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Domination of subcubic planar graphs with large girth*

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Abstract

Since Reed conjectured in 1996 that the domination number of a connected cubic graph of order n is at most $\lceil \frac{1}{3}n \rceil$, the domination number of cubic graphs has been extensively studied. It is now known that the conjecture is false in general, but Henning and Dorbec showed that it holds for graphs with girth at least 9. Zhu and Wu stated an analogous conjecture for 2-connected cubic planar graphs.

In this paper, we present a new upper bound for the domination number of subcubic planar graphs: if G is a subcubic planar graph with girth at least 8, then $\gamma(G) < n_0 + \frac{3}{4}n_1 + \frac{11}{20}n_2 + \frac{7}{20}n_3$, where n_i denotes the number of vertices in G of degree i, for $i \in \{0, 1, 2, 3\}$. We also prove that if G is a subcubic planar graph with girth at least 9, then $\gamma(G) < n_0 + \frac{13}{17}n_1 + \frac{9}{17}n_2 + \frac{6}{17}n_3$.

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1 Introduction

A *dominating set* of G is a set D of vertices of G such that every vertex of G is either in D or is adjacent to a vertex in D. The *domination number* of G is the size of the smallest dominating set of G, and is denoted by $\gamma(G)$.

For a vertex v in G, v dominates the vertices in its closed neighborhood $N_G[v]$, so the larger $\deg_G(v)$, the greater the number of vertices dominated by v. Thus, it is reasonable to infer that the greater $\delta(G)$, the smaller $\gamma(G)$. In 1962, Ore [22] proved that a graph G with $\delta(G) \geq 1$ satisfies $\gamma(G) \leq \frac{1}{2}|V(G)|$. This bound is tight; for example, the equality is attained by an infinite family of graphs obtained by attaching a leaf to each vertex of an arbitrary graph. In 1973, Blank [2] (and in 1989, independently McCuaig and Shepherd [20]) proved that a connected graph with $\delta(G) \geq 2$ and $|V(G)| \geq 8$ satisfies $\gamma(G) \leq \frac{2}{5}|V(G)|$. Furthermore, McCuaig and Shepherd [20] also characterized those connected graphs G with $\delta(G) \geq 2$ such that any dominating set contains at least $\frac{2}{5}|V(G)|$ vertices. In 1996, Reed [23] proved that a graph with $\delta(G) \geq 3$ satisfies $\gamma(G) \leq \frac{3}{8}|V(G)|$. Cubic graphs attaining the bound can be found in [10, 23], but no arbitrarily large family of connected graphs that attain the bound has been found.

In [23], Reed also conjectured that a connected cubic graph G satisfies $\gamma(G) \leq \lceil \frac{1}{3}|V(G)| \rceil$. This conjecture attracted much attention and has been a popular topic for a long time. However, in 2005, Kostochka and Stodolsky [15] disproved the conjecture by constructing an infinite family of connected cubic graphs $\{G_k\}_{k=1}^{\infty}$ such that $|V(G_k)| = 46k$ and $\gamma(G) \geq 16k$, which implies $\frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{1}{3} + \frac{1}{69}$ for all $k \geq 1$. The graphs in this family have girth 4 and are not 2-connected. However, Reed's conjecture is also false for 2-connected cubic graphs; counterexamples were presented independently by Stodolsky [24] and Kelmans [13]. Note that these counterexamples have girth 4. These counterexamples raised two main questions:

- Which is the correct bound for the domination number of cubic graphs?
- Which bound holds if an additional condition on the girth of the graph is given?

For cubic graph with order greater than 8, $\gamma(G) \leq \frac{5}{14}|V(G)|$ was proved by Kostochka and Stocker [14], and later improved by Cho, Choi, Kwon, and Park [3] to independent domination when there is no 4-cycle in a graph. These results are tight, as shown by a generalized Petersen graph on 14 vertices.

Obtaining good bounds for the domination number of planar graphs has been of separate interest. MacGillivray and Seyffarth [19] showed that determining the domination number of a graph is NP-hard even for planar graphs. Upper bounds for the domination number of planar graphs of diameter 2 and 3 have been studied in [6, 9, 19]. Zhu and Wu [26] made a conjecture on the domination number of cubic planar graphs.

Conjecture 1.1 ([26]). *If G is a 2-connected cubic planar graph, then $\gamma(G) \leq \frac{|V(G)|}{3}$.*

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We do not know of any counterexamples to the conjecture when the condition of 2-connectedness is removed. The conjecture might be true without any hypothesis on the graph connectivity.

The problem has also been studied for total domination, and Henning and McCoy [11] showed that planar graphs of diameter 2 have total domination number 3. More general upper bounds were obtained for the prefect Italian domination number of planar graphs [17], the global domination number of planar graphs [7], the power domination number of maximal planar graphs [4], and the independent domination number of 2-connected planar graphs [1]. Moreover, Abrishami, Henning, and Rahbarnia [1] proposed a weaker version of Conjecture 1.1. Note that a bipartite cubic graph is 2-connected.

Conjecture 1.2 ([1])**.** *If G is a bipartite cubic planar graph, then* $\gamma(G) \le \frac{|V(G)|}{3}$.

Conjecture 1.2 is known to be best possible and is true for graphs on at most 24 vertices [1].

Favaron [8] showed that if T is a tree on $n \ge 2$ vertices with q leaves, then $\gamma(T) \le \frac{n+q}{3}$. Applying this result to a subcubic tree T gives the following bound:

$$\gamma(T) \le \frac{2}{3}n_1(T) + \frac{1}{3}n_2(T) + \frac{1}{3}n_3(T),$$

where $n_i(G)$ denotes the number of vertices of degree i in G.

Finding a girth condition which implies Reed's upper bound has been of great interest to many mathematicians. In 2006, Kawarabayashi, Plummer, and Saito [12] proved that every 2-edge-connected cubic graph with girth at least g (where g is divisible by 3) satisfies $\gamma(G) \le (\frac{1}{3} + \frac{1}{3g+3})|V(G)|$. In 2009, Kostochka and Stodolsky [16] proved that a cubic graph with order greater than 8 and girth at least g has $\gamma(G) \le (\frac{1}{3} + \frac{8}{3g^2})|V(G)|$. In 2008, Löwenstein and Rautenbach [18] proved that a cubic graph with girth at least $g \ge 6$ satisfies $\gamma(G) \le (\frac{44}{135} + \frac{82}{135g})|V(G)|$, which proves that the Reed's conjecture is true for graphs with girth at least 83.

Very recently, further progress towards solving Reed's conjecture for graphs of higher girth has been made. Dorbec and Henning [5] have proved that if G is a cubic graph on n vertices of girth at least 6 with no 7-cycle and no 8-cycle, then $\gamma(G) \le \frac{1}{3}n$. Additionally, they also prove that if G is a cubic bipartite graph on n vertices with no 4-cycle and no 8-cycle, then $\gamma(G) \le \frac{1}{3}n$. However, we are not aware of progress made specifically for planar graphs or graphs containing 8-cycles.

In this paper, we made progress on bounding the domination number of subcubic planar graphs with girth at least 8 by proving the following theorem.

Theorem 1.3. *If G is a subcubic planar graph with girth at least 8, then*

$$20\gamma(G) < 20n_0(G) + 15n_1(G) + 11n_2(G) + 7n_3(G).$$

Note that since the domination number is an integer, $\gamma(G)$ can be bounded from above by the largest integer *strictly smaller* than

$$\frac{1}{20}(20n_0(G) + 15n_1(G) + 11n_2(G) + 7n_3(G)) = n_0(G) + \frac{3}{4}n_1(G) + \frac{11}{20}n_2(G) + \frac{7}{20}n_3(G).$$

We present here two examples of subcubic planar graphs of girth at least 8 whose domination number is equal to the largest integer strictly smaller than the bound of Theorem 1.3.

Let $C_8^\star$ and $C_9^\star$ be graphs obtained by attaching a leaf to each vertex of an 8-cycle and a 9-cycle, respectively. Clearly, $\gamma(C_8^\star) = 8$, and $n_0(C_8^\star) + \frac{3}{4}n_1(C_8^\star) + \frac{11}{20}n_2(C_8^\star) + \frac{7}{20}n_3(C_8^\star) = \frac{44}{5} = 8.8$. Similarly, we can see that $\gamma(C_9^\star) = 9$, and Theorem 1.3 gives an upper bound of $\frac{99}{10} = 9.9$.

The rest of the paper is organized as follows. The notation needed is listed in Section 2, and the general framework of the proof of Theorem 1.3 is explained in Section 3. For clarity, the proof itself is given in two parts, first we prove Theorem 1.3 for graphs with girth at least 9 in Section 4, and only then we proceed with the more complicated proof for girth at least 8 in Section 5. In both cases, the proof consists of two parts, listing reducible configurations and finishing with a discharging argument.

In Section 6, we present an alternative bound for the domination number of subcubic planar graphs, and in Section 7, we conclude the paper by discussing possibilities for an upper bound with a different girth condition.

2 Notation

Throughout this paper, let G be a finite, simple, undirected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, we use $\deg_G(v)$ and $N_G(v)$ to denote the *degree* and the *neighborhood*, respectively, of v in G. Let $N_G[v] = N_G(v) \cup \{v\}$ be the *closed neighborhood* of v in G. For $S \subseteq V(G)$, we define $N_G(S) = \bigcup_{v \in S} N_G(v)$ and $N_G[S] = \bigcup_{v \in S} N_G[v]$. For $S = \{v_1, \dots, v_n\}$, we write $N_G[v_1, \dots, v_n]$ instead of $N_G[\{v_1, \dots, v_n\}]$ for simplicity. Let $\delta(G)$ be the *minimum degree* of G, which is the minimum degree among all the vertices in G. Let $g(G)$ be the *girth* of G, which is the length of the shortest cycle of G. When G is a plane graph, we use $F(G)$ to denote the set of faces of G, and $\ell(f)$ to denote the length of $f \in F(G)$.

For a positive integer k, we use $[k]$ to denote the set $\{1, \dots, k\}$. A vertex of degree k is called a *k-vertex*, and a vertex of degree at least k is called a *k^+-vertex*. A face of length k is called a *k-face*, and a face of length at least k is called a *k^+-face*. Let $n_i(G)$ be the number of i-vertices in G. A *facial walk* of a k-face f is a walk in G of length k along the boundary of the face f. Note that a facial walk of f visits each vertex incident with f several times. For a positive integer k, a *path* P_k in G with k vertices is a sequence of distinct vertices $v_1, \dots, v_k$ such that $v_i v_{i+1} \in E(G)$ for $i \in [k-1]$.

For a graph G and $S \subseteq V(G)$, let $G[S]$ be the graph induced by S. Also, let $G - S = G[V(G) \setminus S]$.

Let G be a subcubic graph. A *sequence* $(i_1, \dots, i_k)$ in G is a path $v_1, \dots, v_k$ in G such that $\deg_G(v_j) = i_j$ for all $j \in [k]$. Additionally, a *sequence* $(i_1, \dots, i_{k-1}, 3_p^q)$ in G is a sequence $(i_1, \dots, i_{k-1}, 3)$ formed by $v_1, \dots, v_k$, where $N_G(v_k) = \{v_{k-1}, x, y\}$, $\deg_G(x) = p$, and $\deg_G(y) = q$. For an example, see Figure 1.

Figure 1: An example of the sequence $(2,3,2,3,1)$ and an example of the sequence $(2,2,3_3^2)$.

3 General framework

The main idea of the proof of Theorem 1.3 is to first find several reducible configurations, and then use the discharging method. The working frame is developed in the rest of this section, the reducible configurations are presented in Sections 4.1 and 5.1, and the discharging process is presented in Sections 4.2 and 5.2.

For each $v \in V(G)$, let $w_G(v)$ be the *weight* of v on G, defined as follows:

$$w_G(v) = \begin{cases} 20 & \text{if } \deg_G(v) = 0, \\ 15 & \text{if } \deg_G(v) = 1, \\ 11 & \text{if } \deg_G(v) = 2, \\ 7 & \text{if } \deg_G(v) = 3. \end{cases}$$

For $\emptyset \neq S \subseteq V(G)$ and a subgraph H of G, let $w_G(S) = \sum_{v \in S} w_G(v)$, $w_G(H) = w_G(V(H))$, and $c_G(S) = w_{G-S}(G-S) - w_G(G-S)$. Note that $c_G(S) \geq 0$ and $w_G(G) = 20n_0(G) + 15n_1(G) + 11n_2(G) + 7n_3(G)$.

From now on, let G be a minimal counterexample (with respect to the number of vertices) to the Theorem 1.3. Thus, clearly $20\gamma(G) \geq w_G(G)$, and $20\gamma(H) < w_H(H)$ for every proper subgraph H of G.

Lemma 3.1 (Key Lemma)**.** *For* $\emptyset \neq S \subseteq V(G)$*,* $20\gamma(G[S]) > w_G(S) - c_G(S)$*.*

Proof. By the minimality of G, we have $20\gamma(G-S) < w_{G-S}(G-S)$. Note also the following three facts:

1. $20\gamma(G) \geq w_G(G)$,
2. $\gamma(G) \leq \gamma(G-S) + \gamma(G[S])$,
3. $c_G(S) = w_{G-S}(G-S) - w_G(G-S) = w_{G-S}(G-S) - (w_G(G) - w_G(S))$.

These give us

$$\begin{aligned} w_{G-S}(G-S) - c_G(S) + w_G(S) = w_G(G) &\leq 20\gamma(G) \\ &\leq 20\gamma(G-S) + 20\gamma(G[S]) \\ &< w_{G-S}(G-S) + 20\gamma(G[S]), \end{aligned}$$

from which the desired inequality follows. □

The Key Lemma is our main tool when proving reducible configurations. The idea is to find a suitable set S, determine the $\gamma(G[S])$ (trivial since S will be small), calculate the weight $w_G(S)$ of S (by definition, we simply add up the respective weights that are in one-to-one correspondence with the degree of the vertices in S), and find a bound on the cost $c_G(S)$ of S. The cost represents how much we can afford to change the weight of the graph by removing S. Each vertex v in $N_G(S) \setminus S$ will contribute to the cost of S, and this contribution is determined by observing the degree change of the vertex v. By definition, it equals $w_{G-S}(v) - w_G(v)$. Thus, it can be calculated from $\deg_G(v)$ and $|N_G(v) \cap S|$; see Table 1. Observe that the change in weight is the largest when a 0-vertex is created. Thus, we also need to keep track of the change in degree for vertices in $G - S$. If for each $v \in N_G(S) \setminus S$, $|N_G(v) \cap S| = 1$ holds, then the cost of S depends on the number of

$\deg_G(v)$ \ $\lvert N_G(v) \cap S\rvert$	0	1	2	3
0	0	/	/	/
1	0	5	/	/
2	0	4	9	/
3	0	4	8	13

Table 1: The amount a vertex v contributes to $c_G(S)$.

1-vertices and 2^+-vertices in $N_G(S) \setminus S$. Note that the condition that girth is at least 8 will often imply that for each $v \in N_G(S) \setminus S$, $|N_G(v) \cap S| = 1$ for small sets S.

Sometimes, we will bound the cost of S by considering the edges leaving S instead of considering the vertices in $N_G(S) \setminus S$. The possibility of several edges being incident to the same vertex in $N_G(S) \setminus S$ must be taken into account. This is formalized in the following lemma.

Lemma 3.2. *Let $\emptyset \neq S \subseteq V(G)$ be such that there are k edges between S and $G - S$. If all vertices in $N_G(S) \setminus S$ are of degree 2 or 3, then $c_G(S) \leq 4k + \lfloor \frac{k}{2} \rfloor$. More precisely, if there are at most ℓ edges between S and $G - S$ that are incident with a 2-vertex in $N_G(S) \setminus S$, then $c_G(S) \leq 4k + \max\left\{\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k-\ell}{3} \rfloor, \lfloor \frac{\ell-1}{2} \rfloor + \lfloor \frac{k-\ell+1}{3} \rfloor\right\}$.*

Proof. Let x be the number of edges between S and $G - S$ that are incident with a 2-vertex in $N_G(S) \setminus S$. Then $c_G(S) \leq 4k + \lfloor \frac{x}{2} \rfloor + \lfloor \frac{k-x}{3} \rfloor$. Note that each edge between S and $G - S$ basically increases the cost of $N_G(S) \setminus S$ by 4, and whenever there is an isolated vertex in $N_G(S) \setminus S$, there is an additional increase in cost by 1. Observing that $\max\left\{\lfloor \frac{x}{2} \rfloor + \lfloor \frac{k-x}{3} \rfloor; \ 0 \leq x \leq \ell\right\} = \max\left\{\lfloor \frac{\ell}{2} \rfloor + \lfloor \frac{k-\ell}{3} \rfloor, \lfloor \frac{\ell-1}{2} \rfloor + \lfloor \frac{k-\ell+1}{3} \rfloor\right\} \leq \lfloor \frac{k}{2} \rfloor$ concludes the proof. □

Note that the more precise version of the lemma is not needed in the following, and we have included it for completeness only.

4 Graphs with girth at least 9

To make the proof more readable, we first present the proof of Theorem 1.3 only for graphs with girth at least 9. Reducible configurations are listed in Section 4.1, and the discharging method is presented in Section 4.2. The complete proof of the result (including graphs containing 8-faces) is presented in Section 5.

4.1 Reducible configurations

As above, G is a minimal counterexample to the Theorem 1.3. From now on, assume that G is embedded in the plane and this embedding is fixed. From the minimality of G we immediately infer that G is connected.

Lemma 4.1. *The graph G does not contain a sequence $(1, 2)$.*

Proof. Let v, u be a sequence $(1, 2)$ in G; see Figure 2.

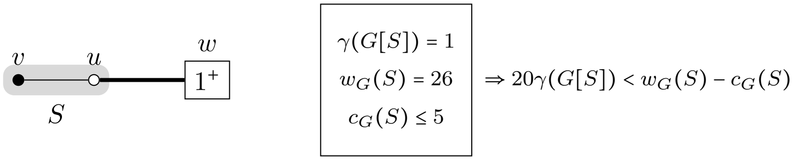


Figure 2: The reducible configuration $(1, 2)$ from Lemma 4.1.

Let $S = \{v, u\} \subseteq N_G[u]$. Clearly, $\gamma(G[S]) = 1$ and $w_G(S) = 11 + 15 = 26$. Denote $N_G(u) = \{v, w\}$. Then $c_G(S) = w_{G-S}(G - S) - w_G(G - S) = \sum_{x \in V(G-S)} w_{G-S}(x) - \sum_{x \in V(G-S)} w_G(x) = w_{G-S}(w) - w_G(w) \in \{4, 5\}$, so $c_G(S) \leq 5$. (Note that since $|N_G(S) \setminus S| = 1$, $c_G(S) = 5$ if and only if $\deg_G(w) = 1$; see Table 1.)

Thus, by the Key Lemma, we obtain $20 = 20\gamma(G[S]) > w_G(S) - c_G(S) \geq 26 - 5 = 21$, which is a contradiction. □

Before we continue to establish further reducible configurations, we consider Figure 2 again, since the same style will be used for the rest of the paper. Vertices forming set S are marked with a grey shade, and the vertices dominating $G[S]$ are colored white (the remaining vertices of S are given color black). The edges between S and $G - S$ are drawn thicker. The vertices in $N_G(S) \setminus S$ are drawn as square boxes. The value k^+ (resp. k) inside the box means that this vertex has degree at least k (resp. exactly k). Sometimes, even a degree of a vertex in S is not unique – in this case the edge to a possibly non-existing neighbor will be marked with a dashed line (see for example Figure 8).

Lemma 4.2. *The graph G does not contain a sequence* $(1, 3, 1)$.

Proof. Let v_1, u, v_2 be a sequence $(1, 3, 1)$ in G; see Figure 3.

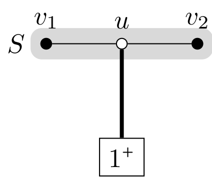


Figure 3: The reducible configuration $(1, 3, 1)$ from Lemma 4.2.

Let $S = \{u, v_1, v_2\} \subseteq N_G[u]$. Clearly, $\gamma(G[S]) = 1$, $w_G(S) = 2 \cdot 15 + 7 = 37$, and $c_G(S) \leq 5$. By the Key Lemma, we get $20 > 37 - 5 = 32$, which is a contradiction. □

Lemma 4.3. *The graph G does not contain a sequence* $(1, 3, 2)$.

Proof. Let v, u, w be a sequence $(1, 3, 2)$ in G; see Figure 4.

Take $S = \{v, u, w\} \subseteq N_G[u]$. We have $\gamma(G[S]) = 1$, $w_G(S) = 15 + 7 + 11 = 33$, and $c_G(S) \leq 2 \cdot 5 = 10$. By the Key Lemma, we obtain $20 > 33 - 10 = 23$, which is a contradiction. □

Lemma 4.4. *The graph G does not contain a sequence* $(1, 3, 3, 1)$.

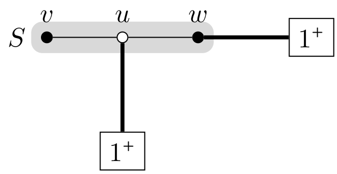


Figure 4: The reducible configuration $(1,3,2)$ from Lemma 4.3.

Proof. Let u_1, u, v, v_1 be a sequence $(1,3,3,1)$ in G. Since u, v are of degree 3, they each have another neighbor, u_2, v_2, respectively. By Lemmas 4.2 and 4.3, we know that $\deg_G(u_2) = \deg_G(v_2) = 3$; see Figure 5.

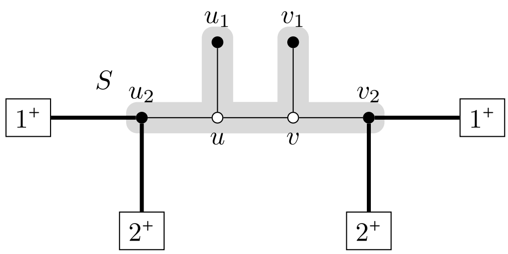


Figure 5: The reducible configuration $(1,3,3,1)$ from Lemma 4.4.

Take $S = \{u, u_1, u_2, v, v_1, v_2\} = N_G[u, v]$. We have $\gamma(G[S]) = 2$ and $w_G(S) = 2\cdot15 + 4\cdot7 = 58$. Since G has girth at least 8, $N_G(S) \setminus S$ has exactly 4 vertices. By Lemma 4.2, at most two of the vertices in $N_G(S) \setminus S$ are of degree 1. Thus, $c_G(S) \le 2\cdot5 + 2\cdot4 = 18$. By the Key Lemma, we obtain $40 = 2\cdot20 > 58 - 18 = 40$, which is a contradiction. □

Lemma 4.5. *The graph G has no 1-vertex.*

Proof. Suppose that G contains a 1-vertex v with a neighbor u. By Lemma 4.1, $\deg_G(u) = 3$. Let w_1 and w_2 be the other neighbors of u; see Figure 6. By Lemmas 4.2 and 4.3, both w_1 and w_2 must be of degree 3.

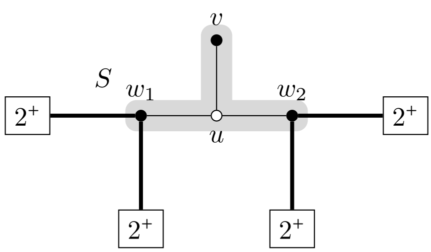


Figure 6: The reducible configuration from Lemma 4.5 that finalizes the argument that G has no 1-vertex.

Take $S = \{v, u, w_1, w_2\} = N_G[u]$. We have $\gamma(G[S]) = 1$ and $w_G(S) = 15 + 3\cdot7 = 36$. Since G has girth at least 8, no vertex in $N_G(S) \setminus S$ has at least two neighbors in S. By Lemma 4.4, none of the vertices in $N_G(S) \setminus S$ are of degree 1. Thus $c_G(S) \le 4\cdot4 = 16$. By the Key Lemma, we obtain $20 > 36 - 16 = 20$, which is a contradiction. □

Lemma 4.6. *The graph G does not contain a sequence* $(2,2,2)$.

Proof. Let u_1, u_2, u_3 be a sequence $(2,2,2)$ in G; see Figure 7.

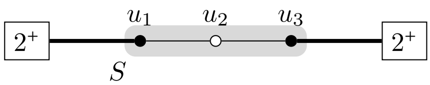


Figure 7: The reducible configuration $(2,2,2)$ from Lemma 4.6.

Take $S = \{u_1, u_2, u_3\} = N_G[u_2]$. We have $\gamma(G[S]) = 1$ and $w_G(S) = 3 \cdot 11 = 33$. Since G has girth at least 8, no vertex in $N_G(S) \setminus S$ has at least two neighbors in S, and we get $c_G(S) \leq 2 \cdot 5 = 10$. By the Key Lemma, we obtain $20 > 33 - 10 = 23$, which is a contradiction. □

Lemma 4.7. *The graph G contains no sequence* $(2,3,2)$.

Proof. Let v_1, v_2, v_3 be a sequence $(2,3,2)$ in G; see Figure 8. Let $N_G(v_2) = \{v_1, w_2, v_3\}$.

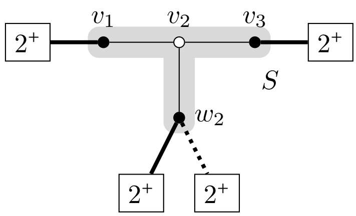


Figure 8: The reducible configuration $(2,3,2)$ from Lemma 4.7.

Take $S = \{v_1, v_2, v_3, w_2\} = N_G[v_2]$. We have $\gamma(G[S]) = 1$ and $w_G(S) \geq 2 \cdot 11 + 2 \cdot 7 = 36$. Since G has girth at least 8, no vertex in $N_G(S) \setminus S$ has at least two neighbors in S. Also, by Lemma 4.5, all the vertices in $N_G(S) \setminus S$ have degree at least 2, and we get $c_G(S) \leq 4 \cdot 4 = 16$. By the Key Lemma, we obtain $20 > w_G(S) - c_G(S) \geq 36 - 16 = 20$, which is a contradiction. □

Lemma 4.8. *Each P_6 in G contains at most three* 2*-vertices.*

Proof. By Lemma 4.6, G cannot contain a P_6 with three consecutive 2-vertices. Thus G can contain a P_6 with at most four 2-vertices.

Suppose that G contains a P_6 of vertices $v_1, \dots, v_6$, where exactly four of them are of degree 2, and the remaining two are of degree 3. Applying Lemma 4.7 to a short case-analysis, we obtain that the only possibility is that $\deg_G(v_1) = \deg_G(v_2) = \deg_G(v_5) = \deg_G(v_6) = 2$ and $\deg_G(v_3) = \deg_G(v_4) = 3$; see Figure 9.

Take $S = \{v_1, \dots, v_6\} = N[v_2, v_5]$. We have $\gamma(G[S]) = 2$ and $w_G(S) = 4 \cdot 11 + 2 \cdot 7 = 58$. Since G has girth at least 8, no vertex in $N_G(S) \setminus S$ has at least two neighbors in S. Also, by Lemma 4.5, all the vertices in $N_G(S) \setminus S$ have degree at least 2, and we get $c_G(S) \leq 4 \cdot 4 = 16$. By the Key Lemma, we obtain $40 > 58 - 16 = 42$, which is a contradiction. □

For the following, observe that since the girth of G is at least 8, every consecutive k vertices, $1 \leq k \leq 6$, along a facial walk in G form a P_k.

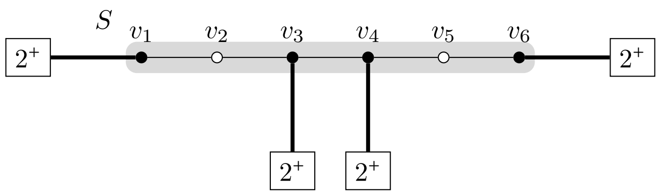


Figure 9: An example of a reducible configuration $(2,2,3,3,2,2)$ from Lemma 4.8.

Lemma 4.9. *For $i \in [5]$, each face in G contains a P_i with at most $\lfloor \frac{i}{2} \rfloor$ 2-vertices.*

Proof. Let f be a face in G of length k. Since G has girth at least 8, $k \geq 8$. By Lemma 4.6, f contains a P_1 with zero 2-vertices, and a P_2 with at most one 2-vertex. By Lemma 4.7, f contains a P_3 with at most one 2-vertex. By Lemmas 4.6 and 4.7, f contains a P_4 with at most two 2-vertices. If f contains no two consecutive 2-vertices, then f has a P_5 with at most two 2-vertices. Otherwise, f contains at least two consecutive 2-vertices, say v_3, v_4. Let $v_1, \ldots, v_6$ be a P_6 on f. By Lemmas 4.6 and 4.7, $\deg_G(v_j) = 3$ for all $j \in \{1,2,5,6\}$. Thus $v_1, \ldots, v_5$ is a P_5 on f with at most two 2-vertices. □

Lemma 4.10. *For $k \geq 11$, in the facial walk of each k-face in G there are at most $\lfloor \frac{k}{2} \rfloor$ 2-vertices.*

Proof. Let f be a face in G of length $k \geq 11$. Let $k \equiv i \pmod 6$, $i \in [6]$. By Lemma 4.8 or 4.9, the face f contains a P_i with at most $\lfloor \frac{i}{2} \rfloor$ 2-vertices. The remaining vertices on the facial walk can be covered by $\frac{k-i}{6}$ P_6s, and by Lemma 4.8, each of them contains at most three 2-vertices. Altogether, the number of 2-vertices appearing on the facial walk of f is at most $\lfloor \frac{i}{2} \rfloor + \frac{k-i}{6} \cdot 3 \in \{\frac{k-1}{2}, \frac{k}{2}\}$. Thus, the facial walk of f contains at most $\lfloor \frac{k}{2} \rfloor$ 2-vertices. □

Note that if G is 2-connected, we can replace "facial walk of the k-face f" above with simply "k-face f". But if G is not 2-connected, it is possible that some vertices are incident with only the face f, and thus these vertices appear on the facial walk of f twice.

Lemma 4.11. *Every 10-face in G is incident with at most four 2-vertices.*

Proof. Let f be a 10-face $v_1, \ldots, v_{10}$ in G. If f contains no two consecutive 2-vertices, then by Lemma 4.7, f contains at most three 2-vertices. For example, see Figure 10.

Let f contain two consecutive 2-vertices; without loss of generality, let $\deg_G(v_1) = \deg_G(v_2) = 2$. By Lemmas 4.6 and 4.7, vertices v_3, v_4, v_{10}, v_9 are all of degree 3. By Lemma 4.8, at most one of v_5, v_6 is a 2-vertex, and at most one of v_8, v_7 is a 2-vertex. Thus, f contains at most four 2-vertices. For example, see Figure 11. □

Note that since the girth of G is at least 8 and there are no 1-vertices, the facial walk of a 10-face is always a cycle.

Lemma 4.12. *Every 9-face in G is incident with at most three 2-vertices.*

Proof. Let f be a 9-face $v_1, \ldots, v_9$ in G. If f has no two consecutive 2-vertices, then by Lemma 4.7, f contains at most three 2-vertices (if $\deg_G(v_1) = 2$, then $\deg_G(v_2) =$

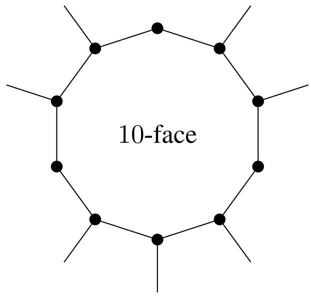


Figure 10: An example of a 10-face with no two consecutive 2-vertices.

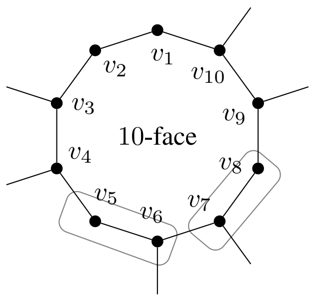


Figure 11: An example of a 10-face with two consecutive 2-vertices.

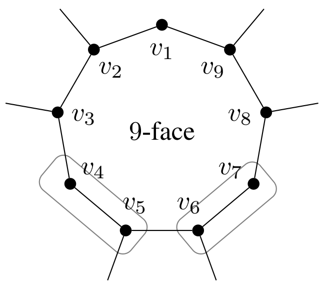


Figure 12: An example of a 9-face with no two consecutive 2-vertices.

$\deg_G(v_3) = \deg_G(v_9) = \deg_G(v_8) = 3$, and since there are no two consecutive 2-vertices on f, at most two of the v_4, v_5, v_6, v_7 can be 2-vertices). For example, see Figure 12.

Suppose that f contains two consecutive 2-vertices; without loss of generality, let $\deg_G(v_1) = \deg_G(v_2) = 2$. Then by Lemmas 4.6 and 4.7, $\deg_G(v_3) = \deg_G(v_4) = \deg_G(v_9) = \deg_G(v_8) = 3$. By Lemma 4.8, at most one of v_5, v_6 can be a 2-vertex and at most one of v_6, v_7 can be a 2-vertex. But since v_5 and v_7 cannot both be 2-vertices by Lemma 4.7, at most one of v_5, v_6, v_7 is a 2-vertex. This implies that f again contains at most three 2-vertices. For example, see Figure 13. □

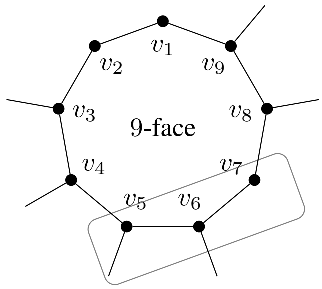


Figure 13: An example of a 9-face with two consecutive 2-vertices.

4.2 Discharging

For $x \in V(G) \cup F(G)$, we define the following initial charge:

$$\mu(x) = \begin{cases} 2\deg_G(x) - 6 & \text{if } x \in V(G), \\ \ell(x) - 6 & \text{if } x \in F(G). \end{cases}$$

Since G is planar, Euler's formula implies that the initial charge of G is

$$\mu(G) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -12.$$

We use the following discharging rule:

- Every face f sends charge 1 to each of the 2-vertices on the facial walk of f, counted with multiplicities.

Let $\mu^*(x)$ denote the charge of vertices and faces after applying the discharging rule.

Lemma 4.13. *For every $x \in V(G) \cup F(G)$, $\mu^*(x) \geq 0$.*

Proof. By Lemma 4.5, G has no vertex of degree 1. A 2-vertex that is incident with two different faces receives charge 1 from each of them. A 2-vertex that is incident with only one face f appears on the facial walk of f twice, so it receives charge 2 from f. So $\mu^*(v) = (2 \cdot 2 - 6) + 2 \cdot 1 = 0$. If v is a vertex of degree 3, then $\mu^*(v) = \mu(v) = 0$.

If f is a k-face, $k \geq 11$, then by Lemma 4.10, the facial walk of f contains at most $\lfloor \frac{k}{2} \rfloor$ 2-vertices. Thus,

$$\mu^*(f) \geq (k-6) - \left\lfloor \frac{k}{2} \right\rfloor = \left\lceil \frac{k}{2} \right\rceil - 6 \geq 0$$

since $k \geq 11$.

If f is a 10-face, then by Lemma 4.11, f is incident with at most four 2-vertices. Hence, $\mu^*(f) \geq (10-6) - 4 = 0$. If f is a 9-face, then by Lemma 4.12, f is incident with at most three 2-vertices. Hence, $\mu^*(f) \geq (9-6) - 3 = 0$. □

Clearly it holds that

$$-12 = \mu(G) = \mu^*(G) \geq 0,$$

which is a contradiction. This concludes the proof of Theorem 1.3 for graphs with girth at least 9.

5 Graphs with girth at least 8

In this section, we give the full proof of Theorem 1.3, so proving the desired bound for all subcubic planar graphs with girth at least 8. The structure of the proof is the same as for graphs with girth at least 9, but additional care is needed to deal with 8-faces. The discharging method used in the last step of the proof is also similar as in the case of graphs with girth at least 9, but we include it in full for completeness.

5.1 Reducible configurations

A minimal counterexample G in this case satisfies all reducible configurations presented in Section 4.1. Thus, Lemmas 4.1-4.12 still hold. In the rest of this section, we present additional reducible configurations with respect to 8-faces.

Definition 5.1. A face f in G is a *bad* 8*-face* if it is an 8-face $v_1, \dots, v_8$ with vertices of degrees $(2, 2, 3, 3, 2, 3, 3, 3)$ (see Figure 14). If the 8-face is not bad, we say it is a *good* 8*-face*. The two consecutive 3-vertices on a bad 8-face f that both have a neighboring 2-vertex on f (vertices v_3 and v_4 in the above notation) are called *contributing vertices* of f. A face adjacent to the bad 8-face f that contains the two contributing vertices of f is called the *contributing face* of f. (Note that by definition the contributing face of f is unique.)

Recall that a vertex v is incident with a face f (or, v belongs to f) if v is one of the vertices on f. For a contributing face f, let the *set of bad* 8*-faces incident with* f, $\mathcal{C}(f)$, be the set of all bad 8-faces whose two contributing vertices are on f, i.e. $g \in \mathcal{C}(f)$ means that g is a bad 8-face, and the two contributing vertices of g belong to f. Shortly, we will say that a bad 8-face g is incident with its contributing face f.

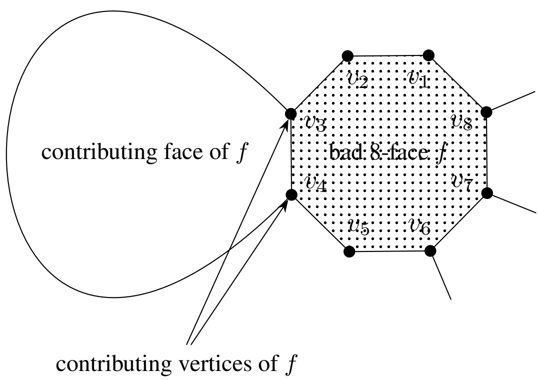


Figure 14: A bad 8-face f. The dotted background in the figures in this section is used only to help the reader recognize the face of the main focus.

Lemma 5.2. *Every* 8*-face in* G *is either incident with at most two* 2*-vertices or it is a bad* 8*-face.*

Proof. Let f be an 8-face $v_1, \dots, v_8$ in G. If f has no two consecutive 2-vertices, then by Lemma 4.7, f contains at most two 2-vertices (if $\deg_G(v_1) = 2$, then $\deg_G(v_2) =$

$\deg_G(v_3) = \deg_G(v_8) = \deg_G(v_7) = 3$, and since there are no two consecutive 2-vertices on f, at most one of v_4, v_5, v_6 can be a 2-vertex).

Suppose that f contains two consecutive 2-vertices; without loss of generality, $\deg_G(v_1) = \deg_G(v_2) = 2$. Then by Lemma 4.7, $\deg_G(v_3) = \deg_G(v_4) = \deg_G(v_8) = \deg_G(v_7) = 3$. By Lemma 4.8, at most one of v_5, v_6 can be a 2-vertex. Thus, f contains at most three 2-vertices. If f contains exactly three 2-vertices, then by the above argument, f is a bad 8-face. □

Note that from now on, even though the girth of the graph is at least 8, it may happen that for some $v \in N_G(S) \setminus S$, $|N_G(v) \cap S| \geq 2$. Thus planarity of G and $g(G) \geq 8$ will both be used to limit the number of common neighbors of vertices from S.

Lemma 5.3. *Both contributing vertices of a bad 8-face are adjacent to two 3-vertices on the contributing face. Additionally, one of the two contributing vertices of a bad 8-face is adjacent to a 3-vertex whose neighbors are all 3-vertices.*

Proof. Let $v_1, \ldots, v_8$ be a bad 8-face on G with $\deg_G(v_i) = 2$ for $i \in \{1, 2, 5\}$, so v_3 and v_4 are its contributing vertices. Let w_3, w_4 be the other neighbors of v_3, v_4, respectively. By Lemma 4.7, $\deg_G(w_3) = \deg_G(w_4) = 3$. Thus v_3 is adjacent to two 3-vertices on the contributing face (v_4 and w_3), and so is v_4 which is adjacent to v_3 and w_4. Suppose that both w_3 and w_4 have a neighbor of degree 2, say x_3 and x_4, respectively. See Figure 15 for example.

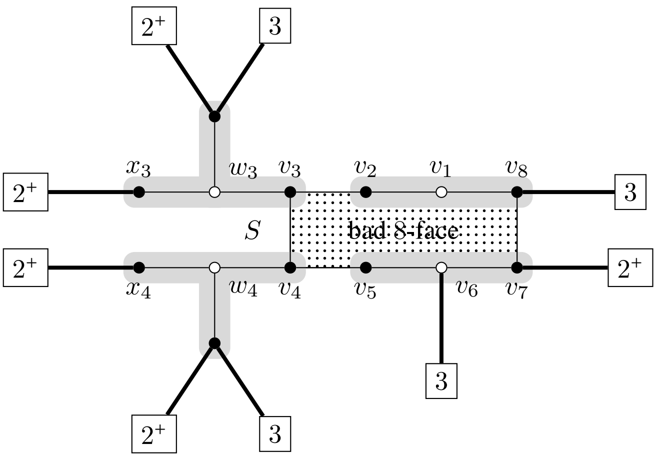


Figure 15: The configuration from Lemma 5.3.

Let $S = N_G[v_1, w_3, w_4] \cup \{v_5, v_6, v_7\} \subseteq N_G[v_1, v_6, w_3, w_4]$. There are at most nine edges between S and $G - S$. We have $\gamma(G[S]) = 4$, $w_G(S) \geq 5 \cdot 11 + 9 \cdot 7 = 118$, and $c_G(S) \leq 7 \cdot 4 + 9 = 37$ (by planarity and the girth condition, at most one pair of vertices in S can have a common neighbor in $N_G(S) \setminus S$; for example, x_3 and v_7, or x_4 and v_8, can have a common neighbor in $N_G(S) \setminus S$). By the Key Lemma, we obtain $80 > 118 - 37 = 81$, which is a contradiction. □

Lemma 5.4. *No two adjacent bad 8-faces of G share a contributing vertex.*

Proof. Let $v_1, \dots, v_8$ be a bad 8-face f on G with $\deg_G(v_i) = 2$ for $i \in \{1, 2, 5\}$. Recall that by Lemma 5.3, two bad 8-faces cannot have both v_3 and v_4 in common.

First suppose that the vertex v_3 belongs to another bad 8-face f' of G. Let its vertices be v_1, v_2, v_3, w_4, w_5, w_6, w_7, v_8. See Figure 16 for example. Since f' is a bad 8-face and v_3 is one of its contributing vertices, w_4 is also a contributing vertex, and thus $\deg_G(w_5) = 2$. Let z be the other neighbor of w_6. By Lemma 4.7, $\deg_G(z) = 3$. Let the other neighbor of v_6 be denoted by y. By Lemma 4.7, $\deg_G(y) = 3$.

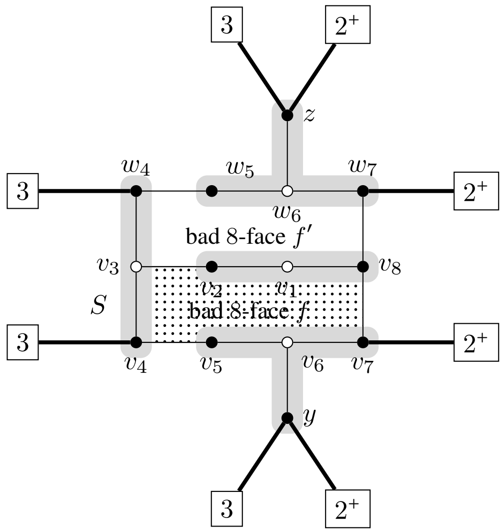


Figure 16: Two bad 8-faces have vertex v_3 in common.

Take $S = \{v_1, \dots, v_8, w_4, \dots, w_7, y, z\} = N_G[v_1, v_3, v_6, w_6]$. By above, there are exactly eight edges between S and $G - S$. We have $\gamma(G[S]) = 4$, $w_G(S) = 4 \cdot 11 + 10 \cdot 7 = 114$, and $c_G(S) \le 6 \cdot 4 + 9 = 33$ (since y and z could have a common neighbor). By the Key Lemma, we obtain $80 > 114 - 33 = 81$, which is a contradiction.

Next suppose that the vertex v_4 belongs to another bad 8-face f'' of G. Let its vertices be w_1, w_2, w_3, v_4, v_5, v_6, w_7, w_8. See Figure 17 for example. Since f'' is a bad 8-face, w_1 and w_2 are 2-vertices, and v_4 and w_3 are contributing vertices of f''.

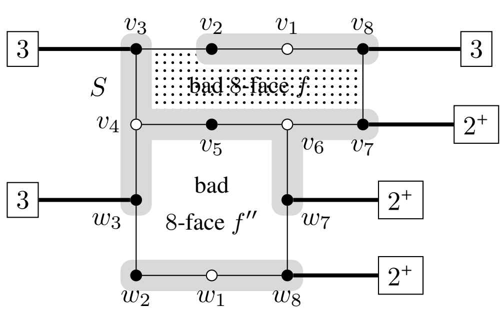


Figure 17: Two bad 8-faces have vertex v_4 in common.

Take $S = \{v_1, \dots, v_8, w_1, w_2, w_3, w_7, w_8\} = N_G[v_1, v_4, v_6, w_1]$. Clearly there are exactly six edges between S and $G - S$. We have $\gamma(G[S]) = 4$, $w_G(S) = 5 \cdot 11 + 8 \cdot 7 = 111$, and $c_G(S) \leq 6 \cdot 4 = 24$. By the Key Lemma, we obtain $80 > 111 - 24 = 87$, which is a contradiction. $\square$

Lemma 5.5. *Let $k \geq 11$ and let f be a k-face. If a facial walk of f contains x 2-vertices and f is incident with y bad 8-faces, then $x + y \leq \lfloor \frac{k}{2} \rfloor$.*

Proof. Observe that vertices incident with f are 2-vertices, contributing 3-vertices, or non-contributing 3-vertices. Let A be the multiset of 2-vertices on f, B be the multiset of contributing 3-vertices on f, and C be the multiset of the remaining 3-vertices of f. Note that A, B and C are multisets since a vertex is included multiple times if it appears more than once in the facial walk of f. Clearly, $|A| + |B| + |C| = k$.

By Lemma 5.4 and since each bad 8-face incident with f has two contributing 3-vertices in common with f, $y \leq \frac{|B|}{2}$. Clearly, $x = |A|$. By definition of a contributing 3-vertex and by Lemmas 4.7 and 5.4, the neighbors of contributing 3-vertices on f are 3-vertices. This, together with Lemma 4.6, implies that each 2-vertex on f is adjacent to at least one non-contributing 3-vertex on f. Also, by Lemma 4.7, each non-contributing 3-vertex on f is adjacent to at most one 2-vertex on f. Thus there is an injection from A to C, which maps each $v \in A$ to $w \in C \cap N_G(v)$, so $|A| \leq |C|$.

Hence,

$$x + y \leq |A| + \frac{|B|}{2} \leq \frac{|A| + |B| + |C|}{2} = \frac{k}{2},$$

thus also $x + y \leq \lfloor \frac{k}{2} \rfloor$. $\square$

Note that this bound holds for all k-faces, $k \geq 8$, but the obtained bound is not strong enough for our later use in the discharging process. Thus we consider cases $k \in \{8, 9, 10\}$ separately.

Lemma 5.6. *If a contributing 8-face f is incident with x 2-vertices and y bad 8-faces, then $x + y \leq 2$.*

Proof. Since f is a contributing 8-face, by Lemma 5.3, there are at least five consecutive 3-vertices on the facial walk of f. Thus, f is not a bad 8-face and so by Lemma 5.2, $x \leq 2$. Suppose that $x + y \geq 3$. We distinguish between the following cases.

Case 1: $x = 2$ and $y \geq 1$.

Let the vertices of f be $v_1, \dots, v_8$. Since $y \geq 1$, we may assume that v_1 and v_2 are the contributing vertices that f shares with a bad 8-face f'. Let the vertices of f' be $v_1, v_2, u_3, \dots, u_8$, where u_3, u_4, u_8 are 2-vertices; see Figure 18.

By Lemma 4.7, $\deg_G(v_3) = \deg_G(v_8) = 3$. By Lemma 4.8, at most one of v_4, v_5 is a 2-vertex. By Lemma 5.3, at most one of v_4, v_7 is a 2-vertex.

If $\deg_G(v_4) = 2$, then $\deg_G(v_5) = \deg_G(v_7) = 3$. By Lemma 4.7, $\deg_G(v_6) = 3$. This is a contradiction that $x = 2$. Thus $\deg_G(v_4) = 3$.

Case 1a: $\deg_G(v_5) = 2$.

Since $x = 2$, exactly one of v_6 and v_7 is a 2-vertex. By Lemma 4.7, $\deg_G(v_6) = 2$ and $\deg_G(v_7) = 3$. Let the neighbor of a 3-vertex v_i that lies on neither f nor f' be denoted by v_i', and similarly, the other neighbor of u_i is u_i'. See Figure 18.

Let $S = \{v_1, \dots, v_8, u_3, \dots, u_8, v_4', v_7', u_5', u_7'\} = N_G[v_4, v_7, v_2, u_5, u_7]$.

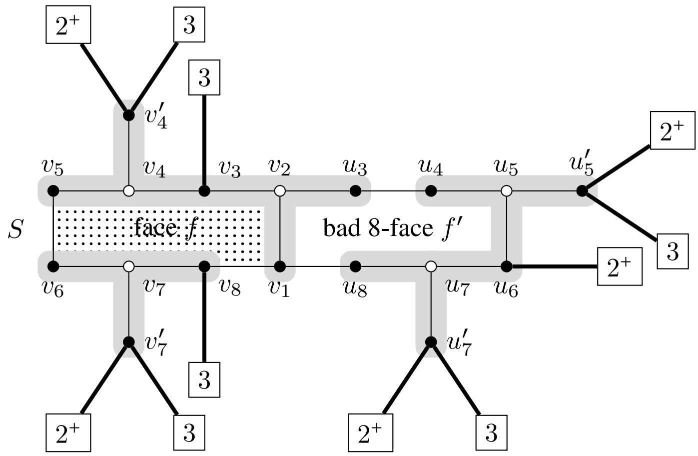


Figure 18: The configuration from Lemma 5.6, Case 1a.

Claim 5.7. *$G - S$ has at most two isolated vertices.*

Proof of Claim 5.7. Suppose $\deg_{G-S}(w) = 0$ for some $w \in V(G) \setminus S$. Since G has girth at least 8, $N_G(w)$ can contain at most one vertex from $\{v_3, v_4', v_7', v_8\}$, and at most one vertex from $\{u_5', u_6, u_7'\}$. Thus, $\deg_G(w) = 2$. Also, since G has girth at least 8, the following are the only possibilities for $N_G(w)$.

(1) $N_G(w) = \{v_4', u_6\}$.

(2) $N_G(w) = \{v_7', u_6\}$.

(3) $N_G(w) = \{v_4', u_7'\}$.

(4) $N_G(w) = \{v_7', u_7'\}$.

(5) $N_G(w) = \{v_4', u_5'\}$.

(6) $N_G(w) = \{v_7', u_5'\}$.

(7) $N_G(w) = \{v_8, u_5'\}$.

Suppose on the contrary that $\deg_{G-S}(w_1) = \deg_{G-S}(w_2) = \deg_{G-S}(w_3) = 0$ for $w_1, w_2, w_3 \in V(G) \setminus S$.

Suppose w_1 satisfies (1) so that $N_G(w_1) = \{v_4', u_6\}$. Since $\deg_G(u_6) = 3$, w_2 and w_3 cannot satisfy (2). By Lemma 4.7, w_2 and w_3 cannot satisfy (3) and (5). By the planarity of G, w_2 and w_3 cannot satisfy (6) and (7). Thus, w_2 and w_3 both satisfy (4), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (1).

Suppose w_1 satisfies (2) so that $N_G(w_1) = \{v_7', u_6\}$. By the planarity of G, w_2 and w_3 cannot satisfy (3) and (7). By Lemma 4.7, w_2 and w_3 cannot satisfy (4) and (6). Thus, w_2 and w_3 both satisfy (5), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (2).

Suppose w_1 satisfies (3) so that $N_G(w_1) = \{v_4', u_7'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (4) and (5). By the planarity of G, w_2 and w_3 cannot satisfy (6) and (7). Thus,

w_2 and w_3 satisfy none of the conditions (1) through (7), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (3).

Suppose w_1 satisfies (4) so that $N_G(w_1) = \{v'_7, u'_7\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (6). By the planarity of G, w_2 and w_3 cannot satisfy (7). Thus, w_2 and w_3 both satisfy (5), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (4).

Suppose w_1 satisfies (5) so that $N_G(w_1) = \{v'_4, u'_5\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (6) and (7). Thus, w_2 and w_3 satisfy none of the conditions (1) through (7), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (5).

Now, since w_1, w_2, w_3 cannot satisfy conditions (1) through (5), all of w_1, w_2, w_3 satisfy (6) or (7), which is a contradiction. $\square$

By Claim 5.7, $c_G(S) \le 11 \cdot 4 + 2 = 46$. Since $w_G(S) \ge 5 \cdot 11 + 13 \cdot 7 = 146$ and $\gamma(G[S]) = 5$, by the Key Lemma, $5 \cdot 20 = 100 > 146 - 46 = 100$, which is a contradiction.

Case 1b: $\deg_G(v_5) = 3$.
Since $x = 2$, v_6, v_7 are 2-vertices; see Figure 19.

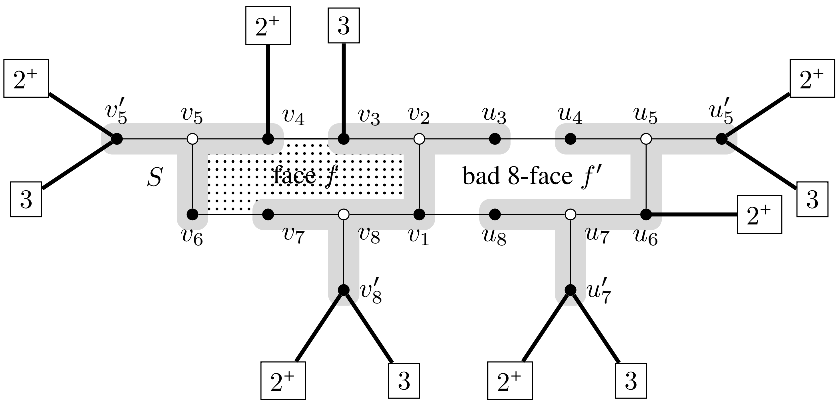


Figure 19: The configuration from Lemma 5.6, Case 1b.

Let $S = \{v_1, \dots, v_8, u_3, \dots, u_8, v'_5, v'_8, u'_5, u'_7\} = N_G[v_2, v_5, v_8, u_5, u_7]$.

Claim 5.8. *$G - S$ has at most two isolated vertices.*

Proof of Claim 5.8. Suppose $\deg_{G-S}(w) = 0$ for some $w \in V(G) \setminus S$. Since G has girth at least 8, $N_G(w)$ can contain at most one vertex from $\{v_3, v_4, v'_5, v'_8\}$, and at most one vertex from $\{u'_5, u_6, u'_7\}$. Thus $\deg_G(w) = 2$. Also, since G has girth at least 8, the following are the only possibilities for $N_G(w)$.

(1) $N_G(w) = \{v'_8, u'_5\}$.

(2) $N_G(w) = \{v_4, u'_5\}$.

(3) $N_G(w) = \{v'_5, u'_5\}$.

(4) $N_G(w) = \{v_4, u'_7\}$.

(5) $N_G(w) = \{v'_5, u'_7\}$.

(6) $N_G(w) = \{v_4, u_6\}$.

(7) $N_G(w) = \{v_5', u_6\}$.

Suppose on the contrary that $\deg_{G-S}(w_1) = \deg_{G-S}(w_2) = \deg_{G-S}(w_3) = 0$ for $w_1, w_2, w_3 \in V(G) \setminus S$.

Suppose w_1 satisfies (1) so that $N_G(w_1) = \{v_8', u_5'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (2) and (3). By the planarity of G, w_2 and w_3 cannot satisfy (4), (5), (6), and (7). Thus, w_2 and w_3 satisfy none of the conditions (1) through (7), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (1).

Suppose w_1 satisfies (2) so that $N_G(w_1) = \{v_4, u_5'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (3). Since $\deg_G(v_4) = 3$, w_2 and w_3 cannot satisfy (4) and (6). Thus, w_2 and w_3 satisfy (5) and (7) each, which is a contradiction to Lemma 4.7. By symmetry, none of w_1, w_2, w_3 satisfy (2).

Suppose w_1 satisfies (3) so that $N_G(w_1) = \{v_5', u_5'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (5) and (7). By the planarity of G, x_2 and w_3 cannot satisfy (4) and (6). Thus, w_2 and w_3 satisfy none of the conditions (1) through (7), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (3).

Suppose w_1 satisfies (4) so that $N_G(w_1) = \{v_4, u_7'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (5). Since $\deg_G(v_4) = 3$, w_2 and w_3 cannot satisfy (6). Thus, w_2 and w_3 both satisfy (7), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (4).

Suppose w_1 satisfies (5) so that $N_G(w_1) = \{v_5', u_7'\}$. By Lemma 4.7, w_2 and w_3 cannot satisfy (7). Thus, w_2 and w_3 both satisfy (6), which is a contradiction. By symmetry, none of w_1, w_2, w_3 satisfy (5).

Now, since w_1, w_2, and w_3 cannot satisfy conditions (1) through (5), all of w_1, w_2 and w_3 satisfy (6) or (7), which is a contradiction. □

By Claim 5.8, $c_G(S) \leq 11 \cdot 4 + 2 = 46$. Since $w_G(S) = 5 \cdot 11 + 13 \cdot 7 = 146$ and $\gamma(G[S]) = 5$, by the Key Lemma, $100 > 146 - 46 = 100$, which is a contradiction.

Case 2: $x = 1$ and $y \geq 2$.
Let the vertices of f be $v_1, \ldots, v_8$. Let $\deg_G(v_1) = 2$. Since $x = 1$, all other vertices on f are of degree 3. By Lemma 5.3, there are only four possible edges that f can have in common with a bad 8-face (v_3v_4, v_4v_5, v_5v_6, v_6v_7). Since bad 8-faces cannot share a contributing vertex by Lemma 5.4, there are only two different configurations to consider. However, if v_3, v_4 and v_5, v_6 are contributing vertices of f, then neither of v_3, v_4 has a neighbor on f that is a 3-vertex with only degree 3 neighbors; see Figure 20. Since this contradicts Lemma 5.3, there is only one configuration left to consider.

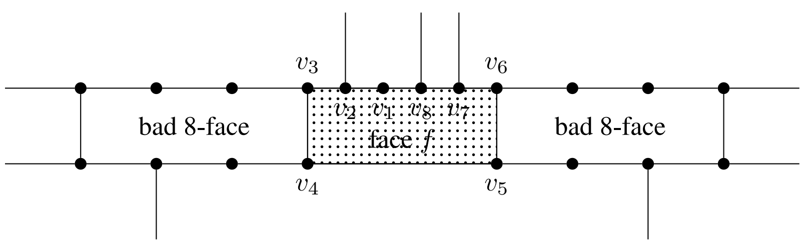


Figure 20: An example of the impossible configuration from Lemma 5.6, Case 2.

The remaining configuration is sketched in Figure 21.

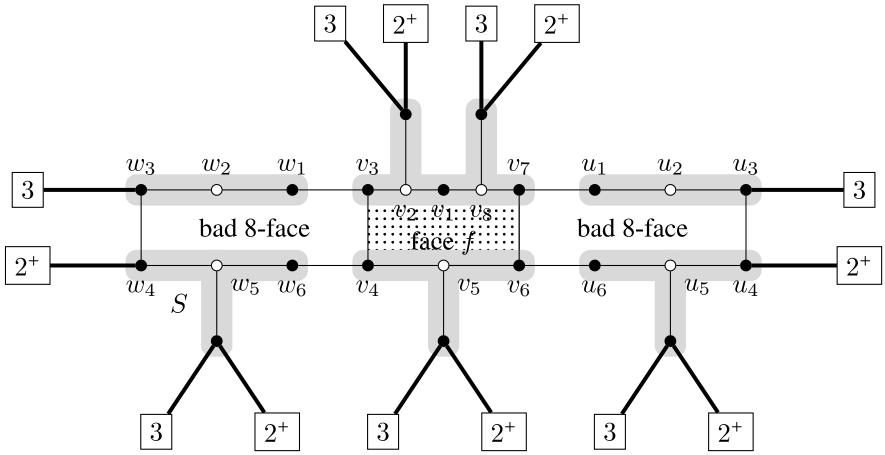


Figure 21: The remaining configuration from Lemma 5.6, Case 2.

Let $S = N_G[v_2, v_5, v_8, u_2, u_5, w_2, w_5]$. Observe that there are seven 2-vertices and eighteen 3-vertices in S, and that the selection of S is independent of the positions of 2-vertices on bad 8-faces. Without loss of generality, let $\deg_G(w_5) = \deg_G(u_5) = 3$. By Lemma 4.7, we see that the neighbors of w_5 or u_5 that are not on a bad face are of degree 3. Note that v_5 and all of its neighbors are of degree 3 by Lemma 5.3. Thus $\gamma(G[S]) = 7$ and $w_G(S) \geq 7 \cdot 11 + 18 \cdot 7 = 203$. There are 14 edges between S and $G - S$. Thus, by Lemma 3.2, $c_G(S) \leq 14 \cdot 4 + \lfloor \frac{14}{2} \rfloor = 63$. The Key Lemma gives $7 \cdot 20 = 140 > 203 - 63 = 140$, which is a contradiction.

Case 3: $x = 0$ and $y \geq 3$.
Let the vertices of f be $v_1, \ldots, v_8$. Since $x = 0$, all vertices on f are of degree 3. Lemmas 5.3 and 5.4 imply that in fact $y \leq 3$, and there is only one possible configuration; see Figure 22.

Let S be as in the Figure 22, and notice that it does not depend on the positions of the 2-vertices on the bad 8-faces. Thus $\gamma(G[S]) = 9$ and $w_G(S) \geq 9 \cdot 11 + 22 \cdot 7 = 253$. There are 16 edges between S and $G - S$. Thus, by Lemma 3.2, $c_G(S) \leq 16 \cdot 4 + \lfloor \frac{16}{2} \rfloor = 72$. The Key Lemma gives $9 \cdot 20 = 180 > 253 - 72 = 181$, which is a contradiction. This concludes the proof of Lemma 5.6. □

Lemma 5.9. *If a 9-face f is incident with x 2-vertices and y bad 8-faces, then $x + y \leq 3$.*

Proof. Since f is a 9-face, $x \leq 3$ by Lemma 4.12. Suppose that $x + y \geq 4$. We distinguish between the following cases. Let the vertices of f be $v_1, \ldots, v_9$.

Case 1: $x = 3$ and $y \geq 1$.
Since $y \geq 1$, we may assume that v_1, v_2 are contributing vertices of f that belong to a bad 8-face; see Figure 23. By Lemma 5.3, both v_3 and v_9 are 3-vertices, and (without loss of generality) every neighbor of v_3 has degree 3, so $\deg_G(v_4) = 3$. Now f has at least three 2-vertices among v_5, v_6, v_7, v_8, which is not possible by Lemmas 4.6 and 4.7.

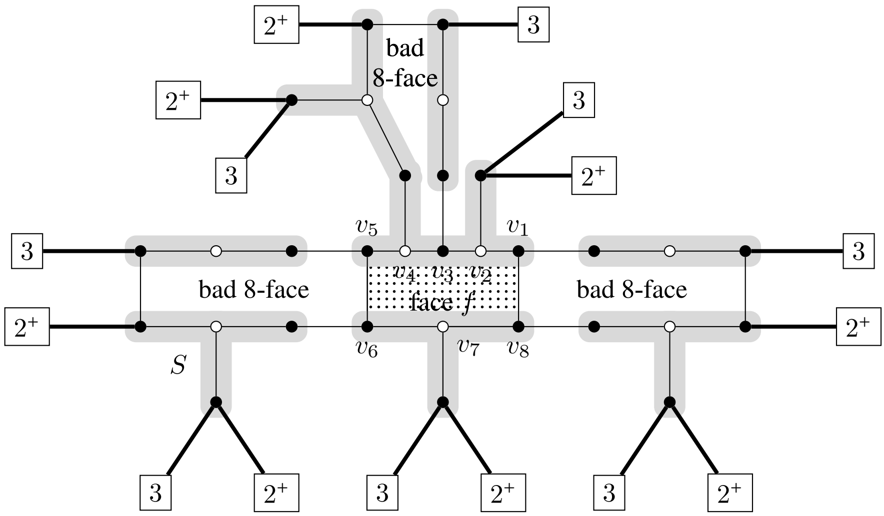


Figure 22: The configuration from Lemma 5.6, Case 3.

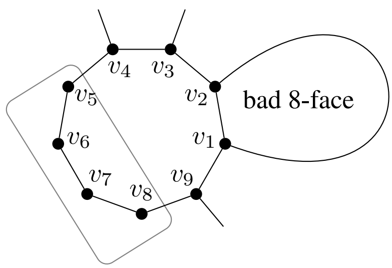


Figure 23: The impossible configuration from Case 1 of Lemma 5.9.

Case 2: $x = 2$ and $y \geq 2$.
Since $y \geq 2$, f is incident with at least two bad 8-faces. If v_1, v_2 and v_3, v_4 are contributing vertices, then by Lemma 5.3, v_5, v_6, v_9, v_8 are all 3-vertices. Thus f can have at most one 2-vertex, v_7; see the left of Figure 24. Similarly, if v_1, v_2 and v_5, v_6 are contributing vertices, then by Lemma 4.7, v_3, v_4, v_7, v_9 are all 3-vertices, and f can again have at most one 2-vertex, v_8; see the right of Figure 24.

By symmetry, the only remaining option is that v_1, v_2 and v_4, v_5 are contributing vertices. By Lemma 5.3, v_3, v_6, v_9 are 3-vertices. Since $x = 2$, it follows that $\deg_G(v_7) = \deg_G(v_8) = 2$. Thus by Lemma 4.7, v_6 and v_9 have a neighbor of degree 3 outside f, and by Lemma 5.3, v_3 has a neighbor of degree 3 outside f. Let S be as in Figure 25. Notice that S does not depend on the position of 2-vertices in the bad 8-faces. (Lemma 4.8 actually forces the position as in the figure, but we do not need this.)

Clearly, $\gamma(G[S]) = 7$ and $w_G(S) \geq 8 \cdot 11 + 18 \cdot 7 = 214$. There are 14 edges between S and $V(G) \setminus S$. Thus, by Lemma 3.2, $c_G(S) \leq 14 \cdot 4 + \lfloor \frac{14}{2} \rfloor = 63$. The Key Lemma gives $7 \cdot 20 = 140 > 214 - 63 = 151$, which is a contradiction.

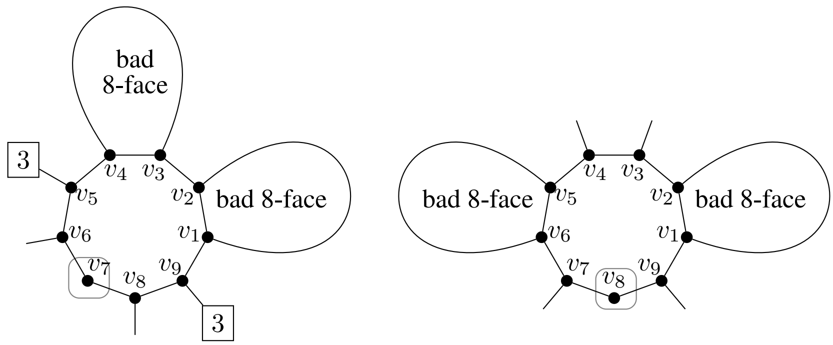


Figure 24: The impossible configurations from Case 2 of Lemma 5.9.

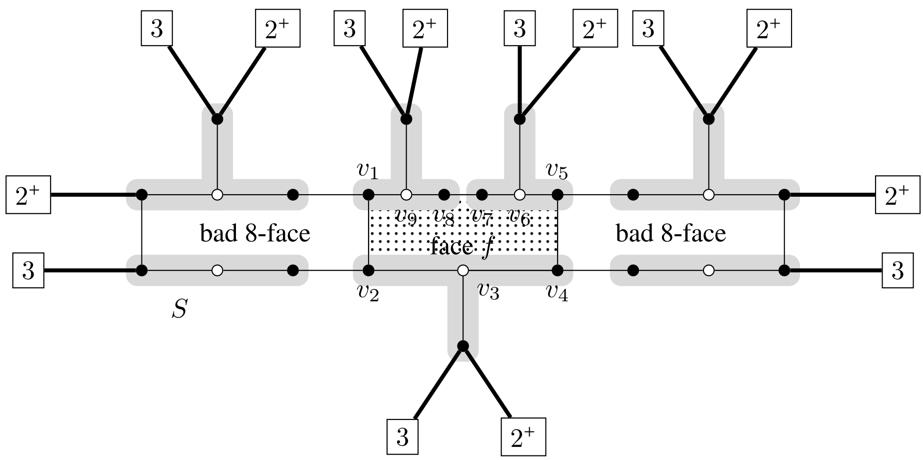


Figure 25: The configuration from Lemma 5.9, Case 2.

Case 3: $x = 1$ and $y \geq 3$.
Since $x = 1$, we may assume that $\deg_G(v_1) = 2$, and this is the only 2-vertex on f. By Lemma 4.7, v_2 and v_9 are adjacent to a 3-vertex outside f. Thus only pairs $\{v_3, v_4\}$, $\{v_4, v_5\}$, $\{v_5, v_6\}$, $\{v_6, v_7\}$, and $\{v_7, v_8\}$ can form a pair of contributing vertices of f. But by Lemma 5.4 and since $y \geq 3$, the unique option is that $\{v_3, v_4\}$, $\{v_5, v_6\}$, and $\{v_7, v_8\}$ are pairs of contributing vertices. But this gives a contradiction with Lemma 5.3 since for example v_2 and v_5 are both adjacent to some 2-vertex; see Figure 26.

Case 4: $x = 0$ and $y \geq 4$.
Since $y \geq 4$ and Lemma 5.4, there is at least one pair of bad 8-faces such that there is only one edge of f between the bad 8-faces. We may thus assume that $\{v_1, v_2\}$ and $\{v_3, v_4\}$ are pairs of contributing vertices. By Lemma 5.3, vertices v_5 and v_9 are adjacent to a 3-vertex outside of f; see Figure 27. Thus only $\{v_6, v_7\}$ and $\{v_7, v_8\}$ remain as possible pairs of contributing vertices, but by Lemma 5.4, this is impossible. Thus $y \leq 3$, which is a contradiction.

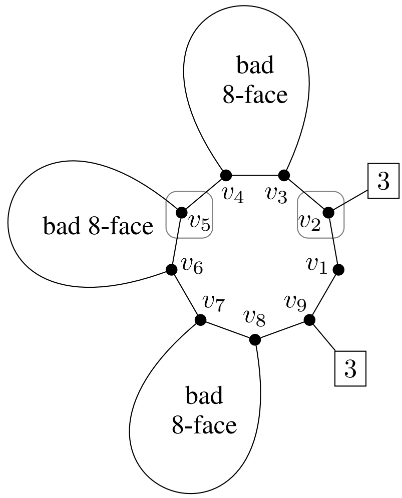


Figure 26: The impossible configuration from Case 3 of Lemma 5.9.

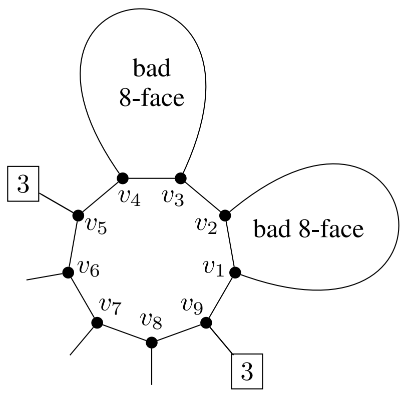


Figure 27: The impossible configuration from Case 4 of Lemma 5.9.

This concludes the proof of Lemma 5.9. □

Lemma 5.10. *If a* 10*-face* f *is incident with* x 2*-vertices and* y *bad* 8*-faces, then* $x + y \leq 4$.

Proof. Let the vertices of f be $v_1, \ldots, v_{10}$. Since f is a 10-face, $x \leq 4$ by Lemma 4.11. Suppose that $x + y \geq 5$. We distinguish between the following cases.

Case 1: $x = 4$ and $y \geq 1$.
Since $y \geq 1$, we may assume that $\{v_1, v_2\}$ is a pair of contributing vertices. Thus, by Lemma 5.3, v_3 and v_{10} are 3-vertices; see Figure 28. Among the remaining six consecutive vertices of f, four must be 2-vertices (since $x = 4$). But this contradicts Lemma 4.8.

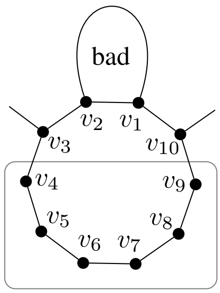


Figure 28: The contradictory configuration from Case 1 of Lemma 5.10. To keep the figure clearer, we write "bad" instead of "bad 8-face".

Case 2: $x = 3$ and $y \geq 2$.
Since $y \geq 2$, we may assume that $\{v_1, v_2\}$ is a pair of contributing vertices. Thus, by Lemma 5.3, v_3 and v_{10} are 3-vertices, as is one of v_4, v_9. Without loss of generality, let v_4 be a 3-vertex; see Figure 29. Since $x = 3$, there are three 2-vertices among $v_5, \ldots, v_9$, and since $y \geq 2$, there is another bad 8-face incident with f. The only possible contributing pairs are $\{v_i, v_{i+1}\}$, $i \in \{4, \ldots, 9\}$.

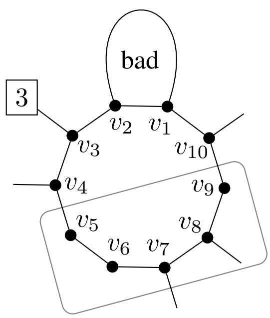


Figure 29: The contradictory configuration from Case 2 of Lemma 5.10.

It follows from Lemmas 4.6 and 4.7 that $(2, 2, 3, 3, 2)$ and $(2, 3, 3, 2, 2)$ are the only possible degree sequences of vertices $v_5, \ldots, v_9$. So $\deg_G(v_5) = \deg_G(v_9) = 2$, $\deg_G(v_7) = 3$, and one of v_6, v_8 is a 2-vertex and the other a 3-vertex. The following argument is analogous for both cases, so we assume that $\deg_G(v_6) = 2$. Thus, the only remaining possible pair of contributing vertices is $\{v_7, v_8\}$, but since both of these vertices have a 2-vertex as a neighbor, they cannot be contributing vertices. Thus $y = 1$, which is a contradiction.

Case 3: $x = 2$ and $y \geq 3$.
Since $x = 2$, the two 2-vertices can either be adjacent or not. Note that by Lemma 4.7 in the second case, the 2-vertices are at distance at least 3.

Case 3a: The 2-vertices are adjacent.
Without loss of generality, let v_1 and v_2 be the 2-vertices of f. By Lemma 4.7, all neighbors of v_3 and v_{10} are 3-vertices. Thus, there is only five remaining possible contributing pairs:

$\{v_i, v_{i+1}\}$, $i \in \{4, \dots, 8\}$; see Figure 30. Since $y \geq 3$ and by Lemma 5.4, the position of the bad 8-faces must be on $\{v_4, v_5\}, \{v_6, v_7\}, \{v_8, v_9\}$. However, this gives a contradiction with Lemma 5.3, since for example, both neighbors of the pair $\{v_6, v_7\}$ on f are adjacent to a 2-vertex.

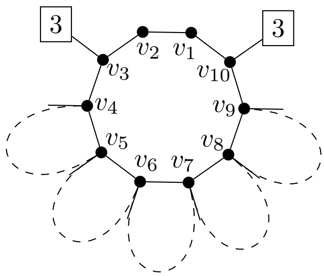
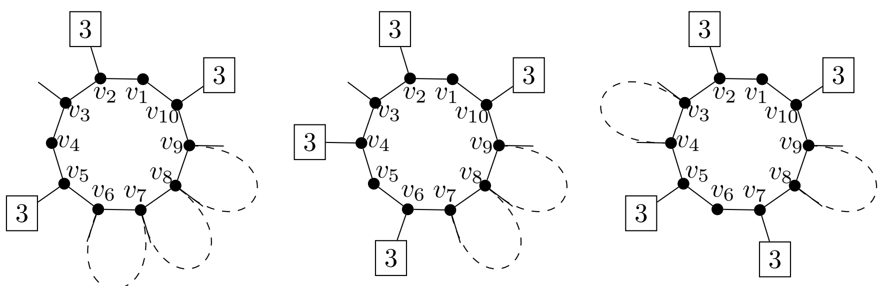


Figure 30: The contradictory configuration from Case 3.a of Lemma 5.10.

Case 3b: The 2-vertices are not adjacent.
Without loss of generality, let v_1 be a 2-vertex. By Lemma 4.7, all other neighbors of v_2 and v_{10} are 3-vertices. Thus, using symmetry, we only need to consider the cases when the other 2-vertex is one of v_4, v_5, v_6.

If $\deg_G(v_4) = 2$, then by Lemma 4.7, all other neighbors of v_5 are 3-vertices and thus, the only possible contributing pairs are $\{v_6, v_7\}, \{v_7, v_8\}, \{v_8, v_9\}$, so by Lemma 5.4, $y \leq 2$, which is a contradiction; see Figure 31(i).



(i) (ii) (iii)

Figure 31: The contradictory configurations from Case 3.b of Lemma 5.10.

If $\deg_G(v_5) = 2$, then by Lemma 4.7, all other neighbors of v_4 and v_6 are 3-vertices and thus, the only possible contributing pairs are $\{v_7, v_8\}, \{v_8, v_9\}$, so by Lemma 5.4, $y \leq 1$, which is a contradiction; see Figure 31(ii).

If $\deg_G(v_6) = 2$, then by Lemma 4.7, all other neighbors of v_5 and v_7 are 3-vertices and thus, the only possible contributing pairs are $\{v_3, v_4\}, \{v_8, v_9\}$, so by Lemma 5.4, $y \leq 2$, which is a contradiction; see Figure 31(iii).

Case 4: $x = 1$ and $y \geq 4$.
Without loss of generality, let v_1 be the only 2-vertex of f (since $x = 1$). By Lemma 5.3, the

pairs $\{v_2, v_3\}$ and $\{v_9, v_{10}\}$ cannot be pairs of contributing vertices; see Figure 32. Thus, the only possible pairs of contributing vertices are the six pairs $\{v_i, v_{i+1}\}$, $i \in \{3, \dots, 8\}$. Since by Lemma 5.4, no two bad 8-faces share a contributing vertex, we have $y \leq 3$, which is a contradiction.

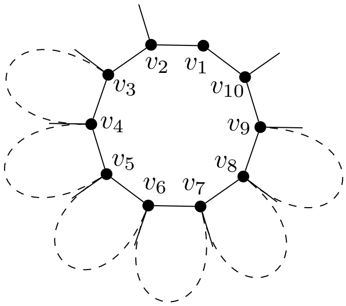


Figure 32: The contradictory configuration from Case 4 of Lemma 5.10.

Case 5: $x = 0$ and $y \geq 5$.
Since $x = 0$, all vertices of f are 3-vertices. By Lemma 5.4, no two bad 8-faces share a vertex on f. Thus $y \leq 5$, which imples $y = 5$, and without loss of generality, $\{v_1, v_2\}$, $\{v_3, v_4\}$, $\{v_5, v_6\}$, $\{v_7, v_8\}$, $\{v_9, v_{10}\}$ are pairs of contributing vertices; see Figure 33. But this contradicts Lemma 5.3, since for example, both neighbors of the pair $\{v_1, v_2\}$ on f are adjacent to one 2-vertex.

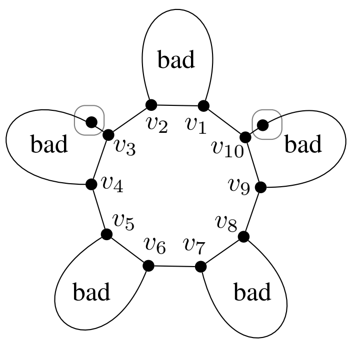


Figure 33: The contradictory configuration from Case 5 of Lemma 5.10.

This concludes the proof of Lemma 5.10. □

5.2 Discharging

For $x \in V(G) \cup F(G)$, we define the following initial charge:

$$\mu(x) = \begin{cases} 2\deg_G(x) - 6 & \text{if } x \in V(G), \\ \ell(x) - 6 & \text{if } x \in F(G). \end{cases}$$

Since G is planar, Euler's formula implies that the initial charge of G is

$$\mu(G) = \sum_{x \in V(G) \cup F(G)} \mu(x) = -12.$$

We use the following two discharging rules:

- Every face f sends charge 1 to each of the 2-vertices on the facial walk of f, counted with multiplicities.
- Every face sends charge 1 to each of the bad 8-faces incident with it.

Let $\mu^*(x)$ denote the charge of vertices and faces after applying the discharging rules.

Lemma 5.11. *For every $x \in V(G) \cup F(G)$, $\mu^*(x) \geq 0$.*

Proof. By Lemma 4.5, G has no vertex of degree 1. A 2-vertex that is incident with two different faces receives charge 1 from each of them. A 2-vertex that is incident with only one face f appears on the facial walk of f twice, so it receives charge 2 from f. Thus $\mu^*(v) = (2 \cdot 2 - 6) + 2 \cdot 1 = 0$. If v is a vertex of degree 3, then $\mu^*(v) = \mu(v) = 0$.

If f is a k-face, $k \geq 11$, then by Lemma 5.5, the facial walk of f contains at most $\lfloor \frac{k}{2} \rfloor$ 2-vertices and bad 8-faces. Thus

$$\mu^*(f) \geq (k-6) - \left\lfloor \frac{k}{2} \right\rfloor = \left\lceil \frac{k}{2} \right\rceil - 6 \geq 0$$

since $k \geq 11$.

If f is a 10-face, then by Lemma 5.10, f is incident with at most four 2-vertices and bad 8-faces. Hence, $\mu^*(f) \geq (10-6) - 4 = 0$. If f is a 9-face, then by Lemma 5.9, f is incident with at most three 2-vertices and bad 8-faces. Hence, $\mu^*(f) \geq (9-6) - 3 = 0$. If f is a good 8-face, then by Lemma 5.6, f is incident with at most two 2-vertices and bad 8-faces. Hence, $\mu^*(f) \geq (8-6) - 2 = 0$.

If f is a bad 8-face, then it is incident with three 2-vertices and one contributing face. By Lemma 5.4, it follows that f cannot be the contributing face of another bad 8-face. So $\mu^*(f) = (8-6) - 3 + 1 = 0$. □

Clearly it holds that

$$-12 = \mu(G) = \mu^*(G) \geq 0,$$

which is a contradiction. This concludes the proof of Theorem 1.3.

6 Different weights

It is not clear how to measure how good a certain bound for the domination number of subcubic planar graphs is. The result in Theorem 1.3 minimizes the weight of 3-vertices, but there are other bounds we can prove with the same method. One of them is the following.

Theorem 6.1. *If G is a planar subcubic graph with girth at least 9, then*

$$17\gamma(G) < 17n_0(G) + 13n_1(G) + 9n_2(G) + 6n_3(G).$$

The proof of Theorem 6.1 uses the same ideas as the proof of Theorem 1.3 with the following modifications. Let w_i be the coefficient of $n_i(G)$ in

$$\gamma(G) \leq w_0 n_0(G) + w_1 n_1(G) + w_2 n_2(G) + w_3 n_3(G).$$

(1) Lemmas 4.1 – 4.6 (and their proofs) stay the same.

(2) To prove Lemma 4.5, additional configurations must be considered first, to deduce more information about the degree of the vertices in $N_G(S) \setminus S$. In particular, we first prove that the sequences $(2, 2, 3, 3_3^1)$ and $({}_2^23, 3_3^1)$ are reducible, and only then proceed to proving Lemma 4.5. The reason for this lies in the fact that in Theorem 6.1 $w_2 - w_3 < w_1 - w_2$, while in Theorem 1.3 these two differences are the same.

(3) Lemma 4.7 *does not hold*. Instead, we first prove that there is no 3-vertex with all three neighbors of degree 2, and then we prove that $({}_2^33, 2, 2, 3_2^3)$, $(2, 2, 3, 2, 2, 3)$, $({}_2^23, 3_2^2)$, $(2, 2, 3, 2, 3, 2)$, $(2, 2, 3, 3, 2, 2)$, and $(3, 2, 3, 2, 3)$ are reducible, to conclude the result in Lemma 4.8.

(4) Lemma 4.9 is modified to consider only 9^+-faces. Since we do not have Lemma 4.7, the proof becomes much more technical.

(5) Lemma 4.10 stays the same.

(6) Lemmas 4.11 and 4.12 stay the same, but the proofs are slightly different (since we do not have Lemma 4.7).

(7) Since in Theorem 6.1, we only consider graphs with girth at least 9, Lemmas 5.2 – 5.10 are not needed.

(8) The discharging remains the same, except that we do not consider bad 8-faces and we restrict our focus to 9^+-faces only (dealing with 8-faces in this case seems to be more tricky than in Theorem 1.3).

Writing down the conditions needed to satisfy the steps in our proof using weights $w_0, \dots, w_3$ instead of concrete values for weights, we obtain a family of bounds instead of only the bound from Theorem 1.3 (this can be easily done using a computer). As already mentioned, minimizing w_3 gives the set of weights from Theorem 1.3, and to obtain the weights from Theorem 6.1, we can for example minimize $w_2 + 6w_3$.

7 Conclusion

For subcubic planar graphs with a smaller girth restriction, it is not clear what the best upper bound on the domination number is. The method of Theorem 1.3 might be able to be pushed to include girths 6 and 7 with significantly more work, but there seem to be barriers of our method to handling cycles of length less than 6.

For subcubic planar graphs with no girth restriction, the upper bound of Theorem 1.3 no longer holds. Through an exhaustive computer search of subcubic planar graphs on at most 19 vertices (using SageMath [25] and a customized version of `geng` from the `nauty` package [21]), we found two graphs (with girth 3 or 4) that exceed the upper bound of Theorem 1.3 (see Figure 34). Thus any theorem proving the upper bound of Theorem 1.3 must require at least girth 5.

However, we believe that these are sporadic small examples. The question remains of what the best asymptotic upper bound is for subcubic planar graphs with no girth restriction. By considering disjoint copies of K_1, K_2, C_4, or $K_3 \square K_2$, no upper bound can improve the individual coefficients in $\gamma(G) \le n_0(G) + \frac{1}{2}n_1(G) + \frac{1}{2}n_2(G) + \frac{1}{3}n_3(G)$. However, this bound is not possible in general, as can be seen by the following two infinite families.

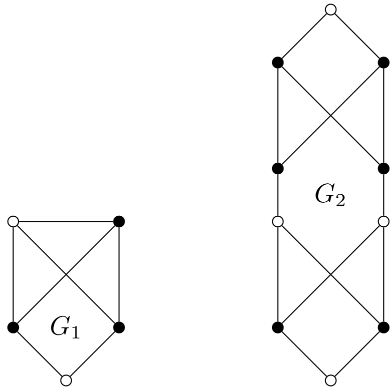


Figure 34: The only subcubic planar graphs with no girth restriction and at most 19 vertices that exceed the upper bound of Theorem 1.3. The graph on the left has girth 3, domination number $\gamma(G_1) = 2$, and the upper bound of Theorem 1.3 gives $\frac{39}{20}$. The graph on the right has girth 4, domination number $\gamma(G_2) = 4$, and the upper bound gives $\frac{39}{10}$. Examples of minimal dominating sets are marked with white vertices. Note that both graphs are planar.

For $n \geq 3$, let $C_n^\star$ be the graph obtained by attaching a leaf to each vertex of C_n. The graph $C_n^\star$ has $2n$ vertices (n 1-vertices and n 3-vertices) and $\gamma(C_n^\star) = n$. Then the family $\{C_n^\star\}_{n=3}^\infty$ shows that an upper bound on the domination number cannot simultaneously have a coefficient of $\frac{1}{2}$ on $n_1(G)$ and a coefficient of $\frac{1}{3}$ on $n_3(G)$. More precisely, the coefficient on $n_1(G)$ has to be at least $1 - w_3$, where w_3 is the coefficient on $n_3(G)$. For example, if $w_3 = \frac{7}{20}$, the best we can hope for is a coefficient of $\frac{13}{20}$ on $n_1(G)$.

For $n \geq 3$, let H_n be the graph obtained from the cycle C_n by attaching the graph G_2 from Figure 34 to each vertex with an edge (see Figure 35 for a picture of H_6). The graph H_n has $11n$ vertices: n 2-vertices and $10n$ 3-vertices. Since $\gamma(G_2) = 4$ (see Figure 34) and vertices adjacent to the cycle C_n can be in the dominating set, we have $\gamma(H_n) \leq 4n$. To see that $\gamma(H_n) \geq 4n$, notice that each gadget consisting of G_2 and the adjacent vertex on C_n contains at least four vertices from every minimal dominating set of H_n. Thus, $\gamma(H_n) = 4n$. Note that the family $\{H_n\}_{n=3}^\infty$ shows that an upper bound on the domination number cannot simultaneously have a coefficient of $\frac{1}{2}$ on $n_2(G)$ and a coefficient of $\frac{1}{3}$ on $n_3(G)$.

The upper bound from Theorem 1.3 gives $\gamma(H_n) < \frac{81n}{20}$. The bound would become tight for H_n if the coefficient on $n_2(G)$ is lowered to $\frac{1}{2}$. We conjecture that this is optimal:

Conjecture 7.1. *There exists a constant N such that any connected subcubic planar graph with at least N vertices satisfies*

$$\gamma(G) \leq n_0(G) + \frac{13}{20}n_1(G) + \frac{1}{2}n_2(G) + \frac{7}{20}n_3(G).$$

Since there are no cubic planar graphs with girth at least 8, the bound in Theorem 1.3 does not apply to cubic planar graphs. To make progress towards Conjectures 1.1 and 1.2, and considering the new result [5], we are left to consider cubic planar graphs with small girth. Perhaps a partial result could be obtained by allowing small cycles of certain lengths (such as 3-cycles) and forbidding other small cycles (such as cycle lengths 4 through 8).

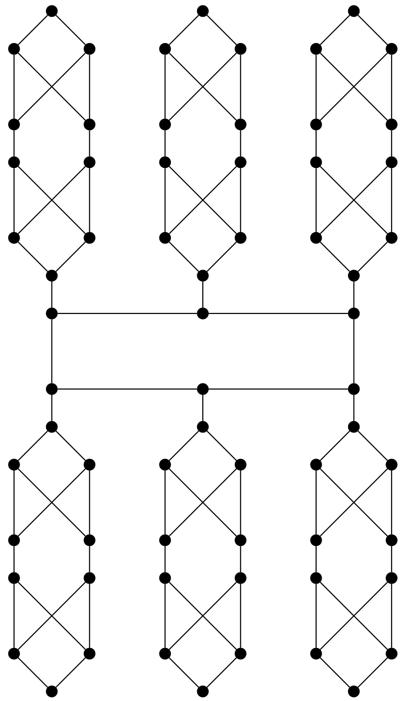


Figure 35: The graph H_6. Note that the graph is planar; it is drawn with edge-crossings only to emphasize the symmetry present.

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