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The exact region determined by Spearman's footrule, Gini's gamma and Kendall's tau

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A B S T R A C T

Concordance measures are used to express the degree of association between random variables. Practitioners may use several distinct concordance measures to narrow down the space of possible dependence structures. Consequently, the relations between different (weak) concordance measures have been extensively studied in recent years. The goal of this paper is to study the relation between Kendall's tau, Gini's gamma and Spearman's footrule. In particular, we describe the exact region determined by these three measures, using shuffles of M and ordinal sums of copulas. We also provide the formulas for five main (weak) concordance measures and Chatterjee's xi of ordinal sums of copulas.

1. Introduction

Various concordance measures can be used to describe the association between random variables. Given two random variables X and Y, the concordance measure between them depends only on the copula associated with their joint distribution. Traditionally, Spearman's rho and Kendall's tau are the most commonly used in practice and there is an abundance of literature discussing their properties. Later, several other (weak) concordance measures were introduced, such as Gini's gamma, Blomqvist's beta and Spearman's footrule. The central interest of this paper is the relation between Kendall's tau, Gini's gamma and Spearman's footrule, therefore, we provide some references for these three. Kendall's tau was investigated in [1–4], Gini's gamma in [5–7], and Spearman's footrule in [6,8–10].

Each concordance measure captures a specific aspect of association, so practitioners tend to use several of them to narrow the space of possible dependence structures. Hence, the relations between different (weak) concordance measures of copulas are of interest and have been extensively studied.

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The problem of the exact region determined by Spearman's rho and Kendall's tau was open since1960′s and was solved in [11]. The relation between Blomqvist's beta and other (weak) concordance measures is relatively easy to describe (see [12]), while relations between others are more challenging. The region determined by Gini's gamma and Spearman's footrule is characterized in [13]. The problem of the region determined by Spearman's rho and Spearman's footrule is partially solved in [14], see also [15]. The lower bound for this region is determined exactly, while for the upper bound only a good approximation is known. The relations between Kendall's tau and other (weak) concordance measures are considered in [16]. The region determined by Spearman's rho and Gini's gamma is still open.

In 2021, Chatterjee's coefficient of rank correlation (Chatterjee's xi) was introduced in [17] and became extremely popular. It quantifies the strength of functional dependence between random variables X and Y. Even though this is not a concordance measure, it has still been compared to common measures of concordance, see [18–20].

The paper [19] emphasizes the importance of studying higher-dimensional versions of the problem, namely, capturing the connections between more than two concordance measures simultaneously. The first treatment of a triplet of concordance measures, Blomqvist's beta, Spearman's footrule and Gini's gamma, was presented in [21]. The goal of the present paper is to study the relation between Kendall's tau (denoted by τ), Spearman's footrule (denoted by ϕ) and Gini's gamma (denoted by γ). We first prove the lower and upper bound for Kendall's tau of a copula if its Spearman's footrule and Gini's gamma are given. Then we give several examples to demonstrate that these bounds are attained by shuffles of M. It turns out that the set of all triplets $(\phi(C), \gamma(C), \tau(C))$ for all copulas C is a polyhedron.

To prove our results, we use ordinal sums of copulas, hence, we need the formulas expressing concordance measures of an ordinal sum of copulas in terms of concordance measures of its summands. We derive these formulas for all five most common (weak) concordance measures and Chatterjee's xi. These results may be of independent interest to the reader. Some of them already appeared in the literature without proofs.

The paper is structured as follows. In Section 2 we introduce the notation, give the basic definitions and recall some results that we need later. In Section 3 we prove the formulas for concordance measures and Chatterjee's xi of an ordinal sum of copulas. In Section 4 we present our main result, i.e., we describe the exact region determined by Spearman's footrule, Gini's gamma and Kendall's tau. Moreover, we provide examples of copulas attaining the boundary of the obtained region. We end the paper with some concluding remarks, including a discussion on the possible range of Kendall's tau when Spearman's footrule and Gini's gamma of a copula are given.

2. Preliminaries

Let $\mathbb{I}$ be the unit interval $[0,1] \subseteq \mathbb{R}$ and $u_1, u_2, v_1, v_2 \in \mathbb{I}$ be such that $u_1 \le u_2$ and $v_1 \le v_2$. The Cartesian product of intervals $R = [u_1, u_2] \times [v_1, v_2]$ is called a *rectangle* in $\mathbb{I}^2$. Let $H : \mathbb{I}^2 \to \mathbb{R}$ be a real function. We define the H-volume of R as $V_H(R) = H(u_2, v_2) - H(u_2, v_1) - H(u_1, v_2) + H(u_1, v_1)$. A *bivariate copula* is a function $C : \mathbb{I}^2 \to \mathbb{I}$ with the following properties:

- $C(0, v) = C(u, 0) = 0$ for all $u, v \in \mathbb{I}$ (C is *grounded*),
- $C(u, 1) = u$ and $C(1, v) = v$ for all $u, v \in \mathbb{I}$ (C has *uniform marginals*), and
- $V_C(R) \ge 0$ for every rectangle $R \subseteq \mathbb{I}^2$ (C is 2*–increasing*).

Bivariate copulas are functions of two variables which couple bivariate distribution functions with their one-dimensional marginal distribution functions, a famous theorem by Sklar [22]. The set of all bivariate copulas will be denoted by $\mathcal{C}$. It is well known that this set is compact in the sup norm. For any copula C its diagonal will be denoted by δ_C and its opposite diagonal by ω_C, i.e., $\delta_C(u) = C(u,u)$ and $\omega_C(u) = C(u, 1-u)$ for all $u \in \mathbb{I}$.

Given two copulas C and D, we denote $C \le D$ if $C(u,v) \leqslant D(u,v)$ for all $(u,v) \in \mathbb{I}^2$. This is the so-called *pointwise order* of copulas. For any copula C we have $W \le C \le M$, where $W(u,v) = \max\{0, u+v-1\}$ and $M(u,v) = \min\{u,v\}$ are the lower and upper *Fréchet-Hoeffding bounds* for the set of all copulas. With $\Pi(u,v) = uv$ we denote the independence copula. Furthermore, any copula C induces reflected copulas C^{σ_1} and C^{σ_2} defined by $C^{\sigma_1}(u,v) = v - C(1-u, v)$ and $C^{\sigma_2}(u,v) = u - C(u, 1-v)$ for all $(u,v) \in \mathbb{I}^2$. Note that $\delta_{C^{\sigma_2}}(u) = u - \omega_C(u)$.

Let $h : \mathbb{I} \to \mathbb{I}$ be a measure preserving bijection, where $\mathbb{I}$ is equipped with the Lebesgue measure λ. Then the function defined by

$$C(u,v) = \lambda\big(\{t \in \mathbb{I} \mid t \le u, h(t) \le v\}\big) \tag{1}$$

is a copula whose mass is concentrated on the graph of h, i.e., $\mu_C\big(\{(t, h(t)) \mid t \in \mathbb{I}\}\big) = 1$. Particular examples of such copulas are the so-called *shuffles of* M. A shuffle of M

$$C = M(n, J, \pi, \omega)$$

is determined by a positive integer n, a partition $J = \{J_1, J_2, \dots, J_n\}$ of the interval $\mathbb{I}$ into n pieces, where $J_i = [u_{i-1}, u_i]$ and $0 = u_0 \le u_1 \le u_2 \le \dots \le u_{n-1} \le u_n = 1$, shortly written as $(n-1)$-tuple of splitting points $J = (u_1, u_2, \dots, u_{n-1})$, a permutation $\pi \in S_n$, written as n-tuple of images $\pi = (\pi(1), \pi(2), \dots, \pi(n))$, and a mapping $\omega : \{1, 2, \dots, n\} \to \{-1, 1\}$, written as n-tuple of images $\omega = (\omega(1), \omega(2), \dots, \omega(n))$. The mass of C is concentrated on the main diagonals of the squares $[u_{i-1}, u_i] \times [v_{\pi(i)-1} \times v_{\pi(i)}]$ where $0 = v_0 \le v_1 \le v_2 \le \dots \le v_{n-1} \le v_n = 1$. Hence, C is defined by the measure preserving bijection $h_C : \mathbb{I} \to \mathbb{I}$, given by

$$h_C(u) = \begin{cases} u + v_{\pi(i)-1} - u_{i-1}; & u \in (u_{i-1}, u_i),\ \omega(i) = 1, \\ v_{\pi(i)} + u_{i-1} - u; & u \in (u_{i-1}, u_i),\ \omega(i) = -1, \\ v_i; & u = u_i. \end{cases}$$

Furthermore, C is the copula of random variables U and $h_C(U)$, which are uniformly distributed on $\mathbb{I}$. For more details see [23, §3.2.3].

In [24] Scarsini introduced formal axioms for *concordance measures*. These are mappings that assign to each copula a real number in $[-1,1]$ and are meant to measure the degree of concordance/discordance between the components of random vectors. Recall that two observations (x,y) and (x',y') from a random vector (X,Y) are concordant if $(x-x')(y-y')>0$ and discordant if $(x-x')(y-y')<0$. For the formal definition of concordance measures and further details we refer the reader to [25] and [23]. Here we give the properties of concordance measures, which we will need in the sequel: if κ is a concordance measure, then $\kappa(M)=1$, $\kappa(C^{\sigma_1})=\kappa(C^{\sigma_2})=-\kappa(C)$ for any copula $C\in\mathcal{C}$, κ is continuous with respect to the pointwise convergence, and κ is monotone increasing with respect to the pointwise order.

Many of the most important bivariate concordance measures, including Kendall's tau and Gini's gamma, can be expressed with the so-called *concordance function* $\mathcal{Q}$, introduced by Kruskal [26]. If (X_1,Y_1) and (X_2,Y_2) are pairs of continuous random variables, then the concordance function of random vectors (X_1,Y_1) and (X_2,Y_2) depends only on the corresponding copulas C_1 and C_2 and is given by (see [23, Theorem 5.1.1])

$$\mathcal{Q}=\mathcal{Q}(C_1,C_2)=4\int_{\mathbb{I}^2} C_2(u,v)\,dC_1(u,v)-1. \tag{2}$$

It turns out that the concordance function is symmetric in its arguments, i.e., $\mathcal{Q}(C_1,C_2)=\mathcal{Q}(C_2,C_1)$, and has several other useful properties, see [23, Corollary 5.1.2] and [27, §3].

The most important concordance measures include Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta. With the concordance function at hand, *Spearman's rho* can be defined by

$$\rho(C)=3\mathcal{Q}(C,\Pi)=12\int_{\mathbb{I}^2} C(u,v)\,du\,dv-3, \tag{3}$$

Kendall's tau by

$$\tau(C)=\mathcal{Q}(C,C)=4\int_{\mathbb{I}^2} C(u,v)\,dC(u,v)-1, \tag{4}$$

and *Gini's gamma* by

$$\gamma(C)=\mathcal{Q}(C,M)+\mathcal{Q}(C,W)=4\int_0^1 \delta_C(u)\,du+4\int_0^1 \omega_C(u)\,du-2. \tag{5}$$

Blomqvist's beta is defined as

$$\beta(C)=4C(\tfrac{1}{2},\tfrac{1}{2})-1. \tag{6}$$

If $h_C:\mathbb{I}\to\mathbb{I}$ is a measure preserving bijection and C a copula whose mass is concentrated on the graph of h_C, then $\tau(C)$ can be expressed as

$$\tau(C)=4\int_0^1 C(u,h_C(u))\,du-1. \tag{7}$$

In [28] Liebscher considered *weak concordance measures*, which are slightly more general mappings than concordance measures (the formal definition can be found in Liebscher's paper). The most important example of a weak concordance measure is *Spearman's footrule* defined by

$$\phi(C)=\tfrac{1}{2}(3\mathcal{Q}(C,M)-1)=6\int_0^1 \delta_C(u)\,du-2. \tag{8}$$

Spearman's footrule is not a true concordance measure since its range is $[-\frac{1}{2},1]$.

Finally, we recall the definition of Chatterjee's xi, which is not a concordance measure, but rather a nonsymmetric coefficient of correlation. Every copula $C\in\mathcal{C}$ is increasing in each variable, hence partially differentiable almost everywhere. We denote by $\partial_1 C(u,v)$ the partial derivative of C with respect to the first variable. One of the equivalent definitions of Chatterjee's coefficient of rank correlation is the following (see [20,29])

$$\xi(C)=6\int_{\mathbb{I}^2}(\partial_1 C(u,v))^2\,du\,dv-2. \tag{9}$$

Since $\partial_1 C(u,v)$ exist almost everywhere, the above integral is well defined in the Lebesgue sense.

In the next proposition we recall the relations between Kendall's tau, Gini's gamma and Spearman's footrule, which will be needed in the sequel.

Proposition 1 ([13,16]). *For any copula $C\in\mathcal{C}$ we have*

$$\tfrac{4}{3}\phi(C)-\tfrac{1}{3}\le\gamma(C)\le\min\left\{\tfrac{4}{3}\phi(C)+\tfrac{1}{6},\tfrac{2}{3}\phi(C)+\tfrac{1}{3}\right\}, \tag{10}$$

$$\tfrac{4}{3}\phi(C)-\tfrac{1}{3}\le\tau(C)\le\tfrac{2}{3}\phi(C)+\tfrac{1}{3}, \tag{11}$$

$$\max\left\{\tfrac{2}{3}\gamma(C)-\tfrac{1}{3},2\gamma(C)-1\right\}\le\tau(C)\le\min\left\{\tfrac{2}{3}\gamma(C)+\tfrac{1}{3},2\gamma(C)+1\right\}. \tag{12}$$

The bounds are attained by shuffles of M.

3. Concordance measures of an ordinal sum

In this section we give the formulas for five (weak) concordance measures (Spearman's rho, Kendall's tau, Spearman's footrule, Gini's gamma, and Blomqvist's beta), and Chatterjee's xi of an ordinal sum of copulas.

Let $\{(a_k, b_k) \mid k = 1, 2, \dots, n\}$ be a finite family of disjoint open subintervals of $\mathbb{I}$ and $\{B_k \mid k = 1, 2, \dots, n\}$ a family of copulas. Then the *ordinal sum* (sometimes also called *M-ordinal sum*) B of $\{B_k \mid k = 1, 2, \dots, n\}$ with respect to $\{(a_k, b_k) \mid k = 1, 2, \dots, n\}$ is a copula defined by

$$B(u, v) = \begin{cases} a_k + (b_k - a_k) B_k\left(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}\right); & (u, v) \in [a_k, b_k]^2, k = 1, 2, \dots, n, \\ \min\{u, v\}; & \text{otherwise,} \end{cases} \tag{13}$$

(see [30], compare also [23, §3.2.2]).

We start with the formula for Spearman's rho. It already appeared in [14], but without a proof, so we give the proof here.

Proposition 2. *Spearman's rho of an ordinal sum B, given in (13), is*

$$\rho(B) = 1 - \sum_{k=1}^{n} (b_k - a_k)^3 (1 - \rho(B_k)). \tag{14}$$

Proof. Since $B(u, v) = M(u, v)$ outside the union of squares $[a_k, b_k]^2$, we have

$$\begin{aligned} \int_{\mathbb{I}^2} B(u, v)\, du\, dv &= \int_{\mathbb{I}^2} M(u, v)\, du\, dv + \int_{\mathbb{I}^2} (B(u, v) - M(u, v))\, du\, dv \\ &= \tfrac{1}{3} + \sum_{k=1}^{n} \int_{[a_k,b_k]^2} (B(u, v) - M(u, v))\, du\, dv \\ &= \tfrac{1}{3} + \sum_{k=1}^{n} \int_{[a_k,b_k]^2} B(u, v)\, du\, dv - \sum_{k=1}^{n} \int_{[a_k,b_k]^2} M(u, v)\, du\, dv \\ &= \tfrac{1}{3} + \sum_{k=1}^{n} \int_{[a_k,b_k]^2} B(u, v)\, du\, dv - \sum_{k=1}^{n} (b_k - a_k)^2 (\tfrac{2}{3} a_k + \tfrac{1}{3} b_k). \end{aligned}$$

Applying the change of variables $u = (b_k - a_k)t + a_k$ and $v = (b_k - a_k)s + a_k$ with the Jacobian determinant $(b_k - a_k)^2$, we calculate

$$\begin{aligned} \int_{[a_k,b_k]^2} B(u, v)\, du\, dv &= \int_{[a_k,b_k]^2} \left(a_k + (b_k - a_k) B_k\left(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}\right)\right) du\, dv \\ &= a_k (b_k - a_k)^2 + (b_k - a_k)^3 \int_{\mathbb{I}^2} B_k(t, s)\, dt\, ds \\ &= a_k (b_k - a_k)^2 + (b_k - a_k)^3 \cdot \frac{\rho(B_k) + 3}{12}. \end{aligned}$$

Hence, we obtain

$$\begin{aligned} \rho(B) &= 12 \int_{\mathbb{I}^2} B(u, v)\, du\, dv - 3 \\ &= 1 + 12 \sum_{k=1}^{n} \int_{[a_k,b_k]^2} B(u, v)\, du\, dv - 12 \sum_{k=1}^{n} (b_k - a_k)^2 (\tfrac{2}{3} a_k + \tfrac{1}{3} b_k) \\ &= 1 + \sum_{k=1}^{n} \left(12 a_k (b_k - a_k)^2 - (b_k - a_k)^2 (8a_k + 4b_k) + (b_k - a_k)^3 (\rho(B_k) + 3)\right) \\ &= 1 + \sum_{k=1}^{n} \left(-4(b_k - a_k)^3 + (b_k - a_k)^3 (\rho(B_k) + 3)\right) \\ &= 1 - \sum_{k=1}^{n} (b_k - a_k)^3 (1 - \rho(B_k)), \end{aligned}$$

which finishes the proof. □

We continue with the formula for Kendall's tau of an ordinal sum.

Proposition 3. *Kendall's tau of an ordinal sum B, given in (13), is*

$$\tau(B) = 1 - \sum_{k=1}^{n} (b_k - a_k)^2 (1 - \tau(B_k)). \tag{15}$$

Proof. Since $B(u, v) = M(u, v)$ outside the union of squares $[a_k, b_k]^2$, we calculate

$$\tau(B) = -1 + 4 \int_{\mathbb{I}^2} B(u, v)\, dB(u, v)$$

$$= -1 + 4\sum_{k=1}^{n}\int_{[a_k,b_k]^2} B(u,v)\,dB(u,v) + 4\int_{\mathbb{I}\setminus\cup_{k=1}^{n}(a_k,b_k)} u\,du$$
$$= -1 + 4\sum_{k=1}^{n}\int_{[a_k,b_k]^2} B(u,v)\,dB(u,v) + 4\int_0^1 u\,du - 4\sum_{k=1}^{n}\int_{a_k}^{b_k} u\,du$$
$$= 1 + 4\sum_{k=1}^{n}\int_{[a_k,b_k]^2} B(u,v)\,dB(u,v) - 2\sum_{k=1}^{n}(b_k^2 - a_k^2).$$

Note that for $(u,v)\in[a_k,b_k]^2$ we have $dB(u,v) = (b_k - a_k)\,dB_k(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k})$. Using this and applying the same change of variables as in the proof of Proposition 2, we calculate

$$\int_{[a_k,b_k]^2} B(u,v)\,dB(u,v) = \int_{[a_k,b_k]^2}\left(a_k + (b_k-a_k)B_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k})\right)dB(u,v)$$
$$= \int_{[a_k,b_k]^2} a_k\,dB(u,v) + (b_k-a_k)\int_{[a_k,b_k]^2} B_k\left(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}\right)dB(u,v)$$
$$= \int_{[a_k,b_k]^2} a_k(b_k-a_k)\,dB_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}) + (b_k-a_k)^2\int_{[a_k,b_k]^2} B_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k})\,dB_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k})$$
$$= a_k(b_k-a_k)\int_{\mathbb{I}^2} dB_k(t,s) + (b_k-a_k)^2\int_{\mathbb{I}^2} B_k(t,s)\,dB_k(t,s)$$
$$= a_k(b_k-a_k) + (b_k-a_k)^2\cdot\frac{\tau(B_k)+1}{4}.$$

Hence, we obtain

$$\tau(B) = 1 + 4\sum_{k=1}^{n} a_k(b_k-a_k) + 4\sum_{k=1}^{n}(b_k-a_k)^2\cdot\frac{\tau(B_k)+1}{4} - 2\sum_{k=1}^{n}(b_k^2-a_k^2)$$
$$= 1 - 2\sum_{k=1}^{n}(b_k-a_k)^2 + \sum_{k=1}^{n}(b_k-a_k)^2(\tau(B_k)+1)$$
$$= 1 - \sum_{k=1}^{n}(b_k-a_k)^2(1-\tau(B_k)),$$

which finishes the proof. □

A version of the formula for Spearman's footrule of an ordinal sum appeared already in [15], under the assumption that the union of the family of closed subintervals $\{[a_k,b_k] \mid k = 1,2,\dots,n\}$ equals the whole interval $\mathbb{I}$. Here we prove it for the general case.

Proposition 4. *Spearman's footrule of an ordinal sum B, given in (13), is*

$$\phi(B) = 1 - \sum_{k=1}^{n}(b_k-a_k)^2(1-\phi(B_k)). \tag{16}$$

Proof. Note that the diagonal section δ_B of copula B is equal to

$$\delta_B(u) = \begin{cases} a_k + (b_k-a_k)\delta_{B_k}(\frac{u-a_k}{b_k-a_k}); & u\in[a_k,b_k], k=1,2,\dots,n,\\ u; & \text{otherwise}\end{cases}$$
$$= u + \begin{cases} a_k - u + (b_k-a_k)\delta_{B_k}(\frac{u-a_k}{b_k-a_k}); & u\in[a_k,b_k], k=1,2,\dots,n,\\ 0; & \text{otherwise,}\end{cases}$$

so that, using the change of variable,

$$\int_0^1 \delta_B(u)\,du = \int_0^1 u\,du + \sum_{k=1}^{n}\int_{a_k}^{b_k}\left(a_k - u + (b_k-a_k)\delta_{B_k}(\tfrac{u-a_k}{b_k-a_k})\right)du$$
$$= \tfrac{1}{2} + \sum_{k=1}^{n}\left(-\tfrac{1}{2}(b_k-a_k)^2 + (b_k-a_k)^2\int_0^1\delta_{B_k}(t)\,dt\right)$$
$$= \tfrac{1}{2} + \sum_{k=1}^{n}\left(-\tfrac{1}{2}(b_k-a_k)^2 + (b_k-a_k)^2\cdot\frac{\phi(B_k)+2}{6}\right)$$
$$= \tfrac{1}{2} + \sum_{k=1}^{n}(b_k-a_k)^2\cdot\frac{\phi(B_k)-1}{6},$$

which implies

$$\phi(B) = 3 + \sum_{k=1}^{n}(b_k-a_k)^2(\phi(B_k)-1) - 2 = 1 - \sum_{k=1}^{n}(b_k-a_k)^2(1-\phi(B_k))$$

and finishes the proof. □

Note that in the formulas for Spearman's rho, Kendall's tau, and Spearman's footrule of an ordinal sum the position of the subintervals (a_k, b_k) is not important, only their size $b_k - a_k$ matters. For Gini's gamma this is no longer true. In order to give the result for Gini's gamma, we first prove a simple lemma.

Lemma 1. *For any copula $C \in \mathcal{C}$ we have*

$$\gamma(C) = \tfrac{2}{3}(\phi(C) - \phi(C^{\sigma_2})).$$

Proof. For any copula C we have

$$\int_0^1 \delta_{C^{\sigma_2}}(u)\,du = \int_0^1 C^{\sigma_2}(u,u)\,du = \frac{1}{2} - \int_0^1 C(u,1-u)\,du = \frac{1}{2} - \int_0^1 \omega_C(u)\,du,$$

hence

$$\begin{aligned}\gamma(C) &= 4\int_0^1 \delta_C(u)\,du + 4\left(\frac{1}{2} - \int_0^1 \delta_{C^{\sigma_2}}(u)\,du\right) - 2\\ &= 4\cdot\frac{\phi(C)+2}{6} - 4\cdot\frac{\phi(C^{\sigma_2})+2}{6}\\ &= \tfrac{2}{3}(\phi(C) - \phi(C^{\sigma_2})),\end{aligned}$$

which finishes the proof. □

Proposition 5. *Let B be an ordinal sum given in (13). If $\frac{1}{2}$ does not belong to any of the subintervals (a_k, b_k), then*

$$\gamma(B) = 1 - \tfrac{2}{3}\sum_{k=1}^{n}(b_k - a_k)^2(1 - \phi(B_k)). \tag{17}$$

If $\frac{1}{2} \in (a_j, b_j)$ for some $j \in \{1, 2, \dots, n\}$ then

$$\gamma(B) = 4(b_j - a_j)^2 \max\{0,c\}^2 + 4a_j - 4a_j^2 - \tfrac{2}{3}\sum_{k=1}^{n}(b_k - a_k)^2(1 - \phi(B_k)) + 4(b_j - a_j)^2 \int_{\max\{0,c\}}^{1+\min\{0,c\}} B_j\big(t, 1 + c - t\big)\,dt,$$

where $c = \frac{1-a_j-b_j}{b_j-a_j} \in (-1, 1)$. In particular, if $a_j + b_j = 1$ for some $j \in \{1, 2, \dots, n\}$ (i.e., $\frac{1}{2}$ lies at the middle of the subinterval (a_j, b_j)), then

$$\gamma(B) = 1 - (b_j - a_j)^2(1 - \gamma(B_j)) - \tfrac{2}{3}\sum_{\substack{k=1\\k\neq j}}^{n}(b_k - a_k)^2(1 - \phi(B_k)). \tag{18}$$

Proof. If $\frac{1}{2} \notin \cup_{k=1}^{n}(a_k, b_k)$ then $B(\frac{1}{2}, \frac{1}{2}) = \frac{1}{2}$ and $\omega_B = \omega_M$. (Note that this also covers the case when $a_j = \frac{1}{2}$ or $b_j = \frac{1}{2}$ for some j.) It follows that $\delta_{B^{\sigma_2}} = \delta_{M^{\sigma_2}} = \delta_W$ and $\phi(B^{\sigma_2}) = \phi(W) = -\frac{1}{2}$. By Lemma 1 and Proposition 4 we have

$$\begin{aligned}\gamma(B) &= \tfrac{2}{3}\phi(B) - \tfrac{2}{3}\phi(B^{\sigma_2}) = \tfrac{2}{3} - \tfrac{2}{3}\sum_{k=1}^{n}(b_k - a_k)^2(1 - \phi(B_k)) + \tfrac{1}{3}\\ &= 1 - \tfrac{2}{3}\sum_{k=1}^{n}(b_k - a_k)^2(1 - \phi(B_k)).\end{aligned}$$

Suppose now that $\frac{1}{2} \in (a_j, b_j)$. We consider two cases. Suppose first that $a_j \geq 1 - b_j$. Then

$$\omega_B(u) = \begin{cases} u; & 0 \leq u \leq a_j,\\ a_j + (b_j - a_j)B_j\big(\frac{u-a_j}{b_j-a_j}, \frac{1-u-a_j}{b_j-a_j}\big); & a_j < u \leq 1 - a_j,\\ 1 - u; & 1 - a_j < u \leq 1,\end{cases}$$

and thus

$$\delta_{B^{\sigma_2}}(u) = \begin{cases} 0; & 0 \leq u \leq a_j,\\ u - a_j - (b_j - a_j)B_j\big(\frac{u-a_j}{b_j-a_j}, \frac{1-u-a_j}{b_j-a_j}\big); & a_j < u \leq 1 - a_j,\\ 2u - 1; & 1 - a_j < u \leq 1.\end{cases}$$

It follows that

$$\begin{aligned}\phi(B^{\sigma_2}) &= 6\int_0^1 \delta_{B^{\sigma_2}}(u)\,du - 2\\ &= -2 + 6\int_{a_j}^{1-a_j} \big(u - a_j - (b_j - a_j)B_j\big(\tfrac{u-a_j}{b_j-a_j}, \tfrac{1-u-a_j}{b_j-a_j}\big)\big)\,du + 6\int_{1-a_j}^{1}(2u - 1)\,du\end{aligned}$$

$$= 6a_j^2 - 6a_j + 1 - 6(b_j - a_j) \int_{a_j}^{1-a_j} B_j\left(\tfrac{u-a_j}{b_j-a_j}, \tfrac{1-u-a_j}{b_j-a_j}\right) du$$
$$= 6a_j^2 - 6a_j + 1 - 6(b_j - a_j)^2 \int_0^{\frac{1-2a_j}{b_j-a_j}} B_j\left(t, \tfrac{1-2a_j}{b_j-a_j} - t\right) dt.$$

Suppose now that $a_j < 1 - b_j$. In this case

$$\delta_{B^{\sigma_2}}(u) = \begin{cases} 0; & 0 \le u \le 1 - b_j, \\ u - a_j - (b_j - a_j) B_j\left(\tfrac{u-a_j}{b_j-a_j}, \tfrac{1-u-a_j}{b_k-a_j}\right); & 1 - b_j < u \le b_j, \\ 2u - 1; & b_j < u \le 1, \end{cases}$$

and a similar calculation as above gives

$$\phi(B^{\sigma_2}) = 12b_j - 6b_j^2 - 12a_j b_j + 6a_j - 5 - 6(b_j - a_j)^2 \int_{\frac{1-a_j-b_j}{b_j-a_j}}^1 B_j\left(t, \tfrac{1-2a_j}{b_j-a_j} - t\right) dt$$
$$= 6a_j^2 - 6a_j + 1 - 6(1 - a_j - b_j)^2 - 6(b_j - a_j)^2 \int_{\frac{1-a_j-b_j}{b_j-a_j}}^1 B_j\left(t, \tfrac{1-2a_j}{b_j-a_j} - t\right) dt.$$

Putting both cases together we obtain

$$\phi(B^{\sigma_2}) = 6a_j^2 - 6a_j + 1 - 6\max\{0, 1 - a_j - b_j\}^2 - 6(b_j - a_j)^2 \int_{\max\left\{0, \frac{1-a_j-b_j}{b_j-a_j}\right\}}^{\min\left\{1, \frac{1-2a_j}{b_j-a_j}\right\}} B_j\left(t, \tfrac{1-2a_j}{b_j-a_j} - t\right) dt$$
$$= 6a_j^2 - 6a_j + 1 - 6(b_j - a_j)^2 \max\{0, c\}^2 - 6(b_j - a_j)^2 \int_{\max\{0,c\}}^{1+\min\{0,c\}} B_j\left(t, 1 + c - t\right) dt,$$

where $c = \frac{1-a_j-b_j}{b_j-a_j} \in (-1, 1)$. By Lemma 1 and Proposition 4 we have

$$\gamma(B) = \tfrac{2}{3}\phi(B) - \tfrac{2}{3}\phi(B^{\sigma_2})$$
$$= \tfrac{2}{3} - \tfrac{2}{3}\sum_{k=1}^n (b_k - a_k)^2 (1 - \phi(B_k)) - 4a_j^2 + 4a_j - \tfrac{2}{3} + 4(b_j - a_j)^2 \max\{0, c\}^2 + 4(b_j - a_j)^2 \int_{\max\{0,c\}}^{1+\min\{0,c\}} B_j\left(t, 1 + c - t\right) dt$$
$$= 4(b_j - a_j)^2 \max\{0, c\}^2 + 4a_j - 4a_j^2 - \tfrac{2}{3}\sum_{k=1}^n (b_k - a_k)^2 (1 - \phi(B_k)) + 4(b_j - a_j)^2 \int_{\max\{0,c\}}^{1+\min\{0,c\}} B_j\left(t, 1 + c - t\right) dt.$$

In particular, if $a_j + b_j = 1$ this simplifies into

$$\gamma(B) = 4a_j - 4a_j^2 - \tfrac{2}{3}\sum_{k=1}^n (b_k - a_k)^2 (1 - \phi(B_k)) + 4(b_j - a_j)^2 \int_0^1 B_j(t, 1 - t)\, dt.$$

Since

$$\int_0^1 B_j(t, 1 - t)\, dt = \int_0^1 \omega_{B_j}(t)\, dt = \int_0^1 (t - \delta_{B_j^{\sigma_2}}(t))\, dt = \tfrac{1}{2} - \int_0^1 \delta_{B_j^{\sigma_2}}(t)\, dt$$
$$= \tfrac{1}{2} - \tfrac{1}{6}(\phi(B_j^{\sigma_2}) + 2) = \tfrac{1}{6}(1 - \phi(B_j^{\sigma_2})),$$

we obtain

$$\gamma(B) = 4a_j - 4a_j^2 - \tfrac{2}{3}\sum_{k=1}^n (b_k - a_k)^2 (1 - \phi(B_k)) + \tfrac{2}{3}(b_j - a_j)^2 (1 - \phi(B_j^{\sigma_2}))$$
$$= 4a_j - 4a_j^2 + \tfrac{2}{3}(b_j - a_j)^2 (\phi(B_j) - \phi(B_j^{\sigma_2})) - \tfrac{2}{3}\sum_{\substack{k=1\\k\neq j}}^n (b_k - a_k)^2 (1 - \phi(B_k))$$
$$= 4a_j - 4a_j^2 + (b_j - a_j)^2 \gamma(B_j) - \tfrac{2}{3}\sum_{\substack{k=1\\k\neq j}}^n (b_k - a_k)^2 (1 - \phi(B_k))$$
$$= 1 - (b_j - a_j)^2 (1 - \gamma(B_j)) - \tfrac{2}{3}\sum_{\substack{k=1\\k\neq j}}^n (b_k - a_k)^2 (1 - \phi(B_k)),$$

which finishes the proof. □

Next, we consider Blomqvist's beta of an ordinal sum. The proof is straightforward, so we omit it.

Proposition 6. *Let B be an ordinal sum given in (13). If $\frac{1}{2}$ does not belong to any of the subintervals (a_k, b_k), then $\beta(B) = 1$. If $\frac{1}{2} \in (a_j, b_j)$ for some $j \in \{1, 2, \dots, n\}$ then*

$$\beta(B) = 4a_j + 4(b_j - a_j)\delta_{B_j}\left(\frac{\frac{1}{2}-a_j}{b_j-a_j}\right) - 1.$$

In particular, if $a_j + b_j = 1$ for some $j \in \{1, 2, \dots, n\}$ (i.e., $\frac{1}{2}$ lies in the middle of the subinterval (a_j, b_j)), then

$$\beta(B) = 1 - (b_j - a_j)(1 - \beta(B_j)).$$

Finally, we consider Chatterjee's xi of an ordinal sum of copulas.

Proposition 7. *Chatterjee's xi of an ordinal sum B, given in (13), is*

$$\xi(B) = 1 - \sum_{k=1}^{n}(b_k - a_k)^2(1 - \xi(B_k)). \tag{19}$$

Proof. Note that for almost all $(u, v) \in \mathbb{I}^2$ the partial derivative $\partial_1 B$ of copula B with respect to the first variable is equal to

$$\partial_1 B(u, v) = \begin{cases} \partial_1 B_k(\frac{u-a_k}{b_k-a_k}, \frac{v-a_k}{b_k-a_k}); & (u, v) \in (a_k, b_k)^2, k = 1, 2, \dots, n, \\ 1; & v > u, (u, v) \notin \cup_{k=1}^n [a_k, b_k]^2, \\ 0; & v < u, (u, v) \notin \cup_{k=1}^n [a_k, b_k]^2, \end{cases}$$

so that

$$\begin{aligned} \int_{\mathbb{I}^2}(\partial_1 B(u, v))^2\, du\, dv &= \sum_{k=1}^{n}\int_{[a_k,b_k]^2}(\partial_1 B_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}))^2\, du\, dv + \int_{\{(u,v)\in\mathbb{I}^2|v\geq u\}\setminus\cup_{k=1}^n[a_k,b_k]^2} du\, dv \\ &= \sum_{k=1}^{n}\int_{[a_k,b_k]^2}(\partial_1 B_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}))^2\, du\, dv + \tfrac{1}{2} - \sum_{k=1}^{n}\tfrac{1}{2}(b_k - a_k)^2. \end{aligned}$$

Since

$$\int_{[a_k,b_k]^2}(\partial_1 B_k(\tfrac{u-a_k}{b_k-a_k}, \tfrac{v-a_k}{b_k-a_k}))^2\, du\, dv = (b_k - a_k)^2\int_{\mathbb{I}^2}(\partial_1 B_k(s, t))^2\, ds\, dt = (b_k - a_k)^2 \cdot \frac{\xi(B_k) + 2}{6},$$

we obtain

$$\begin{aligned} \xi(B) &= 6\int_{\mathbb{I}^2}(\partial_1 B(u, v))^2\, du\, dv - 2 \\ &= \sum_{k=1}^{n}(b_k - a_k)^2(\xi(B_k) + 2) + 3 - \sum_{k=1}^{n}3(b_k - a_k)^2 - 2 \\ &= 1 - \sum_{k=1}^{n}(b_k - a_k)^2(1 - \xi(B_k)), \end{aligned}$$

which finishes the proof. □

We finish this section by looking at a special simple ordinal sum, containing just one copula in the middle of the unit square, which will be needed later.

Lemma 2. *For any $a \in [0, \frac{1}{2}]$ and copula $C \in \mathcal{C}$ let $B_{a,C}$ be an ordinal sum of $\{C\}$ with respect to the interval $\{(a, 1 - a)\}$. Then*

$$\begin{aligned} \rho(B_{a,C}) &= (1 - 2a)^3\rho(C) + 8a^3 - 12a^2 + 6a, \\ \tau(B_{a,C}) &= (1 - 2a)^2\tau(C) + 4a - 4a^2, \\ \phi(B_{a,C}) &= (1 - 2a)^2\phi(C) + 4a - 4a^2, \\ \gamma(B_{a,C}) &= (1 - 2a)^2\gamma(C) + 4a - 4a^2, \\ \beta(B_{a,C}) &= (1 - 2a)\beta(C) + 2a, \qquad \textit{and} \\ \xi(B_{a,C}) &= (1 - 2a)^2\xi(C) + 4a - 4a^2. \end{aligned}$$

Proof. If $a = \frac{1}{2}$ then $B_{a,C} = M$ and the formulas hold. So, we may assume without loss of generality that $a < \frac{1}{2}$. By Propositions 2–7, respectively, we have

$$\begin{aligned} \rho(B_{a,C}) &= 1 - (1 - 2a)^3(1 - \rho(C)) = (1 - 2a)^3\rho(C) + 8a^3 - 12a^2 + 6a, \\ \tau(B_{a,C}) &= 1 - (1 - 2a)^2(1 - \tau(C)) = (1 - 2a)^2\tau(C) + 4a - 4a^2, \\ \phi(B_{a,C}) &= 1 - (1 - 2a)^2(1 - \phi(C)) = (1 - 2a)^2\phi(C) + 4a - 4a^2, \\ \gamma(B_{a,C}) &= 1 - (1 - 2a)^2(1 - \gamma(C)) = (1 - 2a)^2\gamma(C) + 4a - 4a^2, \\ \beta(B_{a,C}) &= 1 - (1 - 2a)(1 - \beta(C)) = (1 - 2a)\beta(C) + 2a, \qquad \text{and} \\ \xi(B_{a,C}) &= 1 - (1 - 2a)^2(1 - \xi(C)) = (1 - 2a)^2\xi(C) + 4a - 4a^2, \end{aligned}$$

respectively, which finishes the proof. □

4. Exact region phi-gamma-tau

In this section we find the exact region determined by Spearman's footrule, Gini's gamma and Kendall's tau

$$\Omega_{\phi,\gamma,\tau} = \left\{(\phi(C), \gamma(C), \tau(C)) \in [-\tfrac{1}{2}, 1] \times [-1,1] \times [-1,1] \mid C \in \mathcal{C}\right\}.$$

The first lemma gives the bounds for Kendall's tau of a copula, if its Spearman's footrule and Gini's gamma are known.

Lemma 3. *For any copula $C \in \mathcal{C}$ we have*

$$\max\left\{\tfrac{4}{3}\phi(C) - \tfrac{1}{3},\ -\tfrac{2}{3}\phi(C) + \gamma(C) - \tfrac{1}{3}\right\} \le \tau(C) \le \min\left\{\tfrac{2}{3}\phi(C) + \tfrac{1}{3},\ -\tfrac{4}{3}\phi(C) + 2\gamma(C) + \tfrac{1}{3}\right\}.$$

Proof. By (11) we need to prove only

$$-\tfrac{2}{3}\phi(C) + \gamma(C) - \tfrac{1}{3} \le \tau(C) \le -\tfrac{4}{3}\phi(C) + 2\gamma(C) + \tfrac{1}{3}.$$

For any copula $C \in \mathcal{C}$ it follows from Lemma 1 that

$$\phi(C^{\sigma_2}) = \phi(C) - \tfrac{3}{2}\gamma(C). \tag{20}$$

Using (11) again, we infer

$$\tau(C) = -\tau(C^{\sigma_2}) \le -(\tfrac{4}{3}\phi(C^{\sigma_2}) - \tfrac{1}{3}) = -\tfrac{4}{3}(\phi(C) - \tfrac{3}{2}\gamma(C)) + \tfrac{1}{3} = -\tfrac{4}{3}\phi(C) + 2\gamma(C) + \tfrac{1}{3},$$

and

$$\tau(C) = -\tau(C^{\sigma_2}) \ge -(\tfrac{2}{3}\phi(C^{\sigma_2}) + \tfrac{1}{3}) = -\tfrac{2}{3}(\phi(C) - \tfrac{3}{2}\gamma(C)) - \tfrac{1}{3} = -\tfrac{2}{3}\phi(C) + \gamma(C) - \tfrac{1}{3},$$

which finishes the proof. □

We define a polyhedron Ω in $\mathbb{R}^3$ by

$$\begin{aligned} \Omega = \{(\phi,\gamma,\tau) \in \mathbb{R}^3 \mid -2\phi &\le 1, \\ -8\phi + 6\gamma &\le 1, \\ -2\phi + 3\gamma &\le 1, \\ 4\phi - 3\tau &\le 1, \\ -2\phi + 3\tau &\le 1, \\ -2\phi + 3\gamma - 3\tau &\le 1, \\ 4\phi - 6\gamma + 3\tau &\le 1\}. \end{aligned} \tag{21}$$

It follows from the fact that $\phi(C) \ge -\frac{1}{2}$, Eq. (10) and Lemma 3 that for any copula $C \in \mathcal{C}$ the point $(\phi(C), \gamma(C), \tau(C))$ lies in the polyhedron Ω. Note that the polyhedron Ω has 6 vertices

$$P_1(-\tfrac{1}{2}, -1, -1),\ P_2(-\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}),\ P_3(-\tfrac{1}{2}, -\tfrac{1}{2}, 0),\ P_4(1,1,1),\ P_5(\tfrac{1}{4}, \tfrac{1}{2}, 0),\ P_6(\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}),$$

7 faces

$$\begin{aligned} F_1 &:= P_1P_2P_3 = \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi = 1\}, \\ F_2 &:= P_1P_2P_5 = \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\gamma - 3\tau = 1\}, \\ F_3 &:= P_1P_3P_4 = \{(\phi,\gamma,\tau) \in \Omega \mid 4\phi - 6\gamma + 3\tau = 1\}, \\ F_4 &:= P_1P_4P_5 = \{(\phi,\gamma,\tau) \in \Omega \mid 4\phi - 3\tau = 1\}, \\ F_5 &:= P_2P_3P_5P_6 = \{(\phi,\gamma,\tau) \in \Omega \mid -8\phi + 6\gamma = 1\}, \\ F_6 &:= P_3P_4P_6 = \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\tau = 1\}, \\ F_7 &:= P_4P_5P_6 = \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\gamma = 1\}, \end{aligned}$$

and 11 edges

$$\begin{aligned} P_1P_2 &= \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi = 1,\ -2\phi + 3\gamma - 3\tau = 1\}, \\ P_1P_3 &= \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi = 1,\ 4\phi - 6\gamma + 3\tau = 1\}, \\ P_1P_4 &= \{(\phi,\gamma,\tau) \in \Omega \mid 4\phi - 3\tau = 1,\ 4\phi - 6\gamma + 3\tau = 1\}, \\ P_1P_5 &= \{(\phi,\gamma,\tau) \in \Omega \mid 4\phi - 3\tau = 1,\ -2\phi + 3\gamma - 3\tau = 1\}, \\ P_2P_3 &= \{(\phi,\gamma,\tau) \in \Omega \mid \phi = -\tfrac{1}{2},\ \gamma = -\tfrac{1}{2}\}, \\ P_2P_5 &= \{(\phi,\gamma,\tau) \in \Omega \mid -8\phi + 6\gamma = 1,\ -2\phi + 3\gamma - 3\tau = 1\}, \\ P_3P_4 &= \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\tau = 1,\ 4\phi - 6\gamma + 3\tau = 1\}, \\ P_3P_6 &= \{(\phi,\gamma,\tau) \in \Omega \mid -8\phi + 6\gamma = 1,\ -2\phi + 3\tau = 1\}, \\ P_4P_5 &= \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\gamma = 1,\ 4\phi - 3\tau = 1\}, \end{aligned}$$

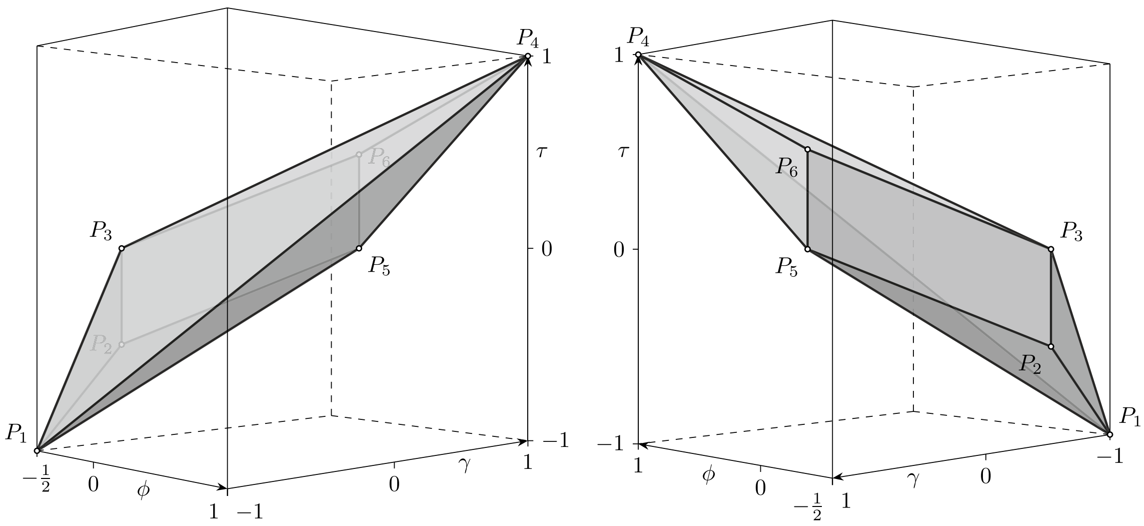

Fig. 1. The polyhedron Ω defined in (21). Theorem 1 shows that Ω is precisely the region determined by ϕ, γ and τ.

$$P_4P_6 = \{(\phi,\gamma,\tau) \in \Omega \mid -2\phi + 3\gamma = 1,\ -2\phi + 3\tau = 1\},$$
$$P_5P_6 = \{(\phi,\gamma,\tau) \in \Omega \mid \phi = \tfrac{1}{4},\ \gamma = \tfrac{1}{2}\}.$$

The polyhedron Ω is depicted in Fig. 1 from two distinct points of view.

Note also that the projection of the polyhedron Ω to the $\phi\gamma$-plane is the quadrilateral $P_1'(-\frac{1}{2},-1)P_4'(1,1)P_5'(\frac{1}{4},\frac{1}{2})P_2'(-\frac{1}{2},-\frac{1}{2})$, the exact region determined by Spearman's footrule and Gini's gamma (see [13]), the projection of the polyhedron Ω to the $\phi\tau$-plane is the triangle $P_1''(-\frac{1}{2},-1)P_4''(1,1)P_3''(-\frac{1}{2},0)$, the exact region determined by Spearman's footrule and Kendall's tau (see [16]), and the projection of the polyhedron Ω to the $\gamma\tau$-plane is the quadrilateral $P_1'''(-1,-1)P_3'''(-\frac{1}{2},0)P_4'''(1,1)P_5'''(\frac{1}{2},0)$, the exact region determined by Gini's gamma and Kendall's tau (see [16]).

Since γ and τ are concordance measures, we have

$$\gamma(C^{\sigma_2}) = -\gamma(C) \qquad \text{and} \qquad \tau(C^{\sigma_2}) = -\tau(C).$$

Furthermore, Lemma 1 implies $\phi(C^{\sigma_2}) = \phi(C) - \frac{3}{2}\gamma(C)$. Thus, the reflection mapping $C \mapsto C^{\sigma_2}$ induces a linear involution

$$\mathcal{A} : (\phi,\gamma,\tau) \mapsto (\phi - \tfrac{3}{2}\gamma, -\gamma, -\tau).$$

In particular, for any copula $C \in \mathcal{C}$ we have

$$\mathcal{A}(\phi(C),\gamma(C),\tau(C)) = (\phi(C^{\sigma_2}),\gamma(C^{\sigma_2}),\tau(C^{\sigma_2})). \tag{22}$$

The following lemma is easy to check.

Lemma 4. *The linear mapping $\mathcal{A}$ has the following properties*

$$\begin{aligned}
&\mathcal{A}(P_1) = P_4, && \mathcal{A}(F_1) = F_7,\\
&\mathcal{A}(P_2) = P_6, && \mathcal{A}(F_2) = F_6,\\
&\mathcal{A}(P_3) = P_5, && \mathcal{A}(F_3) = F_4,\\
&\mathcal{A}(P_4) = P_1, && \mathcal{A}(F_4) = F_3,\\
&\mathcal{A}(P_5) = P_3, && \mathcal{A}(F_5) = F_5,\\
&\mathcal{A}(P_6) = P_2, && \mathcal{A}(F_6) = F_2,\\
& && \mathcal{A}(F_7) = F_1.
\end{aligned}$$

Moreover, polyhedron Ω is invariant under the mapping $\mathcal{A}$.

In the following lemmas we show that points on the faces F_2, F_3, F_4, and F_6 are attained by some copulas.

Lemma 5. *For any point T on the face F_6 of the polyhedron Ω*

$$F_6 = P_3P_4P_6 = \left\{(\phi,\gamma,\tau) \in \mathbb{R}^3 \mid -\tfrac{1}{2} \le \phi \le 1,\ \phi \le \gamma \le \min\{\tfrac{4}{3}\phi + \tfrac{1}{6}, \tfrac{2}{3}\phi + \tfrac{1}{3}\},\ \tau = \tfrac{2}{3}\phi + \tfrac{1}{3}\right\}$$

there exists a copula $C \in \mathcal{C}$ such that $(\phi(C),\gamma(C),\tau(C)) = T$.

Proof. Let $b \in [0, \frac{1}{4}]$ and let C_b be a shuffle of M

$$C_b = M\big(6, (b, \tfrac{1}{2} - b, \tfrac{1}{2}, \tfrac{1}{2} + b, 1 - b), (3, 5, 1, 6, 2, 4), (1, 1, 1, 1, 1, 1)\big).$$

The copula C_b is defined by the measure preserving bijection

$$h_{C_b}(u) = \begin{cases} u + \frac{1}{2} - b; & 0 \le u \le b, \\ u + \frac{1}{2}; & b < u \le \frac{1}{2} - b, \\ u - \frac{1}{2} + b; & \frac{1}{2} - b < u \le \frac{1}{2}, \\ u + \frac{1}{2} - b; & \frac{1}{2} < u \le \frac{1}{2} + b, \\ u - \frac{1}{2}; & \frac{1}{2} + b < u \le 1 - b, \\ u - \frac{1}{2} + b; & 1 - b < u \le 1. \end{cases}$$

We have

$$C_b(u, h_{C_b}(u)) = \begin{cases} u; & 0 \le u \le \frac{1}{2} - b, \\ h_{C_b}(u); & \frac{1}{2} - b < u \le \frac{1}{2}, \\ u; & \frac{1}{2} < u \le \frac{1}{2} + b, \\ h_{C_b}(u); & \frac{1}{2} + b < u \le 1. \end{cases}$$

Using Eq. (7), we obtain

$$\tau(C_b) = 8b^2.$$

Furthermore, we have

$$\delta_{C_b}(u) = \begin{cases} 0; & 0 \le u \le \frac{1}{2} - b, \\ 2u - 1 + 2b; & \frac{1}{2} - b < u \le \frac{1}{2}, \\ 2b; & \frac{1}{2} < u \le \frac{1}{2} + b, \\ 2u - 1; & \frac{1}{2} + b < u \le 1, \end{cases}$$

and

$$\omega_{C_b}(u) = \begin{cases} u; & 0 \le u \le \frac{1}{4}, \\ \frac{1}{2} - u; & \frac{1}{4} < u \le \frac{1}{2} - b, \\ u + 2b - \frac{1}{2}; & \frac{1}{2} - b < u \le \frac{1}{2}, \\ 2b + \frac{1}{2} - u; & \frac{1}{2} < u \le \frac{1}{2} + b, \\ u - \frac{1}{2}; & \frac{1}{2} + b < u \le \frac{3}{4}, \\ 1 - u; & \frac{3}{4} < u \le 1, \end{cases}$$

Using Eqs. (5) and (8), we obtain

$$\phi(C_b) = 12b^2 - \tfrac{1}{2}, \quad \text{and} \quad \gamma(C_b) = 16b^2 - \tfrac{1}{2}.$$

The mass distribution of copula C_b and the graphs of functions δ_{C_b} and ω_{C_b} are shown in Fig. 2.

Now let $A_{a,b} = B_{a,C_b}$ be an ordinal sum of $\{C_b\}$ with respect to the interval $\{(a, 1 - a)\}$. By Lemma 2 we have

$$\tau(A_{a,b}) = 8(1 - 2a)^2 b^2 - 4a^2 + 4a,$$
$$\phi(A_{a,b}) = 12(1 - 2a)^2 b^2 - 6a^2 + 6a - \tfrac{1}{2}, \text{ and}$$
$$\gamma(A_{a,b}) = 16(1 - 2a)^2 b^2 - 6a^2 + 6a - \tfrac{1}{2}.$$

Note that

$$\tfrac{2}{3}\phi(A_{a,b}) + \tfrac{1}{3} = \tfrac{2}{3}\big(12(1 - 2a)^2 b^2 - 6a^2 + 6a - \tfrac{1}{2}\big) + \tfrac{1}{3} = 8(1 - 2a)^2 b^2 - 4a^2 + 4a = \tau(A_{a,b}).$$

Since the point $(\phi(A_{a,b}), \gamma(A_{a,b}), \tau(A_{a,b}))$ lies in the polyhedron Ω, this implies that it lies on the face F_6 for any $a \in [0, \frac{1}{2}]$ and $b \in [0, \frac{1}{4}]$. Furthermore, $A_{\frac{1}{2},b} = M$, so that the point $(\phi(A_{\frac{1}{2},b}), \gamma(A_{\frac{1}{2},b}), \tau(A_{\frac{1}{2},b}))$ is the point P_4 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$,

$$\tfrac{4}{3}\phi(A_{0,b}) + \tfrac{1}{6} = \tfrac{4}{3}\phi(C_b) + \tfrac{1}{6} = \tfrac{4}{3}(12b^2 - \tfrac{1}{2}) + \tfrac{1}{6} = 16b^2 - \tfrac{1}{2} = \gamma(C_b) = \gamma(A_{0,b}),$$

so that the point $(\phi(A_{0,b}), \gamma(A_{0,b}), \tau(A_{0,b}))$ lies on the edge P_3P_6 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$,

$$\phi(A_{a,0}) = -6a^2 + 6a - \tfrac{1}{2} = \gamma(A_{a,0}),$$

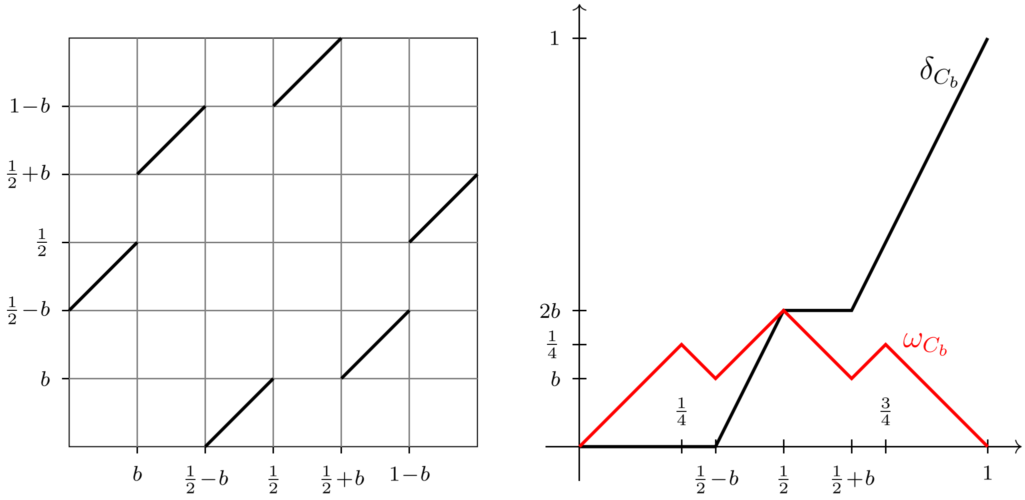

Fig. 2. The mass distribution of copula C_b (left) and the graphs of functions δ_{C_b} and ω_{C_b} (right) from the proof of Lemma 5.

so that the point $(\phi(A_{a,0}), \gamma(A_{a,0}), \tau(A_{a,0}))$ lies on the edge P_3P_4 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$, and

$$\tfrac{2}{3}\phi(A_{a,\frac{1}{4}}) + \tfrac{1}{3} = \tfrac{2}{3}(-3a^2 + 3a + \tfrac{1}{4}) + \tfrac{1}{3} = -2a^2 + 2a + \tfrac{1}{2} = \gamma(A_{a,\frac{1}{4}}),$$

so that the point $(\phi(A_{a,\frac{1}{4}}), \gamma(A_{a,\frac{1}{4}}), \tau(A_{a,\frac{1}{4}}))$ lies on the edge P_4P_6 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$. For any fixed $a \in [0, \frac{1}{2}]$ the set of points

$$\left\{(\phi(A_{a,b}), \gamma(A_{a,b}), \tau(A_{a,b})) \mid b \in [0, \tfrac{1}{4}]\right\}$$

is a curve lying on the face F_6 and connecting a point on the edge P_3P_4 and a point on the edge P_4P_6. By the above and by continuity, when $a = 0$ the curve covers the edge P_3P_6, and when $a = \frac{1}{2}$ the curve degenerates into vertex P_4. Furthermore, the set of the endpoints of all the curves covers the edges P_3P_4 and P_4P_6.

We claim that for any point T on the face F_6 there exist $a \in [0, \frac{1}{2}]$ and $b \in [0, \frac{1}{4}]$ such that $(\phi(A_{a,b}), \gamma(A_{a,b}), \tau(A_{a,b})) = T$. Assume this is not the case and take a point $T_0 \in F_6$ which is not in the image of the mapping $G\colon (a, b) \mapsto (\phi(A_{a,b}), \gamma(A_{a,b}), \tau(A_{a,b}))$. By the above, T_0 lies in the interior of F_6. Since $\phi(A_{a,b}), \gamma(A_{a,b}), \tau(A_{a,b})$ are continuous as functions of a, b, the mapping G induces a homotopy between the functions $f, g\colon [0, 1] \to F_6 \setminus \{T_0\}$, where $f(t) = P_4$ for all $t \in [0, 1]$ and g is a parametrization of the boundary of F_6 with $g(0) = g(1) = P_4$. This contradicts the fact that the fundamental group of the topological space $F_6 \setminus \{T_0\}$ is nontrivial.
□

Lemma 6. *For any point T on the face F_2 of the polyhedron Ω*

$$F_2 = P_1P_2P_5 = \left\{(\phi, \gamma, \tau) \in \mathbb{R}^3 \mid -\tfrac{1}{2} \le \phi \le \tfrac{1}{4},\ 2\phi \le \gamma \le \tfrac{4}{3}\phi + \tfrac{1}{6},\ \tau = -\tfrac{2}{3}\phi + \gamma - \tfrac{1}{3}\right\}$$

there exists a copula $C \in \mathcal{C}$ such that $(\phi(C), \gamma(C), \tau(C)) = T$.

Proof. Let $T(\phi, \gamma, \tau)$ be any point in F_2. Then the point $T'(\phi - \frac{3}{2}\gamma, -\gamma, -\tau) = \mathcal{A}(T)$ lies on the face F_6, by Lemma 4. By applying Lemma 5 to the point T', we can find a copula $C \in \mathcal{C}$ such that $(\phi(C), \gamma(C), \tau(C)) = T'$. By Eq. (22) we get $T = (\phi(C^{\sigma_2}), \gamma(C^{\sigma_2}), \tau(C^{\sigma_2}))$, which finishes the proof. □

Lemma 7. *For any point T on the face F_4 of the polyhedron Ω*

$$F_4 = P_1P_4P_5 = \left\{(\phi, \gamma, \tau) \in \mathbb{R}^3 \mid -\tfrac{1}{2} \le \phi \le 1,\ \tfrac{4}{3}\phi - \tfrac{1}{3} \le \gamma \le \min\{2\phi, \tfrac{2}{3}\phi + \tfrac{1}{3}\},\ \tau = \tfrac{4}{3}\phi - \tfrac{1}{3}\right\}$$

there exists a copula $C \in \mathcal{C}$ such that $(\phi(C), \gamma(C), \tau(C)) = T$.

Proof. Let $b \in [0, \frac{1}{2}]$ and let D_b be a shuffle of M

$$D_b = M\big(3, (b, 1-b), (1, 2, 3), (-1, 1, -1)\big).$$

The copula D_b is defined by the measure preserving bijection

$$h_{D_b}(u) = \begin{cases} b - u; & 0 \le u \le b, \\ u; & b < u \le 1 - b, \\ 2 - b - u; & 1 - b < u \le 1. \end{cases}$$

We have

$$D_b(u, h_{D_b}(u)) = \begin{cases} 0; & 0 \le u \le b, \\ u; & b < u \le 1-b, \\ 1-b; & 1-b < u \le 1. \end{cases}$$

Using Eq. (7), we obtain

$$\tau(D_b) = 1 - 4b^2.$$

Furthermore, we have

$$\delta_{D_b}(u) = \begin{cases} 0; & 0 \le u \le \frac{b}{2}, \\ 2u-b; & \frac{b}{2} < u \le b, \\ u; & b < u \le 1-b, \\ 1-b; & 1-b < u \le 1-\frac{b}{2}, \\ 2u-1; & 1-\frac{b}{2} < u \le 1, \end{cases}$$

and

$$\omega_{D_b}(u) = \begin{cases} u; & 0 \le u \le \frac{1}{2}, \\ 1-u; & \frac{1}{2} < u \le 1, \end{cases}$$

Using Eqs. (8) and (5), we obtain

$$\phi(D_b) = 1 - 3b^2, \quad \text{and} \quad \gamma(D_b) = 1 - 2b^2.$$

The mass distribution of copula D_b and the graphs of functions δ_{D_b} and ω_{D_b} are shown in Fig. 3.

Now let $E_{a,b} = B_{a,D_b^{\sigma_2}}$ be an ordinal sum of $\{D_b^{\sigma_2}\}$ with respect to the interval $\{(a, 1-a)\}$ and $F_{a,b} = E_{a,b}^{\sigma_2}$. Using Lemmas 1 and 2 we obtain

$$\tau(F_{a,b}) = (1-2a)^2\tau(D_b) + 4(a^2 - a) = -4(1-2a)^2b^2 + 8a^2 - 8a + 1,$$
$$\phi(F_{a,b}) = (1-2a)^2\phi(D_b) + 2(a^2 - a) = -3(1-2a)^2b^2 + 6a^2 - 6a + 1, \text{ and}$$
$$\gamma(F_{a,b}) = (1-2a)^2\gamma(D_b) + 4(a^2 - a) = -2(1-2a)^2b^2 + 8a^2 - 8a + 1.$$

Note that

$$\tfrac{4}{3}\phi(F_{a,b}) - \tfrac{1}{3} = \tfrac{4}{3}(-3(1-2a)^2b^2 + 6a^2 - 6a + 1) - \tfrac{1}{3} = -4(1-2a)^2b^2 + 8a^2 - 8a + 1 = \tau(F_{a,b}).$$

Since the point $(\phi(F_{a,b}), \gamma(F_{a,b}), \tau(F_{a,b}))$ lies in the polyhedron Ω, this implies that it lies on the face F_4 for any $a, b \in [0, \frac{1}{2}]$. Furthermore, $F_{\frac{1}{2},b} = W$, so that the point $(\phi(F_{\frac{1}{2},b}), \gamma(F_{\frac{1}{2},b}), \tau(F_{\frac{1}{2},b}))$ is the point P_1 of the polyhedron Ω for any $b \in [0, \frac{1}{2}]$,

$$\tfrac{2}{3}\phi(F_{0,b}) + \tfrac{1}{3} = \tfrac{2}{3}\phi(D_b) + \tfrac{1}{3} = \tfrac{2}{3}(1 - 3b^2) + \tfrac{1}{3} = 1 - 2b^2 = \gamma(D_b) = \gamma(F_{0,b}),$$

so that the point $(\phi(F_{0,b}), \gamma(F_{0,b}), \tau(F_{0,b}))$ lies on the edge P_4P_5 of the polyhedron Ω for any $b \in [0, \frac{1}{2}]$,

$$\tfrac{4}{3}\phi(F_{a,0}) - \tfrac{1}{3} = \tfrac{4}{3}(6a^2 - 6a + 1) - \tfrac{1}{3} = 8a^2 - 8a + 1 = \gamma(F_{a,0}),$$

so that the point $(\phi(F_{a,0}), \gamma(F_{a,0}), \tau(F_{a,0}))$ lies on the edge P_1P_4 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$, and

$$2\phi(F_{a,\frac{1}{2}}) = 6a^2 - 6a + \tfrac{1}{2} = \gamma(F_{a,\frac{1}{2}}),$$

so that the point $(\phi(F_{a,\frac{1}{2}}), \gamma(F_{a,\frac{1}{2}}), \tau(F_{a,\frac{1}{2}}))$ lies on the edge P_1P_5 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$. For any fixed $a \in [0, \frac{1}{2}]$ the set of points

$$\left\{(\phi(F_{a,b}), \gamma(F_{a,b}), \tau(F_{a,b})) \mid b \in [0, \tfrac{1}{2}]\right\}$$

is a curve lying on the face F_4 and connecting a point on the edge P_1P_4 and a point on the edge P_1P_5. By the above and by continuity, when $a = 0$ the curve covers the edge P_4P_5, and when $a = \frac{1}{2}$ the curve degenerates into vertex P_1. Furthermore, the set of the endpoints of all the curves covers the edges P_1P_4 and P_1P_5. A similar argument as in the proof of Lemma 5, shows that for any point T on the face F_4 there exist $a, b \in [0, \frac{1}{2}]$ such that $(\phi(F_{a,b}), \gamma(F_{a,b}), \tau(F_{a,b})) = T$. $\square$

Lemma 8. *For any point T on the face F_3 of the polyhedron Ω*

$$F_3 = P_1P_3P_4 = \left\{(\phi, \gamma, \tau) \in \mathbb{R}^3 \mid -\tfrac{1}{2} \le \phi \le 1,\ \tfrac{4}{3}\phi - \tfrac{1}{3} \le \gamma \le \phi,\ \tau = -\tfrac{4}{3}\phi + 2\gamma + \tfrac{1}{3}\right\}$$

there exists a copula $C \in \mathcal{C}$ such that $(\phi(C), \gamma(C), \tau(C)) = T$.

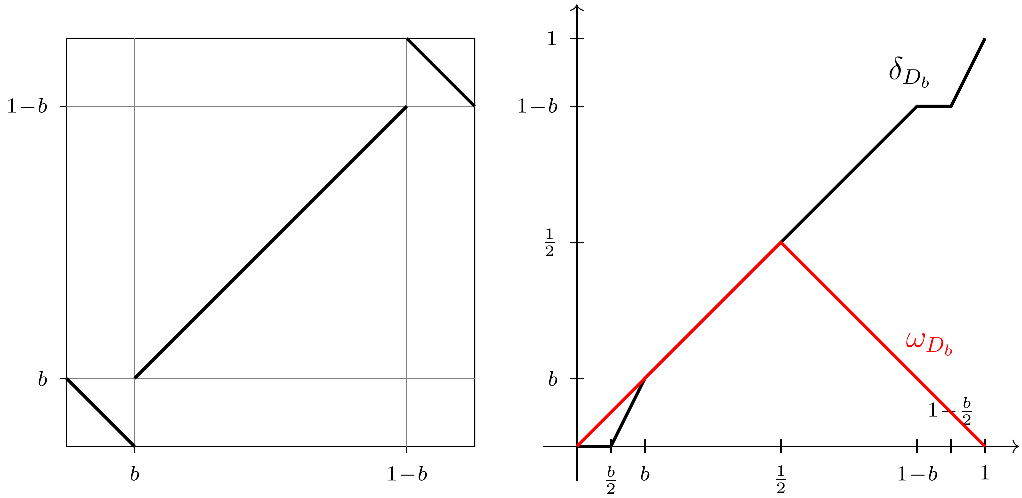

Fig. 3. The mass distribution of copula D_b (left) and the graphs of functions δ_{D_b} and ω_{D_b} (right) from the proof of Lemma 7.

Proof. By Lemma 4, linear mapping $\mathcal{A}$ maps F_3 to F_4, so for any point $T(\phi,\gamma,\tau)$ in F_3 the point $T'(\phi-\frac{3}{2}\gamma,-\gamma,-\tau)=\mathcal{A}(T)$ lies on the face F_4. By Lemma 7, there exists a copula $C\in\mathcal{C}$ such that $(\phi(C),\gamma(C),\tau(C))=T'$. Thus, by Eq. (22) we have $(\phi(C^{\sigma_2}),\gamma(C^{\sigma_2}),\tau(C^{\sigma_2}))=T$. $\square$

We are now ready to prove our main theorem.

Theorem 1. *The exact region determined by Spearman's footrule, Gini's gamma and Kendall's tau*

$$\Omega_{\phi,\gamma,\tau}=\left\{(\phi(C),\gamma(C),\tau(C))\in[-\tfrac{1}{2},1]\times[-1,1]\times[-1,1]\mid C\in\mathcal{C}\right\}$$

is the polyhedron Ω, defined in (21).

Proof. If follows from Eq. (10) and Lemma 3 that $\Omega_{\phi,\gamma,\tau}\subseteq\Omega$. We need to prove that for any point $T\in\Omega$ there exists a copula $C\in\mathcal{C}$ such that $(\phi(C),\gamma(C),\tau(C))=T$. Let $T(a,b,c)\in\Omega$. By Lemma 3 and definition of Ω, the vertical line $\{(a,b,\tau)\mid\tau\in\mathbb{R}\}$ intersects polyhedron Ω in exactly one line segment AB with endpoints determined by inequalities from Lemma 3. Here, the point $A(a,b,\tau_0)$ lies in the union of faces F_2 and F_4, and the point $B(a,b,\tau_1)$ lies in the union of faces F_3 and F_6. By Lemmas 5–8, there exist copulas C_0 and C_1 such that $(\phi(C_0),\gamma(C_0),\tau(C_0))=A(a,b,\tau_0)$ and $(\phi(C_1),\gamma(C_1),\tau(C_1))=B(a,b,\tau_1)$. Now for any $t\in\mathbb{I}$ let C_t be a convex combination of copulas C_0 and C_1, i.e., $C_t=tC_1+(1-t)C_0$. We have $\phi(C_t)=t\phi(C_1)+(1-t)\phi(C_0)=a$ and $\gamma(C_t)=t\gamma(C_1)+(1-t)\gamma(C_0)=b$. Furthermore, $\tau(C_0)=\tau_0\le c\le\tau_1=\tau(C_1)$ and $\tau(C_t)$ is continuous as a function of t, so there exists $t\in\mathbb{I}$ such that $\tau(C_t)=c$, hence $T=(\phi(C_t),\gamma(C_t),\tau(C_t))$. $\square$

Lemmas 5–8 show that any point in the union of faces F_2, F_3, F_4, and F_6 can be attained by a shuffle of M. The following examples demonstrate that also the points on the remaining faces F_1, F_5, and F_7 are attained by shuffles of M.

Example 1. Let $b\in[0,\frac{1}{4}]$ and let G_b be a shuffle of M

$$G_b=M\left(6,(b,\tfrac{1}{2}-b,\tfrac{1}{2},\tfrac{1}{2}+b,1-b),(3,2,1,6,5,4),(1,-1,1,1,-1,1)\right).$$

The copula G_b is defined by the measure preserving bijection

$$h_{G_b}(u)=\begin{cases}u+\frac{1}{2}-b; & 0\le u\le b,\\ \frac{1}{2}-u; & b<u\le\frac{1}{2}-b,\\ u-\frac{1}{2}+b; & \frac{1}{2}-b<u\le\frac{1}{2},\\ u+\frac{1}{2}-b; & \frac{1}{2}<u\le\frac{1}{2}+b,\\ \frac{3}{2}-u; & \frac{1}{2}+b<u\le 1-b,\\ u-\frac{1}{2}+b; & 1-b<u\le 1.\end{cases}$$

We have

$$G_b(u, h_{G_b}(u)) = \begin{cases} u; & 0 \le u \le b, \\ 0; & b < u \le \frac{1}{2} - b, \\ h_{G_b}(u); & \frac{1}{2} - b < u \le \frac{1}{2}, \\ u; & \frac{1}{2} < u \le \frac{1}{2} + b, \\ \frac{1}{2}; & \frac{1}{2} + b < u \le 1 - b, \\ h_{G_b}(u); & 1 - b < u \le 1. \end{cases}$$

Using Eq. (7), we obtain

$$\tau(G_b) = 8b^2.$$

Furthermore, we have

$$\delta_{G_b}(u) = \begin{cases} 0; & 0 \le u \le \frac{1}{4}, \\ 2u - \frac{1}{2}; & \frac{1}{4} < u \le \frac{1}{2}, \\ \frac{1}{2}; & \frac{1}{2} < u \le \frac{3}{4}, \\ 2u - 1; & \frac{3}{4} < u \le 1, \end{cases}$$

and

$$\omega_{G_b}(u) = \begin{cases} u; & 0 \le u \le \frac{1}{2}, \\ 1 - u; & \frac{1}{2} < u \le 1, \end{cases}$$

The mass distribution of copula G_b and the graphs of functions δ_{G_b} and ω_{G_b} are shown in Fig. 4. Using Eqs. (5) and (8), we obtain

$$\phi(G_b) = \tfrac{1}{4}, \quad \text{and} \quad \gamma(G_b) = \tfrac{1}{2}.$$

Now, for any $a \in [0, \frac{1}{2}]$ let $H_{a,b} = B_{a,G_b}$ be an ordinal sum of $\{G_b\}$ with respect to the interval $\{(a, 1-a)\}$. By Lemma 2 we have

$$\tau(H_{a,b}) = 8(1-2a)^2 b^2 - 4a^2 + 4a,$$
$$\phi(H_{a,b}) = -3a^2 + 3a + \tfrac{1}{4}, \text{ and}$$
$$\gamma(H_{a,b}) = -2a^2 + 2a + \tfrac{1}{2}.$$

Note that

$$\tfrac{2}{3}\phi(H_{a,b}) + \tfrac{1}{3} = \tfrac{2}{3}(-3a^2 + 3a + \tfrac{1}{4}) + \tfrac{1}{3} = -2a^2 + 2a + \tfrac{1}{2} = \gamma(H_{a,b}),$$

so the point $(\phi(H_{a,b}), \gamma(H_{a,b}), \tau(H_{a,b}))$ lies on the face F_7 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$ and $b \in [0, \frac{1}{4}]$. Furthermore, $H_{\frac{1}{2},b} = M$, so that the point $(\phi(H_{\frac{1}{2},b}), \gamma(H_{\frac{1}{2},b}), \tau(H_{\frac{1}{2},b}))$ is the point P_4 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$,

$$\phi(H_{0,b}) = \phi(G_b) = \tfrac{1}{4} \quad \text{and} \quad \gamma(H_{0,b}) = \gamma(G_b) = \tfrac{1}{2},$$

so that the point $(\phi(H_{0,b}), \gamma(H_{0,b}), \tau(H_{0,b}))$ lies on the edge P_5P_6 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$,

$$\tfrac{4}{3}\phi(H_{a,0}) - \tfrac{1}{3} = \tfrac{4}{3}(-3a^2 + 3a + \tfrac{1}{4}) - \tfrac{1}{3} = -4a^2 + 4a = \tau(H_{a,0}),$$

so that the point $(\phi(H_{a,0}), \gamma(H_{a,0}), \tau(H_{a,0}))$ lies on the edge P_4P_5 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$, and

$$\tfrac{2}{3}\phi(H_{a,\frac{1}{4}}) + \tfrac{1}{3} = \tfrac{2}{3}(-3a^2 + 3a + \tfrac{1}{4}) + \tfrac{1}{3} = -2a^2 + 2a + \tfrac{1}{2} = \tfrac{1}{2}(1-2a)^2 - 4a^2 + 4a = \tau(H_{a,\frac{1}{4}}),$$

so that the point $(\phi(H_{a,\frac{1}{4}}), \gamma(H_{a,\frac{1}{4}}), \tau(H_{a,\frac{1}{4}}))$ lies on the edge P_4P_6 of the polyhedron Ω for any $a \in [0, \frac{1}{2}]$. Similarly as in the proof of Lemma 5 we show that for any point T on the face F_7 there exist $a \in [0, \frac{1}{2}]$ and $b \in [0, \frac{1}{4}]$ such that $(\phi(H_{a,b}), \gamma(H_{a,b}), \tau(H_{a,b})) = T$.

Example 2. Consider the copula $K_{a,b} = H_{a,b}^{\sigma_2}$, where $H_{a,b}$ is the copula from Example 1. Similarly as in the proof of Lemma 6, we can prove that the set of points

$$\left\{(\phi(K_{a,b}), \gamma(K_{a,b}), \tau(K_{a,b})) \mid (a,b) \in [0, \tfrac{1}{2}] \times [0, \tfrac{1}{4}]\right\}$$

is equal to the face $F_1 = P_1P_2P_3$ of polyhedron Ω.

Example 3. Let $a, b \in [0, \frac{1}{4}]$ with $a \le b$ and let $L_{a,b}$ be a shuffle of M

$$L_{a,b} = M\big(10, (a, b, \tfrac{1}{2} - b, \tfrac{1}{2} - b + a, \tfrac{1}{2}, \tfrac{1}{2} + b - a, \tfrac{1}{2} + b, 1 - b, 1 - a), (7, 5, 8, 10, 2, 9, 1, 3, 6, 4), (-1, 1, 1, -1, 1, 1, -1, 1, 1, -1)\big).$$

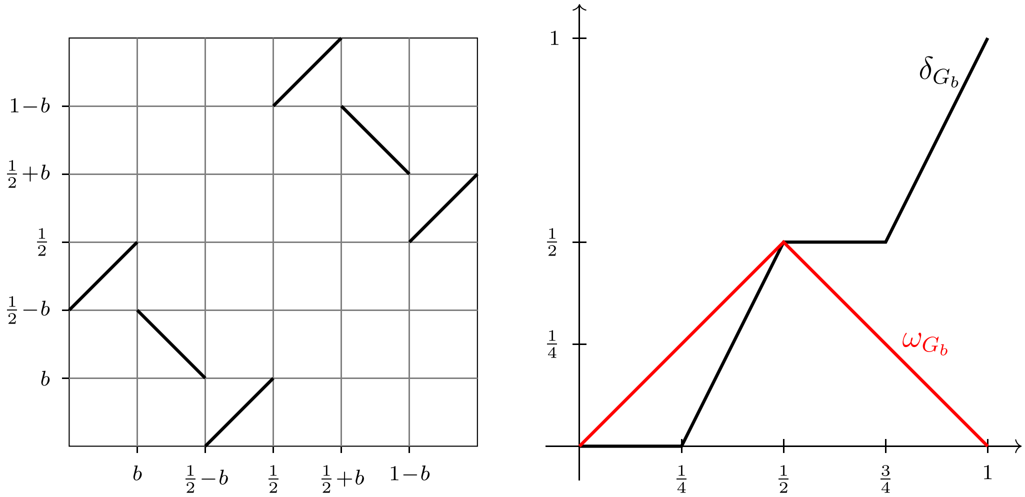

Fig. 4. The mass distribution of copula G_b (left) and the graphs of functions δ_{G_b} and ω_{G_b} (right) from Example 1.

The copula $L_{a,b}$ is defined by the measure preserving bijection

$$h_{L_{a,b}}(u)=\begin{cases}\frac{1}{2}+b-u; & 0\le u\le a,\\ u+\frac{1}{2}-b; & a<u\le b,\\ u+\frac{1}{2}; & b<u\le \frac{1}{2}-b,\\ \frac{3}{2}-b-u; & \frac{1}{2}-b<u\le \frac{1}{2}-b+a,\\ u-\frac{1}{2}+b; & \frac{1}{2}-b+a<u\le \frac{1}{2},\\ u+\frac{1}{2}-b; & \frac{1}{2}<u\le \frac{1}{2}+b-a,\\ \frac{1}{2}+b-u; & \frac{1}{2}+b-a<u\le \frac{1}{2}+b,\\ u-\frac{1}{2}; & \frac{1}{2}+b<u\le 1-b,\\ u-\frac{1}{2}+b; & 1-b<u\le 1-a,\\ \frac{3}{2}-b-u; & 1-a<u\le 1.\end{cases}$$

We have

$$L_{a,b}(u,h_{L_{a,b}}(u))=\begin{cases}0; & 0\le u\le a,\\ u-a; & a<u\le b,\\ u; & b<u\le \frac{1}{2}-b,\\ \frac{1}{2}-b; & \frac{1}{2}-b<u\le \frac{1}{2}-b+a,\\ u-\frac{1}{2}+b-a; & \frac{1}{2}-b+a<u\le \frac{1}{2},\\ u-a; & \frac{1}{2}<u\le \frac{1}{2}+b-a,\\ 0; & \frac{1}{2}+b-a<u\le \frac{1}{2}+b,\\ u-\frac{1}{2}; & \frac{1}{2}+b<u\le 1-b,\\ u-\frac{1}{2}+b-a; & 1-b<u\le 1-a,\\ \frac{1}{2}-b; & 1-a<u\le 1.\end{cases}$$

Using Eq. (7), we obtain

$$\tau(L_{a,b})=16(b-a)^2-8b^2.$$

Furthermore, we have

$$\delta_{L_{a,b}}(u)=\begin{cases}0; & 0\le u\le \frac{1}{2}-b+a,\\ 2u-1+2b-2a; & \frac{1}{2}-b+a<u\le \frac{1}{2},\\ 2b-2a; & \frac{1}{2}<u\le \frac{1}{2}+b-a,\\ 2u-1; & \frac{1}{2}+b-a<u\le 1,\end{cases}$$

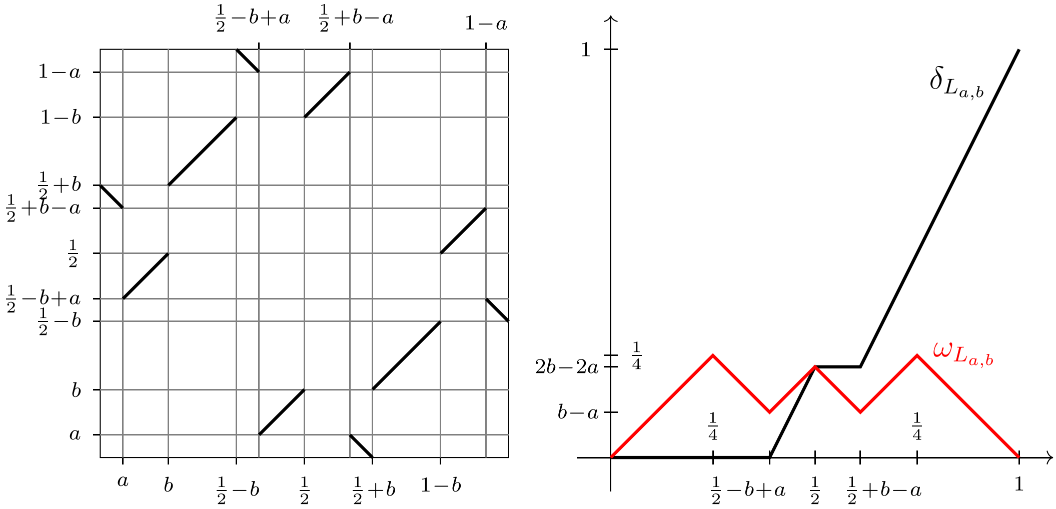

Fig. 5. The mass distribution of copula $L_{a,b}$ (left) and the graphs of functions $\delta_{L_{a,b}}$ and $\omega_{L_{a,b}}$ (right) from Example 3.

and

$$\omega_{L_{a,b}}(u) = \begin{cases} u; & 0 \le u \le \frac{1}{4}, \\ \frac{1}{2} - u; & \frac{1}{4} < u \le \frac{1}{2} - b + a, \\ u + 2b - 2a - \frac{1}{2}; & \frac{1}{2} - b + a < u \le \frac{1}{2}, \\ 2b - 2a + \frac{1}{2} - u; & \frac{1}{2} < u \le \frac{1}{2} + b - a, \\ u - \frac{1}{2}; & \frac{1}{2} + b - a < u \le \frac{3}{4}, \\ 1 - u; & \frac{3}{4} < u \le 1, \end{cases}$$

Using Eq. (5) and (8), we obtain

$$\phi(L_{a,b}) = 12(b-a)^2 - \tfrac{1}{2}, \quad \text{and} \quad \gamma(L_{a,b}) = 16(b-a)^2 - \tfrac{1}{2}.$$

The mass distribution of copula $L_{a,b}$ and the graphs of functions $\delta_{L_{a,b}}$ and $\omega_{L_{a,b}}$ are shown in Fig. 5.

Note that

$$\tfrac{4}{3}\phi(L_{a,b}) + \tfrac{1}{6} = \tfrac{4}{3}\left(12(b-a)^2 - \tfrac{1}{2}\right) + \tfrac{1}{6} = 16(b-a)^2 - \tfrac{1}{2} = \gamma(L_{a,b}),$$

so the point $(\phi(L_{a,b}), \gamma(L_{a,b}), \tau(L_{a,b}))$ lies on the face F_5 for any $a, b \in [0, \frac{1}{4}]$ with $a \le b$. Furthermore, $L_{0,b} = C_b = A_{0,b}$, so the point $(\phi(L_{0,b}), \gamma(L_{0,b}), \tau(L_{0,b}))$ lies on the edge P_3P_6 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$, $\phi(L_{b,b}) = \gamma(L_{b,b}) = -\frac{1}{2}$, so the point $(\phi(L_{b,b}), \gamma(L_{b,b}), \tau(L_{b,b}))$ lies on the edge P_2P_3 of the polyhedron Ω for any $b \in [0, \frac{1}{4}]$, and

$$\gamma(L_{a,\frac{1}{4}}) = 16a^2 - 8a + \tfrac{1}{2} = \tau(L_{a,\frac{1}{4}}),$$

so that the point $(\phi(L_{a,\frac{1}{4}}), \gamma(L_{a,\frac{1}{4}}), \tau(L_{a,\frac{1}{4}}))$ lies on the line segment P_2P_6. For any fixed $b \in [0, \frac{1}{4}]$ the set of points

$$\left\{(\phi(L_{a,b}), \gamma(L_{a,b}), \tau(L_{a,b})) \mid a \in [0, b]\right\}$$

is a curve lying on the face F_5 and connecting a point on the edge P_3P_6 and a point on the edge P_2P_3. By the above and by continuity, when $b = \frac{1}{4}$ the curve covers the line segment P_2P_6. Furthermore, the set of the endpoints of all the curves covers the edges P_2P_3 and P_3P_6. A similar argument as in the proof of Lemma 5, shows that for any point T on the triangle $P_2P_3P_6$ there exist $a, b \in [0, \frac{1}{4}]$ and with $a \le b$ such that $(\phi(L_{a,b}), \gamma(L_{a,b}), \tau(L_{a,b})) = T$.

Finally, let $M_{a,b} = L_{a,b}^{\sigma_2}$. By Lemma 4 the linear mapping $\mathcal{A}$ maps triangle $P_2P_3P_6$ onto triangle $P_6P_5P_2$. Hence, by Eq. (22), for any point T on the triangle $P_2P_5P_6$ there exist $a, b \in [0, \frac{1}{4}]$ with $a \le b$ such that $(\phi(M_{a,b}), \gamma(M_{a,b}), \tau(M_{a,b})) = T$.

5. Concluding remarks

In this paper, we described the exact region $\Omega_{\phi,\gamma,\tau}$ determined by Spearman's footrule, Gini's gamma and Kendall's tau. It is a polyhedron Ω depicted in Fig. 1, with 6 vertices, 11 edges and 7 faces. We proved that every point on the boundary of Ω can be realized using a shuffle of M.

The polyhedron $\Omega_{\phi,\gamma,\tau}$ has volume $\mathrm{vol}(\Omega_{\phi,\gamma,\tau}) = \frac{3}{16} = 0.1875$, which represents $\frac{1}{32} = 0.03125$ of the volume of the (ϕ,γ,τ)-box $[-\frac{1}{2},1]\times[-1,1]\times[-1,1]$. In [21], the authors computed the volume of $\Omega_{\beta,\phi,\gamma}$, i.e., $\mathrm{vol}(\Omega_{\beta,\phi,\gamma}) = \frac{19}{40} = 0.475$, which represents $\frac{19}{240} \approx 0.07916$ of the volume of the (β,ϕ,γ)-box $[-1,1]\times[-\frac{1}{2},1]\times[-1,1]$.

Proposition 1 shows that for any copula C with given value $\phi(C)$, the spread of $\tau(C)$, i.e., the difference between the maximal and the minimal possible value of $\tau(C)$, is at most 1. On average, the spread of $\tau(C)$ given $\phi(C)$ is

$$\frac{\mathrm{area}(\Omega_{\phi,\tau})}{\mathrm{length}([-\frac{1}{2},1])} = \frac{1}{2},$$

since $\mathrm{area}(\Omega_{\phi,\tau}) = \frac{3}{4}$, see [16]. Similarly, given the value of $\gamma(C)$, the spread of $\tau(C)$ is at most $\frac{2}{3}$. On average, the spread of $\tau(C)$ given $\gamma(C)$ is again

$$\frac{\mathrm{area}(\Omega_{\gamma,\tau})}{\mathrm{length}([-1,1])} = \frac{1}{2},$$

since $\mathrm{area}(\Omega_{\gamma,\tau}) = 1$, see [16]. If both $\phi(C)$ and $\gamma(C)$ are given, the maximal spread of $\tau(C)$ is still $\frac{2}{3}$. Indeed, if $\phi(C) = \gamma(C) = 0$, then $\tau(C) \in [-\frac{1}{3},\frac{1}{3}]$ by Lemma 3. The average spread of $\tau(C)$ given $\phi(C)$ and $\gamma(C)$ is

$$\frac{\mathrm{vol}(\Omega_{\phi,\gamma,\tau})}{\mathrm{area}(\Omega_{\phi,\gamma})} = \frac{1}{3},$$

since $\mathrm{area}(\Omega_{\phi,\gamma}) = \frac{9}{16}$, see [13]. Using results from [21], the average spread of $\beta(C)$ given $\phi(C)$ and $\gamma(C)$ is

$$\frac{\mathrm{vol}(\Omega_{\beta,\phi,\gamma})}{\mathrm{area}(\Omega_{\phi,\gamma})} = \frac{38}{45} \approx 0.8444.$$

We conclude that the value of $\tau(C)$ is more closely related to the values of $\phi(C)$ and $\gamma(C)$ than the value of $\beta(C)$ is.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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