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On double Pythagorean-Hodograph curves of degree seven

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HIGHLIGHTS

- Complete analysis of degree seven DPH curves.
- Determination of helical/non-helical properties.
- Determination of number of degrees of freedom for construction.
- Numerical interpolation examples with the derived curves.

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ABSTRACT

This paper presents a comprehensive and constructive analysis of degree 7 double Pythagorean-Hodograph (DPH) curves for the three distinct structural classes. A unified framework is introduced for their construction, particularly useful for interpolation tasks involving prescribed boundary data. The approach explicitly identifies the degrees of freedom available in each class and distinguishes between helical and non-helical curve types. Furthermore, the structure of a specific rational curve in the complex plane that via the normalized Hopf map generates the tangent indicatrix is revealed for all degree 7 DPH curves. This confirms known results for helical curves and extends the interpretation to non-helical cases. The practical applicability of the derived curves is demonstrated through a numerical interpolation example, which also validates the stated number of degrees of freedom.

1. Introduction

A spatial parametric curve $\boldsymbol{p} : [a,b] \to \mathbb{R}^3$, defined by $\boldsymbol{p}(t) = (x(t), y(t), z(t))$, can be equipped with an orthonormal frame – known as the Frenet frame – provided that $\boldsymbol{p}'(t) \neq 0$ and $(\boldsymbol{p}' \times \boldsymbol{p}'')(t) \neq 0$ for all $t \in [a,b]$. The Frenet frame consists of three vector-valued functions $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$,

$$\boldsymbol{t} = \frac{\boldsymbol{p}'}{\|\boldsymbol{p}'\|}, \quad \boldsymbol{n} = \frac{\boldsymbol{p}' \times \boldsymbol{p}''}{\|\boldsymbol{p}' \times \boldsymbol{p}''\|} \times \boldsymbol{t}, \quad \boldsymbol{b} = \frac{\boldsymbol{p}' \times \boldsymbol{p}''}{\|\boldsymbol{p}' \times \boldsymbol{p}''\|},$$

that at every point $\boldsymbol{p}(t)$ along the curve prescribe three orthonormal vectors: the unit tangent vector $\boldsymbol{t}(t)$, the principal normal vector $\boldsymbol{n}(t)$ and the binormal vector $\boldsymbol{b}(t)$.

Orthonormal frames play a crucial role in practical applications such as computer-aided geometric design, robot motion planning, computer graphics, and animation, where they serve as local moving coordinate systems relative to a fixed global coordinate system.

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However, a rational representation of both the curve and its associated orthonormal frame is often desirable. Unfortunately, the Frenet frame does not generally admit a rational form, even when the curve $\boldsymbol{p}$ itself is given by a polynomial parameterization.

The need for rational representation of unit tangent vectors inspired the definition of (polynomial) Pythagorean-hodograph (PH) curves, first introduced in Farouki et al. (1990) and characterized by the property that the norm of the hodograph $\boldsymbol{h} = \boldsymbol{p}'$ is a polynomial, i.e., $\|\boldsymbol{h}\| = \sigma$, where the polynomial σ is called the *parametric speed*. Following their introduction, both planar and spatial PH curves as well as PH curves in other spaces like the Minkowski space have been the subject of extensive study (see Farouki (2008); Choi et al. (2002); Kosinka and Lávička (2014); Farouki et al. (2019), and the references therein). Another key advantage of the polynomial PH curve is that its arc length function is also a polynomial. This property significantly simplifies the computation of PH curves with prescribed length and allows simple real-time interpolator algorithms, which makes PH curves useful in robotics as well.

For spatial PH curves it is well known that their construction from the quaternion-valued polynomials (called preimage polynomials) allows one to equip the curve with rational adapted orthonormal frame known as Euler-Rodrigues (ER) frame (Choi and Han, 2002), where *adapted* stands for frames having the first frame vector equal to the unit tangent vector. The Frenet frame of the PH curve is *adapted*, but not generally rational since the normal $\boldsymbol{n}$ and the binormal $\boldsymbol{b}$ are not rational unless the expression $\|\boldsymbol{p}'\times\boldsymbol{p}''\|$ is a polynomial too. This additional condition defines a subset of PH curves, known as 'double' Pythagorean-hodograph (DPH) curves.

In addition to admitting a rational Frenet frame, another important property of polynomial DPH curves is that the curvature and the torsion

$$\kappa = \frac{\|\boldsymbol{p}'\times\boldsymbol{p}''\|}{\|\boldsymbol{p}'\|^3}, \quad \tau = \frac{(\boldsymbol{p}'\times\boldsymbol{p}'')\cdot\boldsymbol{p}'''}{\|\boldsymbol{p}'\times\boldsymbol{p}''\|^2}$$

are both rational, whereas for general polynomial curves this is true only for the torsion.

The importance of the double PH structure was first highlighted in Beltran and Monterde (2007) in the context of helical polynomial curves, i.e., curves with the constant ratio between the curvature and torsion. The authors observed that every helical polynomial curve is necessarily a PH curve and that $\|\boldsymbol{p}'\times\boldsymbol{p}''\|$ is a polynomial function for such curves implying that all helical polynomial curves are DPH curves. However, the reverse does not hold: there exist non-helical DPH curves of degree 7, as noted in Beltran and Monterde (2007).

Although the DPH condition has a relatively simple algebraic form, constructing a DPH curve involves solving an underdetermined nonlinear system of equations that involves the coefficients of the curve in a chosen representation, such as the Bézier form. This problem was previously addressed in Farouki et al. (2009a,b) for cubic, quintic, and septic (degree 7) DPH curves. In particular, for degree 7, the authors derived complex equations corresponding to three distinct structural cases. Although their work includes important results, particularly on helical DPH curves, they did not provide a general method for expressing the Bézier coefficients in a way that clearly distinguishes the true degrees of freedom from the dependent parameters. As noted in their work, the purely algebraic approach they proposed is not well suited for constructing curves with prescribed geometric properties – such as those stated in interpolation – and they identified the development of more geometrically intuitive construction algorithms as 'work to be done'.

Motivated by this observation, our paper provides a complete and constructive analysis of degree 7 DPH curves for all three distinct structural cases – called classes. We present a framework that enables straightforward construction of these curves and is especially suitable for interpolation problems where the boundary data need to be matched. Our approach explicitly identifies the degrees of freedom available for each class and organizes the construction process so that it distinguishes between helical and non-helical curves. Moreover, the structure of a particular rational curve in the complex plane, that defines through the normalized Hopf map the tangent indicatrix, is revealed for all septic DPH curves, confirming the results known for helical curves and indicating from the perspective of the tangent indicatrix their generalization to non-helical ones. The practical usability of the derived DPH curves for interpolation purposes, by which the reported numbers of degrees of freedom are also numerically validated, is demonstrated through a particular interpolation example.

The paper is organized as follows. Section 2 provides a review of the general theory of PH and DPH curves, along with the notation adopted throughout the paper. Section 3 presents a detailed analysis of degree 7 DPH curves, including theorems that fully characterize their construction across all three classes and specify the corresponding degrees of freedom. Section 4 investigates and proves the helical and non-helical properties of these curves. The paper is concluded with examples given in Section 5 that numerically confirm the derived theoretical results, and future work directions outlined in Section 6.

2. Characterization of PH and DPH curves

2.1. A short comment on notation

In this paper, quaternions are denoted by calligraphic capital letters (e.g., $\mathcal{A}, \mathcal{B}$), complex numbers and complex-valued functions by lower-case letters in a monospace (e.g., $\mathtt{a}, \mathtt{b}$), real numbers and real-valued functions by italic lower-case letters (e.g., a, b) or lower-case Greek letters (e.g., α, β), and vectors by bold letters (e.g., $\boldsymbol{a}, \boldsymbol{b}$).

For a quaternion $\mathcal{A} \in \mathbb{H}$, $\mathcal{A} = \alpha_0\,\mathbf{1} + \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \alpha_3\,\mathbf{k}$, we denote the conjugate quaternion by $\overline{\mathcal{A}} = \alpha_0\,\mathbf{1} - \alpha_1\,\mathbf{i} - \alpha_2\,\mathbf{j} - \alpha_3\,\mathbf{k}$, and define the star operation $\star : \mathbb{H}\times\mathbb{H} \to \mathbb{R}^3$ between two quaternions as

$$\mathcal{A} \star \mathcal{B} = \frac{1}{2}\left(\mathcal{A}\mathbf{i}\overline{\mathcal{B}} + \mathcal{B}\mathbf{i}\overline{\mathcal{A}}\right).$$

Additionally, $\mathcal{A}^{2\star} = \mathcal{A} \star \mathcal{A} = \mathcal{A}\mathbf{i}\overline{\mathcal{A}}$. We denote by $\mathbb{P}_n = \mathbb{P}_n(\mathbb{R})$, $\mathbb{P}_n(\mathbb{C})$ and $\mathbb{P}_n(\mathbb{H})$ real–valued, complex–valued and quaternion–valued polynomials of degree n in a real variable, respectively. For the basis of $\mathbb{P}_n$ we use the Bernstein basis polynomials, defined by

$$B_j^n(t) = \binom{n}{j} t^j (1-t)^{n-j}, \quad j = 0, 1, \ldots, n.$$

The following two subsections provide a summary of the key definitions and properties that will be required in the subsequent analysis (see Farouki (2008) for more details).

2.2. Representation of PH curves of degree $2m+1$

PH curves of degree $2m+1$ can be derived from a preimage polynomial $\mathcal{A}$, which is a quaternion-valued polynomial of degree m, i.e., $\mathcal{A} \in \mathbb{P}_m(\mathbb{H})$,

$$\mathcal{A}(t) = a_0(t)\,\mathbf{1} + a_1(t)\,\mathbf{i} + a_2(t)\,\mathbf{j} + a_3(t)\,\mathbf{k} = \sum_{j=0}^{m} \mathcal{A}_j B_j^m(t), \quad \mathcal{A}_j \in \mathbb{H}, \tag{1}$$

where $a_i \in \mathbb{P}_m$, $i = 0, 1, 2, 3$, are real-valued polynomials of degree $\leq m$ with real coefficients.
By computing the hodograph

$$\boldsymbol{h}(t) = \mathcal{A}^{2\star}(t) = \sum_{j=0}^{2m} \boldsymbol{h}_j B_j^{2m}(t), \qquad \boldsymbol{h}_j = \sum_{\ell=\max(0,j-m)}^{\min(m,j)} \frac{\binom{m}{\ell}\binom{m}{j-\ell}}{\binom{2m}{j}} \mathcal{A}_\ell \star \mathcal{A}_{j-\ell}, \tag{2}$$

and integrating it, we get the PH curve $\boldsymbol{p} : [0,1] \to \mathbb{R}^3$,

$$\boldsymbol{p}(t) = \boldsymbol{p}_0 + \int_0^t \boldsymbol{h}(u)du = \sum_{j=0}^{2m+1} \boldsymbol{p}_j B_j^{2m+1}(t), \quad \text{where} \quad \boldsymbol{p}_{j+1} = \boldsymbol{p}_j + \frac{1}{2m+1} \sum_{\ell=0}^{j} \boldsymbol{h}_\ell, \quad j = 0, 1, \ldots, 2m, \tag{3}$$

and $\boldsymbol{p}_0$ is a free integration constant. Such a curve $\boldsymbol{p}$ is a PH curve for any coefficients $\mathcal{A}_j$, $j = 0, 1, \ldots, m$, since its parametric speed $\sigma = \|\boldsymbol{h}\| = \|\mathcal{A}\|^2$ is a polynomial expression,

$$\sigma(t) = \sum_{j=0}^{2m} \sigma_j B_j^{2m}(t), \qquad \sigma_j = \sum_{\ell=\max(0,j-m)}^{\min(m,j)} \frac{\binom{m}{\ell}\binom{m}{j-\ell}}{\binom{2m}{j}} \frac{1}{2}\left(\mathcal{A}_\ell \overline{\mathcal{A}}_{j-\ell} + \overline{\mathcal{A}}_\ell \mathcal{A}_{j-\ell}\right). \tag{4}$$

Furthermore, $\boldsymbol{p}$ can be equipped with a rational (adapted) orthonormal frame $\mathcal{F}_{\mathrm{ERF}}(\boldsymbol{p};\cdot)$, called the Euler-Rodrigues (ER) frame, defined by

$$\mathcal{F}_{\mathrm{ERF}}(\boldsymbol{p};t) := \chi(\mathcal{A}(t)), \quad \text{where} \quad \chi : \mathbb{H}\backslash\{0\} \to \mathrm{SO}(3), \quad \mathcal{A} \mapsto \chi(\mathcal{A}) = \left(\frac{\mathcal{A}\mathbf{i}\overline{\mathcal{A}}}{\|\mathcal{A}\|^2}, \frac{\mathcal{A}\mathbf{j}\overline{\mathcal{A}}}{\|\mathcal{A}\|^2}, \frac{\mathcal{A}\mathbf{k}\overline{\mathcal{A}}}{\|\mathcal{A}\|^2}\right)$$

is the kinematic mapping that maps nonzero quaternions to rotational matrices whose columns are called Euler-Rodrigues vectors. Its inverse χ^{-1} maps any rotation matrix to two antipodal unit quaternions (see Farin et al. (2002) for more details on its construction).

An equivalent way to derive PH curves is offered by the Hopf map $H : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^3$, defined by

$$H(\mathtt{z},\mathtt{u}) := (|\mathtt{z}|^2 - |\mathtt{u}|^2, 2\,\mathrm{Re}(\mathtt{z}\,\overline{\mathtt{u}}), 2\,\mathrm{Im}(\mathtt{z}\,\overline{\mathtt{u}})).$$

Choosing two complex–valued polynomials $\mathtt{a}, \mathtt{b} \in \mathbb{P}_m(\mathbb{C})$,

$$\mathtt{a}(t) = a_0(t) + \mathtt{i}a_1(t) =: \sum_{j=0}^{m} \mathtt{a}_j B_j^m(t), \quad \mathtt{b}(t) = a_3(t) + \mathtt{i}a_2(t) =: \sum_{j=0}^{m} \mathtt{b}_j B_j^m(t), \quad \mathtt{a}_j, \mathtt{b}_j \in \mathbb{C}, \tag{5}$$

the hodograph (2) can be expressed as $\boldsymbol{h}(t) = H(\mathtt{a}(t), \mathtt{b}(t))$. The conversion between quaternion and complex representation of the preimage is given by

$$\mathcal{A}(t) = \mathtt{a}(t) + \mathbf{k}\mathtt{b}(t) \quad \text{and} \quad \mathtt{a}(t) = \tfrac{1}{2}\left(\mathcal{A}(t) - \mathbf{i}\mathcal{A}(t)\mathbf{i}\right), \quad \mathtt{b}(t) = -\tfrac{1}{2}\mathbf{k}\left(\mathcal{A}(t) + \mathbf{i}\mathcal{A}(t)\mathbf{i}\right),$$

where we identify complex unit $\mathtt{i}$ with a quaternion $\mathbf{i}$, and complex numbers with a subset of quaternions $\mathbb{C} \equiv \mathrm{span}(\mathbf{1}, \mathbf{i}) \subset \mathbb{H}$.

2.3. DPH condition for a PH curve of degree $2m+1$

Define from the preimage (1) two complex–valued polynomials $\mathtt{a}, \mathtt{b} \in \mathbb{P}_m(\mathbb{C})$ as in (5). For the curve to satisfy the DPH condition, specific relations between quaternion coefficients $\mathcal{A}_j$, $j = 0, 1, \ldots, m$, or equivalently between the complex–valued polynomials $\mathtt{a}$ and $\mathtt{b}$, must hold. These relations come from the condition that $\|\boldsymbol{p}' \times \boldsymbol{p}''\|$ is also a polynomial. In particular, this expression is equal to (Farouki et al., 2009a)

$$\|\boldsymbol{p}' \times \boldsymbol{p}''\| = 2\sigma\sqrt{f_1^2 + f_2^2}, \quad f_1 = a_0 a_3' - a_0' a_3 - a_1 a_2' + a_1' a_2, \quad f_2 = a_0 a_2' - a_0' a_2 + a_1 a_3' - a_1' a_3,$$

where f_1 and f_2 are polynomials of degree $\leq 2m-2$. Thus, a PH curve $\boldsymbol{p}$ is a DPH curve if and only if $f_1^2(t)+f_2^2(t)=w^2(t)$ for some polynomial w. By a Kubota theorem (Kubota, 1972), expressed using a complex representation, this condition holds true if and only if

$$\mathtt{f}(t) := f_1(t) + \mathtt{i} f_2(t) = q(t)\mathtt{w}^2(t) \tag{6}$$

for some real–valued polynomial q and some complex–valued polynomial $\mathtt{w}$. Then $w = |q|\,|\mathtt{w}|^2$,

$$\|\boldsymbol{p}' \times \boldsymbol{p}''\| = 2\,\sigma\, w = 2\,\sigma|q|\,|\mathtt{w}|^2, \tag{7}$$

and the curvature simplifies to

$$\kappa = \frac{\|\boldsymbol{p}' \times \boldsymbol{p}''\|}{\|\boldsymbol{p}'\|^3} = \frac{2\,|q|\,|\mathtt{w}|^2}{\sigma^2} = \frac{2\,|q|\,|\mathtt{w}|^2}{\|\mathcal{A}\|^4}. \tag{8}$$

The parameters t for which $q=0$ determine points on the curve, for which the curvature is 0. If $q(t)>0$ for all $t\in[0,1]$, then (6) implies that a planar parametric curve $\mathtt{f}:[0,1]\to\mathbb{C}$, where the points in $\mathbb{R}^2$ are identified with complex numbers, is a regular planar PH curve. Moreover, q is nonzero on $[0,1]$ implies that the curve $\boldsymbol{p}$ has a non-vanishing curvature.
It is straightforward to compute that $\mathtt{a}(t)\mathtt{b}'(t)-\mathtt{a}'(t)\mathtt{b}(t) = f_1(t)+\mathtt{i}f_2(t)$. So, to construct a DPH curve from the preimage polynomial $\mathcal{A}$, it must hold that

$$\mathtt{a}(t)\mathtt{b}'(t)-\mathtt{a}'(t)\mathtt{b}(t) = q(t)\mathtt{w}^2(t), \quad t\in[0,1]. \tag{9}$$

Although the DPH condition is written in a simple form (9), it is the (underdetermined) nonlinear problem to express (1) from chosen q and $\mathtt{w}$. Note that the degree of the left hand side of (9) is $\leq 2m-2$.

3. DPH curves of degree 7

Choose $m=3$ and complex-valued polynomials

$$\mathtt{a}(t) = \sum_{j=0}^{3} \mathtt{a}_j B_j^3(t) \quad \text{and} \quad \mathtt{b}(t) = \sum_{j=0}^{3} \mathtt{b}_j B_j^3(t), \quad \mathtt{a}_j, \mathtt{b}_j \in \mathbb{C}.$$

The left hand side of the DPH condition (9) is a polynomial of degree ≤ 4, so the same must be true for the right hand side. As in Farouki et al. (2009b) we need to distinguish between three substantially different cases regarding the degree of polynomials q and $\mathtt{w}$. Namely, the polynomial in (9) is a quartic polynomial iff one of the following is true :

class 1: $\deg(q)=0$, $\deg(\mathtt{w})=2$,
class 2: $\deg(q)=2$, $\deg(\mathtt{w})=1$,
class 3: $\deg(q)=4$, $\deg(\mathtt{w})=0$.

We call the corresponding curves *class* 1, *class* 2, and *class* 3 DPH curves of degree 7.

In Farouki et al. (2009a,b), the authors focused mainly on the analysis of helical DPH curves and indicated that for a helical DPH curve of degree 7 the coefficients of the preimage (1) must satisfy a specific condition in which $\mathcal{A}_1$ and $\mathcal{A}_2$ are expressed as a certain combination of coefficients $\mathcal{A}_0$ and $\mathcal{A}_3$. In the following sections we show that this is true also for non-helical DPH curves of degree 7, and we provide a construction method suitable for interpolating the prescribed (Hermite) data. The main result is presented in the next theorem.

Theorem 1. *Let a PH curve* $\boldsymbol{p}$ *be derived from a preimage quaternion polynomial* $\mathcal{A}$ *of degree* 3, *i.e.* $\mathcal{A}(t)=\sum_{j=0}^{3}\mathcal{A}_j B_j^3(t)$, $\mathcal{A}_j\in\mathbb{H}$. *Denote* $\mathcal{A}_j = a_j + \mathbf{k}\, b_j$, a_j, $b_j\in\mathbb{C}$, *and assume* $\mathtt{c} := a_0 b_3 - a_3 b_0 \neq 0$. *The PH curve* $\boldsymbol{p}$ *is a DPH curve of degree 7 iff*

$$\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} = \begin{bmatrix} a_0 & a_3 \\ b_0 & b_3 \end{bmatrix} \boldsymbol{M} \tag{10}$$

for a matrix $\boldsymbol{M}\in\mathbb{C}^{2\times 2}$ *with a particular structure. A comprehensive list of matrices* $\boldsymbol{M}$ *that provide all possible ways to derive DPH curves of degree 7 is given in Theorems 2–4, and collected in Tables 1–3. Considering* $\mathbb{C}\equiv span(\mathbf{1},\mathbf{i})\subset\mathbb{H}$, *the PH curve* $\boldsymbol{p}$ *given in a quaternion representation is a DPH curve of degree 7 iff*

$$\begin{bmatrix} \mathcal{A}_1 & \mathcal{A}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{A}_0 & \mathcal{A}_3 \end{bmatrix} \boldsymbol{M}_H \tag{11}$$

where $\boldsymbol{M}_H \equiv \boldsymbol{M}$, $\boldsymbol{M}_H\in\mathbb{H}^{2\times 2}$.

Remark 1. Characterization of the DPH condition given in Theorem 1 is especially suitable for the construction of DPH curves with prescribed geometrical properties, e.g., interpolation of two boundary data points and corresponding boundary frames.

Remark 2. We assume that $\mathcal{A}_0\neq 0$ and $\mathcal{A}_3\neq 0$ to obtain a DPH curve with nonzero boundary tangent vectors. The special case $\mathtt{c}=0$ arises when the boundary tangent vectors have the same direction. This case requires a separate analysis, which is provided in Section 3.4.

Remark 3. In Tables 1–3 the (non-)helical property is noted and the number of degrees of freedom for the construction of DPH curves for each possibility is counted. Explicit expressions for the polynomials q and $\mathtt{w}$ are listed too, from which the polynomial

Table 1
Comprehensive list of possible matrices $\boldsymbol{M}$ in Theorem 1 for degree 7 DPH curves of class 1.

DPH curves of degree 7, class 1	
Case 1 construction: choose $\mathtt{v}_0, \mathtt{v}_2 \in \mathbb{C}$, at least one $\in \mathbb{C}\backslash\mathbb{R}$	Properties
$\boldsymbol{M} = \frac{1}{\mathtt{d}} \begin{bmatrix} 2\mathtt{v}_2 & \mathtt{v}_2^2 \\ \mathtt{v}_0^2 & 2\mathtt{v}_0 \end{bmatrix}$	
`type 1`: $\mathtt{d} = 4 - \mathtt{v}_0\mathtt{v}_2, \quad \mathtt{v}_0\mathtt{v}_2 \neq 4, \quad \mathtt{v}_0\mathtt{v}_2 \neq 1$	`type 1`: 12 DOF, non-helical
`type 2`: $\mathtt{d} = 3\mathtt{v}_0\mathtt{v}_2, \quad \mathtt{v}_0\mathtt{v}_2 \neq 0,$	`type 2`: 12 DOF, helical
$q(t) \equiv 1, \quad \mathtt{w}(t) = \pm\sqrt{\frac{3\mathtt{c}}{\mathtt{d}}} \left(\mathtt{v}_0 B_0^2(t) + B_1^2(t) + \mathtt{v}_2 B_2^2(t)\right),$	
Case 2 construction: choose $\mathtt{v} \in \mathbb{C}\backslash\mathbb{R}$	
$\boldsymbol{M} = \frac{1}{\mathtt{d}} \begin{bmatrix} 0 & \mathtt{v}^2 \\ 1 & 0 \end{bmatrix}$	
`type 1`: $\mathtt{d} = -\mathtt{v}$	`type 1`: 10 DOF, non-helical
`type 2`: $\mathtt{d} = 3\mathtt{v}$	`type 2`: 10 DOF, helical
$q(t) \equiv 1, \quad \mathtt{w}(t) = \pm\sqrt{\frac{3\mathtt{c}}{\mathtt{d}}} \left(B_0^2(t) + \mathtt{v} B_2^2(t)\right),$	

Table 2
Comprehensive list of possible matrices $\boldsymbol{M}$ in Theorem 1, degree 7 DPH curves of class 2.

DPH curves of degree 7, class 2	
Case 1 construction: choose $\mathtt{v} \in \mathbb{C}\backslash\mathbb{R}$ and $\zeta_0, \zeta_2 \in \mathbb{R}$ and $(\zeta_0, \zeta_2) \neq (1, 1)$	Properties
$\boldsymbol{M} = \frac{1}{\mathtt{d}} \begin{bmatrix} \zeta_2 + \mathtt{v} & \zeta_2 \mathtt{v} \\ \zeta_0 \frac{1}{\mathtt{v}} & \zeta_0 + \frac{1}{\mathtt{v}} \end{bmatrix}$	
`type 1`: $\mathtt{d} = \zeta_0 \mathtt{v} + 1 + \zeta_2 \frac{1}{\mathtt{v}}, \ \mathtt{d} \neq 3$	`type 1`: 12 DOF, non-helical
`type 2`: $\mathtt{d} = 3$	`type 2`: 12 DOF, helical
$q(t) = \zeta_0 B_0^2(t) + B_1^2(t) + \zeta_2 B_2^2(t), \quad \mathtt{w}(t) = \pm\sqrt{\frac{3\mathtt{c}}{\mathtt{d}\ \mathtt{v}}} \left(B_0^1(t) + \mathtt{v} B_1^1(t)\right)$	
Case 2 construction: choose $\mathtt{v} \in \mathbb{C}\backslash\mathbb{R}$ and $\zeta \in \mathbb{R}$	
$\boldsymbol{M} = \frac{1}{\mathtt{d}} \begin{bmatrix} \zeta_2 & \zeta_2 \mathtt{v} \\ \zeta_0 \frac{1}{\mathtt{v}} & \zeta_0 \end{bmatrix}, \quad \mathtt{d} = \zeta_0 \mathtt{v} + \zeta_2 \frac{1}{\mathtt{v}}$	
`(a)` $\zeta_0 = 1, \zeta_2 = \zeta$	`(a)` 11 DOF, non-helical
`(b)` $\zeta_0 = 0, \zeta_2 = 1$	`(b)` 10 DOF, non-helical
$q(t) = \zeta_0 B_0^2(t) + \zeta_2 B_2^2(t), \quad \mathtt{w}(t) = \pm\sqrt{\frac{3\mathtt{c}}{\mathtt{d}\ \mathtt{v}}} \left(B_0^1(t) + \mathtt{v} B_1^1(t)\right)$	

Table 3
Comprehensive list of possible matrices $\boldsymbol{M}$ in Theorem 1, DPH curves of class 3.

DPH curves of degree 7, class 3	
`Type 1`: Choose three of $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}$, and $\theta \in (0, 2\pi)\backslash\{\pi\}$	Properties
$\boldsymbol{M} = \mathtt{u}(\theta) \begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{bmatrix}, \quad \mathtt{u}(\theta) = \cos\theta + \mathtt{i}\sin\theta, \quad \zeta_1\zeta_4 - \zeta_2\zeta_3 = \frac{1}{3}$	12 DOF, non-helical
$q(t) = \zeta_3 B_0^4(t) + \frac{1}{2}\zeta_4 B_1^4(t) + \frac{1}{3}\cos\theta \cdot B_2^4(t) + \frac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t)$	
$\mathtt{w}(t) = \pm\sqrt{3\,\mathtt{c}\,\mathtt{u}(\theta)}$	
`Type 2`: Choose $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}$	
$\boldsymbol{M} = \begin{bmatrix} \zeta_1 & \zeta_2 \\ \zeta_3 & \zeta_4 \end{bmatrix}$	12 DOF, helical
$q(t) = \zeta_3 B_0^4(t) + \frac{1}{2}\zeta_4 B_1^4(t) + \frac{1}{6}(1 + 3(\zeta_1\zeta_4 - \zeta_2\zeta_3)) B_2^4(t) + \frac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t)$	
$\mathtt{w}(t) = \pm\sqrt{3\,\mathtt{c}}$	

expression of the norm $\|\boldsymbol{p}' \times \boldsymbol{p}''\|$ follows by (7). The degrees of freedom are counted for the hodograph, meaning up to a translation of the DPH curve. Furthermore, if only the boundary tangent vectors (and not the boundary ER frame vectors) are observed, then we get one less degree of freedom due to the fact that the preimage curves $\mathcal{A}(t)$ and $\mathcal{A}(t)(\cos(\phi) + \mathbf{i}\sin(\phi))$ produce the same DPH curve for any $\phi \in [0, 2\pi]$.

The proof of Theorem 1 will be given in Sections 3.1–3.3 by constructing the matrix $\boldsymbol{M}$ for each class, and the corresponding properties from Tables 1–3 will be derived by analyzing the resulting DPH curves. The helical/non-helical property is proved in Section 4.

3.1. Construction of degree 7 DPH curves of class 1

Let

$$\mathtt{w}(t) = \mathtt{w}_0 B_0^2(t) + \mathtt{w}_1 B_1^2(t) + \mathtt{w}_2 B_2^2(t), \quad \mathtt{w}_0, \mathtt{w}_1, \mathtt{w}_2 \in \mathbb{C}, \tag{12}$$

be a quadratic complex-valued polynomial and let q be a constant polynomial. Without loss of generality we can set $q(t) \equiv 1$. In addition, we require that $\mathtt{w}$ is not a real-valued polynomial multiplied by a fixed complex constant, which ensures that the curves do not belong to class 3.

It is straightforward to see that the polynomial equality (9) is satisfied iff

$$3\left(\mathtt{a}_0\mathtt{b}_1 - \mathtt{a}_1\mathtt{b}_0\right) = \mathtt{w}_0^2, \tag{13a}$$

$$3\left(\mathtt{a}_0\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_0\right) = 2\mathtt{w}_0\mathtt{w}_1, \tag{13b}$$

$$\left(\mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0\right) + 3\left(\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1\right) = \tfrac{2}{3}\left(2\mathtt{w}_1^2 + \mathtt{w}_0\mathtt{w}_2\right), \tag{13c}$$

$$3\left(\mathtt{a}_1\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_1\right) = 2\mathtt{w}_1\mathtt{w}_2, \tag{13d}$$

$$3\left(\mathtt{a}_2\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_2\right) = \mathtt{w}_2^2. \tag{13e}$$

If we denote by $\mathtt{c} := \mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0$, assume that $\mathtt{c} \neq 0$, and we write (13a), (13d) and (13b), (13e) in matrix form

$$\begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_1 \\ \mathtt{b}_1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} \mathtt{w}_0^2 \\ 2\mathtt{w}_1\mathtt{w}_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_2 \\ \mathtt{b}_2 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2\mathtt{w}_0\mathtt{w}_1 \\ \mathtt{w}_2^2 \end{bmatrix}, \tag{14}$$

we get that

$$\begin{bmatrix} \mathtt{a}_1 & \mathtt{a}_2 \\ \mathtt{b}_1 & \mathtt{b}_2 \end{bmatrix} = \frac{1}{3\mathtt{c}} \begin{bmatrix} \mathtt{a}_0 & \mathtt{a}_3 \\ \mathtt{b}_0 & \mathtt{b}_3 \end{bmatrix} \begin{bmatrix} 2\mathtt{w}_1\mathtt{w}_2 & \mathtt{w}_2^2 \\ \mathtt{w}_0^2 & 2\mathtt{w}_0\mathtt{w}_1 \end{bmatrix} \tag{15}$$

which is Eq. (10) for

$$\boldsymbol{M} = \frac{1}{3\mathtt{c}} \begin{bmatrix} 2\mathtt{w}_1\mathtt{w}_2 & \mathtt{w}_2^2 \\ \mathtt{w}_0^2 & 2\mathtt{w}_0\mathtt{w}_1 \end{bmatrix}. \tag{16}$$

However, the coefficients $\mathtt{w}_j$, $j = 0, 1, 2$, are not completely free, but are connected through the remaining Eq. (13c). Therefore we compute the determinant of (15)

$$\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1 = \tfrac{1}{9\mathtt{c}}\mathtt{w}_0\mathtt{w}_2\left(4\mathtt{w}_1^2 - \mathtt{w}_0\mathtt{w}_2\right).$$

The remaining Eq. (13c) then equals

$$\mathtt{c} + \tfrac{1}{3\mathtt{c}}\mathtt{w}_0\mathtt{w}_2\left(4\mathtt{w}_1^2 - \mathtt{w}_0\mathtt{w}_2\right) = \tfrac{2}{3}\left(2\mathtt{w}_1^2 + \mathtt{w}_0\mathtt{w}_2\right),$$

and by some elementary computations, it simplifies to

$$\frac{1}{3\mathtt{c}}\left(\mathtt{c} - \mathtt{w}_0\mathtt{w}_2\right)\left(3\mathtt{c} + \mathtt{w}_0\mathtt{w}_2 - 4\mathtt{w}_1^2\right) = 0, \tag{17}$$

which gives two solutions

$$\mathtt{type}\ 1: \quad 3\mathtt{c} = 4\mathtt{w}_1^2 - \mathtt{w}_0\mathtt{w}_2, \tag{18a}$$

$$\mathtt{type}\ 2: \quad \mathtt{c} = \mathtt{w}_0\mathtt{w}_2. \tag{18b}$$

Based on this derivation, the next theorem provides sufficient and necessary conditions for the construction of DPH curves of class 1 and describes their properties.

Theorem 2. *Suppose that the assumptions of Theorem 1 hold true and that two quaternions $\mathcal{A}_0$ and $\mathcal{A}_3$ are given. If the quaternions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal to (11) with the matrix $\boldsymbol{M}$ of the form stated in Table 1, Case 1 (type 1 or type 2) then the PH curve $\boldsymbol{p}$ is a DPH curve of class 1, and the polynomials in (9) are equal to*

$$q(t) \equiv 1, \quad \mathtt{w}(t) = \pm\sqrt{\tfrac{3c}{d}}\left(v_0 B_0^2(t) + B_1^2(t) + v_2 B_2^2(t)\right)$$

for $v_0, v_2 \in \mathbb{C}$ (and not both in $\mathbb{R}$). The construction has 12 degrees of freedom.

If the quaternions $\mathcal{A}_1$ and $\mathcal{A}_2$ are equal to (11) with the matrix $\boldsymbol{M}$ of the form stated in Table 1, Case 2 (type 1 or type 2) then the PH curve $\boldsymbol{p}$ is a DPH curve of class 1, and the polynomials in (9) are equal to

$$q(t) \equiv 1, \quad \mathtt{w}(t) = \pm\sqrt{\tfrac{3c}{d}}\left(B_0^2(t) + v B_2^2(t)\right)$$

for $v \in \mathbb{C}\backslash\mathbb{R}$. The resulting curve has 10 degrees of freedom (DOF) for the construction. The assumptions on $\mathcal{A}_1$ and $\mathcal{A}_2$ for both cases together provide a complete characterization of degree 7 DPH curves of class 1.

Proof. Considering type 1 solution stated in (18a), assume first that $\mathtt{w}_1 \neq 0$, and denote $\mathtt{v}_0 = \frac{\mathtt{w}_0}{\mathtt{w}_1}$ and $\mathtt{v}_2 = \frac{\mathtt{w}_2}{\mathtt{w}_1}$. With some basic calculations we get $3\mathtt{c} = \mathtt{w}_1^2(4 - \mathtt{v}_0\mathtt{v}_2)$ and

$$\boldsymbol{M} = \frac{1}{4 - \mathtt{v}_0\mathtt{v}_2}\begin{bmatrix}2\mathtt{v}_2 & \mathtt{v}_2^2 \\ \mathtt{v}_0^2 & 2\mathtt{v}_0\end{bmatrix}. \tag{19}$$

Denoting $\mathtt{d} = 4 - \mathtt{v}_0\mathtt{v}_2$ leads to Case 1: type 1 solutions of Table 1. Since $\mathtt{c} \neq 0$, there is a constraint for the choice of $\mathtt{v}_0$ and $\mathtt{v}_2$, namely $\mathtt{v}_0\mathtt{v}_2 \neq 4$. If $\mathtt{v}_0\mathtt{v}_2 = 1$, we have $3\mathtt{c} = \mathtt{w}_1^2(4 - \mathtt{v}_0\mathtt{v}_2) = 3\mathtt{w}_1^2$, which gives us Case 1: type 2 for $\mathtt{w}_0 = \mathtt{w}_2 = \mathtt{w}_1$.

Now consider type 1 for $\mathtt{w}_1 = 0$. Then $\mathtt{w}_0\mathtt{w}_2 \neq 0$ is implied by $\mathtt{c} \neq 0$. Denoting $\mathtt{v} = \frac{\mathtt{w}_2}{\mathtt{w}_0}$, we have $3\mathtt{c} = -\mathtt{w}_0^2\mathtt{v}$ and

$$\boldsymbol{M} = -\frac{1}{\mathtt{v}}\begin{bmatrix}0 & \mathtt{v}^2 \\ 1 & 0\end{bmatrix}. \tag{20}$$

This leads to Case 2: type 1 solutions of Table 1 where we denote $\mathtt{d} = -\mathtt{v}$. Since $\mathtt{w}_0\mathtt{w}_2 \neq 0$ also $\mathtt{v} \neq 0$.

For type 2 solution stated in (18b) we have $\mathtt{c} = \mathtt{w}_0\mathtt{w}_2 \neq 0$ and we insert it into (16). If $\mathtt{w}_1 \neq 0$, denoting $\mathtt{v}_0 = \frac{\mathtt{w}_0}{\mathtt{w}_1}$ and $\mathtt{v}_2 = \frac{\mathtt{w}_2}{\mathtt{w}_1}$, the matrix simplifies to

$$\boldsymbol{M} = \frac{1}{3\mathtt{v}_0\mathtt{v}_2}\begin{bmatrix}2\mathtt{v}_2 & \mathtt{v}_2^2 \\ \mathtt{v}_0^2 & 2\mathtt{v}_0\end{bmatrix} \tag{21}$$

which is Case 1: type 2 of Table 1 with $\mathtt{d} = 3\mathtt{v}_0\mathtt{v}_2$. As before, $\mathtt{c} \neq 0$ implies $\mathtt{v}_0\mathtt{v}_2 \neq 0$.
For the remaining possibility $\mathtt{w}_1 = 0$, denote again $\mathtt{v} = \frac{\mathtt{w}_2}{\mathtt{w}_0}$ and insert it into (16). The resulting matrix equals

$$\boldsymbol{M} = \frac{1}{3\mathtt{v}}\begin{bmatrix}0 & \mathtt{v}^2 \\ 1 & 0\end{bmatrix}, \tag{22}$$

which gives Case 2: type 2 of Table 1, where $\mathtt{d} = 3\mathtt{v}$ and $\mathtt{c} \neq 0$ implies $\mathtt{v} \neq 0$.

The expressions for $q(t)$ and $\mathtt{w}(t)$ follow from the class 1 definition $\mathtt{w}(t) = \mathtt{w}_0 B_0^2(t) + \mathtt{w}_1 B_1^2(t) + \mathtt{w}_2 B_2^2(t)$ and $q(t) \equiv 1$. For Case 1 we obtain from the construction that $\mathtt{w}_1^2 = \frac{3\mathtt{c}}{\mathtt{d}}$, and $\mathtt{w}(t) = \mathtt{w}_1\left(\mathtt{v}_0 B_0^2(t) + B_1^2(t) + \mathtt{v}_2 B_2^2(t)\right)$. Note that choosing $\mathtt{v}_0$ and $\mathtt{v}_2$ both in $\mathbb{R}$ would imply the polynomial $\mathtt{w}(t)$ to be real-valued multiplied by a complex constant and this would produce DPH curves of class 3.

For Case 2, we know that $\mathtt{w}_1 = 0$, therefore $\mathtt{w}(t) = \mathtt{w}_0\left(B_0^2(t) + \mathtt{v}B_2^2(t)\right)$, and from the construction $\mathtt{w}_0^2 = \frac{3\mathtt{c}}{\mathtt{d}}$. Moreover, $\mathtt{v}$ must be in $\mathbb{C}\backslash\mathbb{R}$ to ensure that $\mathtt{w}(t)$ is not a real-valued polynomial multiplied by a complex constant.

There are 12 degrees of freedom for constructing the preimage of a DPH curve of class 1, case 1 ($2 \cdot 4$ for $\mathcal{A}_0$, $\mathcal{A}_3$ and $2 \cdot 2$ for $\mathtt{v}_0$, $\mathtt{v}_2$). For case 2 the number of degrees of freedom reduces to 10 ($2 \cdot 4$ for $\mathcal{A}_0$, $\mathcal{A}_3$ and 2 for $\mathtt{v}$).

The last statement of the theorem holds true since the condition (9) is sufficient and necessary, and all cases are covered by the construction. □

Remark 4. The restrictions on the values of the free parameters listed in Table 1 ensure that $\mathtt{w}$ cannot have two real zeros, i.e., curves cannot be of class 3. Moreover, it is straightforward to verify that $\mathtt{w}$ has no real zeros in Case 2, while a single real zero may occur in Case 1. More precisely, one can check that $\mathtt{v}_0 B_0^2(\xi) + B_1^2(\xi) + \mathtt{v}_2 B_2^2(\xi) = 0$ for some real ξ if

$$\mathtt{v}_2 = \tfrac{(1-\xi)^2}{\xi^2}\left(\tfrac{2\xi}{\xi-1} - \mathtt{v}_0\right), \quad \xi \in \mathbb{R}\backslash\{0,1\} \tag{23}$$

or if $\mathtt{v}_0 = 0, \mathtt{v}_2 \in \mathbb{C}$ when $\xi = 0$, or $\mathtt{v}_2 = 0, \mathtt{v}_0 \in \mathbb{C}$ when $\xi = 1$. For the values $\mathtt{v}_0$ and $\mathtt{v}_2$ in Table 1 (Case 1) that additionally satisfy one of these conditions the corresponding DPH curves also belong to class 2 with linear polynomial $\mathtt{w}$ and $q(t) = (t - \xi)^2$, $q(t) = t^2$ or $q(t) = (1 - t)^2$.

Example 3.1. Choosing

$$\mathtt{a}_0 = 5\,\mathtt{i}, \quad \mathtt{b}_0 = 1 - \mathtt{i}, \quad \mathtt{a}_3 = 1 + \mathtt{i}, \quad \mathtt{b}_3 = 2 + 5\,\mathtt{i}, \quad \mathtt{v}_0 = \mathtt{v}_2 = \tfrac{1}{2} - \tfrac{1}{2}\,\mathtt{i},$$

gives

$$\text{Case 1, type 1}: \quad \mathtt{a}_1 = \tfrac{97}{65} + \tfrac{61}{65}\,\mathtt{i}, \quad \mathtt{b}_1 = \tfrac{34}{65} - \tfrac{53}{65}\,\mathtt{i}, \quad \mathtt{a}_2 = \tfrac{72}{65} - \tfrac{9}{65}\,\mathtt{i}, \quad \mathtt{b}_2 = \tfrac{109}{65} + \tfrac{27}{65}\,\mathtt{i},$$

$$\text{Case 1, type 2}: \quad \mathtt{a}_1 = -3 + \tfrac{11}{3}\,\mathtt{i}, \quad \mathtt{b}_1 = 2 + \tfrac{5}{3}\,\mathtt{i}, \quad \mathtt{a}_2 = 3\,\mathtt{i}, \quad \mathtt{b}_2 = -\tfrac{5}{3} + \tfrac{13}{3}\,\mathtt{i},$$

$$\text{Case 2, type 1}: \quad \mathtt{a}_1 = -2\,\mathtt{i}, \quad \mathtt{b}_1 = 3 - 7\,\mathtt{i}, \quad \mathtt{a}_2 = -\tfrac{5}{2} - \tfrac{5}{2}\,\mathtt{i}, \quad \mathtt{b}_2 = \mathtt{i},$$

$$\text{Case 2, type 2}: \quad \mathtt{a}_1 = \tfrac{2}{3}\,\mathtt{i}, \quad \mathtt{b}_1 = -1 + \tfrac{7}{3}\,\mathtt{i}, \quad \mathtt{a}_2 = \tfrac{5}{6} + \tfrac{5}{6}\,\mathtt{i}, \quad \mathtt{b}_2 = -\tfrac{1}{3}\,\mathtt{i}.$$

The corresponding DPH curves with $\boldsymbol{p}(0) = (0,0,0)$ are shown in Fig. 1. The graph of $\frac{\mathtt{a}(t)}{\mathtt{b}(t)}$ in a complex plane, which numerically confirms the helical property, is shown in Fig. 2. Note that the curve given by Case 1, type 2 equals the helical DPH curve from (Farouki et al., 2009b, Example 2), and has Bézier coefficients of the hodograph, function σ and $\|\boldsymbol{p}' \times \boldsymbol{p}''\|$ equal to

$$\boldsymbol{h}_0 = (23, -10, 10), \quad \boldsymbol{h}_1 = \left(18, \tfrac{5}{3}, \tfrac{32}{3}\right), \quad \boldsymbol{h}_2 = \left(\tfrac{89}{5}, \tfrac{38}{5}, \tfrac{38}{3}\right), \quad \boldsymbol{h}_3 = \left(\tfrac{36}{5}, \tfrac{129}{5}, \tfrac{64}{5}\right),$$

$$\boldsymbol{h}_4 = \left(-\tfrac{61}{5}, 22, \tfrac{46}{15}\right), \quad \boldsymbol{h}_5 = \left(-\tfrac{46}{3}, \tfrac{53}{3}, 0\right), \quad \boldsymbol{h}_6 = (-27, 14, -6),$$

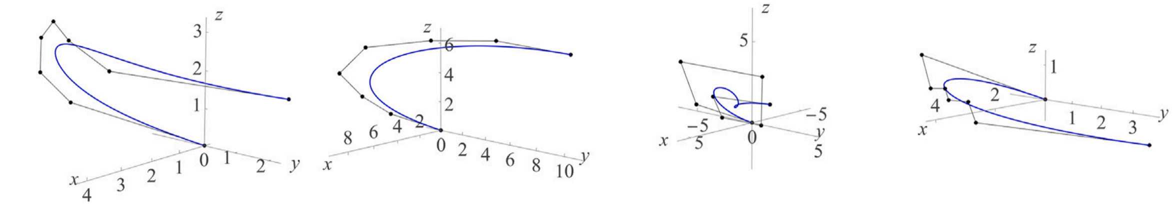

Fig. 1. Examples of degree 7 DPH curves of class 1. From left to right: Case 1, `type` 1 and 2; Case 2, `type` 1 and 2.

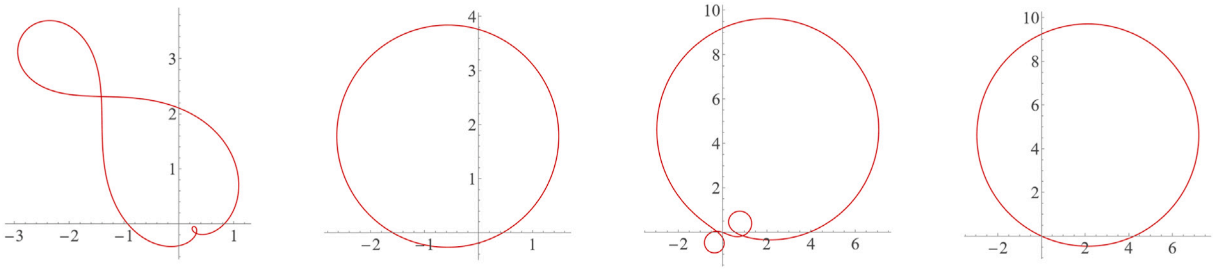

Fig. 2. The graphs of the rational function $t \mapsto \frac{\mathtt{a}(t)}{\mathtt{b}(t)}$, corresponding to the curves in Fig. 1.

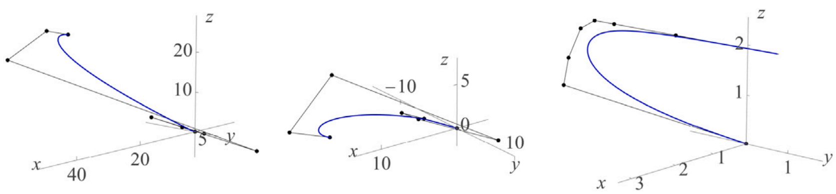

Fig. 3. Degree 7 DPH curves of class 1 from Example 3.2 (from left to right) that are of class 2 as well.

$$\sigma(t) = 216t^6 - 632t^5 + 724t^4 - 416t^3 + 162t^2 - 50t + 27,$$
$$\|\boldsymbol{p}'(t) \times \boldsymbol{p}''(t)\| = 2\,\sigma(t)\sqrt{829}\left(4t^4 - 8t^3 + 4t^2 + 1\right).$$

For a non-helical DPH curve from Case 1, `type` 1 we obtain

$$\boldsymbol{h}_0 = (23, -10, 10), \quad \boldsymbol{h}_1 = \left(\frac{218}{65}, -\frac{229}{65}, \frac{328}{65}\right), \quad \boldsymbol{h}_2 = \left(\frac{13}{25}, \frac{438}{325}, \frac{1882}{325}\right), \quad \boldsymbol{h}_3 = \left(\frac{548}{325}, \frac{1861}{325}, \frac{912}{325}\right),$$
$$\boldsymbol{h}_4 = \left(\frac{371}{325}, \frac{1662}{325}, -\frac{822}{325}\right), \quad \boldsymbol{h}_5 = \left(-\frac{58}{13}, \frac{47}{13}, -\frac{296}{65}\right), \quad \boldsymbol{h}_6 = (-27, 14, -6),$$
$$\sigma(t) = \frac{1}{65}\left(1080t^6 - 3096t^5 + 12564t^4 - 19264t^3 + 17154t^2 - 8178t + 1755\right),$$
$$\|\boldsymbol{p}'(t) \times \boldsymbol{p}''(t)\| = 2\,\sigma(t)\,3\sqrt{\frac{829}{65}}\left(4t^4 - 8t^3 + 4t^2 + 1\right).$$

Example 3.2. In this example we demonstrate the statements of Remark 4. Let us choose $\mathtt{a}_0$, $\mathtt{b}_0$, $\mathtt{a}_3$, $\mathtt{b}_3$ and $\mathtt{v}_0$ as in Example 3.1. Selecting $\xi = \frac{1}{3}$ in (23) gives $\mathtt{v}_2 = -6 + 2\,\mathtt{i}$ and

$$\text{Case 1, type 1}: \quad \mathtt{a}_1 = \frac{125}{52} - \frac{441}{52}\,\mathtt{i}, \quad \mathtt{b}_1 = -\frac{93}{52} + \frac{17}{13}\,\mathtt{i}, \quad \mathtt{a}_2 = \frac{23}{13} + \frac{362}{13}\,\mathtt{i}, \quad \mathtt{b}_2 = \frac{151}{26} - \frac{129}{26}\,\mathtt{i}, \quad \mathtt{d} = 6 - 4\,\mathtt{i},$$
$$\text{Case 1, type 2}: \quad \mathtt{a}_1 = -\frac{203}{60} + \frac{199}{60}\,\mathtt{i}, \quad \mathtt{b}_1 = \frac{71}{60} - \frac{2}{15}\,\mathtt{i}, \quad \mathtt{a}_2 = \frac{33}{5} - \frac{202}{15}\,\mathtt{i}, \quad \mathtt{b}_2 = -\frac{121}{30} + \frac{23}{30}\,\mathtt{i}, \quad \mathtt{d} = -6 + 12\,\mathtt{i}.$$

Furthermore, $\mathtt{c} = -27 + 10\,\mathtt{i}$ and $q(t)\mathtt{w}^2(t) = \frac{3\mathtt{c}}{\mathtt{d}}(1-3t)^2\left(\mathtt{v}_0(1-t) - \frac{1}{2}\mathtt{v}_2\,t\right)^2$, showing that both curves could also be considered as class 2 curves (with quadratic q and linear $\mathtt{w}$). The same holds true for the choice $\mathtt{v}_2 = 0$ which gives

$$\text{Case 1, type 1}: \quad \mathtt{a}_1 = \frac{1}{8} - \frac{1}{8}\,\mathtt{i}, \quad \mathtt{b}_1 = \frac{5}{8} - \frac{1}{4}\,\mathtt{i}, \quad \mathtt{a}_2 = \frac{1}{2} + 0\,\mathtt{i}, \quad \mathtt{b}_2 = \frac{7}{4} + \frac{3}{4}\,\mathtt{i}, \quad \mathtt{d} = 4,$$

and $q(t)\mathtt{w}^2(t) = \frac{3\mathtt{c}}{\mathtt{d}}(1-t)^2\left(\mathtt{v}_0(1-t) + 2\,t\right)^2$. The corresponding DPH curves with $\boldsymbol{p}(0) = (0, 0, 0)$ are shown in Fig. 3. The graph of $\frac{\mathtt{a}(t)}{\mathtt{b}(t)}$ in a complex plane, which numerically confirms the helical property, is shown in Fig. 4.

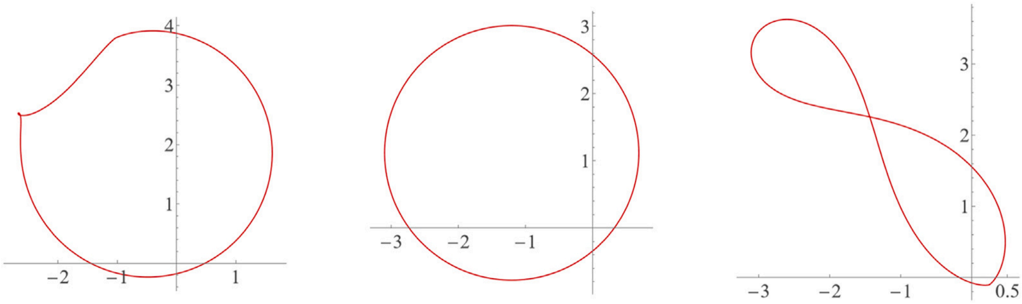

Fig. 4. The graphs of the rational function $t \mapsto \frac{\mathtt{a}(t)}{\mathtt{b}(t)}$, corresponding to the curves in Fig. 3.

3.2. Construction of degree 7 DPH curves of class 2

Let

$$\mathtt{w}(t) = \mathtt{w}_0 B_0^1(t) + \mathtt{w}_1 B_1^1(t), \quad \mathtt{w}_0, \mathtt{w}_1 \in \mathbb{C} \quad \text{and} \quad q(t) = q_0 B_0^2(t) + q_1 B_1^2(t) + q_2 B_2^2(t), \quad q_0, q_1, q_2 \in \mathbb{R}, \tag{24}$$

be complex–valued and real–valued polynomials of degrees 1 and 2, respectively. In order to obtain "true" class 2 curves (and not special cases of class 3 curves), we must assume that $\mathtt{w}_0 \neq 0$, $\mathtt{w}_1 \neq 0$ and $\frac{\mathtt{w}_1}{\mathtt{w}_0} \in \mathbb{C}\backslash\mathbb{R}$. The polynomial equality (9) is satisfied iff

$$3\left(\mathtt{a}_0\mathtt{b}_1 - \mathtt{a}_1\mathtt{b}_0\right) = q_0\mathtt{w}_0^2, \tag{25a}$$

$$3\left(\mathtt{a}_0\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_0\right) = q_0\mathtt{w}_0\mathtt{w}_1 + q_1\mathtt{w}_0^2, \tag{25b}$$

$$\left(\mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0\right) + 3\left(\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1\right) = \tfrac{1}{3}q_0\mathtt{w}_1^2 + \tfrac{4}{3}q_1\mathtt{w}_0\mathtt{w}_1 + \tfrac{1}{3}q_2\mathtt{w}_0^2, \tag{25c}$$

$$3\left(\mathtt{a}_1\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_1\right) = q_1\mathtt{w}_1^2 + q_2\mathtt{w}_0\mathtt{w}_1, \tag{25d}$$

$$3\left(\mathtt{a}_2\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_2\right) = q_2\mathtt{w}_1^2. \tag{25e}$$

Assuming that $\mathtt{c} = \mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0 \neq 0$ we write (25a), (25d) and (25b), (25e) in matrix form

$$\begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_1 \\ \mathtt{b}_1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} q_0\mathtt{w}_0^2 \\ q_1\mathtt{w}_1^2 + q_2\mathtt{w}_0\mathtt{w}_1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_2 \\ \mathtt{b}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} q_0\mathtt{w}_0\mathtt{w}_1 + q_1\mathtt{w}_0^2 \\ q_2\mathtt{w}_1^2 \end{bmatrix} \tag{26}$$

from where we get

$$\begin{bmatrix} \mathtt{a}_1 & \mathtt{a}_2 \\ \mathtt{b}_1 & \mathtt{b}_2 \end{bmatrix} = \frac{1}{3\mathtt{c}} \begin{bmatrix} \mathtt{a}_0 & \mathtt{a}_3 \\ \mathtt{b}_0 & \mathtt{b}_3 \end{bmatrix} \begin{bmatrix} q_1\mathtt{w}_1^2 + q_2\mathtt{w}_0\mathtt{w}_1 & q_2\mathtt{w}_1^2 \\ q_0\mathtt{w}_0^2 & q_0\mathtt{w}_0\mathtt{w}_1 + q_1\mathtt{w}_0^2 \end{bmatrix} \tag{27}$$

which is Eq. (10) for

$$\boldsymbol{M} = \frac{1}{3\mathtt{c}} \begin{bmatrix} q_1\mathtt{w}_1^2 + q_2\mathtt{w}_0\mathtt{w}_1 & q_2\mathtt{w}_1^2 \\ q_0\mathtt{w}_0^2 & q_0\mathtt{w}_0\mathtt{w}_1 + q_1\mathtt{w}_0^2 \end{bmatrix}. \tag{28}$$

The determinant of (27) is equal to

$$\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1 = \tfrac{1}{9\mathtt{c}} q_1\mathtt{w}_0\mathtt{w}_1 \left(q_0\mathtt{w}_1^2 + q_1\mathtt{w}_0\mathtt{w}_1 + q_2\mathtt{w}_0^2\right),$$

and the remaining Eq. (25c) then simplifies to

$$\frac{1}{3\mathtt{c}}\left(\mathtt{c} - q_1\mathtt{w}_0\mathtt{w}_1\right)\left(3\mathtt{c} - q_0\mathtt{w}_1^2 - q_1\mathtt{w}_0\mathtt{w}_1 - q_2\mathtt{w}_0^2\right) = 0, \tag{29}$$

which gives two solutions

$$\mathtt{type\ 1:} \quad 3\mathtt{c} = q_0\mathtt{w}_1^2 + q_1\mathtt{w}_0\mathtt{w}_1 + q_2\mathtt{w}_0^2, \tag{30a}$$

$$\mathtt{type\ 2:} \quad \mathtt{c} = q_1\mathtt{w}_0\mathtt{w}_1. \tag{30b}$$

The next theorem provides sufficient and necessary conditions for the construction of degree 7 DPH curves of class 2 and describes their properties.

Theorem 3. *Suppose that the assumptions of Theorem 1 hold true and that two quaternions $\mathcal{A}_0$ and $\mathcal{A}_3$ are given. If the quaternions $\mathcal{A}_1$, $\mathcal{A}_2$ are equal to (11) with the matrix $\boldsymbol{M}$ of the form stated in Table 2, Case 1, then the PH curve $\boldsymbol{p}$ is a DPH curve of class 2, and the polynomials in (9) are equal to*

$$q(t) = \zeta_0 B_0^2(t) + B_1^2(t) + \zeta_2 B_2^2(t), \quad \mathtt{w}(t) = \pm\sqrt{\tfrac{3\mathtt{c}}{dv}}\left(B_0^1(t) + vB_1^1(t)\right)$$

for any $v \in \mathbb{C}\backslash\mathbb{R}$, and $\zeta_0, \zeta_2 \in \mathbb{R}$ (not both equal to 1). The construction has 12 degrees of freedom.

There are two special cases of class 2 DPH curves as stated in Table 2, Case 2, for any $v \in \mathbb{C}\backslash\mathbb{R}$ and $\zeta \in \mathbb{R}$. The number of free parameters in these two special cases is 11 and 10, respectively.

These results provide a complete characterization of degree 7 DPH curves of class 2.

Proof. Recall that $\mathtt{w}_0 \neq 0$, $\mathtt{w}_1 \neq 0$ and $\frac{\mathtt{w}_1}{\mathtt{w}_0} \in \mathbb{C}\backslash\mathbb{R}$. For the solution of type 2 the condition $\mathtt{c} \neq 0$ implies $q_1 \neq 0$. Setting the same assumption also for type 1 solution, and denoting $\mathtt{v} = \frac{\mathtt{w}_1}{\mathtt{w}_0}$, $\zeta_0 = \frac{q_0}{q_1}$, $\zeta_2 = \frac{q_2}{q_1}$, the matrix M from (28) simplifies to

$$\boldsymbol{M} = \frac{q_1 \mathtt{w}_0^2 \mathtt{v}}{3\mathtt{c}} \begin{bmatrix} \zeta_2 + \mathtt{v} & \zeta_2 \mathtt{v} \\ \zeta_0 \frac{1}{\mathtt{v}} & \zeta_0 + \frac{1}{\mathtt{v}} \end{bmatrix}.$$

Choosing $\mathtt{d} = \zeta_0\mathtt{v}+1+\zeta_2\frac{1}{\mathtt{v}}$ for type 1 and $\mathtt{d} = 3$ for type 2, it follows from (30) that $3\mathtt{c} = q_1\mathtt{w}_0^2\mathtt{v}\mathtt{d}$, which gives the matrix M from Table 2 (Case 1). The expressions for q and $\mathtt{w}$ follow from (24). Namely, we get $q(t) = q_1(\zeta_0 B_0^2(t)+B_1^2(t)+\zeta_2 B_2^2(t))$ and $\mathtt{w}(t) = \mathtt{w}_0(B_0^1(t)+\mathtt{v}B_1^1(t)) = \pm\sqrt{\frac{3\mathtt{c}}{q_1\mathtt{v}\mathtt{d}}}(B_0^1(t) + \mathtt{v}B_1^1(t))$. Since q_1 cancels out when computing $q(t)\mathtt{w}^2(t)$, we can without loss of generality set $q_1 = 1$. In both cases, there are 12 degrees of freedom for the construction of the DPH curve ($2 \cdot 4$ for $\mathcal{A}_0$, $\mathcal{A}_3$, 2 for $\mathtt{v}$ and 2 for ζ_0, ζ_2).

For the solution of type 1 it can happen that $q_1 = 0$ which leads to special cases. In particular, denoting $\mathtt{v} = \frac{\mathtt{w}_1}{\mathtt{w}_0}$ we get from (30a) that $3\mathtt{c} = q_0\mathtt{w}_1^2 + q_2\mathtt{w}_0^2 = \mathtt{w}_0^2\mathtt{v}(q_0\mathtt{v} + q_2\frac{1}{\mathtt{v}})$ and from (28)

$$\boldsymbol{M} = \frac{1}{q_0\mathtt{v} + q_2\frac{1}{\mathtt{v}}} \begin{bmatrix} q_2 & q_2\mathtt{v} \\ q_0\frac{1}{\mathtt{v}} & q_0 \end{bmatrix}.$$

If $q_0 \neq 0$ we denote $\zeta_0 = 1$, $\zeta_2 = \frac{q_2}{q_0}$, $\zeta_2 = \zeta$, and we get the matrix $\boldsymbol{M}$ from the theorem (Case 2 – (a)). The expressions for q and $\mathtt{w}$ follow from (24), setting $q_0 = 1$. There are 11 degrees of freedom ($2 \cdot 4$ for $\mathcal{A}_0$, $\mathcal{A}_3$, 2 for $\mathtt{v}$ and 1 for ζ). If $q_0 = 0$, then q_2 can be set to one and the result follows by assigning $\zeta_0 = 0$ and $\zeta_2 = 1$. The number of degrees of freedom is 10 ($2 \cdot 4$ for $\mathcal{A}_0$, $\mathcal{A}_3$ and 2 for $\mathtt{v}$).

The characterization of class 2 curves is complete because all cases of the condition (9) which is both sufficient and necessary are covered. □

Remark 5. Since the parameter $\mathtt{v}$ in Table 2 is not in $\mathbb{R}$, the derived DPH curves cannot be of class 3. However, in Case 1, choosing $\zeta_0\zeta_2 = 1$ yields

$$q(t) = \frac{1}{\zeta_0}\left(\zeta_0(1-t)+t\right)^2,$$

showing that the corresponding curves are also of class 1 with constant q and a quadratic polynomial $\mathtt{w}$ possessing one real zero (see Remark 4). More precisely, it is straightforward to verify that these curves can also be obtained from Case 1 of Table 1 by choosing $\mathtt{v}_0 = \frac{2\zeta_0}{1+\zeta_0\mathtt{v}}$ and $\mathtt{v}_2 = \frac{2\mathtt{v}}{1+\zeta_0\mathtt{v}}$. Similarly, Case 2 of Table 2 with $\zeta_0 = 1$, $\zeta_2 = 0$ produces the same curves as Case 1 of Table 1 with $\mathtt{v}_0 = \frac{2}{\mathtt{v}}$ and $\mathtt{v}_2 = 0$. Likewise, Case 2 of Table 2 with $\zeta_0 = 0$, $\zeta_2 = 1$ yields the same curves as Case 1 of Table 1 with $\mathtt{v}_0 = 0$ and $\mathtt{v}_2 = 2\mathtt{v}$.

Example 3.3. Let $\mathcal{A}_0$ and $\mathcal{A}_3$ be given by

$$\mathtt{a}_0 = 1,\quad \mathtt{b}_0 = 2,\quad \mathtt{a}_3 = 1+3\,\mathtt{i},\quad \mathtt{b}_3 = 4+2\,\mathtt{i},$$

and let us choose $\mathtt{v} = 1+\mathtt{i}$, $\zeta_0 = 0$ and $\zeta_2 = 4$. This gives

$$\text{Case 1, type 1}:\quad \mathtt{a}_1 = 1+\mathtt{i},\quad \mathtt{b}_1 = 2+2\,\mathtt{i},\quad \mathtt{a}_2 = \tfrac{8}{13}+\tfrac{27}{13}\,\mathtt{i},\quad \mathtt{b}_2 = \tfrac{19}{13}+\tfrac{43}{13}\,\mathtt{i},$$

$$\text{Case 1, type 2}:\quad \mathtt{a}_1 = \tfrac{5}{3}+\tfrac{1}{3}\,\mathtt{i},\quad \mathtt{b}_1 = \tfrac{10}{3}+\tfrac{2}{3}\,\mathtt{i},\quad \mathtt{a}_2 = 2+\tfrac{5}{3}\,\mathtt{i},\quad \mathtt{b}_2 = \tfrac{11}{3}+\tfrac{7}{3}\,\mathtt{i},$$

and two special cases for the same $\mathtt{v}$ and $\zeta = 4$:

$$\text{(a)}\quad \mathtt{a}_1 = \tfrac{17}{10}+\tfrac{9}{10}\,\mathtt{i},\quad \mathtt{b}_1 = \tfrac{17}{5}+\tfrac{4}{5}\,\mathtt{i},\quad \mathtt{a}_2 = \tfrac{4}{5}+\tfrac{13}{5}\,\mathtt{i},\quad \mathtt{b}_2 = \tfrac{13}{5}+\tfrac{21}{5}\,\mathtt{i},$$

$$\text{(b)}\quad \mathtt{a}_1 = 1+\mathtt{i},\quad \mathtt{b}_1 = 2+2\,\mathtt{i},\quad \mathtt{a}_2 = 2\,\mathtt{i},\quad \mathtt{b}_2 = 4\,\mathtt{i}.$$

Note that the curve given by Case 1, type 2 equals the helical DPH curve from (Farouki et al., 2009b, Example 3). The corresponding DPH curves with $\boldsymbol{p}(0) = (0,0,0)$ are shown in Fig. 5. The graph of $\frac{\mathtt{a}(t)}{\mathtt{b}(t)}$ in a complex plane, which confirms the helical property, is shown in Fig. 6. The control points of the hodograph, function σ and $\|\boldsymbol{p}' \times \boldsymbol{p}''\|$ for the first two curves equal

$$\boldsymbol{h}_0 = (-3,4,0),\quad \boldsymbol{h}_1 = (-3,4,0),\quad \boldsymbol{h}_2 = (-\tfrac{294}{65},\tfrac{382}{65},\tfrac{22}{65}),\quad \boldsymbol{h}_3 = (-\tfrac{446}{65},\tfrac{633}{65},\tfrac{89}{65}),$$

$$\boldsymbol{h}_4 = (-\tfrac{107}{13},\tfrac{194}{13},\tfrac{18}{5}),\quad \boldsymbol{h}_5 = (-\tfrac{73}{13},18,\tfrac{106}{13}),\quad \boldsymbol{h}_6 = (-10,20,20),$$

$$\sigma(t) = \tfrac{1}{13}\left(236t^6 - 84t^5 - 210t^4 - 88t^3 + 471t^2 + 65\right),\quad \|\boldsymbol{p}'(t)\times\boldsymbol{p}''(t)\| = 2\,\sigma(t)\,6\sqrt{\tfrac{10}{13}}t\left(t^3+t^2+t+1\right)$$

and

$$\boldsymbol{h}_0 = (-3,4,0),\quad \boldsymbol{h}_1 = (-5,\tfrac{20}{3},0),\quad \boldsymbol{h}_2 = (-\tfrac{22}{3},10,\tfrac{2}{5}),\quad \boldsymbol{h}_3 = (-\tfrac{48}{5},\tfrac{69}{5},\tfrac{9}{5}),$$

$$\boldsymbol{h}_4 = (-\tfrac{181}{15},\tfrac{278}{15},\tfrac{14}{3}),\quad \boldsymbol{h}_5 = (-\tfrac{37}{3},22,\tfrac{34}{3}),\quad \boldsymbol{h}_6 = (-10,20,20),$$

$$\sigma(t) = \left(t^2+1\right)\left(12t^4 - 28t^3 + 6t^2 + 20t + 5\right),\quad \|\boldsymbol{p}'(t)\times\boldsymbol{p}''(t)\| = 2\,\sigma(t)\,2\sqrt{10}t(t+1)\left(t^2+1\right).$$

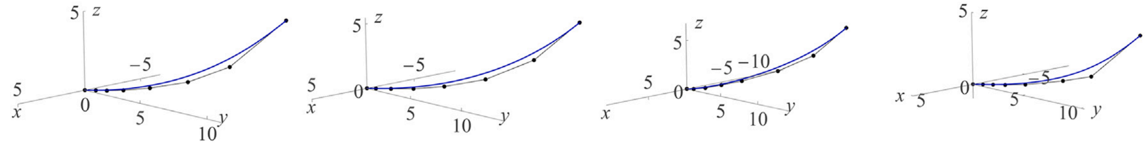

Fig. 5. Examples of degree 7 DPH curves of class 2. From left to right: Case 1, `type` 1 and 2; Case 2, (a) and (b).

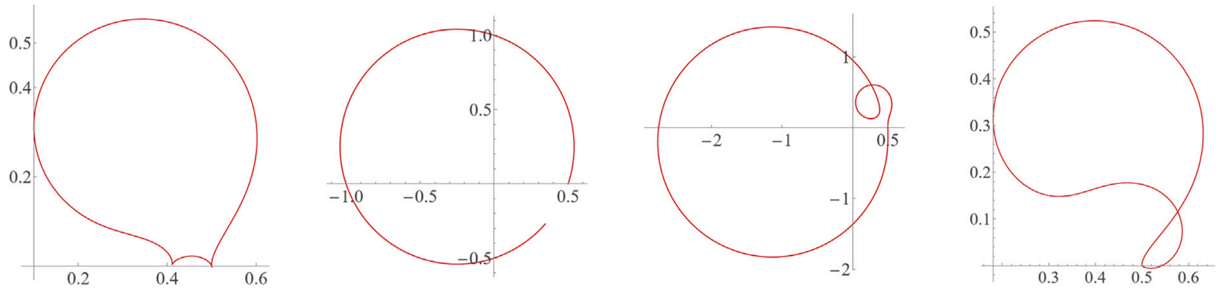

Fig. 6. The graphs of the rational function $t \mapsto \frac{\mathtt{a}(t)}{\mathtt{b}(t)}$, corresponding to the curves in Fig. 5.

3.3. *Construction of degree 7 DPH curves of class* 3

Let

$$\mathtt{w}(t) = \pm\sqrt{\mathtt{w}_0}, \quad \mathtt{w}_0 \in \mathbb{C}, \qquad q(t) = \sum_{j=0}^{4} q_j B_j^4(t), \quad q_j \in \mathbb{R}, \tag{31}$$

be a nonzero constant complex–valued polynomial and a real–valued polynomial of degree ≤ 4, respectively. The polynomial equality (9) is satisfied iff

$$3\left(\mathtt{a}_0\mathtt{b}_1 - \mathtt{a}_1\mathtt{b}_0\right) = q_0\mathtt{w}_0, \tag{32a}$$

$$3\left(\mathtt{a}_0\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_0\right) = 2q_1\mathtt{w}_0, \tag{32b}$$

$$\left(\mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0\right) + 3\left(\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1\right) = 2q_2\mathtt{w}_0 \tag{32c}$$

$$3\left(\mathtt{a}_1\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_1\right) = 2q_3\mathtt{w}_0, \tag{32d}$$

$$3\left(\mathtt{a}_2\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_2\right) = q_4\mathtt{w}_0. \tag{32e}$$

Assuming that $\mathtt{c} = \mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0 \neq 0$ we write (32a), (32d) and (32b), (32e) in a matrix form as

$$\begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_1 \\ \mathtt{b}_1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} q_0\mathtt{w}_0 \\ 2q_3\mathtt{w}_0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -\mathtt{b}_0 & \mathtt{a}_0 \\ \mathtt{b}_3 & -\mathtt{a}_3 \end{bmatrix} \begin{bmatrix} \mathtt{a}_2 \\ \mathtt{b}_2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 2q_1\mathtt{w}_0 \\ q_4\mathtt{w}_0 \end{bmatrix}. \tag{33}$$

From here we get

$$\begin{bmatrix} \mathtt{a}_1 & \mathtt{a}_2 \\ \mathtt{b}_1 & \mathtt{b}_2 \end{bmatrix} = \frac{\mathtt{w}_0}{3\mathtt{c}} \begin{bmatrix} \mathtt{a}_0 & \mathtt{a}_3 \\ \mathtt{b}_0 & \mathtt{b}_3 \end{bmatrix} \begin{bmatrix} 2q_3 & q_4 \\ q_0 & 2q_1 \end{bmatrix} \tag{34}$$

which is Eq. (10) for

$$\boldsymbol{M} = \frac{\mathtt{w}_0}{3\mathtt{c}} \begin{bmatrix} 2q_3 & q_4 \\ q_0 & 2q_1 \end{bmatrix}. \tag{35}$$

Computing the determinant of (34) yields

$$\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1 = \gamma\frac{\mathtt{w}_0^2}{9\mathtt{c}}, \quad \gamma = 4q_1q_3 - q_0q_4.$$

The remaining Eq. (32c) simplifies to $\frac{1}{3\mathtt{c}}\left(3\mathtt{c}^2 - 6q_2\mathtt{w}_0\mathtt{c} + \mathtt{w}_0^2\gamma\right) = 0$ and has solutions of two types:

$$\mathtt{type}\ 1: \quad \text{If} \quad 3q_2^2 < \gamma \quad \text{then} \quad \mathtt{w}_0 = \mathtt{v}\,\mathtt{c}, \quad \mathtt{v} = v_1 + \mathrm{i}v_2 = \frac{3q_2 \pm \sqrt{3}\mathrm{i}\sqrt{\gamma - 3q_2^2}}{\gamma} \in \mathbb{C}, \tag{36a}$$

$$\texttt{type 2}: \quad \text{If} \quad 3q_2^2 \geq \gamma \quad \text{then} \quad \mathtt{w}_0 = \upsilon\,\mathtt{c}, \quad \upsilon = \begin{cases} \frac{3q_2 \pm \sqrt{3}\sqrt{3q_2^2-\gamma}}{\gamma}, & \gamma \neq 0 \\ \frac{1}{2q_2}, & \gamma = 0 \end{cases}. \tag{36b}$$

Note that $\gamma = q_2 = 0$ is not possible since this would by (32c) imply $\mathtt{c} = 0$. The next theorem reveals a constructive way to obtain DPH curves of degree 7, class 3.

Theorem 4. *Suppose that the assumptions of Theorem 1 hold true and that two quaternions $\mathcal{A}_0$ and $\mathcal{A}_3$ are given. If the quaternions $\mathcal{A}_1$, $\mathcal{A}_2$ are equal to (11) with the matrix $\boldsymbol{M}$ of the form stated in Table 3, Type 1, then the PH curve $\boldsymbol{p}$ is a DPH curve of class 3, Type 1, and the polynomials in (9) are equal to*

$$q(t) = \zeta_3 B_0^4(t) + \frac{1}{2}\zeta_4 B_1^4(t) + \frac{1}{3}\cos\theta \cdot B_2^4(t) + \frac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t), \quad \mathtt{w}(t) = \pm\sqrt{3\,\mathtt{c}\,(\cos\theta + \mathtt{i}\sin\theta)}$$

for $\theta \in (0, 2\pi)\backslash\{\pi\}$ and any $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}$ such that $\zeta_1\zeta_4 - \zeta_2\zeta_3 = \frac{1}{3}$. The matrix $\boldsymbol{M}$ of the form stated in Table 3, Type 2 gives DPH curves of class 3, Type 2 with

$$q(t) = \zeta_3 B_0^4(t) + \frac{1}{2}\zeta_4 B_1^4(t) + \frac{1}{6}(1 + 3(\zeta_1\zeta_4 - \zeta_2\zeta_3))B_2^4(t) + \frac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t), \quad \mathtt{w}(t) = \pm\sqrt{3\,\mathtt{c}}$$

for any $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \mathbb{R}$. All class 3 DPH curves of degree 7 can be constructed in this way and the construction has 12 degrees of freedom.

Proof. From the previous analysis, we see that (32) are equivalent to (34) and (36). Consider first type 1 solutions. We note that in (36a), where γ is clearly positive,

$$\mathtt{v} = \sqrt{\frac{3}{\gamma}}\left(\sqrt{\frac{3}{\gamma}}q_2 \pm \mathtt{i}\sqrt{1 - \left(\sqrt{\frac{3}{\gamma}}q_2\right)^2}\right) =: \mathtt{v}_\pm$$

is a non-real point on the complex circle with radius $\sqrt{3/\gamma}$. Thus, there exists a unique value $\theta \in (0, \pi)$ such that $\mathtt{v}_+ = \sqrt{\frac{3}{\gamma}}(\cos\theta + \mathtt{i}\sin\theta)$, $\mathtt{v}_- = \sqrt{\frac{3}{\gamma}}(\cos(2\pi-\theta) + \mathtt{i}\sin(2\pi-\theta))$, and vice versa, i.e., for a fixed $\theta \in (0,\pi)$ we get $q_2 = \sqrt{\frac{\gamma}{3}}\cos\theta$. Since $\cos\theta = \cos(2\pi-\theta)$ and $\sin\theta = -\sin(2\pi-\theta)$, we extend the domain for θ to $(0,2\pi)\backslash\{\pi\}$ and set

$$\mathtt{v} = \sqrt{\frac{3}{\gamma}}(\cos\theta + \mathtt{i}\sin\theta) = \sqrt{\frac{3}{\gamma}}\,\mathtt{u}(\theta), \quad \mathtt{u}(\theta) = \cos\theta + \mathtt{i}\sin\theta, \quad \theta \in (0,2\pi)\backslash\{\pi\}, \quad q_2 = \sqrt{\frac{\gamma}{3}}\cos\theta.$$

By inserting $\mathtt{w}_0 = \mathtt{v}\,\mathtt{c} = \sqrt{\frac{3}{\gamma}}\,\mathtt{u}(\theta)\,\mathtt{c}$ into (35) and introducing new parameters ζ_j as

$$\zeta_1 = \frac{2q_3}{\sqrt{3\gamma}}, \quad \zeta_2 = \frac{q_4}{\sqrt{3\gamma}}, \quad \zeta_3 = \frac{q_0}{\sqrt{3\gamma}}, \quad \zeta_4 = \frac{2q_1}{\sqrt{3\gamma}},$$

we get the matrix $\boldsymbol{M}$ (Type 1) from the theorem. Since $\gamma = 4q_1q_3 - q_0q_4 = (\sqrt{3\gamma})^2(\zeta_1\zeta_4 - \zeta_2\zeta_3)$, the values ζ_j are constrained by $\zeta_1\zeta_4 - \zeta_2\zeta_3 = \frac{1}{3}$, and there is a one-to-one correspondence between q_0, q_1, q_3, q_4 and ζ_j, $j = 1, \ldots, 4$. From (31) we derive

$$\mathtt{w}(t) = \pm\sqrt{\mathtt{w}_0} = \pm\sqrt{\frac{3}{\sqrt{3\gamma}}\,\mathtt{u}(\theta)\,\mathtt{c}}, \quad q(t) = \sqrt{3\gamma}\left(\zeta_3 B_0^4(t) + \tfrac{1}{2}\zeta_4 B_1^4(t) + \tfrac{1}{3}\cos\theta \cdot B_2^4(t) + \tfrac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t)\right).$$

Since γ cancels out when computing $q(t)\mathtt{w}^2(t)$, we can without loss of generality set it to $\frac{1}{3}$ which implies q and $\mathtt{w}$ in Table 3 (Type 1). Because there are bijective correspondences in the construction, the stated conditions are sufficient and necessary. This completes the proof for type 1 solutions.

Suppose now that the solution of (32c) is of type 2 as in (36b). If $\gamma = 0$ then clearly there is a bijective correspondence between nonzero q_2 and nonzero υ, namely $q_2 = \frac{1}{2\upsilon}$. For $\gamma \neq 0$ the condition $3q_2^2 \geq \gamma$ implies that $q_2 \in \mathbb{R}$ for $\gamma < 0$ and $|q_2| \geq \sqrt{\gamma/3}$ for $\gamma > 0$. This prescribes the domain for q_2 of the two functions $q_2 \mapsto \upsilon_\pm(q_2) := \frac{3q_2 \pm \sqrt{3}\sqrt{3q_2^2-\gamma}}{\gamma}$. One can check that both functions are injective and that the union of their co-domains equals $\mathbb{R}\backslash\{0\}$. In more detail, for a fixed nonzero value $\upsilon \in \mathbb{R}$ there is a unique solution $q_2 = \frac{3+\upsilon^2\gamma}{6\upsilon}$ of the equation $\upsilon_\pm(q_2) = \upsilon$ depending on of the sign of $(\upsilon^2\gamma - 3)/\upsilon$, i.e., if $(\upsilon^2\gamma-3)/\upsilon \geq 0$, then $\upsilon_+(q_2) = \upsilon$ else $\upsilon_-(q_2) = \upsilon$. Introducing new parameters $\zeta_1 = \frac{2}{3}\upsilon q_3$, $\zeta_2 = \frac{1}{3}\upsilon q_4$, $\zeta_3 = \frac{1}{3}\upsilon q_0$ and $\zeta_4 = \frac{2}{3}\upsilon q_1$ we get the matrix $\boldsymbol{M}$ from the theorem. Moreover, $\gamma = \frac{9}{\upsilon^2}\left(\zeta_1\zeta_4 - \zeta_2\zeta_3\right)$ and $q_2 = \frac{1}{2\upsilon}\left(1 + \frac{1}{3}\upsilon\gamma\right) = \frac{1}{2\upsilon}(1 + 3(\zeta_1\zeta_4 - \zeta_2\zeta_3))$. This expression for q_2 holds also for the case when $\gamma = 0$. Inserting the new parameters and q_2 into (31), we obtain

$$q(t) = \frac{3}{\upsilon}\left(\zeta_3 B_0^4(t) + \frac{1}{2}\zeta_4 B_1^4(t) + \frac{1}{6}(1 + 3(\zeta_1\zeta_4 - \zeta_2\zeta_3))B_2^4(t) + \frac{1}{2}\zeta_1 B_3^4(t) + \zeta_2 B_4^4(t)\right), \quad \mathtt{w}(t) = \pm\sqrt{\upsilon\mathtt{c}}.$$

Since υ cancels out when computing $q(t)\mathtt{w}^2(t)$, we can without loss of generality set it to $\upsilon = 3$ to get q and $\mathtt{w}$ in Table 3 (Type 2).

We covered all the cases and constructed bijective correspondences between parameters, therefore all the class 3 curves are included in this construction. The number of free parameters is 12 ($2 \cdot 4$ for $\mathcal{A}_0$ and $\mathcal{A}_3$ and for type 1 curves we add 3 for ζ_j and 1 for θ; or for type 2 curves add 4 for ζ_j). □

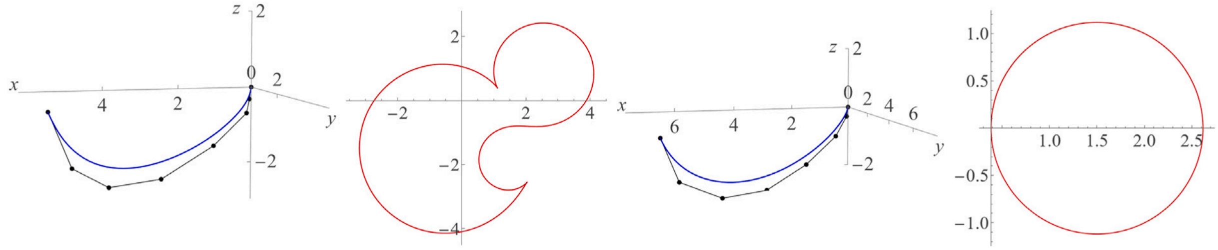

Fig. 7. Examples of degree 7 DPH curves of class 3, type 1 and 2; and the corresponding graphs of the rational function $t \mapsto \frac{\mathtt{a}(t)}{\mathtt{b}(t)}$.

Example 3.4. Let us choose

$$\mathtt{a}_0 = 1 + \mathtt{i}, \quad \mathtt{b}_0 = \mathtt{i}, \quad \mathtt{a}_3 = 3 + \mathtt{i}, \quad \mathtt{b}_3 = 2 - \mathtt{i}, \quad \text{and} \quad \zeta_1 = 2, \quad \zeta_2 = \tfrac{3}{2}, \quad \zeta_3 = \tfrac{4}{9}, \quad \zeta_4 = \tfrac{1}{2}, \quad \theta = \tfrac{\pi}{3}.$$

Note that $\zeta_1\zeta_4 - \zeta_2\zeta_3 = \frac{1}{3}$. This gives

$$\texttt{Type 1}: \quad \mathtt{a}_1 = \tfrac{1}{9}\left(15 - 11\sqrt{3}\right) + \left(\tfrac{11}{9} + \tfrac{5\sqrt{3}}{3}\right)\mathtt{i}, \quad \mathtt{b}_1 = \tfrac{1}{9}\left(4 - 7\sqrt{3}\right) + \tfrac{1}{9}\left(7 + 4\sqrt{3}\right)\mathtt{i},$$
$$\mathtt{a}_2 = \tfrac{3}{2} - \sqrt{3} + \left(1 + \tfrac{3\sqrt{3}}{2}\right)\mathtt{i}, \quad \mathtt{b}_2 = \tfrac{1}{2}\left(1 - \sqrt{3}\right) + \tfrac{1}{2}\left(1 + \sqrt{3}\right)\mathtt{i},$$

$$\texttt{Type 2}: \quad \mathtt{a}_1 = \tfrac{10}{3} + \tfrac{22}{9}\mathtt{i}, \quad \mathtt{b}_1 = \tfrac{8}{9} + \frac{14}{9}\mathtt{i}, \quad \mathtt{a}_2 = 3 + 2\,\mathtt{i}, \quad \mathtt{b}_2 = 1 + \mathtt{i}.$$

The corresponding DPH curves with $\boldsymbol{p}(0) = (0,0,0)$ are shown in Fig. 7 together with graphs of $\frac{\mathtt{a}(t)}{\mathtt{b}(t)}$ in a complex plane, which confirm the helical property. Bézier coefficients of the hodograph, function σ and $\|\boldsymbol{p}' \times \boldsymbol{p}''\|$ are equal to

$$\boldsymbol{h}_0 = (1,2,-2), \quad \boldsymbol{h}_1 = \left(\tfrac{19}{9}, \tfrac{22}{9} + \tfrac{4}{\sqrt{3}}, -2\right), \quad \boldsymbol{h}_2 = \left(\tfrac{1232}{135}, \tfrac{1204}{135} + \tfrac{3\sqrt{3}}{5}, -\tfrac{569}{135}\right),$$
$$\boldsymbol{h}_3 = \left(\tfrac{117}{10}, \tfrac{53}{5}, -\tfrac{17}{5}\right), \quad \boldsymbol{h}_4 = \left(\tfrac{407}{45}, \tfrac{346}{45} - \tfrac{12\sqrt{3}}{5}, -\tfrac{14}{45}\right), \quad \boldsymbol{h}_5 = \left(5, 4 - \tfrac{9\sqrt{3}}{2}, \tfrac{5}{2}\right), \quad \boldsymbol{h}_6 = (5,10,10),$$
$$\sigma(t) = \frac{1}{3}\left(\left(47 - 16\sqrt{3}\right)t^6 + \left(98\sqrt{3} - 200\right)t^5 - 124\left(\sqrt{3} - 6\right)t^4 + \left(88\sqrt{3} - 974\right)t^3 + \left(407 - 62\sqrt{3}\right)t^2\right.$$
$$\left. +4\left(3 + 4\sqrt{3}\right)t + 9\right), \qquad \|\boldsymbol{p}'(t) \times \boldsymbol{p}''(t)\| = 2\,\sigma(t)\,6\sqrt{5}\left(-\tfrac{37t^4}{18} + \tfrac{29t^3}{9} + \tfrac{2t^2}{3} - \tfrac{7t}{9} + \tfrac{4}{9}\right)$$

for Type 1 and

$$\boldsymbol{h}_0 = (1,2,-2), \quad \boldsymbol{h}_1 = \left(\tfrac{38}{9}, \tfrac{44}{9}, -4\right), \quad \boldsymbol{h}_2 = \left(\tfrac{268}{27}, \tfrac{1312}{135}, -\tfrac{130}{27}\right),$$
$$\boldsymbol{h}_3 = \left(\tfrac{117}{10}, \tfrac{53}{5}, -\tfrac{17}{5}\right), \quad \boldsymbol{h}_4 = \left(\tfrac{517}{45}, \tfrac{422}{45}, \tfrac{26}{45}\right), \quad \boldsymbol{h}_5 = (10,8,5), \quad \boldsymbol{h}_6 = (5,10,10),$$
$$\sigma(t) = \tfrac{1}{3}\left(41t^6 - 206t^5 + 522t^4 - 530t^3 + 131t^2 + 78t + 9\right),$$
$$\|\boldsymbol{p}'(t) \times \boldsymbol{p}''(t)\| = 2\,\sigma(t)\,\tfrac{1}{3}\sqrt{5}\left(-19t^4 + 22t^3 + 30t^2 - 14t + 8\right)$$

for Type 2. Replacing ζ_3 by $\zeta_3 = 0$ for Type 2 gives $\mathtt{a}_1 = 2 + 2\,\mathtt{i}$, $\mathtt{b}_1 = 2\,\mathtt{i}$, $\mathtt{a}_2 = 3 + 2\,\mathtt{i}$, $\mathtt{b}_2 = 1 + \mathtt{i}$, and the corresponding curve is the helical DPH curve from (Farouki et al., 2009b, Example 1).

3.4. Construction of DPH curves for $\mathtt{c} = 0$

It remains to construct DPH curves for the case when $\mathtt{c} = \mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0 = 0$. We assume that $\mathcal{A}_0 = \mathtt{a}_0 + \mathbf{k}\mathtt{b}_0 \neq 0$ and $\mathcal{A}_3 = \mathtt{a}_3 + \mathbf{k}\mathtt{b}_3 \neq 0$ to get a DPH curve with nonzero boundary tangent vectors. The value $\mathtt{c} = \mathtt{a}_0\mathtt{b}_3 - \mathtt{a}_3\mathtt{b}_0$ equals zero iff

$$\mathtt{a}_3 = \mathtt{a}_0\,\mathtt{d}, \quad \mathtt{b}_3 = \mathtt{b}_0\,\mathtt{d} \quad \text{for some nozero} \quad \mathtt{d} \in \mathbb{C}. \tag{37}$$

This implies that $\boldsymbol{p}'(1) = \mathcal{A}_3^{2\star} = |\mathtt{d}|^2\mathcal{A}_0^{2\star} = |\mathtt{d}|^2\boldsymbol{p}'(0)$, i.e., boundary tangent vectors have the same direction. Let us denote the Bernstein coefficients of the quartic polynomial $q(t)\mathtt{w}^2(t)$ by $\mathtt{g}_i$, $i = 0,1,\ldots,4$. Using the equalities (37) in the condition (9) gives

$$\begin{aligned} &3\left(\mathtt{a}_0\mathtt{b}_1 - \mathtt{a}_1\mathtt{b}_0\right) = \mathtt{g}_0, \quad \tfrac{3}{2}\left(\mathtt{a}_0\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_0\right) = \mathtt{g}_1, \quad \tfrac{3}{2}\left(\mathtt{a}_1\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_1\right) = \mathtt{g}_2, \\ &-\tfrac{3}{2}\mathtt{d}\left(\mathtt{a}_0\mathtt{b}_1 - \mathtt{a}_1\mathtt{b}_0\right) = \mathtt{g}_3, \quad -3\mathtt{d}\left(\mathtt{a}_0\mathtt{b}_2 - \mathtt{a}_2\mathtt{b}_0\right) = \mathtt{g}_4, \end{aligned} \tag{38}$$

which implies two conditions on the coefficients $\mathtt{g}_i$:

$$\mathtt{g}_3 = -\tfrac{\mathtt{d}}{2}\,\mathtt{g}_0, \quad \mathtt{g}_4 = -2\,\mathtt{d}\,\mathtt{g}_1. \tag{39}$$

Furthermore, assuming that (39) holds true, only the first three equations in (38) must be satisfied, and the remaining two are then fulfilled automatically. From the first two equations we derive

$$\text{if } \mathtt{a}_0 \neq 0: \quad \mathtt{b}_1 = \frac{3\mathtt{a}_1\mathtt{b}_0 + \mathtt{g}_0}{3\mathtt{a}_0},\ \mathtt{b}_2 = \frac{3\mathtt{a}_2\mathtt{b}_0 + 2\mathtt{g}_1}{3\mathtt{a}_0}; \quad \text{else} \quad \mathtt{a}_1 = -\frac{\mathtt{g}_0}{3\mathtt{b}_0},\ \mathtt{a}_2 = -\frac{2\mathtt{g}_1}{3\mathtt{b}_0},$$

which simplifies the third equation to

$$\text{if } \mathtt{a}_0 \neq 0: \quad 2\mathtt{g}_1\mathtt{a}_1 - \mathtt{g}_0\mathtt{a}_2 = 2\mathtt{g}_2\mathtt{a}_0; \quad \text{else} \quad 2\mathtt{g}_1\mathtt{b}_1 - \mathtt{g}_0\mathtt{b}_2 = 2\mathtt{g}_2\mathtt{b}_0.$$

Therefore, in the case when at least one of $\mathtt{g}_0$ and $\mathtt{g}_1$ is nonzero, the coefficients $\mathtt{a}_1$, $\mathtt{b}_1$, $\mathtt{a}_2$, $\mathtt{b}_2$ can be expressed with $\mathtt{a}_0$, $\mathtt{b}_0$, $(\mathtt{g}_i)_{i=0}^2$ and one additional free complex coefficient.

If $\mathtt{g}_0 = \mathtt{g}_1 = 0$, then also $\mathtt{g}_2$ must be zero, which can be true only if $q\,\mathtt{w}^2$ is identically zero. This special case produces a curve (expressed with $\mathtt{a}_0$, $\mathtt{b}_0$, $\mathtt{d}$ and two additional free complex coefficients) with a zero curvature, which can only be a reparameterized straight line.

Assume now that $q\,\mathtt{w}^2 \not\equiv 0$ and that $\mathtt{g}_0$ and $\mathtt{g}_1$ are not both equal to zero. The following lemma reveals the solutions of (39) for all three classes of DPH curves showing also the degrees of freedom.

Lemma 1. *Assume that $q(t) \not\equiv 0$, $w(t) \not\equiv 0$ and let $q(t)w^2(t) = \sum_{j=0}^3 g_j B_j^3(t)$. Furthermore, let us write d from (37) in a polar form $d = r_d(\cos\varphi_d + i\sin\varphi_d)$, $r_d > 0$, $\varphi_d \in [0, 2\pi)$, and let*

$$e = \sqrt[3]{d} = \sqrt[3]{r_d}(\cos\varphi_e + i\sin\varphi_e), \quad \varphi_e = \tfrac{\varphi_d + 2\ell\pi}{3}, \quad \ell = 0, 1, 2,$$

denote the solutions of the equation $e^3 = d$. Assuming in addition that g_0 and g_1 are not both equal to zero, conditions (39) are satisfied iff

$$\textit{class } 1: \quad w_0 = -2e^{-1}v,\ w_1 = v,\ w_2 = -2ev \tag{40a}$$

$$\textit{class } 2: \quad w_0 = v,\ w_1 = ev,\ q_0 = -\tfrac{1}{2}(Re(e))^{-1},\ q_1 = 1,\ q_2 = r_d^{2/3}q_0 \quad (\textit{for} \quad Re(e) \neq 0) \quad \textit{or} \tag{40b}$$

$$w_0 = v,\ w_1 = -\zeta dv,\ q_0 = \zeta,\ q_1 = 0,\ q_2 = 1 \tag{40c}$$

for any nonzero $v \in \mathbb{C}$ and nonzero $\zeta \in \mathbb{R}$. For curves of class 3 the solution exists iff $d = d \in \mathbb{R}$ and

$$w_0 = v,\ q_0 = \zeta_0,\ q_1 = \zeta_1,\ q_2 = 1,\ q_3 = -\tfrac{d}{2}\zeta_0,\ q_4 = -2d\zeta_1, \tag{40d}$$

for any nonzero $v \in \mathbb{C}$ and $\zeta_0, \zeta_1 \in \mathbb{R}$ not both equal to zero.

Proof. Consider first class 1 curves with $q(t) = 1$ and $\mathtt{w}(t)$ from (12). Coefficients $\mathtt{g}_i$ are given by

$$\mathtt{g}_0 = \mathtt{w}_0^2, \quad \mathtt{g}_1 = \mathtt{w}_0\mathtt{w}_1, \quad \mathtt{g}_2 = \tfrac{1}{3}\left(2\mathtt{w}_1^2 + \mathtt{w}_0\mathtt{w}_2\right), \quad \mathtt{g}_3 = \mathtt{w}_1\mathtt{w}_2, \quad \mathtt{g}_4 = \mathtt{w}_2^2,$$

and Eq. (39) are equal to

$$2\mathtt{w}_1\mathtt{w}_2 + \mathtt{d}\,\mathtt{w}_0^2 = 0 \quad \text{and} \quad \mathtt{w}_2^2 + 2\mathtt{d}\,\mathtt{w}_0\mathtt{w}_1 = 0. \tag{41}$$

If $\mathtt{w}_1 = 0$, then also $\mathtt{w}_0 = \mathtt{w}_2 = 0$, which implies $\mathtt{w}(t) \equiv 0$. Assuming $\mathtt{w}_1 \neq 0$, it is straightforward to see that both $\mathtt{w}_0$ and $\mathtt{w}_2$ must also be nonzero, else $\mathtt{g}_0$ and $\mathtt{g}_1$ would both be zero. Multiplying the first equation in (41) by $-\mathtt{d}\,\mathtt{w}_0$ and the second by $\mathtt{w}_2$, their sum gives $\mathtt{w}_2^3 = \mathtt{d}^2\,\mathtt{w}_0^3$. Thus, $\mathtt{w}_2 = \mathtt{e}^2\mathtt{w}_0$, and the conditions (40a) follow directly from (41).

For class 2 curves with q and $\mathtt{w}$ given by (24), coefficients $\mathtt{g}_i$ and Eq. (39) are equal to

$$\mathtt{g}_0 = q_0\mathtt{w}_0^2,\ \mathtt{g}_1 = \frac{1}{2}\mathtt{w}_0\left(q_1\mathtt{w}_0 + q_0\mathtt{w}_1\right),\ \mathtt{g}_2 = \frac{1}{6}\left(q_2\mathtt{w}_0^2 + \mathtt{w}_1\left(4q_1\mathtt{w}_0 + q_0\mathtt{w}_1\right)\right),\ \mathtt{g}_3 = \frac{1}{2}\mathtt{w}_1\left(q_2\mathtt{w}_0 + q_1\mathtt{w}_1\right),\ \mathtt{g}_4 = q_2\mathtt{w}_1^2,$$

and

$$q_1\mathtt{w}_1^2 + q_2\mathtt{w}_0\mathtt{w}_1 + \mathtt{d}\,q_0\mathtt{w}_0^2 = 0, \quad q_2\mathtt{w}_1^2 + \mathtt{d}(q_0\mathtt{w}_0\mathtt{w}_1 + q_1\mathtt{w}_0^2) = 0. \tag{42}$$

Again, it is straightforward to see that both $\mathtt{w}_0$ and $\mathtt{w}_1$ must be nonzero. Multiplying the first equation in (42) by $(-\mathtt{w}_1)$ and the second by $\mathtt{w}_0$, their sum gives $q_1(\mathtt{w}_1^3 - \mathtt{d}\,\mathtt{w}_0^3) = 0$.

If $q_1 \neq 0$, then $\mathtt{w}_1 = \mathtt{e}\mathtt{w}_0$ and (42) both reduce to $\mathtt{e}\,q_0 + q_1 + \mathtt{e}^{-1}q_2 = 0$. Since q_1 does not introduce an additional degree of freedom, we set $q_1 = 1$. Moreover, if $\mathrm{Re}(\mathtt{e}) = 0$, then no solutions for q_0, q_2 exist, the conditions can be expressed as in (40b). Note that in case $\mathtt{e} \in \mathbb{R}$, we obtain one additional degree of freedom, and the curve is a special case of class 3 curves considered below. However, $\mathtt{e}$ can be real only if $\mathtt{d}$ is real.

In the case when $q_1 = 0$, Eq. (42) both reduce to $\mathtt{d}q_0\mathtt{w}_0 + q_2\mathtt{w}_1 = 0$. Values q_0 and q_2 must be nonzero, $\mathtt{g}_0$ and $\mathtt{g}_1$ would be zero. Again, one of the nonzero coefficients in $q(t)$ may be normalized to one, since it does not introduce a new degree of freedom. Choosing $q_2 = 1$ yields (40c).

For curves of class 3 with q and $\mathtt{w}$ given by (31), coefficients $\mathtt{g}_i$ and Eq. (39) are equal to

$$\mathtt{g}_i = q_i\mathtt{w}_0, \quad i = 0, 1, \ldots, 4, \quad \text{and} \quad \mathtt{w}_0(\mathtt{d}\,q_0 + 2q_3) = 0, \quad \mathtt{w}_0(2\mathtt{d}\,q_1 + q_4) = 0.$$

Clearly the solution exists iff $\mathtt{d} = d \in \mathbb{R}$ and equals (40d). This completes the proof. □

A theoretical examination of the helical properties in the special case $\mathtt{c} = 0$ is omitted for brevity. However, numerical experiments suggest that in the specific case (40c) the resulting DPH curves are helical, whereas for $\mathtt{d}$ that is neither real nor purely imaginary the corresponding curves appear to be nonhelical.

4. Helical properties of DPH curves

The last statement in Tables 1–3 reveals the helical property of the DPH curves. A given curve is a helix (Kreyszig, 1959) if its unit tangent $\boldsymbol{t} = \frac{p'}{\|p'\|}$ maintains a constant angle ψ with respect to a fixed unit vector $\boldsymbol{a} \in \mathbb{R}^3$, so that $\boldsymbol{a} \cdot \boldsymbol{t} = \cos\psi$, or equivalently $\frac{\kappa}{\tau} = \tan\psi$, i.e., the ratio between the curvature and torsion is constant. A helical curve can also be characterized by observing the curve that the unit tangent vector $\boldsymbol{t}$ traces on the unit sphere. Namely, for a helical curve, this trace is a circle (Farouki et al., 2004).

It is shown in Farouki et al. (2009a,b) that polynomial helices described as PH curves, with the hodograph computed by the Hopf map of complex-valued polynomials $\mathtt{a}$ and $\mathtt{b}$, are exactly the curves for which $\mathtt{z}(t) = \frac{\mathtt{a}(t)}{\mathtt{b}(t)}$ is a rational parametrization of a line or a circle in $\mathbb{C}$. This means that for helical DPH curves the function $t \mapsto \mathtt{z}(t)$ must be a reparametrized Möbius transformation, i.e., it must be of the form

$$\mathtt{z}(t) = \frac{(\widetilde{\mathtt{a}}_0(1-\varphi(t)) + \widetilde{\mathtt{a}}_3\varphi(t))\widetilde{\mathtt{r}}(t)}{(\widetilde{\mathtt{b}}_0(1-\varphi(t)) + \widetilde{\mathtt{b}}_3\varphi(t))\widetilde{\mathtt{r}}(t)} \tag{43}$$

for some $\widetilde{\mathtt{a}}_0, \widetilde{\mathtt{a}}_3, \widetilde{\mathtt{b}}_0, \widetilde{\mathtt{b}}_3 \in \mathbb{C}$, a real-valued *'reparameterization function'* φ, and a complex-valued *'multiplication function'* $\widetilde{\mathtt{r}}$. This implies

$$\mathtt{a}(t)\mathtt{b}'(t) - \mathtt{a}'(t)\mathtt{b}(t) = (\widetilde{\mathtt{a}}_0\widetilde{\mathtt{b}}_3 - \widetilde{\mathtt{a}}_3\widetilde{\mathtt{b}}_0)\varphi'(t)\widetilde{\mathtt{r}}^2(t). \tag{44}$$

Thus, starting from a line/circle Möbius transformation in the complex plane, multiplication with a quadratic polynomial, quadratic re-parameterization followed by multiplication with a linear polynomial, and cubic reparameterization lead to class 1, class 2 and class 3 helical DPH curves, respectively.

If $\mathtt{z}$ is not of the form (43), the DPH curve is not a helix. The next theorem follows directly using (10).

Theorem 5. *Suppose that the assumptions of Theorem 1 hold true and let the matrix M be denoted as $M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$. Then*

$$z(t) = \frac{a(t)}{b(t)} = \frac{a_0\,(1-t)\,r_0(t) + a_3\,t\,r_3(t)}{b_0\,(1-t)\,r_0(t) + b_3\,t\,r_3(t)}, \tag{45}$$

where

$$r_0(t) = B_0^2(t) + \tfrac{3}{2}\,m_{11}\,B_1^2(t) + 3\,m_{12}\,B_2^2(t), \quad r_3(t) = B_2^2(t) + \tfrac{3}{2}\,m_{22}\,B_1^2(t) + 3\,m_{21}\,B_0^2(t) \tag{46}$$

for all DPH curves of degree 7.

The following lemma and its corollary are further needed.

Lemma 2. *Suppose that polynomials r_0 and r_3 given in (46) do not have a common factor (complex or real valued) and assume $c = a_0 b_3 - a_3 b_0 \neq 0$. Then the same holds true for polynomials a and b in (45).*

Proof. Assuming the opposite, let $\mathtt{k}$ be the non-constant polynomial that divides $\mathtt{a}$ and $\mathtt{b}$, i.e.,

$$\mathtt{a}_0\,(1-t)\,\mathtt{r}_0(t) + \mathtt{a}_3\,t\,\mathtt{r}_3(t) = \mathtt{k}(t)\,\widetilde{\mathtt{a}}(t), \quad \mathtt{b}_0\,(1-t)\,\mathtt{r}_0(t) + \mathtt{b}_3\,t\,\mathtt{r}_3(t) = \mathtt{k}(t)\,\widetilde{\mathtt{b}}(t) \tag{47}$$

for some polynomials $\widetilde{\mathtt{a}}$ and $\widetilde{\mathtt{b}}$. Assumption $\mathtt{c} \neq 0$ implies that $\mathtt{a}_0\mathtt{b}_0 \neq 0$ and $\mathtt{a}_3\mathtt{b}_3 \neq 0$. Multiplying Eq. (47) by $\mathtt{b}_3$ and $\mathtt{a}_3$ respectively, their difference gives $\mathtt{c}\,(1-t)\,\mathtt{r}_0(t) = \mathtt{k}(t)(\mathtt{b}_3\widetilde{\mathtt{a}}(t) - \mathtt{a}_3\widetilde{\mathtt{b}}(t))$. Similarly we obtain $\mathtt{c}\,t\,\mathtt{r}_3(t) = \mathtt{k}(t)(\mathtt{a}_0\widetilde{\mathtt{b}}(t) - \mathtt{b}_0\widetilde{\mathtt{a}}(t))$. Since $\mathtt{r}_0$ and $\mathtt{r}_3$ are assumed not to have a common factor, polynomial $\mathtt{k}$ must divide $(1-t)$ and t. This is possible only for constant $\mathtt{k}$ which gives the contradiction, and completes the proof. □

Corollary 1. *Suppose that polynomials r_0 and r_3 given in (46) are linearly independent over $\mathbb{C}$ and $c = a_0 b_3 - a_3 b_0 \neq 0$. Then the rational function $t \mapsto z(t)$ in (45) cannot reduce to a linear rational function (Möbius transformation).*

Theorem 6. *Class 1 DPH curves of degree 7, given by Theorem 2 and Table 1, have*

$$r_0(t) = \tfrac{1}{d}\left(dB_0^2(t) + 3v_2 B_1^2(t) + 3v_2^2 B_2^2(t)\right), \quad r_3(t) = \tfrac{1}{d}\left(dB_2^2(t) + 3v_0 B_1^2(t) + 3v_0^2 B_0^2(t)\right) \tag{48}$$

for Case 1 and

$$r_0(t) = B_0^2(t) + 3\tfrac{v^2}{d}B_2^2(t), \quad r_3(t) = B_2^2(t) + 3\tfrac{1}{d}B_0^2(t) \tag{49}$$

for Case 2. Additionally, all $type$ 2 curves are helical and all $type$ 1 curves are not.

Proof. The expressions (48) and (49) follow directly from (46) and Table 1. For Case 1, polynomials (48) are linearly independent over $\mathbb{C}$ iff $\mathtt{d} \neq 3\mathtt{v}_0\mathtt{v}_2$. Namely, for $\mathtt{k}_0, \mathtt{k}_3 \in \mathbb{C}$, $\mathtt{k}_0\mathtt{k}_3 \neq 0$, the equality $\mathtt{k}_0\mathtt{r}_0(t) + \mathtt{k}_3\mathtt{r}_3(t) \equiv 0$ holds true iff

$$\mathtt{d}\,\mathtt{k}_0 + 3\mathtt{v}_0^2\,\mathtt{k}_3 = 0, \quad 3\mathtt{v}_2\,\mathtt{k}_0 + 3\mathtt{v}_0\,\mathtt{k}_3 = 0, \quad 3\mathtt{v}_2^2\,\mathtt{k}_0 + \mathtt{d}\,\mathtt{k}_3 = 0.$$

Note that at least one of $\mathtt{v}_0, \mathtt{v}_2$ is nonzero. Assuming $\mathtt{v}_0$ is nonzero, multiplying the second equation by $\mathtt{v}_0$ and applying first equality, we obtain $\mathtt{k}_0(\mathtt{d} - 3\mathtt{v}_0\mathtt{v}_2) = 0$. If $\mathtt{d} \neq 3\mathtt{v}_0\mathtt{v}_2$, then $\mathtt{k}_0 = 0$ and from the last equation we see that also $\mathtt{k}_3 = 0$, so (46) are linearly independent over $\mathbb{C}$. The same holds true if $v_0 = 0$ and $v_2 \neq 0$. Since r_0 and r_3 are for Case 1, Type 1, linearly independent over $\mathbb{C}$, these curves are non-helical by Corollary 1 and the fact that class 1 helical curves can be obtained only by a *quadratic multiplication* of some Möbius transformation.

For $d = 3v_0v_2$ we compute that $v_0\, r_0(t) = v_2\, r_3(t)$, and thus

$$z(t) = \frac{a(t)}{b(t)} = \frac{a_0\, v_2\,(1-t) + a_3\, v_0\, t}{b_0\, v_2\,(1-t) + b_3\, v_0\, t}$$

showing that Case 1, Type 2 curves are (monotonic) helical curves.

Consider now polynomials (49). If $d = 3v$, then $r_0(t) = v r_3(t)$ and $z(t) = \frac{a_0\, v\,(1-t)+a_3\, t}{b_0\, v\,(1-t)+b_3\, t}$ showing that Case 2, Type 2 curves are helical curves. For $d = -v$ it is straightforward to check that (49) are linearly independent and therefore (by Corollary 1) non-helical. □

Theorem 7. *Class 2 DPH curves of degree 7, given by Theorem 3 and Table 2, have*

$$r_0(t) = \frac{1}{d}\left(dB_0^2(t) + \frac{3}{2}(\zeta_2 + v)B_1^2(t) + 3\zeta_2 v B_2^2(t)\right),\quad r_3(t) = \frac{1}{d}\left(dB_2^2(t) + \frac{3}{2}(\zeta_0 + \frac{1}{v})B_1^2(t) + \frac{3\zeta_0}{v}B_0^2(t)\right) \tag{50a}$$

for Case 1 and

$$(a):\quad r_0(t) = B_0^2(t) + \frac{3}{2}\frac{v\zeta}{v^2+\zeta}B_1^2(t) + \frac{3\zeta v^2}{v^2+\zeta}B_2^2(t),\qquad r_3(t) = B_2^2(t) + \frac{3}{2}\frac{v}{v^2+\zeta}B_1^2(t) + \frac{3}{v^2+\zeta}B_0^2(t), \tag{50b}$$

$$(b):\quad r_0(t) = B_0^2(t) + \frac{3}{2}vB_1^2(t) + 3v^2B_2^2(t),\qquad r_3(t) = B_2^2(t). \tag{50c}$$

for Case 2. Additionally, Case 1, type 2 curves are helical and all other curves are non-helical.

Proof. The expressions (50) follow directly from (46) and Table 2. Consider first Case 1, Type 2 solutions for which $d = 3$. It is easy to check that (50a) simplifies to

$$r_0(t) = (B_0^1(t) + \zeta_2 B_1^1(t))(B_0^1(t) + vB_1^1(t)),\quad r_3(t) = \frac{1}{v}(\zeta_0 B_0^1(t) + B_1^1(t))(B_0^1(t) + vB_1^1(t))$$

and thus

$$z(t) = \frac{a_0(B_0^2(t) + \frac{\zeta_2}{2}B_1^2(t)) + \frac{a_3}{v}(B_2^2(t) + \frac{\zeta_0}{2}B_1^2(t))}{b_0(B_0^2(t) + \frac{\zeta_2}{2}B_1^2(t)) + \frac{b_3}{v}(B_2^2(t) + \frac{\zeta_0}{2}B_1^2(t))} = \frac{a_0 v(1-\rho(t)) + a_3\rho(t)}{b_0 v(1-\rho(t)) + b_3\rho(t)},$$

where

$$\rho(t) = \frac{\frac{1}{2}\zeta_0 B_1^2(t) + B_2^2(t)}{B_0^2(t) + \frac{1}{2}(\zeta_0 + \zeta_2)B_1^2(t) + B_2^2(t)},$$

which proves that these type 2 DPH curves are helical curves, obtained by *'multiplication and quadratic reparameterization'* of the quotient $\frac{v a_0(1-t)+a_3\, t}{v b_0(1-t)+b_3\, t}$.

To prove that all other class 2 curves are non-helical, it is sufficient to check that r_0 and r_3 cannot have a common linear complex-valued factor. If the polynomials had a common quadratic factor, the resulting curve would be class 1.

For Case 1, Type 1, for which $d = \zeta_0 v + 1 + \zeta_2\frac{1}{v}$, $d \neq 3$, let us prove that (50a) cannot have a common linear complex-valued factor. Note that $d = 3$ iff $\zeta_0 v^2 - 2v + \zeta_2 = 0$. Assuming that $\frac{r_0(t)}{r_3(t)} = \frac{k_1(1-t)+k_2\, t}{k_3(1-t)+k_4\, t}$ for some $k_j \in \mathbb{C}$ leads to a following homogeneous matrix system

$$\begin{bmatrix} -3\zeta_0 & 0 & d\,v & 0 \\ -\zeta_0 v - 1 & -\zeta_0 & v\left(\zeta_2 + v\right) & \frac{1}{3}d\,v \\ -\frac{1}{3}d\,v & -\zeta_0 v - 1 & \zeta_2 v^2 & v\left(\zeta_2 + v\right) \\ 0 & -d\,v & 0 & 3\zeta_2 v^2 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The determinant of the matrix equals $\frac{1}{9}\left(\zeta_0 v^2 - 2v + \zeta_2\right)^3 d\,v$ and it is nonzero, since $d \neq 0$, $d \neq 3$ and $v \neq 0$. Thus, $k_j = 0$ for $j = 1, 2, 3, 4$, which proves the statement. □

Theorem 8. *Class 3 DPH curves of degree 7, given by Theorem 4 and Table 3, have*

$$r_0(t) = B_0^2(t) + 3u(\theta)\left(\frac{1}{2}\zeta_1 B_1^2(t) + \zeta_2 B_2^2(t)\right),\quad r_3(t) = B_2^2(t) + 3u(\theta)\left(\frac{1}{2}\zeta_4 B_1^2(t) + \zeta_3 B_0^2(t)\right). \tag{51}$$

Additionally, Type 2 curves are helical and Type 1 are non-helical.

Proof. The expressions (51) follow directly from (46) and Table 3. For type 2, for which $u(0) = 1$, one can check that

$$z(t) = \frac{a_0(1-\rho(t)) + a_3\rho(t)}{b_0(1-\rho(t)) + b_3\rho(t)} \quad \text{where} \quad \rho(t) = \frac{\zeta_3 B_1^3(t) + \zeta_4 B_2^3(t) + B_3^3(t)}{B_0^3(t) + (\zeta_1 + \zeta_3)B_1^3(t) + (\zeta_2 + \zeta_4)B_2^3(t) + B_3^3(t)},$$

which proves that these type 2 DPH curves are again helical curves, obtained by *'cubic reparameterization'*. In Farouki et al. (2009b), Proposition 4 states that a class 3 DPH curve is non-helical if $\delta = 9q_2^2 + 3q_0q_4 - 12q_1q_3$ is negative. We compute δ for type 1 curves and get $\delta = \cos^2\theta + 3\zeta_3\zeta_2 - 3\zeta_1\zeta_4 = \cos^2\theta - 1 < 0$ since $\theta \notin \{0, \pi\}$. Thus, all type 1 curves are non-helical. □

5. Interpolation examples

In this section, we demonstrate the use of the derived DPH curves of degree 7 for interpolation purposes, through which we numerically validate the degrees of freedom reported in Tables 1–3. Specifically, we consider the following interpolation problem: given two boundary data points $\boldsymbol{P}_0, \boldsymbol{P}_1 \in \mathbb{R}^3$, each associated with an orthonormal frame $\mathcal{F}_0, \mathcal{F}_1 \in SO(3)$ and a curvature $\kappa_0, \kappa_1 \in \mathbb{R}$, along with a prescribed curve's length L, we aim to construct the DPH curve $\boldsymbol{p} : [0,1] \to \mathbb{R}^3$ of the form (3) with the hodograph (2) and the preimage (1), that satisfies

$$\boldsymbol{p}(\ell) = \boldsymbol{P}_\ell, \quad \mathcal{F}_{\mathrm{ERF}}(\boldsymbol{p};\ell) = \mathcal{F}_\ell, \quad \kappa(\ell) = \kappa_\ell, \quad \ell = 0,1, \quad \text{and} \quad \mathrm{length}(\boldsymbol{p};[0,1]) = \int_0^1 \sigma(t)dt = L.$$

Here $\mathcal{F}_{\mathrm{ERF}}(\boldsymbol{p};\cdot)$ denotes the Euler-Rodrigues frame, defined in Section 2.2, and κ is the curvature.

From $\mathcal{F}_0$ and $\mathcal{F}_1$ we compute $\mathcal{U}_0 = \pm\chi^{-1}(\mathcal{F}_0)$ and $\mathcal{U}_1 = \pm\chi^{-1}(\mathcal{F}_1)$, where from the four pairs of solutions we select such a pair that $\langle \mathcal{U}_0, \mathcal{U}_1 \rangle \geq 0$. Using the quaternion form (1) of the preimage, we see that the frame interpolation condition is satisfied iff

$$\mathcal{A}_0 = \lambda_0 \mathcal{U}_0, \quad \mathcal{A}_3 = \lambda_1 \mathcal{U}_1 \tag{52}$$

for some nonzero $\lambda_0, \lambda_1 \in \mathbb{R}$. From (3) one can see that the boundary points are interpolated iff $\boldsymbol{p}_0 = \boldsymbol{P}_0$ and

$$\Delta \boldsymbol{P}_0 = \frac{1}{7}\sum_{\ell=0}^{6} \boldsymbol{h}_\ell, \quad \text{where} \quad \Delta \boldsymbol{P}_0 = \boldsymbol{P}_1 - \boldsymbol{P}_0. \tag{53}$$

The length constraint and the curvature interpolation conditions give by (4) and (8) using (52) three scalar equations

$$L = \frac{1}{7}\sum_{\ell=0}^{6} \sigma_\ell, \quad \kappa_0 = \frac{1}{\lambda_0^4}|q(0)||\mathtt{w}(0)|, \quad \kappa_1 = \frac{1}{\lambda_1^4}|q(1)||\mathtt{w}(1)|. \tag{54}$$

Thus (53) and (54) together form a system of six nonlinear equations involving the unknowns λ_0, λ_1, and four additional ones listed as free parameters in Tables 1–3. These correspond to cases with 12 degrees of freedom, of which 8 are already determined by the boundary quaternions $\mathcal{A}_0$ and $\mathcal{A}_3$. For constructions with fewer than 12 degrees of freedom, some interpolation conditions must be omitted, as explained in the following discussion.

A more detailed analysis of the existence of the solution of (53)–(54) is beyond the scope of this paper and is left for future research. Our aim here is to numerically demonstrate, through specific examples, that the stated interpolation problem can be effectively addressed using the derived DPH curves. This approach confirms that the stated free parameters indeed represent the degrees of freedom. Since the system (53)–(54) is too complex to be solved directly using some algebraic solvers – such as the `NSolve` function in `Wolfram Mathematica` – we employ the iterative Newton-Raphson method to compute solutions from selected initial values. In particular, we use the `FindRoot` function in `Wolfram Mathematica`. Clearly, in this way we compute only the solutions to which the initial values converge.

Table 4
Initial values and computed solutions for interpolating DPH curves (with maximal degrees of freedom) from Example 5.1.

	initial values	**solution**
Class 1, Case 1		
Type 1	$\lambda_0 = 1,\ \lambda_1 = 2,$ $\mathtt{v}_0 = 0.5 + 0.5\mathtt{i},\ \mathtt{v}_2 = 2 - 0.5\mathtt{i},$	$\lambda_0 = 1.13102,\ \lambda_1 = 2.49189,$ $\mathtt{v}_0 = 0.407158 + 0.152967\mathtt{i},\ \mathtt{v}_2 = 1.57971 - 1.40076\mathtt{i}$
Type 2	$\lambda_0 = 1,\ \lambda_1 = 2,$ $\mathtt{v}_0 = 0.5 + 0.25\mathtt{i},\ \mathtt{v}_2 = 0.5 - 3\mathtt{i},$	$\lambda_0 = 1.15034,\ \lambda_1 = 2.81003,$ $\mathtt{v}_0 = 0.746844 - 0.0130410\mathtt{i},\ \mathtt{v}_2 = 0.944972 - 4.35589\mathtt{i}$
Class 2, Case 1		
Type 1	$\lambda_0 = 1,\ \lambda_1 = 2,$ $\mathtt{v} = 0.5 - 0.5\mathtt{i},\ \zeta_0 = 0.5,\ \zeta_2 = 1,$	$\lambda_0 = 0.987066,\ \lambda_1 = 2.61150,$ $\mathtt{v} = 0.883892 - 1.17340\mathtt{i},\ \zeta_0 = 0.260658,\ \zeta_2 = 5.91795$
Type 2	$\lambda_0 = 1,\ \lambda_1 = -2,$ $\mathtt{v} = -0.5 + 2\mathtt{i},\ \zeta_0 = 0.5,\ \zeta_2 = 2,$	$\lambda_0 = 2.44006,\ \lambda_1 = -0.321034,$ $\mathtt{v} = -0.518558 + 2.20387\mathtt{i},\ \zeta_0 = 31.6958,\ \zeta_2 = 0.0018535$
Class 3		
Type 1	$\lambda_0 = 1,\ \lambda_1 = 3,\ \theta = \frac{\pi}{2},$ $\zeta_1 = 1,\ \zeta_2 = 2,\ \zeta_3 = 1,$	$\lambda_0 = 1.21048,\ \lambda_1 = 2.89545,\ \theta = -1.73249,$ $\zeta_1 = 0.635320,\ \zeta_2 = 2.06792,\ \zeta_3 = -0.0631681$

Example 5.1. As a particular example, let us choose

$$\boldsymbol{P}_0 = (0,0,0), \quad \boldsymbol{P}_1 = (2,0,0), \quad \mathcal{F}_0 = \begin{pmatrix} \frac{23}{49} & \frac{24}{49} & \frac{36}{49} \\ \frac{24}{49} & -\frac{41}{49} & \frac{12}{49} \\ \frac{36}{49} & \frac{12}{49} & -\frac{31}{49} \end{pmatrix}, \quad \mathcal{F}_1 = \begin{pmatrix} \frac{3}{5} & \frac{16}{25} & -\frac{12}{25} \\ 0 & -\frac{3}{5} & -\frac{4}{5} \\ -\frac{4}{5} & \frac{12}{25} & -\frac{9}{25} \end{pmatrix}, \tag{55}$$

and $\kappa_0 = \frac{1}{2}$, $\kappa_1 = \frac{1}{2}$, $L = 2.5$. The inverse kinematic mapping gives $\mathcal{V}_0 = \frac{1}{7}(6\,\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k})$ and $\mathcal{V}_1 = \frac{1}{5}(2\,\mathbf{1} + 4\,\mathbf{i} + \mathbf{j} - 2\,\mathbf{k})$. Table 4 shows the initial values and the computed solutions for class 1 (Case 1), class 2 (Case 1) and class 3 (Type 1) DPH curves, illustrated in Fig. 8 together with boundary frame vectors and control polygons.

For DPH curves of class 1 (Case 2) and class 2 (Case 2 – (b)), the number of free parameters equals 10. Consequently, we omit the two curvature interpolation equations and obtain a nonlinear system of four equations for four unknowns. In these cases, the system can be solved numerically using the `NSolve` function (which computes all solutions). However, for class 1 (Case 2, Type 1), no real solution exists for $L = 2.5$. By choosing $L = 8$, we obtain two distinct solutions, shown in Fig. 10 (left). For the other two cases with 10 degrees of freedom and $L = 2.5$, the system admits four distinct solutions, illustrated in Fig. 9.

An interesting situation arises with helical class 3 DPH curves. Although these curves possess 12 degrees of freedom, the formulated interpolation problem cannot be solved. Specifically, solving the three nonlinear equations (53) algebraically for ζ_1, ζ_2, and ζ_3 yields four solutions expressed with ζ_4, λ_0 and λ_1. However, substituting them into the length constraint causes these remaining unknowns to cancel out, resulting in a fixed value of 5.1 for the curve's length. This indicates that the curve's length cannot be freely prescribed. To overcome this limitation, we replace the length constraint with the condition $\kappa(0.5) = 1$, which leads to a well-defined interpolation problem. Starting from initial values $\lambda_0 = -2$, $\lambda_1 = 4$, $\zeta_1 = 1$, $\zeta_2 = -2$, $\zeta_3 = -1$, $\zeta_4 = -2$, we obtain the solution with $\lambda_0 = -2.88491$, $\lambda_1 = 4.32000$, $\zeta_1 = 0.564971$, $\zeta_2 = -2.88178$, $\zeta_3 = -0.573129$, $\zeta_4 = -1.72181$, illustrated in Fig. 10 (right).

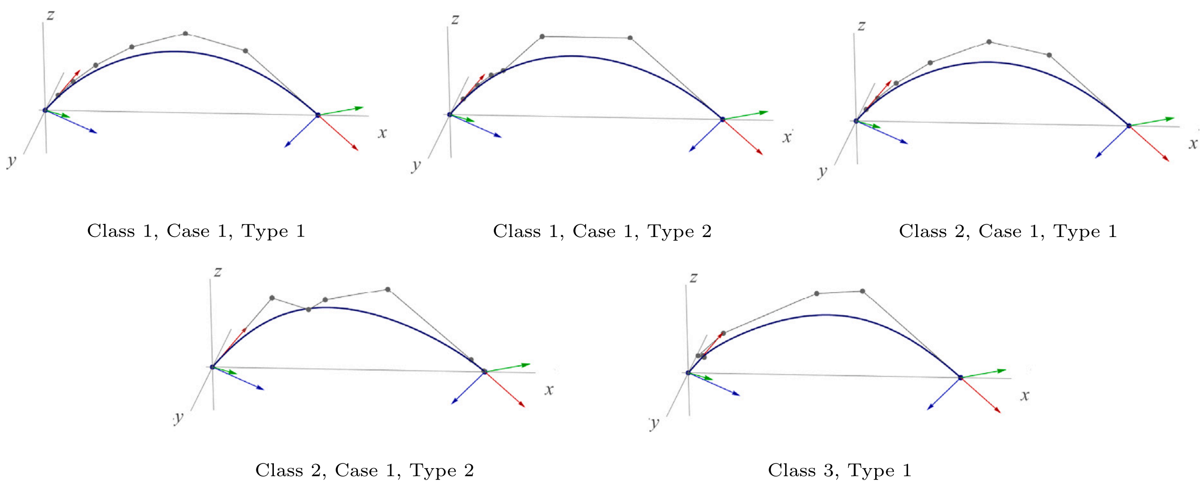

Fig. 8. Interpolating DPH curves from Example 5.1 corresponding to values given in Table 4.

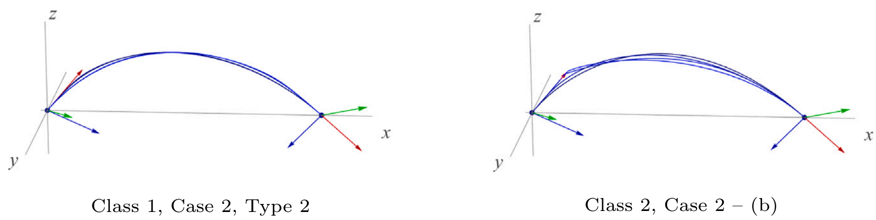

Fig. 9. Four distinct solutions of the interpolation problem (with omitted curvature interpolation conditions) from Example 5.1 computed using `NSolve` function.

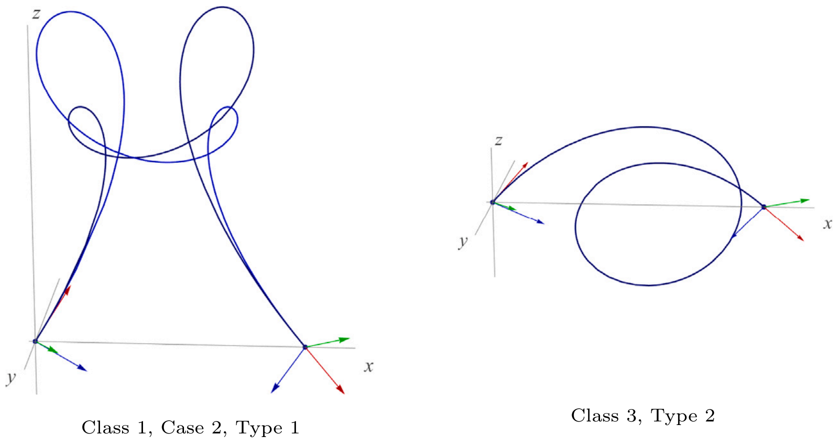

Fig. 10. Left: Two interpolating DPH curves of class 1, Case 2, Type 1 that solve the interpolation problem with omitted curvature interpolation conditions and the length increased to $L = 8$. Right: Interpolating DPH curve of class 3, Type 2 for which the length constraint is replaced with the condition $\kappa(0.5) = 1$. The length of the resulting interpolant equals $L = 5.1$.

6. Conclusions

In this paper, DPH curves of degree 7 are studied and analyzed. Three different classes of these curves are identified and their properties are revealed. In particular, the focus is on the examination of helical properties and on the number of degrees of freedom. Known results about the tangent indicatrix for helical curves are confirmed and extended to non-helical curves. Most importantly, a way to construct helical or non-helical DPH curves of degree seven in a manner that is suitable for computing the interpolating (spline) curves is provided. The degrees of freedom of the derived DPH curves are numerically validated by solving a specific interpolation problem. A theoretical investigation of this problem, with particular emphasis on the existence of interpolants based on the given data, is left for future research. Due to the highly nonlinear nature of the problem, we also plan to explore alternative approaches for deriving simpler interpolation methods, such as those based on biarc DPH curves of degree 7. Another challenge for future work is the classification of non-helical DPH curves of degree greater than 7.

CRediT authorship contribution statement

Marjeta Knez: Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Conceptualization. **Selena Praprotnik:** Writing – review & editing, Writing – original draft, Visualization, Validation, Software, Resources, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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