

## RESEARCH ARTICLE

## Jordan homomorphisms and T-ideals

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## Abstract

Let  $A$  and  $B$  be associative algebras over a field  $F$  with  $\text{char}(F) \neq 2$ . Our first main result states that if  $A$  is unital and equal to its commutator ideal, then every Jordan epimorphism  $\varphi : A \rightarrow B$  is the sum of a homomorphism and an antihomomorphism. Our second main result concerns (not necessarily surjective) Jordan homomorphisms from  $H(A, *)$  to  $B$ , where  $*$  is an involution on  $A$  and  $H(A, *) = \{a \in A \mid a^* = a\}$ . We show that there exists a T-ideal  $G$  having the following two properties: (1) the Jordan homomorphism  $\varphi : H(G(A), *) \rightarrow B$  can be extended to an (associative) homomorphism, subject to the condition that the subalgebra generated by  $\varphi(H(A, *))$  has trivial annihilator, and (2) every element of the T-ideal of identities of the algebra of  $2 \times 2$  matrices is nilpotent modulo  $G$ . A similar statement is true for Jordan homomorphisms from  $A$  to  $B$ . A counter-example shows that the assumption on trivial annihilator cannot be removed.

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## 1 | INTRODUCTION

Let  $A$  and  $B$  be associative algebras over a field  $F$  with  $\text{char}(F) \neq 2$ . Recall that a linear subspace  $J$  of  $A$  that is closed under the Jordan product  $x \circ y = xy + yx$  is called a special Jordan algebra. A

linear mapping  $\varphi : J \rightarrow B$  is called a **Jordan homomorphism** if

$$\varphi(x \circ y) = \varphi(x) \circ \varphi(y) \quad (1.1)$$

for all  $x, y \in J$ . We remark that while  $J$  can be any special Jordan algebra in this definition, the most studied and arguably most important cases are when  $J = A$  and  $J = H(A, *)$ , the latter being the set of symmetric elements in the algebra  $A$  with an involution  $*$ :  $A \rightarrow A$ . We will also confine ourselves to these two cases.

The restrictions of (associative) homomorphisms and antihomomorphisms  $A \rightarrow B$  to  $J$  are obviously Jordan homomorphisms. Finding conditions under which a Jordan homomorphism  $J \rightarrow B$  arises from homomorphisms and antihomomorphisms is a classical problem that originated in the 1940–1950s in the works of Ancochea [2, 3], Kaplansky [15], Hua [9], Jacobson and Rickart [12, 13], Kadison [14], and Herstein [8]. This problem is natural from the point of view of the Jordan theory, and is independently important since Jordan homomorphisms naturally appear in a variety of mathematical topics.

For more information about the history and background of the study of Jordan homomorphisms we refer the reader to [4, 5, 16, 18]. In the next few lines, we will briefly describe the results from two papers [4, 16] that are relevant for our work.

In [21], the second author constructed the so-called “tetrad-eating” T-ideal  $T$  of the free Jordan algebra. In [16], McCrimmon showed that if  $J$  is a unital special Jordan algebra such that  $J = T(J)$ , then all Jordan homomorphisms  $J \rightarrow B$  are of the usual form.

In [4], the first author proved that if a Jordan homomorphism  $\varphi : A \rightarrow B$  has the property that the ideal generated by all  $\varphi(xy) - \varphi(x)\varphi(y)$ ,  $x, y \in A$ , has trivial intersection with the ideal generated by all  $\varphi(xy) - \varphi(y)\varphi(x)$ ,  $x, y \in A$  (such a Jordan homomorphism is called **splittable**), then the restriction of  $\varphi$  to the **commutator ideal**  $K(A)$  (= the ideal generated by all commutators  $[x, y] = xy - yx$ ,  $x, y \in A$ ) has the usual form. The restriction to  $K(A)$  is justified by examples.

We will extend the results from [4, 16] in two directions. To describe our results, we first introduce some terminology.

We will refer to Jordan homomorphisms having the usual form as “standard”. To define this precisely, we separate the case where  $J = A$  and the case where  $J = H(A, *)$ .

- (a) A Jordan homomorphism  $\varphi : A \rightarrow B$  is **standard** if there exist a homomorphism  $\varphi_1 : A \rightarrow B$  and an antihomomorphism  $\varphi_2 : A \rightarrow B$  such that

$$\varphi = \varphi_1 + \varphi_2 \quad \text{and} \quad \varphi_1(A)\varphi_2(A) = \varphi_2(A)\varphi_1(A) = (0).$$

- (b) A Jordan homomorphism  $\varphi : H(A, *) \rightarrow B$  is **standard** if it can be extended to a homomorphism from  $A$  to  $B$ .

(Note that if  $\Phi : A \rightarrow B$  is an antihomomorphism, then  $x \mapsto \Phi(x^*)$  is a homomorphism that coincides with  $\Phi$  on  $H(A, *)$ . That is why there are no antihomomorphisms in (b).)

In Section 2, we show that if  $A$  is any unital algebra such that  $A = K(A)$ , then every Jordan homomorphism from  $A$  onto an arbitrary algebra  $B$  is standard (Theorem 2.10). Thus, unlike in [4], no condition on  $\varphi$  is required. Also, condition  $A = T(A)$  from [16] obviously implies  $A = K(A)$ , but not vice versa. Moreover, the final result of Section 2 (Corollary 2.14) gives some evidence that condition  $A = K(A)$  is natural and in some sense optimal. We also mention that our method of proof in Section 2 is different from those in [4, 16].

For a subset  $S$  of  $A$ , we write  $\langle S \rangle$  for the (associative) subalgebra of  $A$  generated by  $S$ . In Section 3, we prove the existence of a T-ideal  $G$  such that every element of the T-ideal of identities of the algebra of  $2 \times 2$  matrices is nilpotent modulo  $G$ , and every Jordan homomorphism  $A \rightarrow B$  (resp.  $H(A, *) \rightarrow B$ ) is standard on  $G(A)$  (resp.  $H(G(A), *)$ ), provided that the subalgebra  $\langle \varphi(A) \rangle$  (resp.  $\langle \varphi(H(A, *)) \rangle$ ) has trivial annihilator (Theorem 3.1). The latter restriction on  $\varphi$  is very mild and easy to handle compared to the restriction from [4] that  $\varphi$  is splittable, and, moreover, we show that it is necessary (Proposition 3.12). Of course, the main difference and advantage when compared with the result from [16] is that  $G(A)$  (resp.  $H(G(A), *)$ ) is not assumed to be equal to  $A$  (resp.  $H(A, *)$ ). As in [16], the main idea of the proof is to use the result from [21]. However, the realization of this idea is somewhat different.

## 2 | JORDAN HOMOMORPHISMS ON ALGEBRAS EQUAL TO THEIR COMMUTATOR IDEALS

In this section, we consider a Jordan homomorphism  $\varphi : A \rightarrow B$  where  $A$  and  $B$  are associative algebras over a field  $F$ .

Recall our convention that  $\text{char}(F) \neq 2$ . We assume this without further mention. In some of the results—specifically, in Lemmas 2.1, 2.3, 2.4, 2.6, and 2.7—this assumption actually is not needed. Here, it should be mentioned that a Jordan homomorphism between algebras over a field with characteristic 2 is defined as a linear map  $\varphi$  satisfying two conditions,

$$\varphi(x^2) = \varphi(x)^2 \quad \text{and} \quad \varphi(xyx) = \varphi(x)\varphi(y)\varphi(x). \tag{2.1}$$

In our setting where  $\text{char}(F) \neq 2$  the first condition is readily equivalent to (1.1) and implies the second condition since  $xyx = \frac{1}{2}(x \circ (y \circ x) - x^2 \circ y)$ .

We will now record a few elementary observations regarding any Jordan homomorphism  $\varphi : A \rightarrow B$ . First, from (2.1) we infer that

$$\varphi(xyz + zyx) = \varphi(x)\varphi(y)\varphi(z) + \varphi(z)\varphi(y)\varphi(x) \quad \text{and} \quad \varphi(y^i) = \varphi(y)^i$$

for all  $i \geq 1$ . Hence,

$$\varphi(y^i w y^j + y^j w y^i) = \varphi(y)^i \varphi(w) \varphi(y)^j + \varphi(y)^j \varphi(w) \varphi(y)^i$$

and therefore

$$\varphi\left(\sum_{i=0}^m y^i w y^{m-i}\right) = \sum_{i=0}^m \varphi(y)^i \varphi(w) \varphi(y)^{m-i} \tag{2.2}$$

for every  $m \in \mathbb{N}$ .

Another standard equation,

$$\varphi([x, [u, v]]) = [\varphi(x), [\varphi(u), \varphi(v)]], \tag{2.3}$$

is an immediate consequence of

$$[x, [u, v]] = (x \circ u) \circ v - (x \circ v) \circ u. \quad (2.4)$$

Take  $u, v \in A$  and write

$$a = \varphi(uv) - \varphi(u)\varphi(v) \quad \text{and} \quad b = \varphi(uv) - \varphi(v)\varphi(u). \quad (2.5)$$

The following two equations are less obvious but also well known:

$$ab = ba = 0 \quad (2.6)$$

and

$$a\varphi(x)b + b\varphi(x)a = 0 \quad (2.7)$$

for every  $x \in A$ . The first one was discovered in [12] and the second one in [20].

**Lemma 2.1.** *Let  $\varphi : A \rightarrow B$  be a Jordan homomorphism. Let  $u, v \in A$  and write  $y = [u, v]$ ,  $\tilde{y} = [\varphi(u), \varphi(v)]$ . Then*

$$\varphi([x, y^n]) = [\varphi(x), \varphi(y)^{n-1}\tilde{y}]$$

for all  $x \in A$  and all  $n \in \mathbb{N} \setminus \{2\}$ .

*Proof.* By (2.2), we have

$$\varphi([x, y^n]) = \varphi\left(\sum_{i=0}^{n-1} y^i [x, y] y^{n-i-1}\right) = \sum_{i=0}^{n-1} \varphi(y)^i \varphi([x, y]) \varphi(y)^{n-i-1}. \quad (2.8)$$

From (2.3) we see that  $\varphi([x, y]) = [\varphi(x), \tilde{y}]$ , that is, the assertion of the lemma is true for  $n = 1$ . Hence, (2.8) becomes

$$\varphi([x, y^n]) = \sum_{i=0}^{n-1} \varphi(y)^i [\varphi(x), \tilde{y}] \varphi(y)^{n-i-1}. \quad (2.9)$$

Using the notation introduced in (2.5), we have

$$\begin{aligned} b - a &= [\varphi(u), \varphi(v)] = \tilde{y}, \\ b + a &= 2\varphi(uv) - \varphi(u \circ v) = \varphi(y). \end{aligned}$$

In light of (2.6), we see that

$$\begin{aligned} \varphi(y)^k &= (b + a)^k = b^k + a^k, \\ \varphi(y)^k \tilde{y} &= \tilde{y} \varphi(y)^k = (b^k + a^k)(b - a) = (b - a)(b^k + a^k) = b^{k+1} - a^{k+1} \end{aligned}$$

for every  $k \geq 1$ .

Assume now that  $n \geq 3$ . We can rewrite (2.9) as

$$\begin{aligned} \varphi([x, y^n]) &= [\varphi(x), (b - a)](b^{n-1} + a^{n-1}) \\ &+ \sum_{i=1}^{n-2} (b^i + a^i)[\varphi(x), (b - a)] (b^{n-i-1} + a^{n-i-1}) \\ &+ (b^{n-1} + a^{n-1})[\varphi(x), (b - a)] \\ &= \varphi(x)(b^n - a^n) - (b - a)\varphi(x) (b^{n-1} + a^{n-1}) \\ &+ \sum_{i=1}^{n-2} (b^i + a^i)\varphi(x) (b^{n-i} - a^{n-i}) - \sum_{i=1}^{n-2} (b^{i+1} - a^{i+1})\varphi(x) (b^{n-i-1} + a^{n-i-1}) \\ &+ (b^{n-1} + a^{n-1}) \varphi(x)(b - a) - (b^n - a^n)\varphi(x). \end{aligned}$$

Observe that (2.6) and (2.7) imply

$$b^j \varphi(x) a^{n-j} = a^j \varphi(x) b^{n-j} = 0, \quad 1 \leq j \leq n - 1.$$

Using this together with  $b^n - a^n = \varphi(y)^{n-1} \bar{y}$ , it follows that

$$\begin{aligned} &\varphi([x, y^n]) - [\varphi(x), \varphi(y)^{n-1} \bar{y}] \\ &= a\varphi(x)a^{n-1} - \sum_{i=1}^{n-2} a^i \varphi(x) a^{n-i} + \sum_{i=1}^{n-2} a^{i+1} \varphi(x) a^{n-i-1} - a^{n-1} \varphi(x) a \\ &\quad - b\varphi(x)b^{n-1} + \sum_{i=1}^{n-2} b^i \varphi(x) b^{n-i} - \sum_{i=1}^{n-2} b^{i+1} \varphi(x) b^{n-i-1} + b^{n-1} \varphi(x) b \\ &= 0, \end{aligned}$$

as desired. □

We continue by introducing an auxiliary notion that will naturally appear in what follows. We will say that a Jordan homomorphism  $\varphi : A \rightarrow B$  is **nearly-standard** if there exists mappings  $\varphi_1, \varphi_2 : A \rightarrow B$  such that

$$\varphi = \varphi_1 + \varphi_2 \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi_1(y) + \varphi_2(y)\varphi(x) \tag{2.10}$$

for all  $x, y \in A$  (we are not assuming that  $\varphi_1$  is a homomorphism and  $\varphi_2$  an antihomomorphism).

Observe that a standard Jordan homomorphism is nearly-standard. If  $A$  is commutative, then every Jordan homomorphism on  $A$  is automatically nearly-standard (with  $\varphi_1 = \varphi_2 = \frac{\varphi}{2}$ ), but not always standard, as we will see in Example 2.12. An example of a noncommutative algebra  $A$  and a non-standard but nearly-standard Jordan homomorphism  $\varphi : A \rightarrow B$  will be given in Example 2.13.

**Lemma 2.2.** *If an algebra  $A$  is equal to the linear span of its commutators and their  $n$ th powers for  $n \geq 3$ , then every Jordan homomorphism  $\varphi : A \rightarrow B$ , where  $B$  is any  $F$ -algebra, is nearly-standard.*

*Proof.* From Lemma 2.1, we see that our assumption on  $A$  implies that there is a function  $\psi : A \rightarrow B$  such that

$$\varphi([x, y]) = [\varphi(x), \psi(y)]$$

for all  $x, y \in A$ . Since  $xy = \frac{1}{2}x \circ y + \frac{1}{2}[x, y]$ ,

$$\varphi_1 = \frac{1}{2}(\varphi + \psi) \quad \text{and} \quad \varphi_2 = \frac{1}{2}(\varphi - \psi)$$

satisfy (2.10). □

Our next goal is to replace the assumption on  $A$  in Lemma 2.2 by a more transparent assumption that  $A$  is unital and equal to its commutator ideal  $K(A)$  (Proposition 2.5). To this end, we need two lemmas.

**Lemma 2.3.** *Let  $n \in \mathbb{N}$  be such that  $\text{char}(F)$  does not divide  $n$ . If a linear subspace  $V$  of an algebra  $A$  contains all commutators and their  $n$ th powers, then  $V$  contains the ideal of  $A$  generated by elements*

$$\left[ u, \sum_{\sigma \in S_{n-1}} [u, v_{\sigma(1)}] \cdots [u, v_{\sigma(n-1)}] \right],$$

$u, v_1, \dots, v_{n-1} \in A$ .

*Proof.* By assumption,  $[u, v]^n \in V$  for all  $u, v \in A$ . Linearizing in  $v$ , we obtain

$$\sum_{\sigma \in S_n} [u, v_{\sigma(1)}] \cdots [u, v_{\sigma(n)}] \in V \tag{2.11}$$

for all  $u, v_1, \dots, v_n \in A$ . Write  $w = v_n$  and

$$\bar{v}_{n-1} = \sum_{\sigma \in S_{n-1}} [u, v_{\sigma(1)}] \cdots [u, v_{\sigma(n-1)}].$$

Since  $V$  contains the commutators, (2.11) implies  $n[u, w]\bar{v}_{n-1} \in V$ , and therefore

$$[u, w]\bar{v}_{n-1} \in V$$

by the characteristic assumption. Consequently, for any  $u, w, v_i \in A$ ,

$$w[u, \bar{v}_{n-1}] = [u, w\bar{v}_{n-1}] - [u, w]\bar{v}_{n-1} \in V,$$

and hence also

$$w[u, \bar{v}_{n-1}]z = [w[u, \bar{v}_{n-1}], z] + (zw)[u, \bar{v}_{n-1}] \in V$$

for all  $u, w, z, v_i \in A$ . This proves the lemma. □

Let  $X = \{x_1, x_2, \dots\}$  and let  $F\langle X \rangle$  be the free algebra on the set  $X$ . For an element  $f \in F\langle X \rangle$ , let  $(f)^T$  denote the T-ideal generated by  $f$ , that is,  $(f)^T = F\langle X \rangle f (F\langle X \rangle) F\langle X \rangle$ .

We call a T-ideal  $T$  of  $F\langle X \rangle$  a **nonmatrix commutator T-ideal** if  $T \subseteq ([x_1, x_2])^T$  and  $T(M_2(F)) \neq (0)$ . We also assume that for any of its elements,  $T$  also contains all its partial linearizations. This automatically holds if the field  $F$  is infinite.

**Lemma 2.4.** *Let  $T_1, T_2$  be two nonmatrix commutator T-ideals. If  $A$  is any unital algebra, then  $A = T_1(A)$  if and only if  $A = T_2(A)$ .*

*Proof.* Without loss of generality, we may assume that the field  $F$  is algebraically closed and  $\text{card}(F) > \dim_F(A)$ .

Let  $A = T_1(A)$ . Consider the factor algebra  $\bar{A} = A/T_2(A)$ . Our goal is to show that  $\bar{A} = (0)$ . We remark that  $\bar{A} = T_1(\bar{A})$ .

Let  $\bar{A}'$  be a primitive homomorphic image of  $\bar{A}$ . By Kaplansky's theorem (see, e.g., [19, Corollary 23.31]),  $\bar{A}' \cong M_k(D)$  where  $D$  is a finite-dimensional division algebra over  $Z(D)$ . Since  $\text{card}(F) > \dim_F(D)$  and the field  $F$  is algebraically closed,  $D = F$  by the result by Amitsur (see, e.g., [19, Corollary 15A.8]). The algebra  $M_k(F)$  satisfies all identities from  $T_2$ . Hence,  $k = 1$  and  $\bar{A}' \cong F$ .

Let  $J(\bar{A})$  be the Jacobson radical of the algebra  $\bar{A}$ . We showed that the factor algebra  $\bar{A}/J(\bar{A})$  is commutative. Now,  $\bar{A} = T_1(\bar{A})$  and  $T_1 \subseteq ([x_1, x_2])^T$  imply  $\bar{A} = J(\bar{A})$ . Since  $1 \in \bar{A}$  we conclude that  $\bar{A} = (0)$ , as desired. □

**Proposition 2.5.** *If  $A$  is a unital algebra such that  $A = K(A)$ , then every Jordan homomorphism  $\varphi : A \rightarrow B$ , where  $B$  is any  $F$ -algebra, is nearly-standard.*

*Proof.* Note that

$$T_1 = \left( \left[ x_4, \sum_{\sigma \in S_3} [x_4, x_{\sigma(1)}] \cdots [x_4, x_{\sigma(3)}] \right] \right)^T$$

is a nonmatrix commutator T-ideal. Obviously, so is  $T_2 = ([x_1, x_2])^T$ . Lemma 2.4 thus shows that  $A = K(A) = T_2(A)$  implies  $A = T_1(A)$ . The  $n = 4$  case of Lemma 2.3 further shows that in this case  $A$  is equal to the linear span of commutators and their 4th powers. We can now apply Lemma 2.2. □

Our next goal is to provide conditions under which nearly-standard Jordan homomorphisms are standard. We need the following definition.

Let  $B$  be an algebra. A mapping  $\Delta : B \times B \rightarrow B$  is called a **biderivation** if it is a derivation in each argument. A biderivation  $\Delta$  is said to be **inner** if there exists an element  $\lambda \in Z(B)$  such that

$$\Delta(r, s) = \lambda[r, s]$$

for all  $r, s \in B$ .

**Lemma 2.6.** *Let  $B$  be a unital algebra such that every biderivation of  $B$  is inner and for every  $\alpha \in Z(B)$ ,  $\alpha[[B, B], B] = (0)$  implies  $\alpha = 0$ . Then, every nearly-standard Jordan isomorphism  $\varphi : A \rightarrow B$ , where  $A$  is any algebra, is standard. Moreover, there exists an idempotent  $\varepsilon \in Z(B)$  such that  $x \mapsto \varepsilon\varphi(x)$  is a homomorphism and  $x \mapsto (1 - \varepsilon)\varphi(x)$  is an antihomomorphism.*

*Proof.* Let  $\varphi_1$  and  $\varphi_2$  be mappings satisfying

$$\varphi = \varphi_1 + \varphi_2 \quad \text{and} \quad \varphi(xy) = \varphi(x)\varphi_1(y) + \varphi_2(y)\varphi(x).$$

These equations imply

$$\varphi(xy) - \varphi(x)\varphi(y) = [\varphi_2(y), \varphi(x)].$$

Replacing the roles of  $x$  and  $y$ , we obtain

$$\varphi(yx) - \varphi(y)\varphi(x) = [\varphi_2(x), \varphi(y)].$$

The sum of the left-hand sides is 0 since  $\varphi$  is a Jordan homomorphism. Therefore,

$$[\varphi(x), \varphi_2(y)] = [\varphi_2(x), \varphi(y)].$$

Here,  $x$  and  $y$  are arbitrary elements in  $A$ . Writing  $r = \varphi(x)$  and  $s = \varphi(y)$ , we obtain

$$[r, \varphi_2(\varphi^{-1}(s))] = [\varphi_2(\varphi^{-1}(r)), s]$$

for all  $r, s \in B$ . The mapping  $(r, s) \mapsto [r, \varphi_2(\varphi^{-1}(s))]$  is therefore a biderivation of  $B$ . By assumption, there exists a  $\lambda \in Z(B)$  such that

$$[r, \varphi_2(\varphi^{-1}(s))] = \lambda[r, s]$$

for all  $r, s \in B$ . Hence,

$$\varphi_2(\varphi^{-1}(s)) - \lambda s \in Z(B)$$

for every  $s \in B$ . Writing  $s = \varphi(y)$  and

$$\mu(y) = \varphi_2(y) - \lambda\varphi(y),$$

we see that  $\mu(y) \in Z(B)$  for every  $y \in A$ . Since  $\varphi = \varphi_1 + \varphi_2$ , we have

$$\varphi_1(y) = (1 - \lambda)\varphi(y) - \mu(y).$$

Set  $\varepsilon = 1 - \lambda$  and note that  $\varphi(xy) = \varphi(x)\varphi_1(y) + \varphi_2(y)\varphi(x)$  can now be written as

$$\varphi(xy) = \varepsilon\varphi(x)\varphi(y) + (1 - \varepsilon)\varphi(y)\varphi(x) \tag{2.12}$$

for all  $x, y \in A$ . Consequently,

$$\begin{aligned} \varphi(xyz) &= \varphi(x(yz)) \\ &= \varepsilon\varphi(x)\varphi(yz) + (1 - \varepsilon)\varphi(yz)\varphi(x) \\ &= \varepsilon^2\varphi(x)\varphi(y)\varphi(z) + \varepsilon(1 - \varepsilon)\varphi(x)\varphi(z)\varphi(y) \\ &\quad + \varepsilon(1 - \varepsilon)\varphi(y)\varphi(z)\varphi(x) + (1 - \varepsilon)^2\varphi(z)\varphi(y)\varphi(x). \end{aligned}$$

On the other hand,

$$\begin{aligned} \varphi(xyz) &= \varphi((xy)z) \\ &= \varepsilon\varphi(xy)\varphi(z) + (1 - \varepsilon)\varphi(z)\varphi(xy) \\ &= \varepsilon^2\varphi(x)\varphi(y)\varphi(z) + \varepsilon(1 - \varepsilon)\varphi(y)\varphi(x)\varphi(z) \\ &\quad + \varepsilon(1 - \varepsilon)\varphi(z)\varphi(x)\varphi(y) + (1 - \varepsilon)^2\varphi(z)\varphi(y)\varphi(x). \end{aligned}$$

Comparing both expressions, we obtain

$$\varepsilon(1 - \varepsilon)[[\varphi(x), \varphi(z)], \varphi(y)] = 0$$

for all  $x, y, z \in A$ . By our assumption, this implies  $\varepsilon(1 - \varepsilon) = 0$ . Thus,  $\varepsilon$  is a central idempotent. Now, multiplying (2.12) by  $\varepsilon$  we see that  $x \mapsto \varepsilon\varphi(x)$  is a homomorphism, and multiplying (2.12) by  $1 - \varepsilon$  we see that  $x \mapsto (1 - \varepsilon)\varphi(x)$  is an antihomomorphism.  $\square$

We need three more lemmas to obtain the main result of this section. The first one is known from the theory of functional identities, see [5, Example 1.5].

**Lemma 2.7.** *If  $B$  is a unital algebra such that  $B = K(B)$ , then every biderivation of  $B$  is inner.*

The second lemma is a byproduct of Lemma 2.4.

**Lemma 2.8.** *If  $A$  is a unital algebra such that  $A = K(A)$ , then  $A = [[A, A], A] \circ A$ .*

*Proof.* Let  $T_1 = ([[x_1, x_2], x_3])^T$ . Clearly,  $T_1$  is a nonmatrix commutator T-ideal. Hence by Lemma 2.4,  $T_1(A) = A$ , that is,  $A[[A, A], A]A = A$ . Since  $[[A, A], A]$  is a Lie ideal (i.e.,  $[[[A, A], A], A] \subseteq [[A, A], A]$ ), we can write this as  $A = [[A, A], A]A$ . Using  $\text{char}(F) \neq 2$ , it follows that

$$A = [[[A, A], A], A] + [[A, A], A] \circ A.$$

The first term lies in  $[[A, A], A]$ , and hence  $A = [[A, A], A] \circ A$ .  $\square$

The third lemma is about Jordan ideals of associative algebras; recall that these are linear subspaces  $J$  of  $A$  satisfying  $J \circ A \subseteq J$ .

**Lemma 2.9.** *If  $A$  is a unital algebra such that  $A = K(A)$ , then every Jordan ideal of  $A$  is an ideal.*

*Proof.* By [6, Corollary 2.2], it is enough to show that  $A$  coincides with the subalgebra  $\langle [A, A] \rangle$  generated by  $[A, A]$ . As noticed in the proof of the preceding lemma, to this end it suffices to show that  $\langle [A, A] \rangle$  contains the ideal  $A[[A, A], A]A = [[A, A], A]A$ —and this is true as we see from  $[[x, y], z]w = [[x, y]w, z] - [x, y][w, z]$ .  $\square$

**Theorem 2.10.** *If  $A$  is a unital algebra such that  $A = K(A)$ , then every Jordan epimorphism  $\varphi : A \rightarrow B$ , where  $B$  is any  $F$ -algebra, is standard. Moreover, there exists an idempotent  $\varepsilon \in Z(B)$  such that  $x \mapsto \varepsilon\varphi(x)$  is a homomorphism and  $x \mapsto (1 - \varepsilon)\varphi(x)$  is an antihomomorphism.*

*Proof.* Lemma 2.9 implies that  $\ker \varphi$  is an ideal of  $A$ . Therefore,  $\varphi$  induces a Jordan isomorphism from the algebra  $A/\ker \varphi$  onto the algebra  $B$ . Since  $A/\ker \varphi$  is also unital and coincides with its commutator ideal, we may assume without loss of generality that  $\varphi$  itself is a Jordan isomorphism.

Proposition 2.5 tells us that  $\varphi$  is nearly-standard. It is therefore enough to show that  $B$  satisfies the assumptions of Lemma 2.6.

First, note that  $B$  is also unital [12, Corollary 3]. Since  $A = [[A, A], A] \circ A$  by Lemma 2.8 and since (2.4) shows that  $\varphi$  satisfies

$$\varphi([[a_1, a_2], a_3] \circ a_4) = [[\varphi(a_1), \varphi(a_2)], \varphi(a_3)] \circ \varphi(a_4),$$

we have  $B = [[B, B], B] \circ B$ . In particular,  $B = K(B)$ , so Lemma 2.7 tells us that every biderivation of  $B$  is inner. Finally,  $B = [[B, B], B] \circ B$  also shows that the second assumption of Lemma 2.6 holds. This completes the proof of the theorem.  $\square$

We leave as an open question whether the assumption that  $\varphi$  is surjective is necessary. As the algebra  $B$  is arbitrary, we can rephrase this assumption as that the image of  $\varphi$  is an associative algebra.

Our aim in the rest of the section is to show that the basic assumption of Proposition 2.5 and Theorem 2.10 that  $A = K(A)$  is, from a certain viewpoint, close to necessary. We start with a proposition providing conditions for the existence of non-standard Jordan homomorphisms. It was inspired by [12, Example 1].

**Proposition 2.11.** *Let  $A$  be a unital algebra, let  $d$  and  $g$  be derivations of  $A$ , let  $J$  be an ideal of  $A$  and  $\pi : A \rightarrow A/J$  be the natural projection. Then,  $B = A \times A/J$  becomes an associative algebra under componentwise linear structure and multiplication given by*

$$(x, u) \star (y, v) = (xy, \pi(d(x)g(y)) + \pi(x)v + u\pi(y)).$$

If

$$[g(x), d(x)] \in J \quad \text{for all } x \in A,$$

then  $\varphi : A \rightarrow B$ ,

$$\varphi(x) = \left(x, \frac{1}{2}\pi((dg)(x))\right)$$

is a Jordan homomorphism. Moreover, if there exist  $s, t \in A$  such that

$$g(s)d(t) - d(s)g(t) \notin J,$$

then  $\varphi$  is not standard.

*Proof.* Note first that  $\star$  is an associative operation, so  $B$  is indeed an algebra. It is straightforward to check that  $\varphi$  is a Jordan homomorphism if  $[g(x), d(x)] \in J$  for every  $x \in A$ .

Assume that  $\varphi$  is standard, that is,  $\varphi = \varphi_1 + \varphi_2$  where  $\varphi_1 : A \rightarrow B$  is a homomorphism,  $\varphi_2 : A \rightarrow B$  is an antihomomorphism, and  $\varphi_1(A) \star \varphi_2(A) = \varphi_2(A) \star \varphi_1(A) = (0)$ . Write

$$\varphi_1(x) = (\psi_1(x), \theta_1(x)) \quad \text{and} \quad \varphi_2(x) = (\psi_2(x), \theta_2(x)),$$

where  $\psi_i : A \rightarrow A, \theta_i : A \rightarrow A/J, i = 1, 2$ . It is immediate that  $\psi_1$  is a homomorphism of  $A, \psi_2$  is an antihomomorphism of  $A, \psi_1(A)\psi_2(A) = \psi_2(A)\psi_1(A) = (0)$ , and

$$x = \psi_1(x) + \psi_2(x), \quad \frac{1}{2}\pi((dg)(x)) = \theta_1(x) + \theta_2(x)$$

for every  $x \in A$ . In particular,  $1 = e + f$  where  $e = \psi_1(1)$  and  $f = \psi_2(1)$ . Obviously,  $e$  and  $f$  are orthogonal idempotents and

$$\psi_1(x) = \psi_1(x)e = e\psi_1(x), \quad \psi_2(x) = \psi_2(x)f = f\psi_2(x).$$

From  $(x - \psi_1(x))e = \psi_2(x)\psi_1(1) = 0$ , we see that  $\psi_1(x) = xe$ . Similarly,  $\psi_1(x) = ex$ , so  $e$  is a central idempotent. Accordingly,  $\psi_2(x) = fx$  and, since  $\psi_2$  is an antihomomorphism,  $fA$  is a commutative ideal of  $A$ .

As derivations send central idempotents to 0, it follows from  $\varphi_1(1) \star \varphi_2(x) = 0$  that

$$\pi(e)\theta_2(x) + \theta_1(1)\pi(f)\pi(x) = 0.$$

Multiplying by  $\pi(e)$  we obtain  $\pi(e)\theta_2(x) = 0$ . Therefore,  $\theta_2(x) = \pi(f)\theta_2(x)$ . Similarly,  $\theta_1(x) = \pi(e)\theta_1(x)$ . Consequently,

$$\pi(e)\theta_1(x) + \pi(f)\theta_2(x) = \frac{1}{2}\pi((dg)(x)). \tag{2.13}$$

Now take any  $s, t \in A$ . Our goal is to show that  $g(s)d(t) - d(s)g(t) \in J$ . To this end, consider  $\varphi(st)$ . On one hand,

$$\begin{aligned} \varphi(st) &= (st, \frac{1}{2}\pi((dg)(st))) \\ &= (st, \frac{1}{2}\pi((dg)(s))\pi(t) + \frac{1}{2}\pi(d(s)g(t)) + \frac{1}{2}\pi(g(s)d(t)) + \frac{1}{2}\pi(s)\pi((dg)(t))). \end{aligned}$$

On the other hand,

$$\varphi(st) = \varphi_1(st) + \varphi_2(st) = \varphi_1(s) \star \varphi_1(t) + \varphi_2(t) \star \varphi_2(s)$$

$$=(es, \theta_1(s)) \star (et, \theta_1(t)) + (ft, \theta_2(t)) \star (fs, \theta_2(s)).$$

Using the commutativity of  $\pi(f)A/J$  and the fact that  $d$  and  $g$  vanish on central idempotents  $e$  and  $f$ , it follows that

$$\begin{aligned} \varphi(st) &= (est, \pi(ed(s)g(t)) + \pi(e)\pi(s)\theta_1(t) + \pi(e)\theta_1(s)\pi(t)) \\ &\quad + (fst, \pi(fg(s)d(t)) + \pi(f)\theta_2(s)\pi(t) + \pi(f)\pi(s)\theta_2(t)) \\ &= (st, \pi(ed(s)g(t)) + \pi(fg(s)d(t)) \\ &\quad + \pi(s)(\pi(e)\theta_1(t) + \pi(f)\theta_2(t)) + (\pi(e)\theta_1(s) + \pi(f)\theta_2(s))\pi(t)). \end{aligned}$$

Comparing both expressions and using (2.13), we obtain

$$\frac{1}{2}d(s)g(t) + \frac{1}{2}g(s)d(t) - ed(s)g(t) - fg(s)d(t) \in J.$$

Multiplying by  $e$ , it follows that

$$e(g(s)d(t) - d(s)g(t)) \in J,$$

and multiplying by  $f$ , it follows that

$$f(g(s)d(t) - d(s)g(t)) \in J.$$

As  $e + f = 1$ , the desired conclusion follows.  $\square$

**Example 2.12.** Proposition 2.11 shows that all we need for the existence of a non-standard Jordan homomorphism from a commutative algebra  $A$  to some algebra  $B$  is a pair of derivations  $d, g$  of  $A$  that satisfy  $g(s)d(t) \neq d(s)g(t)$  for some  $s, t \in A$ . Concrete examples can be easily found. As already mentioned above, such Jordan homomorphisms are automatically nearly-standard.

Let us also give an example of a non-standard but nearly-standard Jordan homomorphism on a noncommutative algebra.

**Example 2.13.** Let  $A$  be the Grassmann algebra on  $n$  generators  $e_1, \dots, e_n$ , where  $n \geq 4$ . For any  $a, b, x \in A$ , we have

$$[a, x][b, x] = [b, [ax, x]] - [b, [a, x]]x.$$

Since  $[[A, A], A] = (0)$ , this gives  $[a, x][b, x] = 0$ . Thus, every pair of inner derivations  $d$  and  $g$  satisfies

$$d(x)g(x) = 0 = g(x)d(x)$$

for every  $x \in A$ . Taking  $d(x) = [e_1, x]$  and  $g(x) = [e_2, x]$ , we have

$$g(e_3)d(e_4) - d(e_3)g(e_4) = 8e_1e_2e_3e_4 \neq 0.$$

Therefore,  $d$  and  $g$  satisfy conditions of Proposition 2.11 (for  $J = (0)$ ). Since  $dg = 0$ ,  $\varphi(x) = (x, 0)$  is a Jordan homomorphism that is not standard. However,  $\varphi$  is nearly-standard. Indeed, the maps

$$\varphi_1(x) = (x, e_1[e_2, x]) \quad \text{and} \quad \varphi_2(x) = (0, -e_1[e_2, x])$$

satisfy  $\varphi = \varphi_1 + \varphi_2$  and  $\varphi(xy) = \varphi(x) \star \varphi_1(y) + \varphi_2(y) \star \varphi(x)$  for all  $x, y \in A$ .

**Corollary 2.14.** *Let  $A$  be a unital algebra.*

- (a) *If  $A = K(A)$ , then, for any commutative unital  $F$ -algebra  $C$  and any  $F$ -algebra  $B$ , every Jordan epimorphism  $\varphi : A \otimes C \rightarrow B$  is standard.*
- (b) *If  $A = K(A)$ , then, for any commutative unital  $F$ -algebra  $C$  and any  $F$ -algebra  $B$ , every Jordan homomorphism  $\varphi : A \otimes C \rightarrow B$  is nearly-standard.*
- (c) *If  $A \neq K(A)$ , then there exists a commutative unital algebra  $C$ , an  $F$ -algebra  $B$ , and a Jordan homomorphism  $\varphi : A \otimes C \rightarrow B$  that is not standard.*
- (d) *If  $A \neq K(A) + Z(A)$ , then there exists a commutative unital algebra  $C$ , an  $F$ -algebra  $B$ , and a Jordan homomorphism  $\varphi : A \otimes C \rightarrow B$  that is not nearly-standard.*

*Proof.* Observe that  $K(A \otimes C) = K(A) \otimes C$  for any commutative unital algebra  $C$ . Therefore,  $K(A) = A$  implies  $K(A \otimes C) = A \otimes C$ . Thus, (a) follows from Theorem 2.10 and (b) follows from Proposition 2.5.

To prove (c) and (d), take  $C$  to be the polynomial algebra  $F[x_1, x_2]$ . Note that  $d = 1_A \otimes \frac{\partial}{\partial x_1}$  and  $g = 1_A \otimes \frac{\partial}{\partial x_2}$  are derivations of  $A \otimes C$ . Let  $J = K(A \otimes C)$ , let  $\pi : A \otimes C \rightarrow (A \otimes C)/J$  be the natural projection, and let

$$B = (A \otimes C) \times (A \otimes C)/J$$

be the algebra from Proposition 2.11. The condition  $[d(x), g(x)] \in J$ ,  $x \in A \otimes C$ , is fulfilled since  $J$  is the commutator ideal. Therefore,

$$\varphi(x) = (x, \frac{1}{2}\pi((dg)(x)))$$

is a Jordan homomorphism from  $A \otimes C$  to  $B$ .

Assume that  $A \neq K(A)$ . Then,  $J = K(A) \otimes C \neq A \otimes C$ . Write  $s = 1 \otimes x_2$  and  $t = 1 \otimes x_1$ . Then,  $g(s) = d(t) = 1 \otimes 1$  and  $d(s) = g(t) = 0$ . Accordingly,  $g(s)d(t) - d(s)g(t) = 1 \otimes 1 \notin J$  and hence Proposition 2.11 tells us that  $\varphi$  is not standard. This proves (c).

Finally, assume that  $A \neq K(A) + Z(A)$ . Suppose that  $\varphi$  is nearly-standard. As noticed at the beginning of the proof of Lemma 2.6, this implies that there exists a mapping  $\varphi_2 : A \otimes C \rightarrow B$  such that

$$\varphi(xy) - \varphi(x) \star \varphi(y) = \varphi_2(y) \star \varphi(x) - \varphi(x) \star \varphi_2(y) \tag{2.14}$$

for all  $x, y \in A \otimes C$ . A short calculation shows that the left-hand side of (2.14) is equal to

$$\frac{1}{2}(0, \pi(g(x)d(y) - d(x)g(y))).$$

Write  $\varphi_2(y) = (\alpha(y), \beta(y))$ , where  $\alpha : A \otimes C \rightarrow A \otimes C$  and  $\beta : A \otimes C \rightarrow (A \otimes C)/J$ . Using the commutativity of  $(A \otimes C)/J$ , one directly computes that the right-hand side of (2.14) is equal to

$$([\alpha(y), x], \pi(g(x)d(\alpha(y)) - d(x)g(\alpha(y))))).$$

Comparing the above two expressions, we obtain

$$[\alpha(x), y] = 0 \quad \text{and} \quad \pi(g(x)d(y - 2\alpha(y)) - d(x)g(y - 2\alpha(y))) = 0$$

for all  $x, y \in A \otimes C$ . The first equation shows that  $\alpha(x) \in Z(A \otimes C)$  for every  $x \in A \otimes C$ . Write  $1 \otimes x_2$  for  $x$  in the second equation. Then,  $g(x) = 1 \otimes 1$  and  $d(x) = 0$ , so we obtain  $d(y - 2\alpha(y)) \in J$  for every  $y \in A \otimes C$ . Since  $\alpha(y)$  lies in  $Z(A \otimes C)$ , so does  $d(\alpha(y))$ . Therefore,  $d(y) \in Z(A \otimes C) + J$ . In particular,

$$a \otimes 1 = d(a \otimes x_1) \in Z(A \otimes C) + J = Z(A) \otimes C + K(A) \otimes C$$

for every  $a \in A$ . However, this cannot hold for  $a$  not lying in  $K(A) + Z(A)$ . With this contradiction, we have completed the proof of (d).  $\square$

Recall that every Jordan homomorphism on a commutative algebra is nearly-standard. The assumption that  $A \neq K(A) + Z(A)$  in (d) therefore cannot be replaced by  $A \neq K(A)$ .

### 3 | JORDAN HOMOMORPHISMS ON A CERTAIN T-IDEAL

As above, we assume throughout this section that  $A$  and  $B$  are associative  $F$ -algebras, where  $F$  is a field with  $\text{char}(F) \neq 2$ .

Let  $A^{\text{op}}$  denote the opposite algebra of  $A$ . That is,  $A^{\text{op}}$  has the same linear structure, but  $a^{\text{op}}b^{\text{op}} = (ba)^{\text{op}}$  for all  $a, b \in A$ . Consider the natural Jordan homomorphism  $s : A \rightarrow A \oplus A^{\text{op}}$ ,  $s(a) = a + a^{\text{op}}$ . The image of  $s$  is the Jordan algebra of symmetric elements with respect to the exchange involution  $\text{ex} : a + b^{\text{op}} \mapsto b + a^{\text{op}}$ .

It is easy to see that a Jordan homomorphism  $\varphi : A \rightarrow B$  is standard if there exists a homomorphism  $\chi : A \oplus A^{\text{op}} \rightarrow B$  such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{s} & A \oplus A^{\text{op}} \\ & \searrow \varphi & \downarrow \chi \\ & & B \end{array}$$

is commutative.

Let  $J$  be either  $A$  or  $H(A, *)$  and  $\varphi : J \rightarrow B$  be a Jordan homomorphism. Write  $B'$  for  $\langle \varphi(J) \rangle$ , that is,  $B'$  is the subalgebra of  $B$  generated by the image  $\varphi(J)$  in  $B$ . Let

$$\text{Ann}(B') = \{b \in B' \mid bB' = B'b = (0)\}.$$

Clearly,  $\varphi : J \rightarrow B$  gives rise to the Jordan homomorphism  $\bar{\varphi} : J \rightarrow B'/\text{Ann}(B')$ , the composition of  $\varphi$  and the natural projection  $B' \rightarrow B'/\text{Ann}(B')$ . We call  $\varphi$  **annihilator-by-standard** if  $\bar{\varphi}$  is standard.

Every Jordan homomorphism  $\varphi : A \rightarrow B$  gives rise to the Jordan homomorphism  $H(A \oplus A^{\text{op}}, \text{ex}) \rightarrow B$ ,  $a + a^{\text{op}} \mapsto \varphi(a)$ . The Jordan homomorphism  $\varphi$  is standard (resp. annihilator-by-standard) if and only if the Jordan homomorphism  $H(A \oplus A^{\text{op}}, \text{ex}) \rightarrow B$  is standard (resp. annihilator-by-standard).

Let  $X = \{x_1, x_2, \dots\}$ . Let  $F\langle X \rangle$  be the free associative algebra on the set of generators  $X$ . When considering T-ideals of  $F\langle X \rangle$ , we always assume that they are closed with respect to partial linearizations.

Let  $G \triangleleft F\langle X \rangle$  be a T-ideal. If  $J = A$  then  $J_G = G(A)$ , and if  $J = H(A, *)$  then  $J_G = H(G(A), *)$ . It is clear that the ideal  $G(A)$  is invariant with respect to any involution.

The following is the main result of this section.

**Theorem 3.1.** *There exists a T-ideal  $G \triangleleft F\langle X \rangle$  with the following properties:*

- (1) *If  $A, B$  are associative algebras,  $J = A$  or  $J = H(A, *)$ , and  $\varphi : J \rightarrow B$  is a Jordan homomorphism, then the restriction of  $\varphi$  to  $J_G$  is annihilator-by-standard.*
- (2) *Let  $T(M_2)$  be the T-ideal of identities of the algebra of  $2 \times 2$  matrices. Then, every element of  $T(M_2)$  is nilpotent modulo the T-ideal  $G$ .*

The proof will be given after a series of lemmas.

We will consider only the case  $J = H(A, *)$ . The case  $J = A$  is easier and, besides,  $A^{(+)} \cong H(A \oplus A^{\text{op}}, \text{ex})$  (as usual,  $A^{(+)}$  is the Jordan algebra which is equal to  $A$  as a linear space and is equipped with the product  $x \circ y = xy + yx$ ).

Without loss of generality we will assume that the field  $F$  is algebraically closed. Indeed, let  $\bar{F}$  be the algebraic closure of  $F$ . The involution  $*$  extends to the  $\bar{F}$ -involution on  $\bar{A} = A \otimes_F \bar{F}$  and the Jordan homomorphism  $\varphi$  extends to the Jordan homomorphism  $\bar{\varphi} : H(\bar{A}, *) \rightarrow \bar{B} = B \otimes_F \bar{F}$ . We have  $G(\bar{A}) = G(A) \otimes_F \bar{F}$  and  $\text{Ann}(\bar{B}) = \text{Ann}(B) \otimes_F \bar{F}$ . If the homomorphism  $\bar{\varphi}$  is annihilator-by-standard, then so is the homomorphism  $\varphi$ .

In the Jordan algebra  $F\langle X \rangle^{(+)}$ , we consider the Jordan subalgebra  $SJ\langle X \rangle$  generated by  $X$ . We refer to elements from  $SJ\langle X \rangle$  as Jordan polynomials.

Let  $J\langle X \rangle$  be the free Jordan  $F$ -algebra on the set of free generators  $X$ .

**Lemma 3.2.** *For an arbitrary T-ideal  $P$  of the free Jordan algebra  $J\langle X \rangle$  such that  $P(SJ\langle X \rangle) \neq (0)$  there exists a nonzero T-ideal  $\tilde{P}$  of the free associative algebra  $F\langle X \rangle$  such that  $\tilde{P}(A) \subseteq AP(H(A, *))A$ .*

*Proof.* By the result of Amitsur [1], the factor algebra  $A/AP(H(A, *))A$  satisfies a polynomial identity that depends only on the T-ideal  $P$ . This implies the assertion of the lemma. □

In what follows  $\tilde{P}$  denotes the largest T-ideal in  $F\langle X \rangle$  satisfying the conclusion of Lemma 3.2.

It is well known that if  $I$  is an ideal of a Jordan algebra  $J$ , then its cube  $I^3 = (I \circ I) \circ I$  is also an ideal of  $J$  (see [10]).

Let  $P \triangleleft J\langle X \rangle$  be a T-ideal. Denote by  $P'$  the ideal of  $J\langle X \rangle$  generated by all elements of the form  $(x \circ y) \circ z - (x \circ z) \circ y$ , where  $x \in P^3$ ,  $y, z \in P$ . Recall from (2.4) that if  $x, y, z$  are elements of an associative algebra  $A$ , then these elements can be written as  $[x, [y, z]]$ . Hence, for the Jordan algebra  $J = H(A, *)$  we have

$$P'(J) \subseteq A [P(J)^3, [P(J), P(J)]] A.$$

The next lemma will be repeatedly used in the case where  $I = P(J)$ , but we will also need it in another situation.

**Lemma 3.3.** *Let  $A$  be an associative algebra (with multiplication denoted by juxtaposition). Let  $J$  be an arbitrary subalgebra of the Jordan algebra  $A^{(+)}$ , let  $I$  be an ideal of the Jordan algebra  $J$ , and let  $I^3$  be the cube of the ideal  $I$  in the Jordan algebra  $J$ . Then,  $JI^3 \subseteq II$  and  $I^3J \subseteq II$ .*

*Proof.* The space  $I^3$  is spanned by elements  $uvu$ ,  $u, v \in I$ . For an arbitrary element  $a \in J$  we have

$$auvu = (auv + vua)u - v(ua)u \in II.$$

This proves the first inclusion. The proof of the second inclusion is similar.  $\square$

Let

$$A_P = \langle P(J) \rangle,$$

that is,  $A_P$  is the subalgebra of the associative algebra  $A$  generated by  $P(J)$ .

**Lemma 3.4.**  $AP'(J)A \subseteq A_P$ .

*Proof.* Since  $A$  is the direct sum of  $H(A, *)$  and the space of skew-symmetric elements  $K(A, *) = \{a \in A \mid a^* = -a\}$ , it is enough to show that  $aP'(J)b \subseteq A_P$  for any elements  $a, b \in H(A, *) \cup K(A, *)$ . Moreover, to this end it is enough to show that

$$a[P(J)^3, [P(J), P(J)]]b \subseteq A_P$$

for any  $a, b \in H(A, *) \cup K(A, *)$ .

Let us start by showing  $P'(J)b \subseteq A_P$  for an arbitrary element  $b \in H(A, *) \cup K(A, *)$ . If  $b \in H(A, *)$  then the assertion follows from Lemma 3.3. Let  $b \in K(A, *)$ . Choose elements  $p \in P(J)^3$ ,  $p_1, p_2 \in P(J)$ . We have

$$[p_1, p_2]b = \frac{1}{2} ([p_1, p_2] \circ b + [[p_1, p_2], b]) \quad (3.1)$$

The mapping  $J \rightarrow J, x \mapsto [x, b]$ , is a derivation. Since  $P$  is a T-ideal, it follows that  $[P(J), b] \subseteq P(J)$ . Therefore, (3.1) implies that

$$[p_1, p_2]b \in H(A, *) + [P(J), P(J)].$$

Similarly,

$$b[p_1, p_2] \in H(A, *) + [P(J), P(J)].$$

Hence,

$$p[p_1, p_2]b \in P(J)^3H(A, *) + A_P = A_P$$

by Lemma 3.3. Similarly, since  $[p, b] \in P(J)$  implies  $[p_1, p_2][p, b] \in A_p$ , it follows that

$$[p_1, p_2]pb = [p_1, p_2][p, b] + [p_1, p_2]bp \in A_p.$$

Now, choose arbitrary elements  $a, b \in H(A, *) \cup K(A, *)$ . If  $a \in H(A, *)$  then

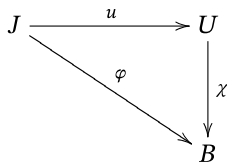
$$aP'(J)b \subseteq (a \circ P'(J))b + P'(J)ab \subseteq P'(J)b + P'(J)ab \subseteq A_p$$

by the above. Let  $a \in K(A, *)$ . Then, the mapping  $J \rightarrow J, x \mapsto [a, x]$  is a derivation of the Jordan algebra  $J$ . Hence,

$$aP'(J)b \subseteq [a, P'(J)]b + P'(J)ab \subseteq P'(J)b + P'(J)ab \subseteq A_p.$$

This completes the proof of the lemma. □

Let  $U, u : J \rightarrow U$ , be the special enveloping algebra of the Jordan algebra  $J$ . This means that there exists a Jordan homomorphism  $u : J \rightarrow U$  such that the associative algebra  $U$  is generated by the image  $u(J)$  and for any Jordan homomorphism  $\varphi : J \rightarrow B$  there exists a homomorphism  $\chi : U \rightarrow B$  of associative algebras such that the diagram



is commutative. The algebra  $U$  is equipped with an involution  $\tau : U \rightarrow U$  leaving all elements  $u(a), a \in J$ , fixed. For more information about universal special algebras, see [11, 17, 22].

Applying the above definition to the Jordan embedding  $H(A, *) \rightarrow A, a \mapsto a$ , we obtain a homomorphism  $t : U \rightarrow A, t(u(a)) = a, a \in H(A, *)$ . Consider the ideal  $I = \ker t$ . Clearly,  $I^\tau = I$ . Denote

$$I_p = I \cap U_p.$$

In [21], the second author constructed the T-ideal  $T$  of the free Jordan algebra  $J\langle X \rangle$  with the following properties:

- (1) If  $a_1 \in T(SJ\langle X \rangle)$  and  $a_2, a_3, a_4 \in SJ\langle X \rangle$ , then

$$a_1 a_2 a_3 a_4 + a_4 a_3 a_2 a_1 \in SJ\langle X \rangle$$

(we call this the **tetrad-eating property** of  $T$ ).

- (2) If  $J$  is a finite-dimensional simple Jordan algebra that is not an algebra of a symmetric bilinear form, then  $T(J) = J$ .

From (1) and the result by Cohn [7] it follows that for arbitrary elements  $a_1, \dots, a_n \in T(SJ\langle X \rangle)$ ,  $n \geq 1$ , we have

$$a_1 \cdots a_n + a_n \cdots a_1 \in T(SJ\langle X \rangle).$$

This implies that

$$H(U_T, \tau) = u(T(J)).$$

Hence,

$$I \cap H(U_T, \tau) = (0).$$

Clearly,  $I_T^\tau = I^\tau \cap U_T^\tau = I_T$  and therefore

$$H(I_T, \tau) = (0);$$

in other words, every element from  $I_T$  is skew-symmetric with respect to  $\tau$ . By Lemma 3.3,  $JT^3(J) \subseteq T(J)T(J)$ , hence  $u(J)U_{T^3} \subseteq U_T$ , and therefore

$$u(J)I_{T^3} \subseteq I_T. \quad (3.2)$$

**Lemma 3.5.**  $I_{T^3}$  lies in the center of the algebra  $U$ .

*Proof.* Choose elements  $x \in u(J)$ ,  $y \in I_{T^3}$ . We have  $y^\tau = -y$ . By (3.2), we also have  $(xy)^\tau = -xy$ . Hence,

$$(xy)^\tau = y^\tau x^\tau = -yx = -xy,$$

which implies the assertion of the lemma.  $\square$

Recall that  $T'$  is the Jordan T-ideal generated by  $(x \circ y) \circ z - (x \circ z) \circ y$ , where  $x \in T^3$ ,  $y, z \in T$ . Clearly,  $T' \subseteq T^3$ .

**Lemma 3.6.**  $I_{T'} U_{T'} = U_{T'} I_{T'} = (0)$ .

*Proof.* From Lemma 3.5, it follows that

$$I_{T^3}[u(T^3(I)), U] = (0).$$

In terms of the associative multiplication in the algebra  $F\langle X \rangle$ , (2.4) holds. Hence,

$$T'(SJ\langle X \rangle) \subseteq F\langle X \rangle[T^3(SJ\langle X \rangle), F\langle X \rangle]F\langle X \rangle.$$

This implies the assertion of the lemma.  $\square$

Now, we are ready to introduce the T-ideal  $G$  and to prove the main theorem.

Let  $T'' = (T')' \triangleleft J\langle X \rangle$  and let

$$G = \widetilde{T''}$$

be the T-ideal of  $F\langle X \rangle$ .

*Proof of Theorem 3.1.* Our goal is to show that the restriction of  $\varphi$  to  $H(G(A), *)$  is annihilator-by-standard. Let  $B'$  be the subalgebra of  $B$  generated by  $\varphi(H(G(A), *))$ . We need to show that the Jordan homomorphism

$$H(G(A), *) \rightarrow B' / \text{Ann}(B')$$

is standard.

Let  $B_{T'}$  be the subalgebra of  $B$  generated by  $\varphi(T'(J))$  or, equivalently,  $B_{T'} = \chi(U_{T'})$ . By Lemma 3.6,  $\chi(I_{T'}) \subseteq \text{Ann}(B_{T'})$ .

As above, we denote the subalgebra of  $A$  generated by  $T'(H(A, *))$  by  $A_{T'}$ . Let us show that the mapping

$$T'(H(A, *)) \rightarrow B_{T'} / \text{Ann}(B_{T'}),$$

which is the composition of the mapping  $u$  and  $U_{T'} \rightarrow B_{T'} / \text{Ann}(B_{T'})$ , extends to a homomorphism of associative algebras

$$\psi : A_{T'} \rightarrow B_{T'} / \text{Ann}(B_{T'}).$$

By the universality of the mapping  $u$  there is a homomorphism  $\mu_1 : U_{T'} \rightarrow B_{T'}$  and there is the mapping  $\mu_2 : U_{T'} \rightarrow A_{T'}$ . The kernel  $\ker \mu_2 = I_{T'}$  is mapped by  $\mu_1$  to the annihilator  $\text{Ann}(B_{T'})$ . Hence, there is a homomorphism  $A_{T'} \rightarrow B_{T'} / \text{Ann}(B_{T'})$  that extends the Jordan homomorphism  $T'(H(A, *)) \rightarrow B_{T'} / \text{Ann}(B_{T'})$ .

By Lemmas 3.2 and 3.4,

$$G(A) \subseteq AT''(J)A \subseteq A_{T'}.$$

By the tetrad-eating property of the ideal  $T$ ,

$$H(G(A), *) \subseteq H(A_{T'}, *) \subseteq T'(H(A, *)).$$

Let  $\psi_G$  be the restriction of the homomorphism  $\psi$  to  $G(A)$ . Let  $B_G$  be the subalgebra of  $B$  generated by the image of  $H(G(A), *)$ . We have

$$B_G \cap \text{Ann}(B_{T'}) \subseteq \text{Ann}(B_G),$$

$$\psi_G : G(A) \rightarrow B_G / B_G \cap \text{Ann}(B_{T'}).$$

The composition

$$G(A) \rightarrow B_G / B_G \cap \text{Ann}(B_{T'}) \rightarrow B_G / \text{Ann}(B_G)$$

extends the Jordan homomorphism  $H(G(A), *) \rightarrow B_G / \text{Ann}(B_G)$ . We proved that the Jordan homomorphism

$$\varphi : H(G(A), *) \rightarrow B_{T'} / \text{Ann}(B_{T'})$$

is standard. This completes the proof of the first part of the theorem.

Recall that the complexity of a T-ideal  $G \triangleleft F\langle X \rangle$  is a maximal  $n \geq 1$  such that the matrix algebra  $M_n(F)$  satisfies all identities from  $G$ . Our aim is to prove that the complexity of the T-ideal  $G$  is equal to 2.

For that we need to recall Amitsur’s proof in [1]. Recall that the field  $F$  is assumed to be algebraically closed. Let  $P$  be a Jordan T-ideal such that  $P(SJ\langle X \rangle) \neq (0)$ .

Let  $(A_i, *_i)$ ,  $1 \leq i \leq m$ , be the list of all finite-dimensional simple involutive associative algebras such that  $P(H(A_i, *_i)) = (0)$ ,  $1 \leq i \leq m$ . If  $f = 0$  is an identity in all algebras  $A_i$ ,  $1 \leq i \leq m$ , then some power  $f^k$  lies in  $\bar{P}$ .

If  $P = T''$ , then the list is short:  $A_1 = F$ ,  $A_2 = F \oplus F$  with the exchange involution,  $A_3 = M_2(F)$  with the orthogonal or symplectic involution,  $A_4 = M_2(F) \oplus M_2(F)^{op}$  with the exchange involution. Indeed, unless the Jordan algebra  $H(A_i, *_i)$  is associative as in the cases of  $A_1$  and  $A_2$ ,  $T''(H(A_i, *_i)) = (0)$  if and only if  $T(H(A_i, *_i)) = (0)$ . In [21], it was shown that the latter equality holds only in the cases  $A_3$  and  $A_4$ .

If  $f \in T(M_2)$  then some power  $f^k$  lies in the T-ideal  $G$ . This completes the proof of the theorem. □

We will now construct an example of an associative algebra  $A$ , an associative algebra  $B$  with an involution  $*$ :  $B \rightarrow B$  and a Jordan isomorphism  $\varphi : A \rightarrow H(B, *)$  such that for an arbitrary T-ideal  $T$ , the restriction of  $\varphi$  to  $T(A)$  is not standard (Proposition 3.12).

We start with some well-known facts concerning presentations of semigroups by generators and relations.

Let  $X = \{x_1, x_2, \dots\}$  be a finite alphabet. Let  $X^*$  be the set of all nonempty words in  $X$ . The semigroup  $X^*$  is referred to as the free semigroup.

An equivalence relation  $\sim$  on  $X^* \times X^*$  is called a **congruence** if  $u_1 \sim v_1, u_2 \sim v_2$  implies  $u_1 u_2 \sim v_1 v_2$ . This makes the set of equivalence classes a semigroup  $X^* / \sim$ .

For a subset  $\sigma \subset X^* \times X^*$ , let  $\langle \sigma \rangle$  be the minimal congruence containing  $\sigma$ . Two words  $u, v$  are equivalent if there exists a sequence of words  $u = u_1, u_2, \dots, u_n = v$  such that each pair  $u_i, u_{i+1}$ ,  $i = 1, \dots, n - 1$ , is of the form  $u_i = u' u'_i v'$ ,  $u_{i+1} = u' u'_{i+1} v'$ ,  $u'_i \times u'_{i+1}$  or  $u'_{i+1} \times u'_i$  lies in  $\sigma$ . In this case, we say that  $u = v$  follows from  $\sigma$ . The semigroup  $S = X^* / \langle \sigma \rangle$  is **presented** by the set of generators  $X$  and the set of relations  $\sigma$ ,  $S = \langle X \mid \sigma \rangle$ .

The semigroup is called **finitely presented** if it has a presentation  $S = \langle X \mid \sigma \rangle$  with both  $X$  and  $\sigma$  finite.

A semigroup  $S$  is **graded** if no two words (in  $X$ ) of different lengths are equal. Similarly, we can talk about **graded congruences**.

Given a graded congruence  $\sim \subset X^* \times X^*$ , we will define a new graded congruence  $\sim_{in} \subset X^* \times X^*$ . Let  $u, v \in X^*$ ,  $\text{length}(u) = \text{length}(v) = n$ . We say that  $u \sim_{in} v$  if  $u = v$  follows from  $\{u' \times v' \mid \text{length}(u') = \text{length}(v') < n, u' \sim v'\}$ . Clearly,

$$\sim_{in} \subset \sim .$$

Let  $S$  be a semigroup. A mapping  $f : S \times S \rightarrow F$  is called a **2-cocycle** if  $f(ab, c) = f(a, bc)$  for any elements  $a, b, c \in S$ .

Let  $S = X^* / \sim$ , where  $\sim$  is a graded congruence. Let  $h : S \times S \rightarrow F$  be a 2-cocycle. Let us lift  $h$  to  $\tilde{h} : X^* \times X^* \rightarrow F$ .

**Lemma 3.7.** *Let  $a, b, c, d \in X^*$ . If  $ab \sim_{in} cd$  then  $\tilde{h}(a, b) = \tilde{h}(c, d)$ .*

*Proof.* We only need to consider the case where  $ab = w'uw''$ ,  $cd = w'vw''$ ,  $u \sim v$ , the words  $w', w''$  are not simultaneously empty. Suppose that the word  $w'$  is not empty,  $w' = x_{i_1} w'_i$ . Then, the word  $a$  starts with  $x_{i_1}$ ,  $a = x_{i_1} a'$ . Hence,  $\tilde{h}(a, b) = \tilde{h}(x_{i_1}, a'b)$ ,  $a'b = w'_i uw''$ . Similarly,  $\tilde{h}(c, d) = \tilde{h}(x_{i_1}, w'_i vw'')$ , the elements  $w'_i uw''$  and  $w'_i vw''$  are equal in  $S$ . This proves the lemma.  $\square$

**Lemma 3.8.** *Let  $\alpha : X^* \rightarrow F$  be a mapping such that  $\alpha(u) = \alpha(v)$  if  $u \sim_{in} v$ . Define the 2-cocycle  $h : (X^* / \sim) \times (X^* / \sim) \rightarrow F$ ,  $h(u, v) = \alpha(uv)$ . The mapping  $h$  is well defined.*

*Proof.* We need to show that  $u \sim u_1, v \sim v_1$  implies  $\alpha(uv) = \alpha(u_1v_1)$ , but  $uv \sim_{in} u_1v_1$ . This proves the lemma.  $\square$

Now, suppose that the field  $F$  is infinite, the mapping  $\alpha : X^* / \sim_{in} \rightarrow F$  is injective. In other words, given words  $u, v \in X^*$ , we have  $\alpha(u) = \alpha(v)$  if and only if  $u \sim_{in} v$ .

Suppose that the graded semigroup  $S = X^* / \sim$  is not finitely presented (there are many such semigroups).

**Lemma 3.9.** *For an arbitrary  $n \geq 1$  there exist words  $u, v \in X^*$  such that  $u \sim v$ ,  $u \not\sim_{in} v$ ,  $\text{length}(u) = \text{length}(v) \geq n$ .*

*Proof.* It is sufficient to notice that since  $S$  is not finitely presented there exist words  $u, v$  such that  $u = v$  in  $S$ ,  $\text{length}(u) = \text{length}(v) \geq n$ , and  $u = v$  does not follow from all equalities  $u' = v'$  in  $S$ ,  $\text{length}(u') = \text{length}(v') < n$ . This completes the proof of the lemma.  $\square$

Let  $A$  be an associative algebra. Let  $f : A \times A \rightarrow F$  be a bilinear form. Consider the algebra  $\hat{A} = A \oplus Fz$  with multiplication  $\hat{A}z = z\hat{A} = (0)$ ,

$$a \cdot b = ab + f(a, b)z,$$

where  $a, b \in A$ ,  $ab$  is the product in the algebra  $A$ , and  $a \cdot b$  is the product in the algebra  $\hat{A}$ . The algebra  $\hat{A}$  is associative if and only if  $f(ab, c) = f(a, bc)$  for arbitrary elements  $a, b, c \in A$ , that is,  $f$  is a (Hochschild) **2-cocycle**. A 2-cocycle is called a **coboundary** if there exists a linear functional  $\lambda : A \rightarrow F$  such that  $f(a, b) = \lambda(ab)$ .

**Proposition 3.10.** *There exists an algebra  $A$  and a 2-cocycle  $f : A \times A \rightarrow F$  with the following properties: for an arbitrary  $T$ -ideal  $P$ , the ideal  $P(A)$  contains elements  $a_1, a_2, b_1, b_2$  such that  $a_1b_1 - a_2b_2 = b_1a_1 - b_2a_2 = 0$ ,  $f(b_1, a_1) = f(b_2, a_2) = 0$ , but  $f(a_1, b_1) \neq f(a_2, b_2)$ . In particular, the restriction  $f : P(A) \times P(A) \rightarrow F$  is not a coboundary.*

*Proof.* Consider the semigroup algebra  $FS$  and the algebra  $A$  of strictly upper triangular infinite matrices over  $FS$ ,

$$A = \sum_{1 \leq i < j} e_{ij}(FS), \quad e_{ij}(a)e_{pq}(b) = \delta_{jp}e_{iq}(ab).$$

We define a 2-cocycle

$$f(e_{ij}(u), e_{pq}(v)) = \begin{cases} h(u, v) & \text{if } i < j, p < q, j = p; u, v \in S \\ 0 & \text{otherwise} \end{cases}$$

Let  $P$  be a T-ideal. There exists a nonzero multilinear element

$$x_1 \dots x_m + \sum_{1 \neq \sigma \in S_m} k_\sigma x_{\sigma(1)} \dots x_{\sigma(m)} \in P, \quad k_\sigma \in F.$$

This implies that

$$e_{ij}(S^m) \subseteq P(A)$$

as soon as  $j - i \geq m$ . Choose elements  $u, v \in X^*$ ,  $\text{length}(u) = \text{length}(v) \geq 2m$ ,  $u \sim v$ , but  $u \not\sim_{\text{in}} v$ . Let  $u = u'u''$ ,  $\text{length}(u), \text{length}(u'') \geq m$ ,  $v = v'v''$ ,  $\text{length}(v), \text{length}(v'') \geq m$ . The elements

$$a_1 = e_{1,1+m}(u'), \quad b_1 = e_{1+m,1+2m}(u''), \quad a_2 = e_{1,1+m}(v'), \quad b_2 = e_{1+m,1+2m}(v''),$$

lie in  $P(A)$ ,  $a_1 b_1 - a_2 b_2 = b_1 a_1 - b_2 a_2 = 0$  in  $A$ ,  $f(b_1, a_1) = f(b_2, a_2) = 0$ , but

$$f(a_1, b_1) = h(u', u'') \neq h(v', v'') = f(a_2, b_2).$$

This completes the proof. □

Let  $A, f : A \times A \rightarrow F$  be the algebra and the 2-cocycle of Proposition 3.10. Consider the direct sum  $\tilde{A} = A \oplus A^{\text{op}}$  and extend the 2-cocycle on  $\tilde{A}$  :

$$f(A, A^{\text{op}}) = f(A^{\text{op}}, A) = (0), \quad f(x^{\text{op}}, y^{\text{op}}) = -f(y, x)$$

for arbitrary elements  $x, y \in A$ . It is straightforward that this extension is a 2-cocycle on the algebra  $\tilde{A}$ .

Consider the annihilator extension

$$B = A \oplus A^{\text{op}} \oplus Fz.$$

### Lemma 3.11.

- (1) The mapping  $*$  :  $B \rightarrow B$ ,  $a + b^{\text{op}} + kz \mapsto b + a^{\text{op}} - kz$ ,  $k \in F$ , is an involution.
- (2) The mapping  $\varphi : A \rightarrow H(B, *)$ ,  $a \mapsto a + a^{\text{op}}$ , is a Jordan isomorphism.

*Proof.* We need to check that

$$((a + b^{\text{op}}) \cdot (c + d^{\text{op}}))^* = (c + d^{\text{op}})^* \cdot (a + b^{\text{op}})^*.$$

The left-hand side is

$$(ac + (db)^{\text{op}} + (f(a, c) - f(d, b))z)^* = db + (ac)^{\text{op}} - (f(a, c) - f(d, b))z,$$

and the right-hand side is

$$(d + c^{\text{op}}) \cdot (b + a^{\text{op}}) = db + (ac)^{\text{op}} + (f(d, b) - f(a, c))z.$$

This completes the proof of (1). Let us prove (2). We have

$$(a + a^{op})^2 = a^2 + (a^2)^{op} + (f(a, a) + f(a^{op}, a^{op}))z = a^2 + (a^2)^{op}.$$

This completes the proof. □

**Proposition 3.12.** *For an arbitrary T-ideal P, the restriction of  $\varphi$  to  $P(A)$  is not standard.*

*Proof.* By Proposition 3.10,  $P(A)$  contains elements  $a_1, b_1, a_2, b_2$  such that  $a_1b_1 - a_2b_2 = b_1a_1 - b_2a_2 = 0$ , but  $f(a_1, b_1) - f(a_2, b_2) \neq 0$ .

Suppose that  $\varphi = \varphi_1 + \varphi_2$ , where  $\varphi_1 : P(A) \rightarrow B$  is a homomorphism,  $\varphi_2 : P(A) \rightarrow B$  is an antihomomorphism, and

$$\varphi_1(P(A))\varphi_2(P(A)) = \varphi_2(P(A))\varphi_1(P(A)) = (0).$$

Then,

$$\varphi_1(a_1)\varphi_1(b_1) - \varphi_1(a_2)\varphi_1(b_2) = \varphi_1(a_1b_1 - a_2b_2) = 0$$

and

$$\varphi_2(a_1)\varphi_2(b_1) - \varphi_2(a_2)\varphi_2(b_2) = \varphi_2(b_1a_1 - b_2a_2) = 0,$$

hence

$$\varphi(a_1)\varphi(b_1) - \varphi(a_2)\varphi(b_2) = 0.$$

However,

$$\begin{aligned} & \varphi(a_1)\varphi(b_1) - \varphi(a_2)\varphi(b_2) \\ &= (a_1 + a_1^{op}) \cdot (b_1 + b_1^{op}) - (a_2 + a_2^{op}) \cdot (b_2 + b_2^{op}) \\ &= a_1b_1 + (b_1a_1)^{op} + (f(a_1, b_1) - f(b_1, a_1))z \\ & \quad - a_2b_2 - (b_2a_2)^{op} - (f(a_2, b_2) - f(b_2, a_2))z \\ &= (f(a_1, b_1) - f(a_2, b_2))z \neq 0. \end{aligned}$$

This completes the proof. □

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