the decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$ the positive operators U and U^2 are of the form

$$U = \begin{pmatrix} 0 & U_{12} & U_{13} \\ 0 & 0 & U_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = U^2 = \begin{pmatrix} 0 & 0 & U_{12}U_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $T_{13} = U_{12}U_{23}$, we have $\text{rank}^+(T_{13}) \leq \dim(L_2)$.

(iii) \Rightarrow (i) Denote $d_i = \dim(L_i)$ for $i = 1, 2, 3$. Since $\text{rank}^+(T_{13}) \leq d_2$, there exist a nonnegative $d_1 \times d_2$ matrix L and a nonnegative $d_2 \times d_3$ matrix R such that $T_{13} = LR$. Define the nonnegative $n \times n$ matrices by

$$M = \begin{pmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}.$$

Then it is easy to see that $T = MN$ and $M^2 = N^2 = NM = 0$. \square

We now find necessary conditions for a nonnegative matrix to be a commutator of nonnegative square-zero matrices.

Theorem 2.2. *Let T be a nonnegative $n \times n$ matrix such that $T = MN - NM$ for some nonnegative $n \times n$ matrices M and N with $M^2 = N^2 = 0$. Then $MT = TM = NT =$*

$TN = 0$ and there exists a decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$ on standard subspaces with respect to which the operator T is of the form

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive operator $T_{13}: L_3 \rightarrow L_1$ with $\text{rank}(T_{13}) \leq \dim(L_2)$.

Proof. Since $M^2 = 0$, we have $0 \leq MT = -MNM \leq 0$, and so $MT = 0 = MNM$. Similarly, since $N^2 = 0$, we have $0 \leq TN = -NMN \leq 0$, and so $TN = 0 = NMN$. It follows that $TM = MNM = 0$ and $NT = NMN = 0$.

Denote $S = M + N$. Since $S^2 = MN + NM$, we have $S^3 = (MN + NM)(M + N) = 0$. Now, define standard subspaces $L_1 = \mathcal{N}(S)$, $L_2 = \mathcal{N}(S)^d \cap \mathcal{N}(S^2)$ and $L_3 = \mathcal{N}(S^2)^d$. With respect to the decomposition $\mathbb{R}^n = L_1 \oplus L_2 \oplus L_3$ the positive operators S and S^2 are of the form

$$S = \begin{pmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad S^2 = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $0 \leq T = MN - NM \leq MN + NM = S^2$, the operator T has the same pattern as the operator S^2 , so that

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive operator $T_{13}: L_3 \rightarrow L_1$. Since $0 \leq M, N \leq S$, the operators M and N have the following forms

$$M = \begin{pmatrix} 0 & M_{12} & M_{13} \\ 0 & 0 & M_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & N_{12} & N_{13} \\ 0 & 0 & N_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

Since

$$T = MN - NM = \begin{pmatrix} 0 & 0 & M_{12}N_{23} - N_{12}M_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

it follows that $T_{13} = M_{12}N_{23} - N_{12}M_{23}$. Since $M^2 = N^2 = 0$, we have $M_{12}M_{23} = 0$ and $N_{12}N_{23} = 0$, and so we can write $T_{13} = (M_{12} - N_{12})(M_{23} + N_{23})$. This implies that $\text{rank}(T_{13}) \leq \dim(L_2)$ as desired. \square

In view of Theorem 2.1, we would expect that the converse implication in Theorem 2.2 also holds. However, we will show in Theorem 2.4 that this is not the case.

Define the 4×4 matrix

$$T_{13} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}. \quad (1)$$

Note that $\text{rank}(T_{13}) = 3$ and $\text{rank}^+(T_{13}) = 4$; see e.g. [5]. In the proof of Theorem 2.4 we will need the following lemma.

Lemma 2.3. *Let $u = (u_1 \ u_2 \ u_3 \ u_4)^T$ and $v = (v_1 \ v_2 \ v_3 \ v_4)^T$ be nonnegative (column) vectors. Then*

$$\text{rank}(T_{13} + uv^T) \geq 3.$$

Proof. We distinguish 2 cases.

Case 1: $u_1 \leq u_3$. We consider the submatrix obtained from $T_{13} + uv^T$ by deleting the first column and the last row, that is the matrix

$$S := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} (v_2 \ v_3 \ v_4).$$

Denoting

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \quad \text{and} \quad \tilde{v} = \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix},$$

by the matrix determinant lemma (see e.g. [18] or [6, Lemma 1.1]), the determinant of S is equal to

$$\begin{aligned} (1 + \tilde{v}^T A^{-1} \tilde{u}) \det(A) &= 1 + (v_2 \ v_3 \ v_4) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \\ &= 1 + v_2 u_1 + v_3 u_2 + v_4 (u_3 - u_1) \geq 1. \end{aligned}$$

This proves that the rank of $T_{13} + uv^T$ is at least 3.

Case 2: $u_3 \leq u_1$. We now consider the submatrix obtained from $T_{13} + uv^T$ by deleting the second row and the last column, that is the matrix

$$R := \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_3 \\ u_4 \end{pmatrix} (v_1 \ v_2 \ v_3).$$

By the matrix determinant lemma again, the determinant of R is equal to

$$1 + (v_1 \quad v_2 \quad v_3) \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_3 \\ u_4 \end{pmatrix} = 1 + v_1(u_1 - u_3) + v_2u_3 + v_3u_4 \geq 1,$$

showing that the rank of $T_{13} + uv^T$ is at least 3. \square

We now show that the necessary conditions of Theorem 2.2 are not sufficient.

Theorem 2.4. *Define the matrix*

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathbb{R}^{11} = \mathbb{R}^4 \oplus \mathbb{R}^3 \oplus \mathbb{R}^4$, where the matrix T_{13} is given by (1). Then T cannot be written as a commutator $MN - NM$, where M and N are nonnegative 11×11 matrices with $M^2 = N^2 = 0$.

Proof. Observe that $\text{rank}(T_{13}) = 3$, and so for T the necessary condition of Theorem 2.2 is satisfied, while $\text{rank}^+(T_{13}) = 4$, so that the assertion (iii) of Theorem 2.1 is not true. Assume to the contrary that $T = MN - NM$, where M and N are nonnegative 11×11 matrices with $M^2 = N^2 = 0$. Since $MT = TM = NT = TN = 0$ by Theorem 2.2 and T_{13} has no zero rows and columns, M and N must have the forms

$$M = \begin{pmatrix} 0 & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & N_{12} & N_{13} \\ 0 & N_{22} & N_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $\mathbb{R}^{11} = \mathbb{R}^4 \oplus \mathbb{R}^3 \oplus \mathbb{R}^4$. Since $T = MN - NM$, we have $T_{13} = M_{12}N_{23} - N_{12}M_{23}$ or $T_{13} + N_{12}M_{23} = M_{12}N_{23}$. Since $T_{13} \neq 0$, it follows that $M_{12}N_{23} \neq 0$, so that $M_{12} \neq 0$. Since $\text{rank}^+(T_{13}) = 4$, it must hold that $N_{12}M_{23} \neq 0$, so that $M_{23} \neq 0$. From $M^2 = 0$ it follows that $M_{12}M_{23} = 0$, and so $1 \leq \text{rank}(M_{12}) \leq 2$ and $1 \leq \text{rank}(M_{23}) \leq 2$. We distinguish 2 cases.

Case 1: $\text{rank}(N_{12}M_{23}) = 1$. Then $N_{12}M_{23} = uv^T$ for some nonnegative vectors u and v . It follows that $\text{rank}(T_{13} + uv^T) = \text{rank}(M_{12}N_{23}) \leq \text{rank}(M_{12}) \leq 2$. This is a contradiction with the conclusion of Lemma 2.3.

Case 2: $\text{rank}(N_{12}M_{23}) = 2$. Then $\text{rank}^+(N_{12}M_{23}) = 2$ by [5, Theorem 4.1], and so there exist nonnegative vectors u, v, w and z such that $N_{12}M_{23} = uv^T + wz^T$. Now, $\text{rank}(M_{23}) = 2$, and so M_{23} has at least 2 non-zero rows. Since $M_{12}M_{23} = 0$, M_{12} is a 4×3 matrix and M_{23} is a 3×4 matrix, we conclude that M_{12} has only one non-zero column, so that $\text{rank}(M_{12}) = 1$. It follows that

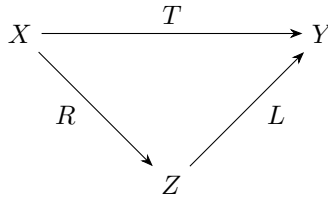
$$\text{rank}(T_{13} + uv^T) = \text{rank}(M_{12}N_{23} - wz^T) \leq \text{rank}(M_{12}N_{23}) + 1 \leq \text{rank}(M_{12}) + 1 = 2.$$

This is again a contradiction with Lemma 2.3. \square

3. The infinite-dimensional case

In this section we consider infinite-dimensional analogues of Theorem 2.1. We first treat positive operators on the atomic Banach lattice ℓ^p ($1 \leq p < \infty$). In this setting we need to extend the notion of a nonnegative rank of a nonnegative matrix.

A positive operator $T: X \rightarrow Y$ between Archimedean vector lattices is said to have a *finite nonnegative rank* if there exists a finite-dimensional Archimedean vector lattice Z and positive operators $L: Z \rightarrow Y$ and $R: X \rightarrow Z$ such that $T = LR$.



Since every finite-dimensional Archimedean vector lattice is lattice isomorphic to \mathbb{R}^k (ordered coordinatewise) for some $k \in \mathbb{N}_0$, in the definition of a finite nonnegative rank we can replace Z with $\mathbb{R}^{\dim Z}$. Therefore, T has a finite nonnegative rank whenever it factors through \mathbb{R}^k (ordered coordinatewise) with positive factors. The *nonnegative rank* $\text{rank}^+(T)$ of T is defined as the minimal $k \in \mathbb{N}_0$ above such that T factors through \mathbb{R}^k with positive factors. Therefore, this definition, in the case when $X = \mathbb{R}^n$ and $Y = \mathbb{R}^m$ coincides with the definition of a nonnegative rank of nonnegative $m \times n$ matrices. We define $\text{rank}^+(T) = \infty$ if T does not admit a factorization through any finite-dimensional Archimedean vector lattice via positive operators.

Lemma 3.1. *Let L_1, L_2 and L_3 be bands in a normed lattice X such that $X = L_1 \oplus L_2 \oplus L_3$. Let $T: X \rightarrow X$ be a positive operator of the form*

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with respect to the decomposition $X = L_1 \oplus L_2 \oplus L_3$. If T_{13} has a finite nonnegative rank and $\text{rank}^+(T_{13}) \leq \dim(L_2)$, then there exist positive operators $M, N: X \rightarrow X$ such that $T = MN$ and $M^2 = N^2 = NM = 0$.

Proof. Since the nonnegative rank $k := \text{rank}^+(T_{13})$ is finite, there exist positive operators $R: L_3 \rightarrow \mathbb{R}^k$ and $L: \mathbb{R}^k \rightarrow L_1$ such that $T_{13} = LR$. Clearly, we may assume that $k \geq 1$.

We claim that L_2 contains k pairwise disjoint non-zero positive vectors. If $\dim L_2 < \infty$, then for some $n \in \mathbb{N}_0$ the vector lattice L_2 is lattice isomorphic to \mathbb{R}^n ordered coordinatewise. Since $k \leq n$, L_2 clearly contains k pairwise disjoint non-zero positive vectors. If $\dim L_2 = \infty$, we use [16, Theorem 26.10] which states that any infinite-dimensional

Archimedean vector lattice contains sets of pairwise disjoint non-zero positive vectors of any finite cardinality.

Let us now pick any set $\{x_1, \dots, x_k\}$ in L_2 of pairwise disjoint non-zero positive vectors. By e_i we denote the i -th standard basis vector of \mathbb{R}^k . The mapping $\iota: e_i \mapsto x_i$ ($1 \leq i \leq k$) can be extended to an injective linear operator (denoted again by ι) $\iota: \mathbb{R}^k \rightarrow L_2$. Since vectors x_1, \dots, x_k are pairwise disjoint, ι is an injective lattice homomorphism.

Now we are going to construct a positive operator $P: L_2 \rightarrow \mathbb{R}^k$ such that $P\iota$ is the identity operator on \mathbb{R}^k as follows. For each $1 \leq i \leq k$, by [19, Theorem 39.3] there exists a positive bounded linear functional $\phi_i: L_2 \rightarrow \mathbb{R}$ such that $\phi_i(x_i) = 1$. For each $1 \leq i \leq k$ we denote by I_i the principal ideal in L_2 generated by x_i . Applying [2, Theorem 1.28] for the ideal I_i , we obtain a positive linear functional $0 \leq \varphi_i \leq \phi_i$ on L_2 such that $\varphi_i(x_j) = \delta_{ij}$ for $1 \leq i, j \leq k$. Since ϕ_i is bounded, by [19, Theorem 25.8] the functional φ_i is bounded as well. If we define $P: L_2 \rightarrow \mathbb{R}^k$ as $\sum_{i=1}^k e_i \otimes \varphi_i$, then P is a positive operator such that $P\iota$ is the identity operator on \mathbb{R}^k .

With respect to the decomposition $X = L_1 \oplus L_2 \oplus L_3$ we define positive operators

$$M = \begin{pmatrix} 0 & LP & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \iota R \\ 0 & 0 & 0 \end{pmatrix}$$

that satisfy $M^2 = N^2 = NM = 0$ and

$$MN = \begin{pmatrix} 0 & 0 & LP\iota R \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & LR \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = T,$$

which concludes the proof. \square

The following theorem is an infinite-dimensional version of Theorem 2.1.

Theorem 3.2. *For a positive operator $T: \ell^p \rightarrow \ell^p$ ($1 \leq p < \infty$) the following statements are equivalent.*

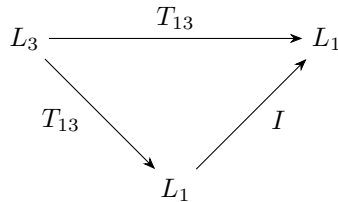
- (i) *There exist positive operators $M, N: \ell^p \rightarrow \ell^p$ such that $T = MN$ and $M^2 = N^2 = NM = 0$.*
- (ii) *There exists a positive operator $U: \ell^p \rightarrow \ell^p$ such that $T = U^2$ and $U^3 = 0$.*
- (iii) *There exists a nontrivial band decomposition $\ell^p = L_1 \oplus L_2 \oplus L_3$ with respect to which the operator T is of the form*

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive operator $T_{13}: L_3 \rightarrow L_1$ with $\text{rank}^+(T_{13}) \leq \dim(L_2)$.

Proof. We may assume that $T \neq 0$. While the proof of (i) \Rightarrow (ii) can be proved in the same way as (i) \Rightarrow (ii) in Theorem 2.1, the proof of (ii) \Rightarrow (iii) requires some explanation. First, using arguments similar to those in Theorem 2.1, one can find nontrivial bands L_1, L_2 and L_3 in ℓ^p such that $\ell^p = L_1 \oplus L_2 \oplus L_3$, and positive operators $U_{12}: L_2 \rightarrow L_1$ and $U_{23}: L_3 \rightarrow L_2$ such T is of the form given in (iii) with $T_{13} = U_{12}U_{23}: L_3 \rightarrow L_1$. If $\dim L_2 = \infty$, then by definition of the nonnegative rank of a positive operator we have $\text{rank}^+(T_{13}) \leq \dim(L_2)$. On the other hand, if $\dim L_2 < \infty$, since T_{13} factors through the finite-dimensional vector lattice L_2 , we have $\text{rank}^+(T_{13}) \leq \dim(L_2)$.

To prove (iii) \Rightarrow (i), note first that we can apply Lemma 3.1 if $\text{rank}^+(T_{13}) < \infty$. Therefore, we may assume that the nonnegative rank of T_{13} is infinite. In particular, L_2 is infinite-dimensional. Since L_2 is a band in ℓ^p , it is lattice isometric to ℓ^p . Moreover, if L_1 were finite-dimensional, then the following diagram



would show that $T_{13}: L_3 \rightarrow L_1$ factors through L_1 . However, this leads to a contradiction with $\text{rank}^+(T_{13}) = \infty$, and we conclude that $\dim L_1 = \infty$. By the same reasoning as above, it follows that L_1 is also lattice isometric to ℓ^p . With respect to the band decomposition $\ell^p = L_1 \oplus L_2 \oplus L_3 \cong \ell^p \oplus \ell^p \oplus L_3$ we define positive operators

$$M = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & T_{13} \\ 0 & 0 & 0 \end{pmatrix}.$$

Now it is easy to prove that $T = MN$ and $M^2 = N^2 = NM = 0$. \square

Finally, we consider operators on the Banach lattice $L^p[0, 1]$ that has no atoms.

Theorem 3.3. *For a positive operator $T: L^p[0, 1] \rightarrow L^p[0, 1]$ ($1 \leq p < \infty$) the following statements are equivalent.*

- (i) $T = MN - NM$ for some positive operators $M, N: L^p[0, 1] \rightarrow L^p[0, 1]$ with $M^2 = N^2 = 0$.
- (ii) There exist positive operators $M, N: L^p[0, 1] \rightarrow L^p[0, 1]$ such that $T = MN$ and $M^2 = N^2 = NM = 0$.
- (iii) There exists a positive operator $U: L^p[0, 1] \rightarrow L^p[0, 1]$ such that $T = U^2$ and $U^3 = 0$.

(iv) *There exists a nontrivial band decomposition $L^p[0, 1] = B_1 \oplus B_2 \oplus B_3$ with respect to which the operator T is of the form*

$$T = \begin{pmatrix} 0 & 0 & T_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some positive operator $T_{13}: B_3 \rightarrow B_1$.

Proof. We may assume that the operator T is non-zero.

(i) \Rightarrow (iv) Let us write $T = MN - NM$ for some positive square-zero operators $M, N: L^p[0, 1] \rightarrow L^p[0, 1]$. Since $0 \leq NM \leq MN$, we have

$$0 \leq (M + N)^3 \leq M^3 + 3MN^2 + 3M^2N + N^3 = 0,$$

so that $(M + N)^3 = 0$. Note that $(M + N)^2 \neq 0$, since otherwise $0 \leq T = MN - NM \leq MN + NM = (M + N)^2 = 0$, and so $T = 0$.

Let us define the bands $B_1 = \mathcal{N}(M + N)$, $B_2 = \mathcal{N}(M + N)^d \cap \mathcal{N}((M + N)^2)$ and $B_3 = \mathcal{N}((M + N)^2)^d$. Since $(M + N)^3 = 0$, the band B_1 is non-zero. Since $(M + N)^2 \neq 0$, the band B_3 is non-zero as well. To prove that the band B_2 is also non-zero, take a non-zero nonnegative function $g \in B_3$. Then $f = (M + N)g$ is a non-zero nonnegative function in B_2 .

With respect to the decomposition $L^p[0, 1] = B_1 \oplus B_2 \oplus B_3$ the operators $M + N$ and $(M + N)^2$ are of the form

$$M + N = \begin{pmatrix} 0 & \star & \star \\ 0 & 0 & \star \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad (M + N)^2 = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $0 \leq T = MN - NM \leq MN + NM = (M + N)^2$, the operator T is also of the form

$$T = \begin{pmatrix} 0 & 0 & \star \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(iv) \Rightarrow (ii) Since $L^p[0, 1]$ is atomless and T is non-zero, the bands B_1, B_2 and B_3 are non-zero atomless separable Banach lattices. By [4, Theorem 7.1], they are all lattice isometric to $L^p[0, 1]$. Hence, without any loss of generality we may assume $B_1 = B_2 = B_3 = L^p[0, 1]$. With respect to the decomposition $L^p[0, 1] = L^p[0, 1] \oplus L^p[0, 1] \oplus L^p[0, 1]$ we define operators M and N as

$$M = \begin{pmatrix} 0 & T_{13} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix}.$$

A direct calculation shows that we have $M^2 = N^2 = NM = 0$ and $MN = T$.

Since it is clear that (ii) implies (i), the assertions (i), (ii) and (iv) are equivalent.

The implication (ii) \Rightarrow (iii) can be proved in the same way as (i) \Rightarrow (ii) in Theorem 2.1. It remains to prove (iii) \Rightarrow (iv). Define the bands $B_1 = \mathcal{N}(U)$, $B_2 = \mathcal{N}(U)^d \cap \mathcal{N}(U^2)$ and $B_3 = \mathcal{N}(U^2)^d$. Since $U^3 = 0$, the band B_1 is non-zero. Since $U^2 \neq 0$, the band B_3 is also non-zero. To prove that the band B_2 is non-zero as well, take a non-zero nonnegative function $g \in B_3$. Then $f = Ug$ is a non-zero nonnegative function in B_2 . With respect to the decomposition $\mathbb{R}^n = B_1 \oplus B_2 \oplus B_3$ the positive operators U and U^2 are of the form

$$U = \begin{pmatrix} 0 & U_{12} & U_{13} \\ 0 & 0 & U_{23} \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad T = U^2 = \begin{pmatrix} 0 & 0 & U_{12}U_{23} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and so $T_{13} = U_{12}U_{23}$. \square

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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