Positivity (2026) 30:30
https://doi.org/10.1007/s11117-026-01190-y

Positivity

RESEARCH

On positive automorphisms of algebras of operators on atomic Archimedean vector lattices

G. Cigler[1,2] **· M. Kandić**[1,2]

Received: 29 January 2026 / Accepted: 3 April 2026

Abstract

Let X be an Archimedean vector lattice. We investigate subalgebras of $\mathscr{L}(X)$ consisting of regular operators that contain all rank-one operators of the form $a \otimes \varphi_b$, where a and b are atoms of X and φ_b denotes the coordinate functional associated with b. Our main result shows that every positive automorphism of such a subalgebra contained in $\mathscr{L}(c_{00}(\Lambda))$, is necessarily spatial, meaning that it is implemented by a transformation of the form

$$T \mapsto PDTD^{-1}P^{-1},$$

where P is a permutation operator and D is a positive diagonal operator. We also use the Kakutani representation theorem to establish that every finite-dimensional vector subspace of X is order closed.

Keywords Vector lattice · Order algebra automorphism · Inner automorphism · Atom · Order continuous operator

Mathematics Subject Classification Primary 46A40 · 47B65; Secondary 16W20 · 47L05

1 Introduction

Let F be a field and let $\Phi\colon M_n(F) \to M_n(F)$ be an algebra automorphism of the algebra $M_n(F)$ of all $n \times n$ matrices over F. A classical result from linear algebra states that Φ is inner; that is, there exists an invertible matrix $A \in M_n(F)$ such that

$$\Phi(T) = ATA^{-1}$$

✉ M. Kandić
marko.kandic@fmf.uni-lj.si

G. Cigler
gregor.cigler@fmf.uni-lj.si

1 Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

2 Institute of Mathematics, Physics, and Mechanics, Jadranska 19, 1000 Ljubljana, Slovenia

for all $T \in M_n(F)$. This follows readily from the Noether–Skolem theorem (see [4, Theorem 3.14]), which asserts that every automorphism of a finite-dimensional central simple F-algebra is inner.

In 1940, Eidelheit established the Banach space analogue (see [3]), proving that for a Banach space X every algebra automorphism of the Banach algebra $\mathscr{B}(X)$ is inner. Later, Sourour extended the investigation to the Banach lattice setting (see [8, Theorem 2]), showing that if X is a Banach lattice, then every order-preserving algebra automorphism of the algebra $\mathscr{L}_r(X)$ of regular operators is inner and is implemented by a lattice isomorphism.

The present paper appears to be the first systematic study of order automorphisms of the algebra $\mathscr{L}_r(X)$ in the context of normed lattices X that are not assumed to be complete. In our main result Theorem 5.8 we assume that X is an Archimedean atomic vector lattice and we provide a description of positive algebra automorphisms for subalgebras of $\mathscr{L}_n(X)$ containing operators $a \otimes \varphi_b$ for all atoms a and b from X. In the special case where $X = c_{00}(\Lambda)$ is the vector lattice of finitely supported functions on a nonempty set Λ, we further obtain that every order-preserving algebra automorphism of $\mathscr{L}(c_{00}(\Lambda))$ or of $\mathscr{B}(c_{00}(\Lambda))$ is inner.

This paper is organized as follows. In Sect. 2 we collect the preliminary material used throughout the paper. In Sect. 3 we apply the Kakutani representation theorem to prove that every finite-dimensional vector subspace of an Archimedean vector lattice is order closed. Although we require this only in the special case of one-dimensional subspaces, the result is of independent interest, as it contributes to the broader understanding of order convergence and order-closed sets in vector lattices. This result will be used in Sect. 4, where we study algebras of regular operators on Archimedean vector lattices generated by operators of the form $a \otimes \varphi_b$, with a and b atoms and φ_b the coordinate functional associated with b.

In Sect. 5 we establish Theorem 5.8, the main theorem of the paper. In Theorem 6.1 we first obtain an operator-theoretic characterization of vector lattices of the form $c_{00}(\Lambda)$ for some nonempty set Λ. In particular, we show that an Archimedean vector lattice X is lattice isomorphic to $c_{00}(\Lambda)$ if and only if

$$\mathscr{L}(X, Y) = \mathscr{L}_b(X, Y) = \mathscr{L}_n(X, Y)$$

for every Archimedean vector lattice Y. When $Y = c_{00}(\Lambda)$, this characterization allows us to rewrite Theorem 5.8 explicitly in that setting, yielding a concrete description of order-preserving automorphisms of the corresponding operator algebras.

Finally, in Sect. 7 we provide concluding remarks concerning order-bounded functionals and the order dual of the lexicographic product of vector lattices.

2 Preliminaries

Let X, Y be vector lattices. A subset of X is *order bounded* if it is contained in some interval. By $\mathscr{L}(X,Y)$, $\mathscr{L}_b(X,Y)$ and $\mathscr{L}_n(X,Y)$ we denote the vector space of all linear operators, the space of all order bounded operators and the space of all order continuous operators from X to Y, respectively. If X and Y are normed lattices, then the space of all bounded operators is denoted by $\mathscr{B}(X,Y)$. If $X=Y$ we write $\mathscr{L}(X)$ instead of $\mathscr{L}(X,Y)$. Similarly, we define the symbols $\mathscr{L}_b(X)$, $\mathscr{L}_n(X)$ and $\mathscr{B}(X)$. If $Y=\mathbb{R}$, we write $X' := \mathscr{L}(X,\mathbb{R})$, $X^\sim := \mathscr{L}_b(X,\mathbb{R})$ and $X_n^\sim := \mathscr{L}_n(X,\mathbb{R})$. If X is a normed lattice, we write $X^* := \mathscr{B}(X,\mathbb{R})$. A net $(x_\alpha)_\alpha$ in X *converges in order* to a vector x whenever there exists another net $(y_\beta)_\beta$ with $y_\beta \searrow 0$ such that for each β there exists α_0 such that for all $\alpha \geq \alpha_0$ we have

$$|x_\alpha - x| \leq y_\beta.$$

In this case, we write $x_\alpha \xrightarrow{o} x$.

The proof of the following well-known lemma is included for the sake of completeness.

Lemma 2.1 *Let u be a nonzero vector of an Archimedean vector lattice X. For a linear functional $\varphi \in X'$ consider the rank one operator $u \otimes \varphi$.*

(i) *$u \otimes \varphi$ is order bounded if and only if φ is order bounded.*
(ii) *$u \otimes \varphi$ is order continuous if and only if φ is order continuous.*

Proof (i) Suppose first that φ is order bounded and choose any interval $[x,y] \subseteq X$. There exists a scalar $\lambda > 0$ such that $\varphi([x,y]) \subseteq [-\lambda,\lambda]$. Hence, for any $z \in [x,y]$ we have

$$|(u\otimes\varphi)(z)| = |\varphi(z)u| = |\varphi(z)|\,|u| \leq \lambda|u|$$

which proves that $u \otimes \varphi$ is order bounded.

Assume now that $u \otimes \varphi$ is order bounded and choose any interval $[x,y] \subseteq X$. There exists a positive vector w such that $(u\otimes\varphi)([x,y]) \subseteq [-w,w]$. Hence, for any $z \in [x,y]$ we have $|\varphi(z)|\,|u| = |\varphi(z)u| = |(u\otimes\varphi)(z)| \leq w$. Since X is Archimedean, the set $\varphi([x,y])$ needs to be bounded.

(ii) Suppose now that $x_\alpha \xrightarrow{o} 0$ in X. If φ is order continuous, then $\varphi(x_\alpha) \to 0$. Since X is Archimedean, we have $(u\otimes\varphi)(x_\alpha) = \varphi(x_\alpha)u \xrightarrow{o} 0$. Conversely, suppose that $u\otimes\varphi$ is order continuous. From $\varphi(x_\alpha)u = (u\otimes\varphi)(x_\alpha) \xrightarrow{o} 0$ it follows $\varphi(x_\alpha) \to 0$. □

A positive vector a of a vector lattice X is called an *atom* whenever it follows from $0 \leq x, y \leq a$ and $x \perp y$ that $x=0$ or $y=0$. The band A generated by all atoms of X is called the *atomic part* of X. If $X=A$, then X is an *atomic vector lattice*. If $X \neq A$, then $C := A^d$ is the *continuous part* of X. By [1, Theorem 1.36] the ideal $A+C$ is always order dense in X. If X is Archimedean and not atomic, then $C^d = A^{dd} = A \neq X$, so that $C \neq \{0\}$. If the principal ideal I_a generated by a equals to the span $\mathbb{R}a$ of a, then a is called a *discrete element*. By [6, Lemma 26.2] it follows that every discrete element is also an atom. Moreover, by [6, Theorem 26.4], in an Archimedean vector lattice the principal band B_a generated by an atom a is a projection band which is equal to $\mathbb{R}a$. In particular, in Archimedean vector lattices, the sets of atoms and discrete elements coincide.

Let a be an atom in an Archimedean vector lattice X. Then the decomposition

$$X = \mathbb{R}a \oplus \{a\}^d$$

is an order decomposition meaning that $x = x_1 + x_2 \in \mathbb{R}a \oplus \{a\}^d$ is positive if and only if x_1 and x_2 are positive in $\mathbb{R}a$ and $\{a\}^d$, respectively. For every $x \in X$ there exists a unique scalar $\varphi_a(x)$ and a unique vector $y_x \perp a$ such that $x = \varphi_a(x)a + y_x$. Since $X = \mathbb{R}a \oplus \{a\}^d$ is an order direct sum, we have that $\varphi_a \colon X \to \mathbb{R}$ is a lattice homomorphism. Moreover, order continuity of the band projection onto the projection band $\mathbb{R}a$ yields order continuity of φ_a.

By Zorn's lemma type argument, every vector lattice with at least one atom admits a maximal family $\mathcal{A}$ of pairwise disjoint atoms. Since for any two atoms a_1 and $a_2 \in X$ in an Archimedean vector lattice we have that either $a_1 \wedge a_2 = 0$ or $a_1 \wedge a_2$ is simultaneously a nonzero multiple of both a_1 and a_2, the span of $\mathcal{A}$ is independent of the choice of $\mathcal{A}$. Therefore, an Archimedean vector lattice X is atomic if and only if the ideal $I_{\mathcal{A}} := \operatorname{span} \mathcal{A}$ generated by $\mathcal{A}$ is order dense in X for any maximal family $\mathcal{A}$ of pairwise disjoint atoms in X. If a and a' are atoms in X such that $a' = \lambda a$ for some $\lambda > 0$, then it is easy to see that $\varphi_{a'} = \frac{1}{\lambda}\varphi_a$.

Suppose now that X is an atomic Archimedean vector lattice and let $\mathcal{A}$ be a maximal family of pairwise disjoint atoms in X. The following lemma characterizes positivity of a given vector $x \in X$.

Lemma 2.2 *A vector x in an atomic Archimedean vector lattice is positive if and only if $\varphi_a(x) \geq 0$ for every $a \in \mathcal{A}$.*

In particular, a vector $x \in X$ is the zero vector if and only if $\varphi_a(x) = 0$ for every $a \in \mathcal{A}$. The family $\mathcal{F}$ of all finite subsets of $\mathcal{A}$ is directed with set inclusion. Since $\operatorname{span} \mathcal{A}$ is order dense in X, for every positive vector $x \in X$ we have

$$x = \sup_{a \in \mathcal{A}} \varphi_a(x)a = \sup_{F \in \mathcal{F}} \sup_{a \in F} \varphi_a(x)a = \sup_{F \in \mathcal{F}} \sum_{a \in F} \varphi_a(x)a. \tag{1}$$

The remaining unexplained facts about vector and normed lattices can be found in [1] and [6].

Let $\mathscr{A}$ be a subalgebra of an algebra $\mathscr{B}$. An automorphism Φ of $\mathscr{A}$ is *inner* if there exists an invertible element $T \in \mathscr{A}$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$. If there exists an invertible element $T \in \mathscr{B}$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathscr{A}$, then Φ is called *spatial*. Spatial automorphism induced by T is denoted by Φ_T. Invertible elements T_1 and T_2 from $\mathscr{B}$ induce the same spatial automorphism of $\mathscr{A}$ if and only if

$$T_2^{-1}T_1A = AT_2^{-1}T_1$$

for all $A \in \mathscr{A}$. The latter is equivalent to the fact that $T_2^{-1}T_1$ belongs to the centralizer

$$C_{\mathscr{B}}(\mathscr{A}) = \{T \in \mathscr{B} : \ AT = TA \text{ for all } A \in \mathscr{A}\}$$

of $\mathscr{A}$ in $\mathscr{B}$. If $\mathscr{A} = \mathscr{B}$, then $C_{\mathscr{B}}(\mathscr{B})$ is called the center of $\mathscr{B}$ and is denoted by $Z(\mathscr{B})$. It is well known that in the case $\mathscr{A} = \mathscr{B} = \mathscr{L}(X)$ operators T_1 and T_2 induce the same inner automorphism if and only if $T_2^{-1}T_1$ is a nonzero scalar multiple of the identity operator.

3 Order closedness of finite-dimensional vector subspaces

In this section we prove (see Theorem 3.2) that every finite-dimensional vector subspace of an Archimedean vector lattice is order closed. The value of this result is twofold. First, we will use it in Sect. 5, where we study positive automorphisms of certain algebras of order continuous operators on atomic Archimedean vector lattices. Second, it is valuable in its own right, since it contributes to the broader development and understanding of concepts related to order convergence.

For the proof of Theorem 3.2 we need the following lemma whose proof is provided for reader's sake.

Lemma 3.1 *Let Ω be a nonempty set and let V be a finite-dimensional subspace of $\mathbb{R}^{\Omega}$. Then there exists a basis $f_1, \dots, f_n$ for V and points $a_1, \dots, a_n$ in Ω such that $f_i(a_j) = \delta_{ij}$.*

Proof We will prove the statement by induction on the dimension of V. The statement is clear if $\dim V = 1$. Assume now that the statement holds for every n-dimensional vector subspace of $\mathbb{R}^{\Omega}$ and let V be an $n+1$-dimensional subspace of $\mathbb{R}^{\Omega}$ with basis $\{h_1, h_2, \dots, h_{n+1}\}$.

Pick $a_1 \in \Omega$ such that $h_1(a_1) \neq 0$ and for $2 \leq i \leq n+1$ define

$$g_i = h_i - \frac{h_i(a_1)}{h_1(a_1)} h_1.$$

Then $g_i(a_1) = 0$ for all $i > 1$ and the set $\{h_1, g_2, \dots, g_{n+1}\}$ is a basis for V. By the induction hypothesis, one can find $a_2, \dots, a_{n+1}$ and $f_2, \dots, f_{n+1}$ such that $f_i(a_j) = \delta_{ij}$ for $2 \leq i, j \leq n+1$ and $\operatorname{span}\{f_2, \dots, f_{n+1}\} = \operatorname{span}\{g_2, \dots, g_{n+1}\}$. Therefore, $f_i(a_1) = 0$ for all $i > 1$. To finish the proof, we first define

$$g_1 = h_1 - \sum_{i=2}^{n+1} h_1(a_i) f_i$$

and then $f_1 = \frac{1}{g_1(a_1)} g_1$. □

To prove Theorem 3.2 we need some more preparation. Let X be a vector lattice. Following [5], a set $A \subseteq X$ is *order closed* in X whenever it consists precisely of those elements $x \in X$ for which there exists a net $(x_\alpha)_\alpha$ in A that converges in order to x in X. Suppose additionally that X is Archimedean. Then, for each positive vector $x \in X$, the principal ideal I_x generated by x admits a lattice norm $\|\cdot\|_x$ defined by

$$\|y\|_x := \inf\{\lambda \geq 0 : \; |y| \leq \lambda x\}.$$

If $(I_x, \|\cdot\|_x)$ is norm complete, then it is an AM-space with a unit x. By the Kakutani representation theorem (see e.g. [1, Theorem 4.29]), I_x is lattice isometric to some $C(K)$ for some compact Hausdorff space K with x being mapped to the constant one function. If $(I_x, \|\cdot\|_x)$ is not norm complete, then its norm completion is an AM-space with a unit x, so that I_x is a norm dense sublattice of some $C(K)$ space.

Theorem 3.2 *Every finite dimensional vector subspace of an Archimedean vector lattice is order closed.*

Proof Let V be a finite dimensional vector subspace of an Archimedean vector lattice X and let $(v_\alpha)_\alpha$ be a net in V which converges in order to some v in X. By passing to a tail, if necessary, we may assume that the net $(v_\alpha)_\alpha$ is order bounded. Therefore, there exists a positive vector y in X such that $|v_\alpha| \leq y$ for every α. We define

$$x = y + \sum_{i=1}^{n} |e_i|$$

where $\{e_1, \ldots, e_n\}$ is some basis of V. Clearly, V is contained in the principal ideal I_x. By the Kakutani representation theorem there exists a compact Hausdorff space K such that $(I_x, \|\cdot\|_x)$ embeds into $C(K)$ as a norm dense sublattice with x being mapped to the constant one function. Hence, without loss of generality we may assume that I_x is contained in $C(K)$.

By Lemma 3.1, there exists a basis $\{f_1, \ldots, f_n\}$ for V and points $a_1, \ldots, a_n$ in K such that $f_i(a_j) = \delta_{ij}$. Hence, for each α we can write

$$v_\alpha = \lambda_\alpha^{(1)} f_1 + \cdots \lambda_\alpha^{(n)} f_n$$

for suitable scalars $\lambda_\alpha^{(1)}, \ldots, \lambda_\alpha^{(n)}$. Since the net $(v_\alpha)_\alpha$ is order bounded in $C(K)$ and for each $1 \leq i \leq n$ the mapping $\varphi_i : C(K) \to \mathbb{R}$ defined by $\varphi(f) = f(a_i)$ is a lattice homomorphism, the net $(\lambda_\alpha^{(i)})_\alpha$ is bounded in $\mathbb{R}$ for each $1 \leq i \leq n$. By passing to appropriate subnets, we may assume that for each $1 \leq i \leq n$ the net $(\lambda_\alpha^{(i)})_\alpha$ already converges to some $\lambda^{(i)} \in \mathbb{R}$. This implies that the net $(v_\alpha)_\alpha$ converges in order in $I_x \subseteq X$ to $\lambda^{(1)} f_1 + \cdots + \lambda^{(n)} f_n \in I_x$. Since order limits are unique, we have $v = \lambda^{(1)} f_1 + \cdots + \lambda^{(n)} f_n \in V$. □

4 Operators on atomic vector lattices

In this section, we study and characterize order continuous operators on an Archimedean vector lattice X belonging to the algebra $\mathscr{A}_0 \subseteq \mathscr{L}_n(X)$ defined as follows. If X

does not contain atoms, we set $\mathscr{A}_0 = \{0\}$. Otherwise, choose a maximal family $\mathcal{A}$ of pairwise disjoint atoms in X and define $\mathscr{A}_0$ to be the algebra generated by all operators of the form $a \otimes \varphi_b$ for all $a, b \in \mathcal{A}$. It is easy to see that

$$\mathscr{A}_0 = \operatorname{span}\{a \otimes \varphi_b : \ a, b \in \mathcal{A}\}. \tag{2}$$

Since each coordinate functional φ_b is order continuous, we indeed have $\mathscr{A}_0 \subseteq \mathscr{L}_n(X)$.

In Proposition 4.2 we prove that an order continuous operator T belongs to $\mathscr{A}_0$ if and only if its “matrix representation" has only finitely many nonzero entries. Moreover, Lemma 4.3 yields that every positive operator $T \in \mathscr{L}_n(X)$ satisfying $TA \subseteq A$ and $T|_C = 0$ is a supremum of an increasing net of positive operators from $\mathscr{A}_0$. In particular, we prove that whenever X is Dedekind complete then $\mathscr{A}_0$ is an order dense vector sublattice of $\mathscr{L}_n(X)$ if and only if X is atomic.

We start with the following simple lemma which will be needed throughout the paper.

Lemma 4.1 *Let $T: X \to Y$ be a linear operator between Archimedean vector lattices and let $\mathcal{A}$ and $\mathcal{B}$ be maximal families of pairwise disjoint atoms in X and Y, respectively.*

- (i) *If X is atomic and T is order continuous, then $T \geq 0$ if and only if $Ta \geq 0$ for every $a \in \mathcal{A}$.*
- (ii) *If Y is atomic, then $T \geq 0$ if and only if $\varphi_b \circ T \geq 0$ for every atom $b \in \mathcal{B}$.*
- (iii) *If X and Y are atomic and T is order continuous, then $T \geq 0$ if and only if $\varphi_b(Ta) \geq 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$.*

Proof Since coordinate functionals are positive, it suffices to prove the “if" statements.

(i) Suppose that $Ta \geq 0$ for every atom $a \in \mathcal{A}$. Then $Tx \geq 0$ for every x that is a positive linear combination of atoms in $\mathcal{A}$. Since X is atomic and T is order continuous, we conclude that T is positive.

(ii) Assume that $\varphi_b \circ T \geq 0$ for every atom $b \in \mathcal{B}$ and pick any $x \in X^+$. Then $\varphi_b(Tx) \geq 0$ for every atom $b \in \mathcal{B}$, so that $Tx \geq 0$ by Lemma 2.2.

(iii) Suppose that $\varphi_b(Ta) \geq 0$ for all $a \in \mathcal{A}$ and $b \in \mathcal{B}$. By Lemma 2.2 we have $Ta \geq 0$ for every $a \in \mathcal{A}$. Now we apply (i). ◻

The following proposition characterizes order continuous operators contained in $\mathscr{A}_0$.

Proposition 4.2 *The following statements are equivalent for an order continuous operator T on an Archimedean vector lattice X.*

- (i) *$T \in \mathscr{A}_0$.*
- (ii) *There exist only finitely many pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that*

$$(a \otimes \varphi_a) T (b \otimes \varphi_b) \neq 0$$

  and T satisfies $TA \subseteq A$ and $T|_C = 0$.
- (iii) *There exist only finitely many pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that $\varphi_a(Tb) \neq 0$ and T satisfies $TA \subseteq A$ and $T|_C = 0$.*

Proof (i)⇒(ii) Pick $T \in \mathscr{A}_0$. Then $T = \sum_{c,d \in F} \lambda_{cd}(c \otimes \varphi_d) \in \mathscr{A}_0$ for some finite set $F \subseteq \mathcal{A}$ and some scalars λ_{cd}. Clearly, $T(\operatorname{span} \mathcal{A}) \subseteq \operatorname{span} \mathcal{A}$ and $T|_C = 0$. Since T is order continuous, we have $TA \subseteq A$. A direct calculation shows that

$$(a \otimes \varphi_a)T(b \otimes \varphi_b) = \sum_{c,d \in F} \lambda_{cd}\varphi_a(c)\varphi_d(b)(a \otimes \varphi_b) = 0$$

for $(a, b) \notin F \times F$.

(ii)⇒(i) Assume now that there exists a finite set $G \subseteq \mathcal{A} \times \mathcal{A}$ such that $(a \otimes \varphi_a)T(b \otimes \varphi_b) = 0$ for $(a, b) \notin G$ and T satisfies $TA \subseteq A$ and $T|_C = 0$.. Then $G \subseteq F \times F$ for some finite subset $F \subseteq \mathcal{A}$. Clearly, if $(a, b) \notin F \times F$, then $(a \otimes \varphi_a)T(b \otimes \varphi_b) = 0$. It follows that $\varphi_a(Tb) = 0$ for $(a, b) \notin F \times F$. We define the order continuous operator

$$S = \sum_{c,d \in F} \varphi_c(Td)(c \otimes \varphi_d) \in \mathscr{A}_0$$

and calculate

$$\varphi_a((T - S)b) = \varphi_a(Tb) - \sum_{c,d \in F} \varphi_c(Td)\varphi_d(b)\varphi_a(c)$$

for $a, b \in \mathcal{A}$. If $a \notin F$, then $\varphi_a(Tb) = 0$ and $\varphi_a(c) = 0$ for every $c \in F$ which clearly implies $\varphi_a((T - S)b) = 0$. On the other hand, if $a \in F$, then $\varphi_a(c) = 0$ for every $c \in \mathcal{A} \setminus \{a\}$ yielding

$$\varphi_a((T - S)b) = \varphi_a(Tb) - \sum_{d \in F} \varphi_a(Td)\varphi_d(b).$$

If $b \notin F$, then every term in the sum above is zero, so that we have $\varphi_a(Tb) = \varphi_a(Sb) = 0$. On the other hand, if $b \in F$, the sum above reduces to $\varphi_a(Tb)$. Therefore, both cases give $\varphi_a((T - S)b) = 0$ for all $a, b \in \mathcal{A}$. Lemma 4.1(iii) implies $T|_A = S|_A$. Since $T|_C = S|_C = 0$, S and T agree on the order dense ideal $A \oplus C$ of X. To conclude the proof we once more use the fact that T and S are order continuous.

(ii)⇔(iii) follows from the identity $(a \otimes \varphi_a)T(b \otimes \varphi_b) = \varphi_a(Tb)(a \otimes \varphi_b)$. □

If a vector lattice X is finite-dimensional and Archimedean, it is atomic and order isomorphic to $\mathbb{R}^n$ ordered coordinatewise where $n = \dim X$. Hence, if $\{a_1, \ldots, a_n\}$ is a maximal family of pairwise disjoint atoms in X, then every linear operator T on X is of the form

$$T = \sum_{i,j=1}^{n} (a_i \otimes \varphi_{a_i})T(a_j \otimes \varphi_{a_j}).$$

Suppose now that X is an atomic Archimedean vector lattice and let $\mathcal{A}$ be a maximal family of pairwise disjoint atoms in X. For a finite subset $F \subseteq \mathcal{A}$ we define the *finite truncation* T_F of T by

$$T_F = \sum_{a,b \in F} (a \otimes \varphi_a)T(b \otimes \varphi_b).$$

If we order the family $\mathcal{F}$ of all finite subsets of $\mathcal{A}$ by set inclusion, $\mathcal{F}$ becomes a directed set, so that we may consider $(T_F)_{F\in\mathcal{F}}$ as a net. The following lemma shows that every positive order continuous operator $T \in \mathscr{L}_n(X)$ on an atomic Archimedean vector lattice can be recovered from its finite truncations.

Lemma 4.3 *Let X be an atomic Archimedean vector lattice. Then for each positive operator $T \in \mathscr{L}_n(X)$ the net $(T_F)_{F\in\mathcal{F}}$ satisfies $T_F \nearrow T$ in $\mathscr{L}_n(X)$. Moreover, for each $x \geq 0$ we have $T_F x \nearrow Tx$.*

Proof Pick any finite subset $F \in \mathcal{F}$ and observe that $0 \leq \sum_{a\in F} a \otimes \varphi_a \leq I$ yields $0 \leq T_F \leq T$. Clearly, positivity of T implies $0 \leq T_{F_1} \leq T_{F_2}$ for each pair $F_1, F_2 \in \mathcal{F}$ with $F_1 \subseteq F_2$. To prove that T is the supremum of $(T_F)_{F\in\mathcal{F}}$ in $\mathscr{L}_n(X)$, let $S \in \mathscr{L}_n(X)$ be any upper bound of $(T_F)_{F\in\mathcal{F}}$. Choose any $c \in \mathcal{A}$, and select $F \in \mathcal{F}$ such that $c \in F$. By assumption, we have $S \geq T_F \geq 0$, and so

$$Sc \geq \sum_{a,b\in F} (a \otimes \varphi_a)T(b \otimes \varphi_b)c = \sum_{a,b\in F} \varphi_b(c)\varphi_a(Tb)a = \sum_{a\in F} \varphi_a(Tc)a.$$

Since

$$Tc = \sup_{F\in\mathcal{F}} \sum_{a\in F} \varphi_a(Tc)a$$

by (1), we have $Sc \geq Tc$. To conclude the proof, we apply Lemma 4.1(i) for the operator $S - T$. ◻

Proposition 4.4 *For an Archimedean vector lattice X the following statements hold.*

(i) *The algebra $\mathscr{A}_0$ is a vector lattice. For $T = \sum_{a,b\in F} \lambda_{ab}(a \otimes \varphi_b) \in \mathscr{A}_0$ the modulus is given by*

$$|T| = \sum_{a,b\in F} |\lambda_{ab}|(a \otimes \varphi_b).$$

(ii) *If X is Dedekind complete, then $\mathscr{A}_0$ is an ideal of $\mathscr{L}_n(X)$.*

Proof We will simultaneously prove (i) and (ii). To this end, pick any $T \in \mathscr{A}_0$ and any maximal set $\mathcal{A}$ of pairwise disjoint atoms in X. Then there exists a finite subset $F \subseteq \mathcal{A}$ such that

$$T = \sum_{a,b\in F} \lambda_{ab}(a \otimes \varphi_b).$$

We claim that the operator

$$S = \sum_{a,b\in F} |\lambda_{ab}|(a \otimes \varphi_b)$$

is the modulus of T in $\mathscr{A}_0$. By definition, S belongs to $\mathscr{A}_0$. Clearly, for every atom $c \in X$ we have $Sc \geq Tc, -Tc$, so that Lemma 4.1 yields $S|_A \geq T|_A, -T|_A$. Furthermore, $S|_C = T|_C = 0$ implies $S|_{A\oplus C} \geq T|_{A\oplus C}, -T|_{A\oplus C}$. Therefore, restrictions of order continuous operators $S - T$ and $S + T$ to the order dense ideal $A \oplus C$ are positive which implies $S - T$ and $S + T$ are positive on X. This yields $S \geq T, -T$.

Let $\widetilde{S} \in \mathscr{L}_n(X)$ be any upper bound for $\{T, -T\}$ and observe that for every $c \in \mathcal{A} \backslash F$ we clearly have $\widetilde{S}c \geq Tc = Sc = 0$. On the other hand, for $c \in F$ we have

$$\widetilde{S}c \geq \sum_{a \in F} \lambda_{ac} a \quad \text{and} \quad \widetilde{S}c \geq -\sum_{a \in \mathcal{F}} \lambda_{ac} a,$$

which implies

$$\widetilde{S}c \geq \left| \sum_{a \in F} \lambda_{ac} a \right| = \sum_{a \in \mathcal{F}} |\lambda_{ac}| a = Sc.$$

By applying Lemma 4.1 once again we obtain $\widetilde{S}|_A \geq S|_A$. Since $S|_C = 0$, we also have $\widetilde{S}|_C \geq S|_C = 0$. Similarly as before, order continuity of S and $\widetilde{S}$ and order density of $A \oplus C$ in X yield $\widetilde{S} \geq S$. This finally proves that S is the supremum of the set $\{T, -T\}$ in both $\mathscr{A}_0$ and $\mathscr{L}_n(X)$. To finish the proof, note that [1, Theorem 1.56] yields that $\mathscr{L}_n(X)$ is a vector lattice whenever X is Dedekind complete.

To prove that $\mathscr{A}_0$ is an ideal in $\mathscr{L}_n(X)$, we first note that a closer examination of the proof of [7, Lemma 3.8] reveals that for all atoms $a, b \in X$ each operator $a \otimes \varphi_b$ is an atom of $\mathscr{L}_n(X)$. Suppose that we have $0 \leq |T| \leq |S|$ for some operators $T \in \mathscr{L}_n(X)$ and $S = \sum_{a,b \in F} \lambda_{ab}(a \otimes \varphi_b) \in \mathscr{A}_0$. Then

$$0 \leq T^+, T^- \leq |T| \leq \sum_{a,b \in F} |\lambda_{ab}|(a \otimes \varphi_b).$$

By the Riesz decomposition property there exist positive operators M_{ab} and N_{ab} such that $T^+ = \sum_{a,b \in F} M_{ab}$, $T^- = \sum_{a,b \in F} N_{ab}$ and $0 \leq M_{ab}, N_{ab} \leq |\lambda_{ab}|(a \otimes \varphi_b)$. Since $a \otimes \varphi_b$ is an atom in $\mathscr{L}_n(X)$, we have that M_{ab} and N_{ab} are positive scalar multiples of $a \otimes \varphi_b$. This yields that $T = T^+ - T^-$ is a linear combination of operators $a \otimes \varphi_b$ for $a, b \in F$ proving that $T \in \mathscr{A}_0$. ◻

We conclude this section with the following operator theoretical characterization of atomic Archimedean vector lattices.

Corollary 4.5 *A nonzero Archimedean vector lattice X is atomic if and only if for every nonzero positive operator $T \in \mathscr{L}_n(X)$ there exists a positive nonzero operator $S \in \mathscr{A}_0$ such that $0 < S \leq T$.*

Proof If X is atomic, then every positive order continuous operator T on S is by Lemma 4.3 the supremum of the increasing net $(T_F)_{F \in \mathcal{F}}$.

To prove the converse statement, observe first that since the identity operator on X is order continuous, the algebra $\mathscr{A}_0$ is, by assumption, nontrivial, yielding that the vector lattice X contains atoms. Assume that X is not atomic. Then there exists a nonzero positive vector x which is disjoint with every atom in X. For an atom c in X we consider the rank-one operator $x \otimes \varphi_c$. By Lemma 2.1, the operator $x \otimes \varphi_c$ is order continuous. By assumption, there exists a nonzero positive operator $S \in \mathscr{A}_0$ such that $0 < S \leq x \otimes \varphi_c$. There exists a finite subset $F \subseteq \mathcal{A}$ and scalars λ_{ab} such that

$$S = \sum_{a,b \in F} \lambda_{ab}(a \otimes \varphi_b).$$

Pick any positive vector $y \in X$ such that $Sy \neq 0$. Then

$$0 < \sum_{a,b \in F} \lambda_{ab} \varphi_b(y) a \leq \varphi_c(y) x$$

shows that x is not disjoint with some atom which proves that the span of $\mathcal{A}$ is order dense in X. □

Corollary 4.6 *A Dedekind complete vector lattice is atomic if and only if $\mathscr{A}_0$ is an order dense ideal in $\mathscr{L}_n(X)$ and $\mathscr{L}_r(X)$.*

5 Automorphisms of algebras of operators on atomic vector lattices

Throughout this section, we assume that $\mathcal{A}$ is a maximal family of pairwise disjoint atoms in an atomic Archimedean vector lattice X. Let $\mathscr{A}$ be an algebra in $\mathscr{L}(X)$ containing $\mathscr{A}_0$ (see Sect. 4) and let $\Phi\colon \mathscr{A} \to \mathscr{A}$ be a positive algebra automorphism. In Theorem 5.6, we prove that every such positive automorphism Φ satisfies $\Phi(\mathscr{A}_0) = \mathscr{A}_0$ whenever $\mathscr{A} \subseteq \mathscr{L}_n(X)$. Furthermore, if $\mathscr{A} \subseteq \mathscr{L}_n(X)$ and X is Dedekind complete, in Theorem 5.8 we conclude that for each positive operator T the restriction of $\Phi(T)$ to the ideal generated by atoms behaves as a “generalized permutation" operator. This result will be extremely useful in Sect. 6 where we consider algebras of operators on vector lattices of the form $c_{00}(\Lambda)$ for some nonempty set Λ. In that case, we prove that every positive automorphism $\Phi\colon \mathscr{A} \to \mathscr{A}$ is of the form $\Phi(T) = PDTD^{-1}P^{-1}$ for some “diagonal" operator D and some “permutation" operator P.

We start with the following characterization of rank-one operators.

Lemma 5.1 *Let T be an order continuous operator on an atomic Archimedean vector lattice X and let $\mathscr{A}$ be any algebra containing $\mathscr{A}_0$. Then T is a rank-one operator if and only if* $\dim(T\mathscr{A}T) = 1$.

Proof Suppose first that the rank of T is one. By Lemma 2.1 we have $T = u \otimes \varphi$ for some nonzero vector $u \in X$ and nonzero functional $\varphi \in X_n^\sim$. Then for any $A \in \mathscr{A}$ we have

$$TAT = (u \otimes \varphi)A(u \otimes \varphi) = \varphi(Au)(u \otimes \varphi),$$

which shows $\dim(T\mathscr{A}T) \leq 1$. Since $u \neq 0$, there exists an atom $b \in \mathcal{A}$ such that $\varphi_b(u) \neq 0$. Furthermore, X being atomic and φ nonzero and order continuous yield the existence of an atom $a \in \mathcal{A}$ such that $\varphi(a) \neq 0$. Therefore, $T(a \otimes \varphi_b)T = \varphi(a)\varphi_b(u)(u \otimes \varphi) \neq 0$ which proves that $\dim(T\mathscr{A}T) = 1$.

For the opposite implication, assume that the rank of T is at least two. We claim that there exist atoms $a, b \in \mathcal{A}$ such that Ta and Tb are linearly independent. If this were not the case, there would exist a nonzero vector $v \in X$ such that for each $a \in \mathcal{A}$ we have $Ta = \lambda_a v$ for some scalar λ_a. If $v = 0$, then Lemma 4.1 yields $T = 0$, which is impossible.

By linearity of T it follows that for every $x \in \operatorname{span} \mathcal{A}$ we have $Tx = \lambda_x v$ for some scalar λ_x. Since $X = X^+ - X^+$ it suffices to prove that for every $x \in X^+$ we have

$Tx = \lambda_x v$ for some scalar λ_x. To this end, since span $\mathcal{A}$ is order dense in X, there exists an increasing net $(x_\alpha)_\alpha$ in span $\mathcal{A}$ of positive vectors such that $0 \le x_\alpha \nearrow x$. Order continuity of T yields $Tx_\alpha \xrightarrow{o} Tx$ and so $\lambda_{x_\alpha} v \xrightarrow{o} Tx$. Since the one-dimensional vector space $\mathbb{R}v$ is order closed by Theorem 3.2, we have $Tx = \lambda v$ for some scalar λ. This shows that T is a rank-one operator which, by assumption, is impossible. Hence, there exist atoms $a, b \in \mathcal{A}$ such that Ta and Tb are linearly independent.

For an arbitrary atom $e \in \mathcal{A}$ we define the rank one operators $X_1 := a \otimes \varphi_e$ and $X_2 = b \otimes \varphi_e$. We claim that $TX_1T = Ta \otimes (\varphi_e \circ T)$ and $TX_2T = Tb \otimes (\varphi_e \circ T)$ are linearly independent for some $e \in \mathcal{A}$. To see this, suppose that

$$0 = \alpha(Ta \otimes (\varphi_e \circ T)) + \beta(Tb \otimes (\varphi_e \circ T)) = (\alpha Ta + \beta Tb) \otimes (\varphi_e \circ T)$$

for some scalars α and β. If $\alpha \neq 0$ or $\beta \neq 0$, then $\varphi_e \circ T = 0$ since Ta and Tb are linearly independent. By Lemma 4.1, we have $T = 0$, which is impossible. Hence, TX_1T and TX_2T are linearly independent elements in $T\mathscr{A}T$. □

The following example shows that the algebra $\mathscr{A}_0$ can be a proper subalgebra of $\mathscr{L}_n(X)$.

Example 5.2 Consider the vector lattice c_{00} of all eventually null sequences. On c_{00} we define the functional φ by $\varphi(x) = \sum_{n=1}^{\infty} x_n$. Then φ is a well-defined positive functional on c_{00}. Choose any net $(x_\alpha)_\alpha$ in c_{00} such that $x_\alpha \searrow 0$ and pick an index α_0. Then for all $\alpha \ge \alpha_0$ we have $0 \le x_\alpha \le x_{\alpha_0}$ so that the net $(x_\alpha)_{\alpha \ge \alpha_0}$ is supported only on finitely many coordinates. Since order convergence on c_{00} is coordinatewise, φ is order continuous. Clearly, for any $u \neq 0$ the operator $u \otimes \varphi$ is not contained in $\mathscr{A}_0$.

In Theorem 6.1 which is needed for the description of positive algebra automorphisms of $\mathscr{L}(c_{00}(\Lambda))$ we are going to prove that every linear functional on c_{00} is order continuous. See also the subsequent corollary.

Let $\mathscr{A}$ be a subalgebra in $\mathscr{L}(X)$ which contains $\mathscr{A}_0$. For a given positive algebra automorphism $\Phi \colon \mathscr{A} \to \mathscr{A}$ and $a, b \in \mathcal{A}$ we define $F_{ab} := \Phi(a \otimes \varphi_b)$.

Lemma 5.3 *If $\Phi(\mathscr{A}_0) \subseteq \mathscr{L}_n(X)$, then the following assertions hold.*

(i) *For all $a, b \in \mathcal{A}$ the operator F_{ab} is a rank-one operator.*
(ii) *For all $a, b \in \mathcal{A}$ there exist positive scalars γ_a, a positive nonzero vector $u_a \in X$ and a positive nonzero order continuous functional ψ_b such that*

$$F_{ab} = \tfrac{\gamma_a}{\gamma_b}(u_a \otimes \psi_b)$$

with $\psi_a(u_a) = 1$ and $\psi_a(u_b) = 0$ for $b \neq a$.
(iii) *For distinct atoms $a, b \in \mathcal{A}$ and an atom e we have $\psi_b(e) = 0$ whenever $u_a \wedge e > 0$.*

Proof (i) By Lemma 5.1, for $a, b \in \mathcal{A}$ the dimension of the vector space $(a \otimes \varphi_b)\mathscr{A}(a \otimes \varphi_b)$ is one. Since Φ is an automorphism of $\mathscr{A}$, the dimension of

$$\Phi\big((a \otimes \varphi_b)\mathscr{A}(a \otimes \varphi_b)\big) = F_{ab}\mathscr{A}F_{ab}$$

is also one. By Lemma 5.1 and order continuity of F_{ab} we conlude that F_{ab} is also a rank-one operator.

(ii) By (i), the operator F_{ab} is a rank-one operator. Therefore, $F_{ab} = x_{ab} \otimes f_{ab}$ for some nonzero vector $x_{ab} \in X$ and nonzero functional f_{ab}. Since F_{ab} is order continuous, f_{ab} is order continuous by Lemma 2.1. Moreover, f_{ab} is order bounded by [2, Theorem 2.1]. Positivity of Φ and [1, Theorem 1.72] yield

$$F_{ab} = |F_{ab}| = |x_{ab} \otimes f_{ab}| = |x_{ab}| \otimes |f_{ab}|.$$

Therefore, we may assume that x_{ab} and f_{ab} are positive. For an atom $a \in \mathcal{A}$ we define $u_a := x_{aa}$ and $\psi_a = f_{aa}$. Now we chose arbitrary atoms a and b from $\mathcal{A}$. Since $(a \otimes \varphi_b)(b \otimes \varphi_a) = a \otimes \varphi_a$, we have

$$F_{ab}F_{ba} = \Phi(a \otimes \varphi_b)\Phi(b \otimes \varphi_a) = \Phi\big((a \otimes \varphi_b)(b \otimes \varphi_a)\big) = \Phi(a \otimes \varphi_a) = F_{aa}.$$

In particular, we get

$$\psi_a(u_a)(u_a \otimes \psi_a) = F_{aa}^2 = F_{aa} = u_a \otimes \psi_a,$$

and therefore, $\psi_a(u_a) = 1$. Moreover,

$$u_a \otimes \psi_a = F_{aa} = F_{ab}F_{ba} = f_{ab}(x_{ba})(x_{ab} \otimes f_{ba})$$

implies that $x_{ab} = \lambda_{ba}u_a$ and $f_{ba} = \mu_{ba}\psi_a$ and so

$$F_{ab} = \gamma_{ab}(u_a \otimes \psi_b)$$

for a suitable positive scalar γ_{ab}. Fix $c \in \mathcal{A}$. Since $\psi_b(u_b) = 1$ for $b \in \mathcal{A}$ we get

$$F_{cc} = F_{cb}F_{bc} = \gamma_{cb}\gamma_{bc}(u_c \otimes \psi_b)(u_b \otimes \psi_c) = \gamma_{cb}\gamma_{bc}F_{cc}$$

yielding $\gamma_{cb} = \frac{1}{\gamma_{bc}}$. For an arbitrary $a \in \mathcal{A}$, the identity $F_{ab} = F_{ac}F_{cb}$ yields $\gamma_{ab} = \gamma_{ac}\gamma_{cb} = \frac{\gamma_{ac}}{\gamma_{bc}}$. Thus, if we define $\gamma_a := \gamma_{ac}$ for all $a \in \mathcal{A}$, we obtain the desired expression

$$F_{ab} = \frac{\gamma_a}{\gamma_b}(u_a \otimes \psi_b).$$

Finally, the fact that for distinct atoms $a, b \in \mathcal{A}$ we have $\frac{\gamma_b^2}{\gamma_a^2}\psi_a(u_b)(u_b \otimes \psi_a) = F_{ba}^2 = 0$ gives us $\psi_a(u_b) = 0$.

(iii) Choose any atom $e \in \mathcal{A}$ with $u_a \wedge e > 0$. Since e is an atom, there exists $\lambda > 0$ such that $u_a \wedge e = \lambda e$. Positivity of ψ_b and the inequality

$$0 = \psi_b(u_a) \geq \psi_b(u_a \wedge e) = \lambda\psi_b(e)$$

yield $\psi_b(e) = 0$. □

For the proof of Proposition 5.5 we need to introduce the sets $\mathcal{U}_a$ and Ψ_a as follows. Pick $a \in \mathcal{A}$, and let u_a and ψ_a be as in Lemma 5.3. We define

$$\mathcal{U}_a = \{e \in \mathcal{A} : \ u_a \wedge e \neq 0\} = \{e \in \mathcal{A} : \ \varphi_e(u_a) \neq 0\}$$

and

$$\Psi_a = \{e \in \mathcal{A} : \ \psi_a(e) \neq 0\}.$$

Lemma 5.4 *For distinct atoms a and b in $\mathcal{A}$ we have*

$$\mathcal{U}_a \cap \mathcal{U}_b = \emptyset \quad \text{and} \quad \Psi_a \cap \Psi_b = \emptyset.$$

Proof We first prove $\mathcal{U}_a \cap \mathcal{U}_b = \emptyset$ for distinct atoms a and b in $\mathcal{A}$. Suppose there exists an atom $e \in \mathcal{U}_a \cap \mathcal{U}_b$. Since e is an atom, there exist positive scalars λ and μ such that $u_a \wedge e = \lambda e$ and $u_b \wedge e = \mu e$. Pick any atom c in X. Then $c \neq a$ or $c \neq b$, so that $0 \leq \lambda\psi_c(e) = \psi_c(u_a \wedge e) \leq \psi_c(u_a) = 0$ or $0 \leq \mu\psi_c(e) = \psi_c(u_b \wedge e) \leq \psi_c(u_b) = 0$ yields $\psi_c(e) = 0$. In particular, we have $F_{dc}(e) = 0$ for arbitrary atoms $c, d \in \mathcal{A}$.

Now we define $S = e \otimes \varphi_e$. Then $F_{dc}S = \frac{\gamma_d}{\gamma_c}(u_d \otimes \psi_c)(e \otimes \varphi_e) = 0$. If we write $T := \Phi^{-1}(S)$, multiplicativity of Φ gives

$$0 = \Phi^{-1}(F_{dc}S) = (d \otimes \varphi_c)T = d \otimes (\varphi_c \circ T)$$

for all atoms $c, d \in \mathcal{A}$. Hence, $\varphi_c \circ T = 0$ for every atom $c \in \mathcal{A}$. By Lemma 4.1(ii) we conclude $T = 0$ which is impossible, so $\mathcal{U}_a \cap \mathcal{U}_b = \emptyset$.

Now we prove $\Psi_a \cap \Psi_b = \emptyset$ for distinct atoms $a, b \in \mathcal{A}$. Suppose there exists an atom $e \in \Psi_a \cap \Psi_b$. Then $\psi_a(e) \neq 0$ and $\psi_b(e) \neq 0$. Since ψ_a and φ_e are positive linear functionals, for an atom $c \in \mathcal{A} \setminus \{e\}$ we have $0 \leq (\psi_a \wedge \varphi_e)(c) \leqslant \varphi_e(c) = 0$ and so $(\psi_a \wedge \varphi_e)(c) = 0$. Moreover, by the Riesz-Kantorovich formula (see e.g. [1, Theorem 1.18]) we have

$$\begin{aligned}
\lambda := (\psi_a \wedge \varphi_e)(e) &= \inf\{\psi_a(te) + \varphi_e((1-t)e) : \ t \in [0, 1]\} \\
&= \inf\{t\psi_a(e) + (1-t) : \ t \in [0, 1]\} \\
&= \min\{1, \psi_a(e)\} > 0.
\end{aligned}$$

Therefore $\psi_a \wedge \varphi_e = \lambda\varphi_e$ as $\psi_a \wedge \varphi_e$ is order continuous and X is atomic. Similarly we get $\psi_b \wedge \varphi_e = \mu\varphi_e$ for some $\mu > 0$.

For every atom $c \in \mathcal{A}$ we have $\psi_a(u_c) = 0$ or $\psi_b(u_c) = 0$ since $c \neq a$ or $c \neq b$. It follows that $0 \leqslant \lambda\varphi_e(u_c) = (\psi_a \wedge \varphi_e)(u_c) \leqslant \psi_a(u_c) = 0$ or $0 \leqslant \mu\varphi_e(u_c) = (\psi_b \wedge \varphi_e)(u_c) \leqslant \psi_b(u_c) = 0$ and so

$$\varphi_e(u_c) = 0$$

for every atom $c \in \mathcal{A}$. The operator $S = e \otimes \varphi_e$ clearly satisfies

$$SF_{cd} = (e \otimes \varphi_e)(u_c \otimes \psi_d) = \varphi_e(u_c)(e \otimes \psi_d) = 0$$

for all atoms $c, d \in \mathcal{A}$. It follows that

$$\Phi^{-1}(S)(c \otimes \varphi_d) = 0$$

for all atoms $c, d \in \mathcal{A}$. By Lemma 4.1(i) we conclude $\Phi^{-1}(S) = 0$ which is impossible. This finally proves the claim. □

Proposition 5.5 *Let $\mathscr{A} \subseteq \mathscr{L}_n(X)$ be any algebra of operators which contains $\mathscr{A}_0$. If Φ is a positive automorphism of $\mathscr{A}$, then for arbitrary $T \in \mathscr{A}$ we have $T \in \mathscr{A}_0$ whenever $\Phi(T) \in \mathscr{A}_0$.*

Proof Suppose that for $T \in \mathscr{A}$ we have $S := \Phi(T) \in \mathscr{A}_0$ whereas $T \notin \mathscr{A}_0$. By Proposition 4.2 there exist infinitely many pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$ such that the product $(a \otimes \varphi_a) T (b \otimes \varphi_b) \neq 0$. Applying Φ we equivalently obtain $F_{aa} S F_{bb} \neq 0$ for infinitely many pairs $(a, b) \in \mathcal{A} \times \mathcal{A}$. Since $S \in \mathscr{A}_0$ we can write

$$S = \sum_{c,d \in F} s_{cd}(c \otimes \varphi_d)$$

for some finite set $F \subseteq \mathcal{A}$. To conclude the proof, we consider two cases.

Case 1: Assume there exist infinitely many $a \in \mathcal{A}$ such that for some $b \in \mathcal{A}$ the product $F_{aa} S F_{bb}$ is nonzero. Since F is finite, using Lemma 5.4 we derive that F intersects Ψ_a for only finitely many $a \in \mathcal{A}$. Therefore, there exists $a \in \mathcal{A}$ such that $F_{aa} S F_{bb} \neq 0$ for some b and $\Psi_a \cap F = \emptyset$. Consequently, since $c \notin \Psi_a$ yields $\psi_a(c) = 0$, we have

$$F_{aa} S = (u_a \otimes \psi_a) \sum_{c,d \in F} s_{cd}(c \otimes \varphi_d) = \sum_{c,d \in F} s_{cd} \psi_a(c)(u_a \otimes \varphi_d) = 0$$

which contradicts $F_{aa} S F_{bb} \neq 0$.

Case 2: There exists only finitely many $a \in \mathcal{A}$ such that for some $b \in \mathcal{A}$ the product $F_{aa} S F_{bb}$ is nonzero and for some $a \in \mathcal{A}$ the product $F_{aa} S F_{bb}$ is nonzero for infinitely many $b \in \mathcal{A}$. A similar argument as in Case 1 shows that there exists $b \in \mathcal{A}$ such that $F_{aa} S F_{bb} \neq 0$ for some a and $\mathcal{U}_b \cap F = \emptyset$. Since $d \notin \mathcal{U}_b$ yields $\varphi_d(u_b) = 0$, we have

$$S F_{bb} = \sum_{c,d \in F} s_{cd}(c \otimes \varphi_d)(u_b \otimes \psi_b) = \sum_{c,d \in F} s_{cd} \varphi_d(u_b)(c \otimes \psi_b) = 0$$

which is again in contradiction with $F_{aa} S F_{bb} \neq 0$. □

Theorem 5.6 *Let $\mathscr{A} \subseteq \mathscr{L}_n(X)$ be any algebra of operators that contains $\mathscr{A}_0$. The following assertions hold for a positive algebra automorphism $\Phi \colon \mathscr{A} \to \mathscr{A}$.*

(i) *There exist a bijection $\pi : \mathcal{A} \to \mathcal{A}$ and a set of positive scalars $\{\delta_a : a \in \mathcal{A}\}$ such that for all $a, b \in \mathcal{A}$ we have*

$$\Phi(a \otimes \varphi_b) = \tfrac{\delta_a}{\delta_b} \left(\pi(a) \otimes \varphi_{\pi(b)} \right) \tag{3}$$

$$\Phi^{-1}(a \otimes \varphi_b) = \tfrac{\delta_{\pi^{-1}(b)}}{\delta_{\pi^{-1}(a)}} \left(\pi^{-1}(a) \otimes \varphi_{\pi^{-1}(b)} \right). \tag{4}$$

In particular, $\Phi(\mathscr{A}_0) = \mathscr{A}_0$ *and* $\Phi(\mathscr{A}_0^+) = \mathscr{A}_0^+$.

(ii) *The inverse* Φ^{-1} *is positive.*

Proof (i) First we prove that for arbitrary $a \in \mathcal{A}$ the set $\mathcal{U}_a$ is a singleton set. Assume on the contrary that for some a the set $\mathcal{U}_a$ contains two different atoms e and f, and pick $b, c \in \mathcal{A}$. Then

$$F_{bc}e = \frac{\gamma_b}{\gamma_c}(u_b \otimes \psi_c)(e) = \frac{\gamma_b}{\gamma_c}\psi_c(e)u_b.$$

If $c \neq a$, by Lemma 5.3(iii) we conclude $\psi_c(e) = 0$ which implies $F_{bc}e = 0$, and similarly, $F_{bc}(f) = 0$.

By Proposition 5.5 there exists a finite set $F \subseteq \mathcal{A}$ which without loss of generality contains a such that $\Phi(T) = e \otimes \varphi_e + f \otimes \varphi_f$ where $T = \sum_{b,c\in F} t_{bc}(b \otimes \varphi_c) \in \mathscr{A}_0$. Then

$$\Phi(T) = \sum_{b,c\in F} t_{bc}F_{bc}.$$

Since for $c \neq a$ we have $F_{bc}e = 0$ and $F_{ba}e = \frac{\gamma_b}{\gamma_a}\psi_a(e)u_b$ it follows that

$$e = \Phi(T)e = \sum_{b,c\in F} t_{bc}F_{bc}e = \sum_{b\in F} t_{ba}F_{ba}e = \tfrac{\gamma_b}{\gamma_a}\psi_a(e)\sum_{b\in \mathcal{F}} t_{ba}u_b.$$

Similarly we get

$$f = \tfrac{\gamma_b}{\gamma_a}\psi_a(f)\sum_{b\in F} t_{ba}u_b.$$

which is a contradiction since e and f are linearly independent. This proves that for each $a \in \mathcal{A}$ the set $\mathcal{U}_a$ is a singleton set which yields an injective mapping $\pi : \mathcal{A} \to \mathcal{A}$ such that $\mathcal{U}_a = \{\pi(a)\}$.

Since $\mathcal{U}_a$ consists of all atoms that are not disjoint with u_a and X is atomic, an application of (1) yields $u_a = \kappa_a\pi(a)$ for some positive scalar κ_a. Hence,

$$\Phi(a \otimes \varphi_b) = \kappa_a\pi(a) \otimes \psi_b.$$

To prove that π is surjective, choose an atom $c \in \mathcal{A}$. Since Φ is an automorphism of $\mathscr{A}$, there exists $T \in \mathscr{A}$ such that $\Phi(T) = c \otimes \varphi_c$. By Proposition 5.5 we conclude $T \in \mathscr{A}_0$. Therefore, there exists a finite set $F \subseteq \mathcal{A}$ such that $T = \sum_{a,b\in F} \lambda_{ab}(a\otimes\varphi_b)$ for some scalars λ_{ab}. Since

$$c = (c \otimes \varphi_c)c = \Phi(T)c = \sum_{a,b\in F} \kappa_a(\pi(a) \otimes \psi_b)c = \sum_{a,b\in F} \kappa_a\psi_b(c)\pi(a),$$

and $\pi(a) \in \mathcal{A}$ for each $a \in F$, we conclude that $c = \pi(a)$ for some $a \in \mathcal{A}$.

For different atoms $a, b \in \mathcal{A}$ we have $\kappa_a\psi_b(\pi(a)) = \psi_b(u_a) = 0$. Since $\kappa_a \neq 0$ we get $\psi_b(\pi(a)) = 0$ for all $a \neq b$. Order continuity of ψ_b implies that

$$\psi_b = \eta_b\varphi_{\pi(b)}$$

for some $\eta_b > 0$, and so, finally $\Phi(a \otimes \varphi_b) = \kappa_a \eta_b (\pi(a) \otimes \varphi_{\pi(b)})$. Since $(\Phi(a \otimes \varphi_a))^2 = \Phi(a \otimes \varphi_a)$ it follows that $\kappa_a \eta_a = 1$, yielding $\eta_a = \frac{1}{\kappa_a}$. By Lemma 5.3 we conclude

$$\Phi(a \otimes \varphi_b) = \tfrac{\gamma_a}{\gamma_b}(u_a \otimes \psi_b) = \tfrac{\gamma_a \kappa_a}{\gamma_b \kappa_b}(\pi(a) \otimes \varphi_{\pi(b)}).$$

By introducing $\delta_a = \gamma_a \kappa_a$ for each $a \in \mathcal{A}$ we obtain (3). One can verify (4) by a direct calculation.

To prove $\Phi(\mathscr{A}_0) = \mathscr{A}_0$ and $\Phi(\mathscr{A}_0^+) = \mathscr{A}_0^+$, note that Proposition 4.4(i), and (3) and (4) imply $\Phi(\mathscr{A}_0^+) \subseteq \mathscr{A}_0^+$ and $\Phi^{-1}(\mathscr{A}_0^+) \subseteq \mathscr{A}_0^+$, respectively, yielding $\Phi(\mathscr{A}_0^+) = \mathscr{A}_0^+$. Since $\mathscr{A}_0 = \mathscr{A}_0^+ - \mathscr{A}_0^+$, we obtain $\Phi(\mathscr{A}_0) = \mathscr{A}_0$.

(ii) Pick a positive operator T in $\mathscr{A}$ and atoms $a, b \in \mathcal{A}$. By (4), there exist atoms $c, d \in \mathcal{A}$ such that $\Phi^{-1}(c \otimes \varphi_c) = b \otimes \varphi_b$ and $\Phi^{-1}(d \otimes \varphi_d) = a \otimes \varphi_a$. Since $(c \otimes \varphi_c)T(d \otimes \varphi_d) = \varphi_c(Td)(c \otimes \varphi_d) \in \mathscr{A}_0^+$ and by (i) we have $\Phi^{-1}(\mathscr{A}_0^+) = \mathscr{A}_0^+$, we conclude that

$$\Phi^{-1}((c \otimes \varphi_c)T(d \otimes \varphi_d))a = (b \otimes \varphi_b)\Phi^{-1}(T)(a \otimes \varphi_a)a = \varphi_b(\Phi^{-1}(T)a)b$$

is a positive vector. In particular, $\varphi_b(\Phi^{-1}(T)a) \geq 0$ for all $a, b \in \mathcal{A}$. To finish the proof we apply the fact that $\Phi^{-1}(T)$ is order continuous and Lemma 4.1. □

Let $\mathscr{A} \subseteq \mathscr{L}_n(X)$ be any algebra of operators which contains $\mathscr{A}_0$ and let $\Phi \colon \mathscr{A} \to \mathscr{A}$ be a positive algebra automorphism. By Theorem 5.6 there exists a bijection $\pi \colon \mathscr{A} \to \mathscr{A}$ and a set of positive scalars $\{\delta_a : a \in \mathcal{A}\}$ such that for all $a, b \in \mathcal{A}$ we have

$$\Phi(a \otimes \varphi_b) = \tfrac{\delta_a}{\delta_b}\left(\pi(a) \otimes \varphi_{\pi(b)}\right).$$

Since $\mathcal{A}$ is a Hamel basis for $I_{\mathcal{A}}$, we can define $P, D \colon I_{\mathcal{A}} \to I_{\mathcal{A}}$ given by $Pa = \pi(a)$ and $Da = \delta_a a$ for each $a \in \mathcal{A}$. Note that P and D are invertible on $I_{\mathcal{A}}$ with their inverses given by $P^{-1}a = \pi^{-1}(a)$ and $D^{-1}a = \frac{1}{\delta_a}a$ for each $a \in \mathcal{A}$. We will call operators P and D a *permutation* and a *diagonal* operator, respectively. Conversely, for any set $\{\delta_a : a \in \mathcal{A}\}$ of real numbers we can define the diagonal operator on $I_{\mathcal{A}}$ as above. The set $\{\delta_a : a \in \mathcal{A}\}$ is called the set of all *diagonal coefficients* of the operator D. The operator $D \colon I_{\mathcal{A}} \to I_{\mathcal{A}}$ is positive if and only if $\delta_a \geq 0$ for every $a \in \mathcal{A}$ and D is bijective if and only if $\delta_a \neq 0$ for every $a \in \mathcal{A}$.

Recall that for a finite subset $F \subseteq \mathcal{A}$ the finite truncation T_F of T is defined by

$$T_F = \sum_{a,b \in F} (a \otimes \varphi_a)T(b \otimes \varphi_b).$$

The following lemma will be needed in the proof of Theorem 5.8.

Lemma 5.7 *Let $\mathscr{A} \subseteq \mathscr{L}_n(X)$ be a subalgebra containing $\mathscr{A}_0$ and $\Phi \colon \mathscr{A} \to \mathscr{A}$ be a positive algebra automorphism. Then for every $T \in \mathscr{A}$ we have*

$$\Phi(T_F) = \Phi(T)_{\pi(F)}$$

for every finite subset $F \subseteq \mathcal{A}$. Moreover, for every positive operator $T \in \mathscr{A}$ and every positive vector $x \in X$ we have $\Phi(T_F)x \nearrow \Phi(T)x$ where F runs over the family $\mathcal{F}$ of all finite subsets of $\mathcal{A}$ ordered by set inclusion.

Proof Let us choose a finite subset $F \subseteq \mathcal{A}$ and write $T_F = \sum_{a,b\in F}(a\otimes\varphi_a)T(b\otimes\varphi_b)$. Since Φ is an algebra homomorphism, by Theorem 5.6 we have

$$\begin{aligned}\Phi(T_F) &= \sum_{a,b\in F} \Phi(a\otimes\varphi_a)\Phi(T)\Phi(b\otimes\varphi_b)\\ &= \sum_{a,b\in F} (\pi(a)\otimes\varphi_{\pi(a)})\Phi(T)(\pi(b)\otimes\varphi_{\pi(b)})\\ &= \sum_{c,d\in\pi(F)} (c\otimes\varphi_c)\Phi(T)(d\otimes\varphi_d) = \Phi(T)_{\pi(F)}.\end{aligned}$$

To prove the moreover statement, note that Lemma 4.3 yields

$$\Phi(T_F)x = \Phi(T)_{\pi(F)}x \nearrow \Phi(T)x$$

since $\pi : \mathcal{A} \to \mathcal{A}$ is a bijection. □

Theorem 5.8 *Let $\mathscr{A} \subseteq \mathscr{L}_n(X)$ be a subalgebra containing $\mathscr{A}_0$ and let $\Phi\colon \mathscr{A} \to \mathscr{A}$ be a positive algebra automorphism. Then for each $T \in \mathscr{A}_0$ the ideal $I_{\mathcal{A}}$ is invariant under $\Phi(T)$ and we have*

$$\Phi(T)|_{I_{\mathcal{A}}} = PDTD^{-1}P^{-1}|_{I_{\mathcal{A}}}.$$

Furthermore, for each positive $T \in \mathscr{A}$ we have

$$\Phi(T)|_{I_{\mathcal{A}}} = \sup\{PDSD^{-1}P^{-1}|_{I_{\mathcal{A}}} \ :\ S \in [0,T]\cap\mathscr{A}_0\}.$$

Proof Since the set $\{a\otimes\varphi_b \ :\ a,b\in\mathcal{A}\}$ spans $\mathscr{A}_0$, by linearity of Φ it suffices to prove $\Phi(a\otimes\varphi_b)|_{I_{\mathcal{A}}} = PD(a\otimes\varphi_b)D^{-1}P^{-1}|_{I_{\mathcal{A}}}$. Pick any $c\in\mathcal{A}$. Then

$$\begin{aligned}PD(a\otimes\varphi_b)D^{-1}P^{-1}c &= PD(a\otimes\varphi_b)D^{-1}\pi^{-1}(c) = \frac{1}{\delta_{\pi^{-1}(c)}}PD(a\otimes\varphi_b)\pi^{-1}(c)\\ &= \frac{\varphi_b(\pi^{-1}(c))}{\delta_{\pi^{-1}(c)}}PDa = \frac{\varphi_b(\pi^{-1}(c))}{\delta_{\pi^{-1}(c)}}\delta_a\,\pi(a).\end{aligned}$$

If $c=\pi(b)$, then $PD(a\otimes\varphi_b)D^{-1}P^{-1}c = \frac{\delta_a}{\delta_b}\pi(a)$. Otherwise, we have $PD(a\otimes\varphi_b)D^{-1}P^{-1}c = 0$ proving $\Phi(T)|_{I_{\mathcal{A}}} = PDTD^{-1}P^{-1}|_{I_{\mathcal{A}}}$.

To prove the second formula, choose a positive operator $T\in\mathscr{A}$. By Lemma 4.3 we have that $T=\sup_{F\in\mathcal{F}}T_F$ is the supremum of the increasing net $(T_F)_{F\in\mathcal{F}}$ where F runs over the family $\mathcal{F}$ of all finite subsets of $\mathcal{A}$ ordered by set inclusion. Clearly, we have $T_F\in[0,T]\cap\mathscr{A}_0$, and for each $S\in[0,T]\cap\mathscr{A}_0$ we can find $F\in\mathcal{F}$

such that $S \leq T_F$. By Lemma 5.7 we have $\Phi(T_F)x \nearrow \Phi(T)x$ for each positive vector $x \in X$. In particular, this holds for every positive vector $x \in I_{\mathcal{A}}$ which yields $\Phi(T_F)|_{I_{\mathcal{A}}} \nearrow \Phi(T)|_{I_{\mathcal{A}}}$. To finish the proof note that we have

$$\begin{aligned}\Phi(T)|_{I_{\mathcal{A}}} &= \sup_{F\in\mathcal{F}} \Phi(T_F)|_{I_{\mathcal{A}}} = \sup\{\Phi(S)|_{I_{\mathcal{A}}} : S \in [0,T] \cap \mathscr{A}_0\} \\ &= \sup\{PDSD^{-1}P^{-1}|_{I_{\mathcal{A}}} : S \in [0,T] \cap \mathscr{A}_0\}.\end{aligned}$$

□

For the converse, assume $\Phi\colon \mathscr{A} \to \mathscr{A}$ is a linear operator such that there exist a positive invertible diagonal operator D and a permutation operator P such that for each $T \in \mathscr{A}_0$ the ideal $I_{\mathcal{A}}$ is invariant under $\Phi(T)$ and we have

$$\Phi(T)|_{I_{\mathcal{A}}} = PDTD^{-1}P^{-1}|_{I_{\mathcal{A}}}.$$

Then for $T, S \in \mathscr{A}$ we have

$$\begin{aligned}\Phi(TS)\big|_{I_{\mathcal{A}}} &= PDTSD^{-1}P^{-1}\big|_{I_{\mathcal{A}}} = PDTD^{-1}P^{-1}PDSD^{-1}P^{-1}\big|_{I_{\mathcal{A}}} \\ &= PDTD^{-1}P^{-1}\big|_{I_{\mathcal{A}}} PDSD^{-1}P^{-1}\big|_{I_{\mathcal{A}}} \\ &= \Phi(T)\big|_{I_{\mathcal{A}}}\Phi(S)\big|_{I_{\mathcal{A}}}\end{aligned}$$

Since operators in $\mathscr{A}$ are order continuous and $I_{\mathcal{A}}$ is order dense in X, we have $\Phi(TS) = \Phi(T)\Phi(S)$ for all $T, S \in \mathscr{A}$.

6 Algebras of operators on $c_{00}(\Lambda)$

Let X be an atomic Archimedean vector lattice. If we fix any maximal set $\mathcal{A}$ of pairwise disjoint atoms in X, then the ideal $I_{\mathcal{A}}$ generated by atoms in $\mathcal{A}$ equals the linear span of $\mathcal{A}$. Hence, $I_{\mathcal{A}}$ can be realized as the vector lattice $c_{00}(\mathcal{A})$ of all finitely supported functions defined on $\mathcal{A}$. In order to better understand the structure of positive automorphisms from Theorem 5.8 we restrict ourselves to the special case where the underlying vector lattice X is the vector lattice $c_{00}(\Lambda)$ equipped with the supremum norm. By e_λ we denote the characteristic function of the set $\{\lambda\}$. When $\Lambda = \mathbb{N}$, we rather write c_{00} instead of $c_{00}(\mathbb{N})$. Since it is standard to interpret elements of c_{00} as eventually null sequences, we can also treat elements of $c_{00}(\Lambda)$ as finitely supported "sequences" indexed by the index set Λ. As special cases of Corollary 6.3 and Corollary 6.6, we deduce that every positive algebra automorphism of $\mathscr{L}(c_{00}(\Lambda))$ and $\mathscr{B}(c_{00}(\Lambda))$ is inner. Since Theorem 5.8 is applicable only to subalgebras contained in the algebra of order continuous operators, it is important to find all order continuous operators in $\mathscr{B}(c_{00}(\Lambda))$. In Theorem 6.1 we provide an operator theoretical characterization of vector lattices that are lattice isomorphic to a vector lattice of the form c_{00}. In particular, it follows that every linear operator on $c_{00}(\Lambda)$ is order continuous, in particular, also order bounded by [2, Theorem 2.1].

Theorem 6.1 *For an Archimedean vector lattice X the following statements are equivalent.*

(i) *X is lattice isomorphic to $c_{00}(\Lambda)$ for some non-empty index set Λ.*
(ii) *$\mathscr{L}(X, Y) = \mathscr{L}_n(X, Y)$ for every Archimedean vector lattice Y.*
(iii) *$\mathscr{L}(X, Y) = \mathscr{L}_b(X, Y)$ for every Archimedean vector lattice Y.*
(iv) *$X^{\sim} = X'$.*

Proof (i) $\Rightarrow$ (ii) Assume first that X is lattice isomorphic to $c_{00}(\Lambda)$ for some non-empty set Λ. Since the inclusion $\mathscr{L}_n(X, Y) \subseteq \mathscr{L}(X, Y)$ always holds, it suffices to prove $\mathscr{L}(X, Y) \subseteq \mathscr{L}_n(X, Y)$.

Pick any lattice isomorphism $T : c_{00}(\Lambda) \to X$. If $S_0 \in \mathscr{L}(X, Y)$, then $S_0 T \in \mathscr{L}(c_{00}(\Lambda), Y)$. We claim that $S := S_0 T \in \mathscr{L}_n(c_{00}(\Lambda), Y)$. To this end, pick any net $(x_\alpha)_\alpha$ in $c_{00}(\Lambda)$ that converges in order to 0. By passing to a tail, if necessary, we may assume that the net $(x_\alpha)_\alpha$ is order bounded. Hence, there exists $x \geq 0$ in $c_{00}(\Lambda)$ such that $0 \leq |x_\alpha| \leq x$ for every α. There exists a finite subset $\Lambda_0 \subseteq \Lambda$ such that $x = \sum_{\lambda \in \Lambda_0} x^{(\lambda)} e_\lambda$. Since $x_\alpha \to 0$ in order, it follows that for every $\lambda \in \Lambda_0$ we have $x_\alpha^{(\lambda)} \to 0$ in $\mathbb{R}$. Pick any $\varepsilon > 0$. Since Λ_0 is finite, there exists an index α_ε such that for all $\alpha \geq \alpha_\varepsilon$ and each $\lambda \in \Lambda_0$ we have $0 \leq |x_\alpha^{(\lambda)}| < \varepsilon$. Let us denote by f the vector $\sum_{\lambda \in \Lambda_0} |Se_\lambda|$. Then for every $\alpha \geq \alpha_\varepsilon$ we have

$$0 \leq |Sx_\alpha| = \left| \sum_{\lambda \in \Lambda_0} x_\alpha^{(\lambda)} Se_\lambda \right| \leq \varepsilon f.$$

Since Y is Archimedean, we have $\varepsilon f \searrow 0$ as $\varepsilon \searrow 0$, so that $Sx_\alpha \to 0$ in order which proves order continuity of S. Since T is a lattice isomorphism, it follows that $S_0 = ST^{-1}$ is order continuous.

(ii) $\Rightarrow$ (iii) Since every order continuous operator is order bounded by [2, Theorem 2.1], we have

$$\mathscr{L}(X, Y) = \mathscr{L}_n(X, Y) \subseteq \mathscr{L}_b(X, Y) \subseteq \mathscr{L}(X, Y).$$

(iii) $\Rightarrow$ (iv) We take $Y = \mathbb{R}$.

(iv) $\Rightarrow$ (i) Pick any positive vector $x \in X$ and consider the principal ideal I_x equipped with the norm $\|\cdot\|_x$. We claim that I_x is finite-dimensional. If this were not the case, then by [6, Theorem 26.10] I_x would contain an infinite sequence $(e_n)_{n \in \mathbb{N}}$ of pairwise disjoint positive vectors. By scaling, if necessary, assume that $0 \leq e_n \leq x$. If we define the vector $y_n := \sum_{k=1}^n e_k$, disjointness and positivity of vectors $e_1, \ldots, e_n$ yields

$$0 \leq y_n = \sum_{k=1}^n e_k = \bigvee_{k=1}^n e_k \leq x.$$

Since disjoint vectors are always linearly independent, the set $\{e_n : n \in \mathbb{N}\}$ can be extended to a basis of X. Pick any linear functional φ with $\varphi(e_n) = 1$ for each $n \in \mathbb{N}$. Since, by assumption, φ is order bounded, by the Riesz-Kantorovich theorem [1, Theorem 1.18] there exists $|\varphi| : X \to \mathbb{R}$. As $|\varphi(z)| \leq |\varphi|(|z|)$ for every vector z, for each $n \in \mathbb{N}$ we get

$$|\varphi|(x) \geq |\varphi|(y_n) \geq \varphi(y_n) = n$$

which is clearly impossible. This contradiction shows that I_x is finite-dimensional. By [6, Theorem 61.4] it follows that X is lattice isomorphic to $c_{00}(\Lambda)$ for some nonempty set Λ. □

Corollary 6.2 *For a nonempty set Λ and for every Archimedean vector lattice Y we have*

$$\mathscr{L}(c_{00}(\Lambda), Y) = \mathscr{L}_b(c_{00}(\Lambda), Y) = \mathscr{L}_n(c_{00}(\Lambda), Y).$$

In particular $c_{00}(\Lambda)' = c_{00}(\Lambda)^{\sim} = c_{00}(\Lambda)^{\sim}_n$.

The following corollary implies that every positive algebra automorphism of $\mathscr{L}(c_{00}(\Lambda))$ is inner.

Corollary 6.3 *Let Λ be a nonempty set and let $\mathscr{A}$ be a subalgebra of $\mathscr{L}(c_{00}(\Lambda))$ which contains $\mathscr{A}_0$. If $\Phi\colon \mathscr{A} \to \mathscr{A}$ is a positive algebra automorphism, then*

$$\Phi(T) = PDTD^{-1}P^{-1}$$

for some permutation operator P and some positive invertible diagonal operator D on $c_{00}(\Lambda)$.

Proof Observe first that by Theorem 6.1 we have $\mathscr{L}_n(c_{00}) = \mathscr{L}(c_{00})$. Since $I_{\mathcal{A}} = c_{00}(\Lambda)$, by Theorem 5.8 we have

$$\Phi(T) = \sup\{PDSD^{-1}P^{-1} : \ S \in [0, T] \cap \mathscr{A}_0\}$$

for each positive operator T on $c_{00}(\Lambda)$. To prove $\Phi(T) = PDTD^{-1}P^{-1}$ we will apply [1, Theorem 1.19]. Note first, that the set $[0, T] \cap \mathscr{A}_0$ can be considered as an increasing net $(T_\alpha)_\alpha$ with supremum T. Hence, for each positive vector x we have $0 \leq T_\alpha x \nearrow Tx$. If we replace x with $D^{-1}P^{-1}x$, we obtain $0 \leq T_\alpha D^{-1}P^{-1}x \nearrow TD^{-1}P^{-1}x$. Order continuity of PD yields $0 \leq PDT_\alpha D^{-1}P^{-1}x \nearrow PDTD^{-1}P^{-1}x$ so that $\Phi(T) = PDTD^{-1}P^{-1}$. □

In the following example, we present an algebra $\mathscr{A} \subseteq \mathscr{L}(c_{00})$ containing $\mathscr{A}_0$ and a positive automorphism $\Phi\colon \mathscr{A} \to \mathscr{A}$ that is not inner.

Example 6.4 Let $\mathscr{A}$ be the algebra in $\mathscr{L}(c_{00})$ generated by the identity operator I and the set $\{e_i \otimes \varphi_{e_j} : \ i, j \in \mathbb{N}\}$. Let P be the permutation operator induced by the bijection π defined as $\pi(2k-1) = 2k$ and $\pi(2k) = 2k-1$. Then it is easy to see that $\Phi\colon T \mapsto PTP^{-1}$ is a positive spatial automorphism of $\mathscr{A}$. We claim that Φ is not inner. If Φ were inner, then there would exist an invertible operator $S \in \mathscr{A}$ such that $\Phi(T) = STS^{-1}$ for all $T \in \mathscr{A}$. This yields that $S^{-1}P$ belongs to the centralizer of $\mathscr{A}$. Since $\mathscr{A}_0$ is order dense in $\mathscr{L}(c_{00}) = \mathscr{L}_n(c_{00})$ and $\mathscr{A}_0 \subseteq \mathscr{A}$, it follows from the order continuity of $S^{-1}P$ that $S^{-1}P$ belongs to the center of $\mathscr{L}(c_{00})$. Consequently, there exists a nonzero scalar λ such that $S^{-1}P = \lambda I$. It then follows that $P \in \mathscr{A}$, which is not true.

Corollary 6.3 is, in particular, applicable to subalgebras $\mathscr{A}$ of bounded operators on $\mathscr{L}(c_{00}(\Lambda))$. One could expect that P and D obtained by Corollary 6.3 are bounded. A direct verification shows that P is indeed bounded, whereas to prove boundedness of D we will need the following lemma which explicitly provides the operator norm of a bounded operator on the normed space $c_{00}(\Lambda)$.

Lemma 6.5 *A linear operator $T : c_{00}(\Lambda) \to c_{00}(\Lambda)$ is bounded if and only if*

$$M := \sup_{\lambda\in\Lambda} \sum_{\mu\in\Lambda} |\varphi_{e_\lambda}(Te_\mu)| < \infty.$$

Moreover, if T is bounded then $\|T\| = M$.

Proof Without loss of generality we may assume that T is nonzero. Assume first that T is bounded and pick $\lambda \in \Lambda$. For a finite subset $F \subseteq \Lambda$ we define

$$x = \sum_{\mu\in F} \operatorname{sgn}(\varphi_{e_\lambda}(Te_\mu))e_\mu \in c_{00}(\Lambda)$$

and observe that $\|x\| \leqslant 1$. Since $Tx \in c_{00}(\Lambda)$, for any finite subset $G \subseteq \Lambda$ containing the support F_{Tx} of Tx we have

$$Tx = \sum_{\lambda\in G}\left(\sum_{\mu\in F} \operatorname{sgn}(\varphi_{e_\lambda}(Te_\mu))\varphi_{e_\lambda}(Te_\mu)\right) e_\lambda.$$

Pick any $\lambda_0 \in \Lambda$ and define $G := F_{Tx} \cup \{\lambda_0\}$. As $\|x\| \leqslant 1$, we obtain

$$\begin{aligned}\|T\| \geq \|Tx\| &= \max_{\lambda\in G}\left|\sum_{\mu\in F} \operatorname{sgn}(\varphi_{e_\lambda}(Te_\mu))\varphi_{e_\lambda}(Te_\mu)\right| = \max_{\lambda\in G}\sum_{\mu\in F}|\varphi_{e_\lambda}(Te_\mu)| \\ &\geq \sum_{\mu\in F}|\varphi_{e_{\lambda_0}}(Te_\mu)|.\end{aligned}$$

By definition of convergence we get $\|T\| \geqslant \sum_{\mu\in\Lambda} |\varphi_{e_{\lambda_0}}(Te_\mu)|$. Since $\lambda_0 \in \Lambda$ was arbitrary, we conclude $\|T\| \geq M$.

To prove the converse statement, assume that $M < \infty$ and pick any unit vector $x = \sum_{\mu\in F_x} x_\mu e_\mu \in c_{00}(\Lambda)$ where $x_\mu = \varphi_{e_\mu}(x)$. Then $|x_\mu| \leq 1$ for every $\mu \in \Lambda$. Pick any $\lambda \in \Lambda$. Then the λ-th component of Tx equals $\varphi_{e_\lambda}(Tx) = \sum_{\mu\in F_x} \varphi_{e_\lambda}(Te_\mu)x_\mu$ and so

$$|\varphi_{e_\lambda}(Tx)| \leq \sum_{\mu\in F_x} |\varphi_{e_\lambda}(Te_\mu)||x_\mu| \leq \sum_{\mu\in F_x} |\varphi_{e_\lambda}(Te_\mu)| \leq M.$$

Hence, $\|Tx\| \leq M$ for every unit vector $x \in c_{00}(\Lambda)$ which finally gives $\|T\| \leq M$.

For the moreover statement, observe that a combination of the conclusions above implies $\|T\| = M$. □

Corollary 6.6 *Let Λ be a nonempty set. Then for every positive automorphism Φ of $\mathscr{B}(c_{00}(\Lambda))$ there exist a permutation operator P and a bounded positive diagonal operator D on $c_{00}(\Lambda)$ with bounded inverse such that*

$$\Phi(T) = PDTD^{-1}P^{-1}$$

for each $T \in \mathscr{B}(c_{00}(\Lambda))$.

In particular, every positive automorphism Φ of $\mathscr{B}(c_{00}(\Lambda))$ is inner.

Proof By Corollary 6.3 there exist a permutation operator P and a positive diagonal operator D such that $\Phi(T) = PDTD^{-1}P^{-1}$ for each $T \in \mathscr{B}(c_{00}(\Lambda))$. We need to prove that D and D^{-1} are bounded. It suffices to see that the sets of diagonal coefficients $\{d_\lambda : \lambda \in \Lambda\}$ and $\{d_\lambda^{-1} : \lambda \in \Lambda\}$ are bounded. To this end, we will apply Lemma 6.5.

By assumption, for every $T \in \mathscr{B}(c_{00}(\Lambda))$ we know that $\Phi(T) = PDTD^{-1}P^{-1}$ belongs to $\mathscr{B}(c_{00}(\Lambda))$ yielding that DTD^{-1} is a bounded operator on $c_{00}(\Lambda)$. Therefore, by Lemma 6.5 we have

$$\|DTD^{-1}\| = \sup_{\lambda\in\Lambda} \sum_{\mu\in\Lambda} |\varphi_{e_\lambda}(DTD^{-1}e_\mu)|.$$

Let us fix $\alpha \in \Lambda$ and choose an absolutely convergent series $\sum_{\mu\in\Lambda} t_\mu$. For each $\mu \in \Lambda$ we define $Te_\mu = t_\mu e_\alpha$. By linearity, T can be uniquely extended to a linear operator (again denoted by T) on $c_{00}(\Lambda)$. Using Lemma 6.5 we see that T is bounded.

Since DTD^{-1} is bounded and $DTD^{-1}e_\mu = d_\alpha d_\mu^{-1} t_\mu e_\alpha$, by Lemma 6.5 we conclude

$$\|DTD^{-1}\| = \sum_{\mu\in\Lambda} |d_\alpha d_\mu^{-1} t_\mu| = d_\alpha \sum_{\mu\in\Lambda} d_\mu^{-1}|t_\mu| < \infty.$$

Therefore, for every absolutely convergent series $\sum_{\mu\in\Lambda} t_\mu$ the series $\sum_{\mu\in\Lambda} d_\mu^{-1}|t_\mu|$ also converges.

For a finite subset $K \subseteq \Lambda$ we define a functional $\varphi_K : \ell^1(\Lambda) \to \mathbb{R}$ as

$$\varphi_K((t_\mu)_\mu) = \sum_{\mu\in K} d_\mu^{-1} t_\mu.$$

Then $\|\varphi_K\| = \max_{\mu\in K} d_\mu^{-1}$. For a fixed $t = (t_\mu)_\mu$, the set

$$\{\varphi_K(t) : \ K \subseteq \Lambda \text{ finite}\}$$

is bounded since $|\varphi_K(t)| \leqslant \sum_{\mu\in\Lambda} d_\mu^{-1}|t_\mu|$. The principle of uniform boundedness implies that the norms $\{\|\varphi_K\| : \ K \subseteq \Lambda \text{ finite}\}$ are bounded yielding that $(d_\mu^{-1})_{\mu\in\Lambda}$ is bounded. Therefore $D^{-1} \in \mathscr{B}(c_{00}(\Lambda))$.

Since, clearly the linear mapping $\Psi : T \mapsto P^{-1}D^{-1}TDP$ is the inverse of Φ, it is a positive automorphism of $\mathscr{B}(c_{00}(\Lambda))$. By the proof above, D is bounded which completes the proof. □

7 Concluding remarks on order bounded functionals

By Theorem 6.1, an Archimedean vector lattice X admits a non-order bounded functional if and only if X is not lattice isomorphic to a vector lattice of the form $c_{00}(\Lambda)$ for some set Λ. In non-Archimedean vector lattices, constructing non-order bounded functionals is simple, as the so-called infinitely small elements come to our help. Recall that a positive vector x of a vector lattice X is *infinitely small* whenever there exists a positive vector y such that for all $n \in \mathbb{N}$ we have $0 \leq nx \leq y$. Clearly, a vector lattice is not Archimedean if and only if it contains nonzero infinitely small positive vectors.

Example 7.1 Let X be a non-Archimedean vector lattice. Then there exist positive nonzero vectors $x, y \in X$ such that for each $n \in \mathbb{N}$ we have $0 < nx \leq y$. Let φ be any linear functional on X such that $\varphi(x) = 1$. Then for each $\alpha > 0$ we have $\alpha = \varphi(\alpha x)$. Since $0 \leq \alpha \leq n$ yields $0 \leq \alpha x \leq nx \leq y$, we conclude $[0, \infty) \subseteq \varphi([0, y])$, and so, φ is not order bounded. In particular, every order bounded functional vanishes on the set of infinitely small vectors.

Recall that the lexicographical product $X \circ Y$ of vector lattices X and Y is the vector space $X \times Y \cong X \oplus Y$ equipped with the lexicographical partial ordering $\leq_{\text{Lex}}$ defined as $(x_1, y_1) \leq_{\text{Lex}} (x_2, y_2)$ if $x_1 < x_2$ or $x_1 = x_2$ and $y_1 \leq y_2$. It is a standard exercise to prove that $X \circ Y$ is a vector lattice. By $\mathbb{R}^n_{\text{Lex}}$ we denote the n-dimensional vector space $\mathbb{R}^n$ of all n-tuples ordered lexicographically. Similarly, $\mathbb{R}^\infty_{\text{Lex}}$ denotes the space of all real sequences ordered lexicographically. Therefore, $\mathbb{R} \circ \mathbb{R}$ is the lexicographically ordered real plane $\mathbb{R}^2_{\text{Lex}}$, and $\mathbb{R} \circ (\mathbb{R} \circ \mathbb{R}) = \mathbb{R} \circ \mathbb{R}^2_{\text{Lex}}$ is the lexicographically ordered real vector space $\mathbb{R}^3$. The positive cones of these examples fall into the scope of the so-called “lexicographic cones" studied by Wortel in [9].

Since the lexicographically ordered real plane $\mathbb{R}^2_{\text{Lex}}$ is not Archimedean, $\mathbb{R}^2_{\text{Lex}}$ admits a non-order bounded linear functional. As $0 \leq ne_2 \leq e_1$ for all $n \in \mathbb{N}$, every order bounded functional φ on $\mathbb{R}^2_{\text{Lex}}$ satisfies $\varphi(e_2) = 0$. Hence, it seems that the order dual of $\mathbb{R}^2_{\text{Lex}}$ is isomorphic to the first copy of $\mathbb{R}$ in the lexicographical product. The remaining part of the paper is devoted to determine order duals of the following lexicographically ordered vector lattices $\mathbb{R}^n_{\text{Lex}}$ and $\mathbb{R}^\infty_{\text{Lex}}$. We start with a more general result.

Theorem 7.2 *For vector lattices X and Y the mapping $\Phi\colon (X \circ Y)^\sim \to X^\sim$ defined as $\varphi \to \varphi|_X$ is a lattice isomorphism.*

In particular, if $X^\sim = \{0\}$, then for every vector lattice Y we have $(X \circ Y)^\sim = \{0\}$.

Proof We claim that the mapping $\Phi\colon (X \circ Y)^\sim \to X^\sim$ defined as $\Phi(\varphi) = \varphi|_X$ is a lattice isomorphism. If φ is order bounded on $X \circ Y$, then $\Phi(\varphi)$ is order bounded on X since every interval $[a, b]_X$ in X is contained in the interval $[a, b]_{X \circ Y}$ in $X \circ Y$ via the natural identification of X and $X \times \{0\}$. Clearly, Φ is a linear operator.

We claim that every order bounded functional $\varphi \in (X \circ Y)^\sim$ is zero on Y. If this were not the case, then there would exist $0 < y \in Y$ such that $\varphi(y) \neq 0$. By replacing φ with $-\varphi$, if necessary, we may suppose that $\varphi(y) > 0$. Pick any $0 \leq x \in X$. Then

$0 \le \lambda y \le x + y$ for all $\lambda \ge 0$ which yields that $\varphi([0, x + y])$ contains $[0, \infty)$. This is in contradiction with order boundedness of φ.

If $\Phi(\varphi) = \varphi|_X = 0$ for some $\varphi \in (X \circ Y)^\sim$, then $\varphi = 0$ as $\varphi|_Y = 0$, which proves injectivity of Φ. Since for every $\varphi \in X^\sim$ we have $\varphi + 0 \in (X \circ Y)^\sim$ it follows that Φ is surjective. To prove that Φ is a lattice isomorphism, observe that $\varphi + 0$ is positive on $X \circ Y$ if and only if φ is positive on X. □

Corollary 7.3 *For $n, m \in \mathbb{N}$ the order dual of $\mathbb{R}^n \circ \mathbb{R}^m$ is lattice isomorphic to $\mathbb{R}^n$. The lattice isomorphism $\Phi\colon (\mathbb{R}^n \circ \mathbb{R}^m)^\sim \to \mathbb{R}^n$ is given as*

$$\Phi(\varphi) = (\varphi(e_1), \ldots, \varphi(e_n)).$$

Corollary 7.4 *For every $n \in \mathbb{N} \cup \{\infty\}$ the order dual of $\mathbb{R}^n_{\text{Lex}}$ is lattice isomorphic to $\mathbb{R}$. The lattice isomorphism $\Phi\colon (\mathbb{R}^n_{\text{Lex}})^\sim \to \mathbb{R}$ is given as $\Phi(\varphi) = \varphi(e_1)$.*

Proof Since for $n \in \mathbb{N}$ we have $\mathbb{R}^n_{\text{Lex}} \cong \mathbb{R} \circ \mathbb{R}^{n-1}_{\text{Lex}}$ and $\mathbb{R}^\infty_{\text{Lex}} \cong \mathbb{R} \circ \mathbb{R}^\infty_{\text{Lex}}$, the order dual of $\mathbb{R}^n_{\text{Lex}}$ is lattice isomorphic to the order dual of the first copy of $\mathbb{R}$ by Theorem 7.2. □

Acknowledgements We would like to thank the anonymous referee for improving Corollary 4.6. The first author was supported by the Slovenian Research and Innovation Agency program P1-0222. The second author was supported by the Slovenian Research and Innovation Agency program P1-0222 and grant J1-50002.

Author Contributions G.C. and M.K. together discussed, proved and wrote all the material together.

Data Availability No datasets were generated or analysed during the current study.

Declarations

Competing interests The authors declare no competing interests.

References

1. Aliprantis, C.D., Burkinshaw, O.: Positive Operators. Springer, Dordrecht (2006). (**Reprint of the 1985 original**)
2. Abramovich, Y., Sirotkin, G.: On order convergence of nets. Positivity **9**, 287–292 (2005)
3. Eidelheit, M.: On isomorphisms of rings of linear operators. Studia Math. **9**, 97–105 (1940)
4. Farb, B., Dennis, R.K.: Noncommutative Algebra, vol. 144. Springer-Verlag, New York (1993). (**Graduate Texts in Mathematics**)
5. Gao, N., Leung, D.H.: Smallest order closed sublattices and option spanning. Proc. Amer. Math. Soc. **146**(2), 705–716 (2018)

6. Luxemburg, W.A.J., Zaanen, A.C.: Riesz Spaces, vol. I. North-Holland Publishing Co., North-Holland Mathematical Library, Amsterdam-London (1971)
7. Muñoz-Lahoz, D., Tradacete, P.: Band projections in spaces of regular operators. Trans. Amer. Math. Soc. **377**(7), 5197–5218 (2024)
8. Sourour, A.R.: Spectrum-preserving linear maps on the algebra of regular operators. In: Aspects of Positivity in Functional Analysis (Tübingen, 1985), North-Holland Math. Stud., vol. 122, pp. 255–259. North-Holland, Amsterdam (1986)
9. Wortel, M.: Lexicographic Cones and the Ordered Projective Tensor Product, Positivity and Noncommutative Analysis, Trends in Mathematics, pp. 601–609. Birkhäuser/Springer, Cham (2019)