



Nut digraphs

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ABSTRACT

A *nut graph* is a simple graph whose kernel is spanned by a single full vector (i.e., the adjacency matrix has a single zero eigenvalue and all non-zero kernel eigenvectors have no zero entry). We classify generalisations of nut graphs to nut digraphs: a digraph whose kernel (resp. co-kernel) is spanned by a full vector is *dextro-nut* (resp. *laevo-nut*); a *bi-nut* digraph is both laevo- and dextro-nut; an *ambi-nut* digraph is a bi-nut digraph where kernel and co-kernel are spanned by the same vector; a digraph is *inter-nut* if the intersection of the kernel and co-kernel is spanned by a full vector. It is known that a nut graph is connected, leafless and non-bipartite. It is shown here that an ambi-nut digraph is strongly connected, non-bipartite (i.e., has a non-bipartite underlying graph) and has minimum in-degree and minimum out-degree of at least 2. Refined notions of core and core-forbidden vertices apply to singular digraphs. Infinite families of nut digraphs and systematic coalescence, crossover and multiplier constructions are introduced. Relevance of nut digraphs to topological physics is discussed.

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1. Introduction

Nut graphs [52] are the graphs that have a one-dimensional nullspace where the non-trivial kernel eigenvector $x = [x_1 \dots x_n]^T \in \ker A(G)$ is *full* (i.e., it has no zero entries). This class of graphs is the subject of an extensive mathematical literature [2–4,6,8,15–21,23,27,28,32,44–48,52,53]. Nut graphs also feature in mathematical chemistry in applications of graph theory to models of electron distribution, radical reactivity and ballistic conduction in molecular π systems [24,26,49–51]. The definition has been extended from graphs to signed graphs [7].

Recently, after a talk on nut graphs given by one of us, a member of the audience raised the question of whether analogues could be found amongst the directed graphs. In other words, the questioner wanted to know whether *nut digraphs* exist. This proved to be a fruitful line of enquiry. The (undirected) nut graphs are connected, leafless (have no vertices of degree 1), non-bipartite, not edge-transitive, and non-trivial examples exist for all orders $n \geq 7$ [6,32]. A natural follow-up question would be on how these properties transfer to non-trivial nut digraphs. The present note reports our

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work to provide a full answer to both questions, which has entailed consideration of alternative definitions of nut digraphs, cataloguing small examples and devising systematic constructions for large nut digraphs.

2. Definitions

Nut graphs sit within the larger class of core graphs. A *core graph* is a singular graph for which every vertex has a non-zero entry in some non-trivial kernel eigenvector. It is traditional to exclude K_1 from both nut and core graph classes, discounting it as a trivial case [52]. Nut graphs are then precisely the core graphs of nullity 1.

A *digraph* is a pair (V, \rightarrow) , where V is a finite non-empty set of vertices and \rightarrow is an arbitrary binary relation on V . The set of vertices of a digraph G is denoted by $V(G)$ and the corresponding relation \rightarrow is denoted by \rightarrow_G . If $u \rightarrow_G v$ for some $u, v \in V(G)$, then we say that u is an *in-neighbour* of v , that v is an *out-neighbour* of u and that (u, v) is an *arc* (also called a *directed edge*) of G . The set of all arcs of G will sometimes be denoted by $E(G)$. For a vertex $u \in V(G)$, let $G^+(u)$ denote the set of all out-neighbours and $G^-(u)$ the set of all in-neighbours of u . The digraph G^R whose vertex set is $V(G)$ and where $u \rightarrow_{G^R} v$ if and only if $v \rightarrow_G u$ is called the *reverse digraph* (sometimes known as the *opposite digraph* or *converse digraph*) of G . We denote the in- and out-degrees of a vertex $u \in V(G)$ by $d^-(u)$ and $d^+(u)$, respectively. A vertex u is called a *source* if $d^-(u) = 0$ and $d^+(u) > 0$. Similarly, a vertex u is called a *sink* if $d^+(u) = 0$ and $d^-(u) > 0$. We use $d(u)$ to denote $d^-(u) + d^+(u)$. The minimal in- and out-degrees of a digraph G will be denoted by $\delta^-(G)$ and $\delta^+(G)$, respectively. The minimal degree of an (undirected) graph Γ will be denoted $\delta(\Gamma)$.

If the relation \rightarrow_G is irreflexive (in the sense that no $v \in V(G)$ satisfies $v \rightarrow_G v$) and symmetric (in the sense that for all $u, v \in V(G)$, $u \rightarrow_G v$ implies $v \rightarrow_G u$), then we call G a (simple) graph and the relation \rightarrow_G becomes the usual adjacency relation. Similarly, if \rightarrow is asymmetric (in the sense that for no pair of possibly equal vertices u and v of G do both $u \rightarrow_G v$ and $v \rightarrow_G u$ hold), then G is said to be an *oriented* (simple) graph, as it can be obtained by orienting each edge of a simple graph. The *underlying graph* of a digraph G is defined as the digraph with the same vertex-set as G and with the adjacency relation being the symmetric closure of the adjacency relation of G . Note that the underlying graph is always a simple graph.

Our definition of bipartite digraphs is the same as the one used by Brualdi [11]. We will call a digraph G *bipartite* if its vertex-set $V(G)$ can be partitioned into subsets V_1 and V_2 , such that for every arc $(u, v) \in E(G)$, either $u \in V_1$ and $v \in V_2$ or $v \in V_1$ and $u \in V_2$. Let U and W be two non-empty disjoint sets of vertices of a digraph G . Then $G[U]$ is the digraph with vertex-set U with $u_1 \rightarrow_{G[U]} u_2$ if and only if $u_1 \rightarrow_G u_2$, i.e., the digraph induced on U . Similarly, $G[U, W]$ is the digraph with vertex-set $U \cup W$ and with $v_1 \rightarrow_{G[U, W]} v_2$ if and only if $|\{v_1, v_2\} \cap U| = |\{v_1, v_2\} \cap W| = 1$ and $v_1 \rightarrow_G v_2$. Note that the digraph $G[U, W]$ is bipartite.

A digraph G can be viewed as a linear operator on the $|V(G)|$ -dimensional vector space \mathbb{R}^V of all functions $\mathbf{x}: V(G) \rightarrow \mathbb{R}$, where the action of G is given by

$$G: \mathbf{x} \mapsto (u \mapsto \sum_{v \in G^+(u)} \mathbf{x}(v)). \tag{1}$$

An element of \mathbb{R}^V is *full* if $\mathbf{x}(v) \neq 0$ for every $v \in V$. For simplicity, we write \mathbf{x}_v instead of $\mathbf{x}(v)$, and when the vertices of G are in some order, say v_0, v_1, \dots, v_{n-1} , then we identify a function $\mathbf{x} \in \mathbb{R}^V$ with the column vector $[\mathbf{x}_{v_0} \ \mathbf{x}_{v_1} \ \dots \ \mathbf{x}_{v_{n-1}}]^T$. Furthermore, for $v_i \in V$, let $\chi_i \in \mathbb{R}^V$ denote the characteristic function of v_i , and observe that $\{\chi_i : i \in \mathbb{Z}_n\}$ is a basis for \mathbb{R}^V . The matrix that corresponds to G in this basis is then precisely the *adjacency matrix* $A(G)$ of G , for which $A(G)_{ij} = 1$ if and only if $v_i \rightarrow_G v_j$ (and is 0 otherwise). Note that

$$A(G^R) = A(G)^T. \tag{2}$$

The kernel $\ker G$ of the digraph G , viewed as an operator on \mathbb{R}^V , consists of all the elements $\mathbf{x} \in \mathbb{R}^V$ satisfying

$$\sum_{u \in G^+(v)} \mathbf{x}(u) = 0 \text{ for all } v \in V(G). \tag{3}$$

Similarly, the co-kernel $\text{coker } G$ of the digraph G , defined as the kernel of the operator G^R , consists of all the elements $\mathbf{x} \in \mathbb{R}^V$ such that

$$\sum_{u \in G^-(v)} \mathbf{x}(u) = 0 \text{ for all } v \in V(G). \tag{4}$$

If $\mathbf{x} \in \mathbb{R}^V$ is identified with the corresponding vector in \mathbb{R}^n , then $\ker G = \{\mathbf{x} \in \mathbb{R}^n : A(G)\mathbf{x} = \mathbf{0}\}$ and $\text{coker } G = \ker G^R = \{\mathbf{x} \in \mathbb{R}^n : A(G)^T\mathbf{x} = \mathbf{0}\} = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T A(G) = \mathbf{0}^T\}$. In other words, the kernel and the co-kernel of a digraph are then precisely the kernel and the co-kernel of the adjacency matrix of G . The nullity of a digraph G , denoted $\eta(G)$, is defined as $\eta(G) = \dim \ker G$. Note that $\eta(G) = \dim \ker G = \dim \text{coker } G$. A digraph G is called *singular* if $\eta(G) > 0$ and *non-singular* otherwise. In other words, G is non-singular if it is invertible as an operator in the sense of (1). Recall that for an undirected graph, its adjacency matrix is symmetric, and the algebraic multiplicity of an eigenvalue matches its geometric multiplicity. In particular, the nullity of an undirected graph equals the algebraic multiplicity of the 0 eigenvalue. In the directed case, this is no longer true. With the definition adopted in this paper, the nullity of a digraph is the geometric multiplicity of the 0 eigenvalue, but not necessarily its algebraic multiplicity.

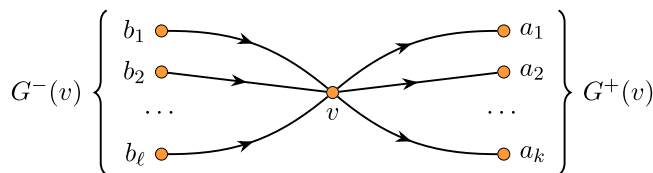


Fig. 1. Local conditions in nut digraphs. Entries on $G^+(v)$ of $\mathbf{x} \in \ker G$ are labelled a_1, \dots, a_k and entries on $G^-(v)$ of $\mathbf{y} \in \text{coker } G$ are labelled b_1, \dots, b_ℓ . For a dextro-nut digraph, the local condition is $a_1 + \dots + a_k = 0$ and for a laevo-nut $b_1 + \dots + b_\ell = 0$. For the underlying graph the local condition would simply be the sum of the two: $a_1 + \dots + a_k + b_1 + \dots + b_\ell = 0$.

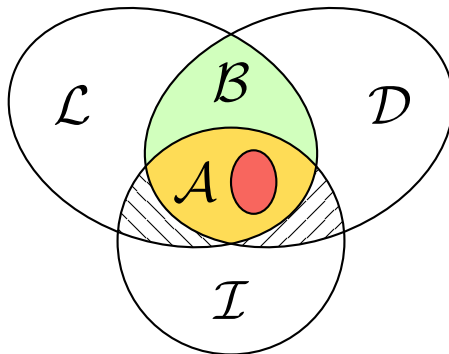


Fig. 2. Venn diagram of relationships amongst notions of nut digraphs. Symbols \mathcal{D} , \mathcal{L} , \mathcal{B} , \mathcal{A} and \mathcal{I} denote dextro-, laevo-, bi-, ambi- and inter-nut digraphs, respectively. Note that the usual nut graphs occupy the small red oval inside the ambi-nut region, and that the hatched regions are empty.

We are now in a position to discuss plausible generalisations of the notion of a nut graph to digraphs. Each of the generalisations that we propose below will have the property that when applied to a digraph which is a graph (that is, the adjacency relation of which is symmetric) it will coincide with the standard definition of a nut graph.

Definition 1. A digraph is called a *dextro-nut* provided it has a one-dimensional kernel spanned by a full vector. Similarly, a digraph is a *laevo-nut* if its co-kernel is one-dimensional and spanned by a full vector.

Observe that G is a laevo-nut digraph if and only if its reverse G^R is a dextro-nut digraph. Note that the conditions (3) and (4) are refinements of the usual *local condition* for kernel eigenvectors in the setting of undirected graphs; see Fig. 1.

Definition 2. A digraph which is both a dextro-nut and a laevo-nut is a *bi-nut* digraph.

Definition 3. A bi-nut digraph whose kernel and the co-kernel are spanned by the same vector is an *ambi-nut* digraph.

Both kernel and co-kernel of a dextro-nut digraph are one-dimensional. Hence, if a dextro-nut digraph is not a bi-nut digraph, the vector that spans its co-kernel must contain some zero entries. Likewise, *mutatis mutandis*, for laevo-nut digraphs.

Finally, one might consider digraphs for which the intersection of the kernel and the co-kernel is one-dimensional and is spanned by a full vector. Such an object could be called an inter-nut digraph.

Definition 4. A digraph G is an *inter-nut* digraph if $\ker G \cap \text{coker } G$ is one-dimensional and spanned by a full vector.

Note that a digraph is an ambi-nut if and only if it is both a bi-nut and an inter-nut. Moreover, an inter-nut that is a dextro-nut is automatically also a laevo-nut and therefore an ambi-nut. Fig. 2 gives an overview of the relations between these nut digraph classes.

In this paper, we will be concerned mainly with the strongest of these generalisations of nut graphs, namely with the ambi-nut digraphs. However, we will prove results in as general a form as possible, thus providing results for the weaker forms of nut digraph *en passant*.

3. Examples and enumeration

In this section, we give small examples for the five distinct notions of nut digraphs introduced above. To begin, we also enumerate nut digraphs of small orders.

Table 1
Enumeration of nut digraphs among oriented graphs on $n \leq 8$ vertices.

n	$ \mathcal{U}_n $	$ \mathcal{O}_n $	$ \mathcal{D}_n $	$ \mathcal{B}_n $	$ \mathcal{A}_n $
3	2	5	0	0	0
4	6	34	1	0	0
5	21	535	4	0	0
6	112	20848	153	2	2
7	853	2120098	17 170	21	1
8	11 117	572849763	5 579 793	9592	104

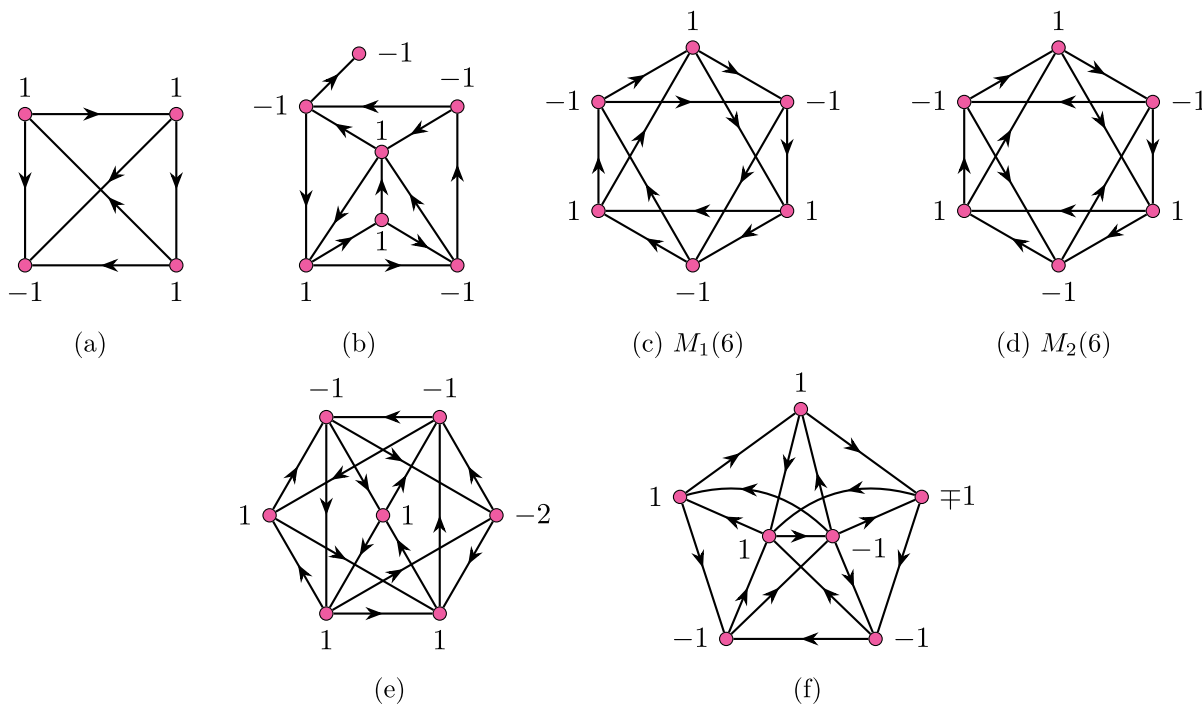


Fig. 3. Examples of nut digraphs among *oriented* graphs: (a) the smallest dextro-nut digraph, (b) a dextro-nut digraph with a leaf, (c) & (d) the two smallest ambi-nut digraphs (labelled $M_1(6)$ and $M_2(6)$), (e) the unique ambi-nut digraph on 7 vertices, (f) one of the 20 bi-nut digraphs on 7 vertices that are not ambi-nuts. In all panels, vertex labels show the kernel vector. In case (f), the vectors from the kernel and co-kernel differ in just one entry (labelled ∓ 1 , with -1 corresponding to the kernel vector).

Let n be the number of vertices of the digraphs under consideration. Then let \mathcal{U}_n be the class of undirected graphs, \mathcal{O}_n the class of oriented graphs, and $\mathcal{D}_n, \mathcal{B}_n$ and \mathcal{A}_n be the classes of dextro-nut, bi-nut and ambi-nut digraphs, respectively, all on n vertices.

Throughout this paper, a digraph will be called *regular* (resp. *k-regular*) if its underlying graph is regular (resp. *k-regular*). We will use \mathcal{O}_n^d to denote the class of d -regular digraphs of order n . Moreover, let $\mathcal{O}(\Gamma)$ denote the class of oriented graphs whose underlying (undirected) graph is Γ . We use a similar notation for the classes of nut digraphs, e.g., $\mathcal{B}(\Gamma)$ is the class of bi-nut digraphs for which the underlying graph is Γ .

Table 1 lists counts of dextro-, bi- and ambi-nut digraphs among *oriented* graphs on up to 8 vertices. These numbers were obtained naïvely, i.e., by generating all oriented graphs on n vertices and filtering out nut digraphs. The counts of laevo-nut digraphs are of course equal to those of the dextro-nut digraphs, order by order. Fig. 3 shows small examples of nut digraphs from Table 1.

Table 2 lists counts of dextro-, bi- and ambi-nut digraphs among 4-regular oriented graphs on up to 11 vertices. Some of these digraphs are shown in Fig. 4. Note that the graphs (e) to (g) in Fig. 4 all have the circulant $\text{Circ}(8, \{1, 2\})$ as their underlying graph. By $\text{Circ}(n, S)$, where $S \subseteq \{1, 2, \dots, n-1\}$, we denote the graph on vertex set \mathbb{Z}_n , such that (i, j) is an edge if and only if $|i - j| \in S$, where addition is performed modulo n . (For further information on circulants see, e.g. [21].) Circulant underlying graphs are also encountered in Fig. 3. This suggests definition of the following families.

Definition 5. For $k \in \{1, 2, 3\}$ and n even, let $M_k(n)$ be a digraph with the vertex set $\{0, 1, \dots, n-1\}$ and the arc set

$$E(M_1(n)) = \{(i, i + 1) \mid 0 \leq i < n\} \cup \{(i, i + 2) \mid 0 \leq i < n\},$$

Table 2
Enumeration of nut digraphs among 4-regular oriented graphs on $n \leq 11$ vertices.

n	$ \mathcal{U}_n^4 $	$ \mathcal{O}_n^4 $	$ \mathcal{D}_n^4 $	$ \mathcal{B}_n^4 $	$ \mathcal{A}_n^4 $
5	1	12	0	0	0
6	1	112	4	2	2
7	2	1602	9	0	0
8	6	32263	202	27	5
9	16	748576	2255	0	0
10	59	19349594	33034	2072	32
11	265	548123668	436947	0	0

$$E(M_2(n)) = \{(i, i + 1) \mid 0 \leq i < n\} \cup \{(i, i + 2) : 0 \leq i < n, i \text{ even}\} \cup \{(i + 2, i) \mid 0 < i < n, i \text{ odd}\},$$

$$E(M_3(n)) = \{(i, i + 1) \mid 0 \leq i < n\} \cup \{(i + 2, i) \mid 0 \leq i < n\},$$

where addition is done modulo n . (See Figs. 3 and 4 for examples.)

Note that all three digraphs $M_1(n)$, $M_2(n)$ and $M_3(n)$ have the circulant $\text{Circ}(n, \{1, 2\})$, the skeleton of the $\frac{n}{2}$ -antiprism, as their underlying graph. Note that the skeleton of a k -antiprism is a 4-regular graph on $2k$ vertices (for details, see [43, Chapter 6]).

Proposition 6. *The digraphs $M_1(n)$ and $M_2(n)$ are ambi-nut digraphs for every even $n \geq 6$. The graph $M_3(n)$ is an ambi-nut digraph for every even $n \geq 6$ that satisfies $n \not\equiv 0 \pmod{6}$.*

Proof. This is a straightforward consequence of local conditions (3) and (4). A vector that spans the (co-)kernel in all three cases is $\mathbf{x}(i) = (-1)^i$. It is easy to verify that $\dim \ker M_3(n) = 3$ if $n \equiv 0 \pmod{6}$ and $\dim \ker M_3(n) = 1$ otherwise, whereas $\dim \ker M_1(n) = \dim \ker M_2(n) = 1$ for all even n . \square

Note that $\text{Circ}(n, \{1, 2\})$ is itself a nut graph for every even $n \geq 6$ that satisfies $n \not\equiv 0 \pmod{6}$. The graph $M_1(8)$ is an ambi-nut digraph and in fact its underlying graph is a nut graph.

The following definition introduces three families that can be considered as directed analogues of Rose Window graphs [59].

Definition 7. For $k \in \{1, 2, 3\}$ and $n \geq 5$, let $D_k(n)$ be a digraph with the vertex set $\{v_0, v_1, \dots, v_{n-1}\} \cup \{u_0, u_1, \dots, u_{n-1}\}$ and the arc set

$$E(D_1(n)) = \{(v_i, v_{i+1}), (u_i, u_{i+2}), (v_i, u_i), (u_i, v_{i+1}) \mid 0 \leq i < n\},$$

$$E(D_2(n)) = \{(v_i, v_{i+1}), (u_i, u_{i+2}), (u_i, v_i), (v_{i+1}, u_i) \mid 0 \leq i < n\},$$

$$E(D_3(n)) = \{(v_i, v_{i+1}), (u_{i+2}, u_i), (v_i, u_i), (u_i, v_{i+1}) \mid 0 \leq i < n\}.$$

where addition is done modulo n . See Fig. 5 for examples.

Proposition 8. *The digraphs $D_1(n)$, $D_2(n)$ and $D_3(n)$ are ambi-nut digraphs for every odd $n \geq 5$.*

Proof. The proof is similar to that of Proposition 6, i.e., local conditions (3) and (4) can be applied. A vector that spans the (co-)kernel in all three cases is $\mathbf{x}(v_i) = 1$ and $\mathbf{x}(u_i) = -1$ for $0 \leq i < n$. It is easy to verify that $\dim \ker D_1(n) = \dim \ker D_2(n) = \dim \ker D_3(n) = 1$ if n is odd. Moreover, $\dim \ker D_1(n) = \dim \ker D_3(n) = 2$ if n is even. In the case of D_2 and even n we have that $\dim \ker D_2(n) = 4$ if $n \equiv 0 \pmod{4}$ and $\dim \ker D_2(n) = 2$ if $n \not\equiv 0 \pmod{4}$. \square

Note that $D_1(n)$, $D_2(n)$ and $D_3(n)$ share the same underlying graph, the Rose Window graph $R_n(1, 2)$ [59]. In [6] it was proved that $R_n(1, 2)$, $n \geq 5$, is a nut graph if and only if $n \not\equiv 0 \pmod{3}$.

Ambi-nut digraphs can also be obtained as cartesian products of certain digraphs which themselves are not ambi-nut digraphs. Let \vec{C}_n denote the directed cycle on n vertices. Then we have:

Proposition 9. *Let $m, n \geq 3$ and let $G = \vec{C}_n \square \vec{C}_m$. The following statements are equivalent:*

- (i) G is a dextro-nut digraph;
- (ii) G is a laevo-nut digraph;
- (iii) G is an ambi-nut digraph;
- (iv) $mn \equiv 0 \pmod{2}$ and $\gcd(m, n) = 1$.

Proof. Recall that $V(G) = \mathbb{Z}_n \times \mathbb{Z}_m$ with arcs of the form $(i, j) \rightarrow (i + 1, j)$ and $(i, j) \rightarrow (i, j + 1)$ where addition in the first coordinate is done modulo n and in the second modulo m .

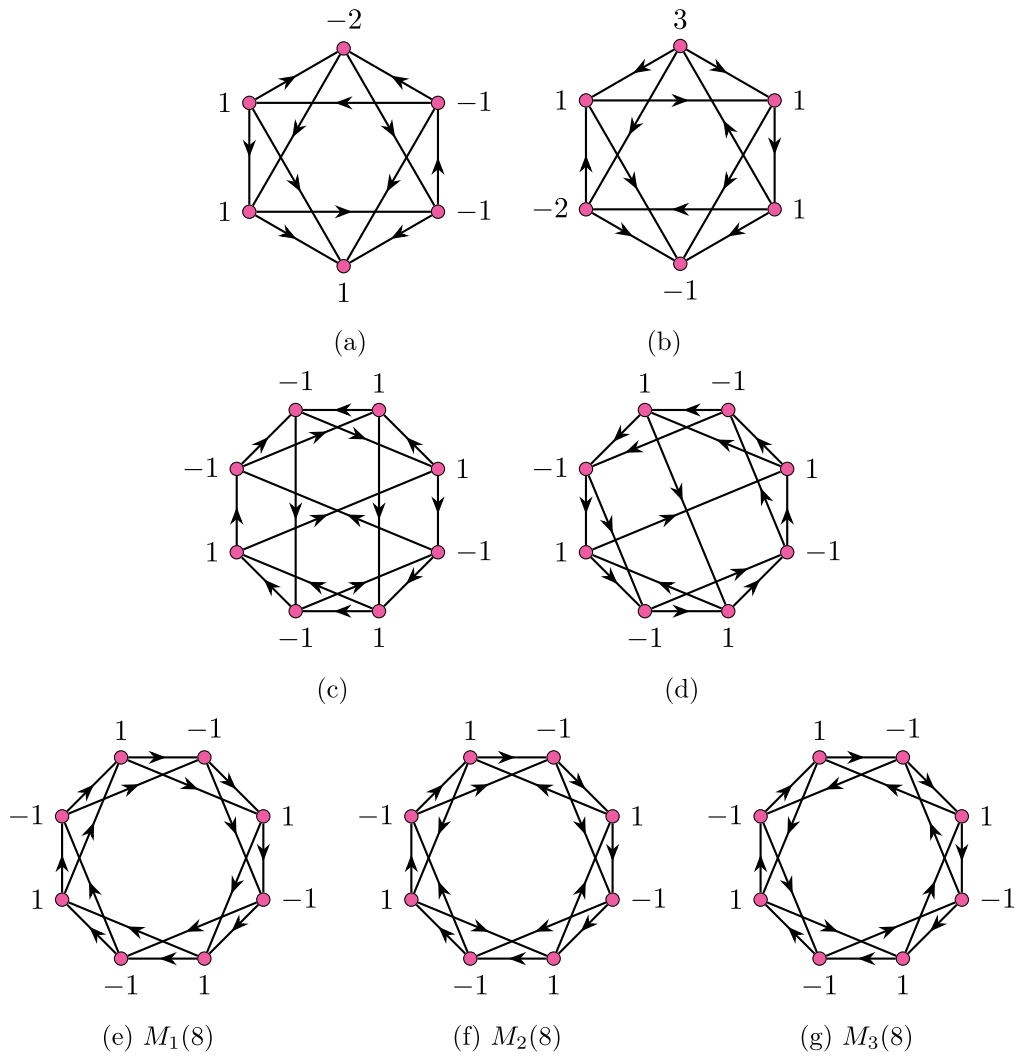


Fig. 4. Small 4-regular nut digraphs. Panels (a) & (b) show the two 4-regular dextro-nut digraphs on 6 vertices that are not bi-nut digraphs. Both have the same underlying graph as $M_1(6)$, and since each of them contains a sink, they are not isomorphic to $M_1(6)$ or $M_2(6)$. Panels (c) to (g) show the full set of five 4-regular ambi-nut digraphs on 8 vertices.

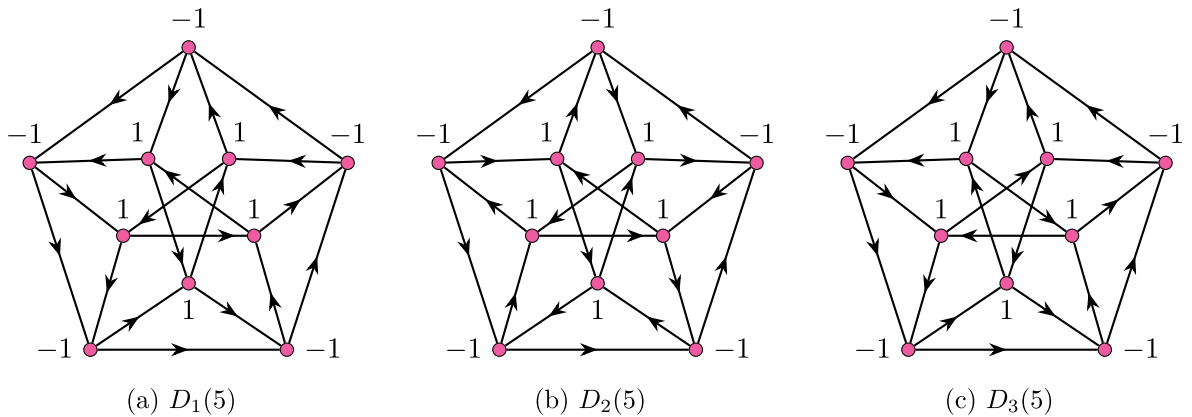


Fig. 5. Small 4-regular ambi-nut digraphs with a Rose Window graph as underlying graph.

Table 3
Enumeration of nut digraphs among tournaments on $n \leq 11$ vertices.

n	$ \mathcal{O}(K_n) $	$ \mathcal{D}(K_n) $	$ \mathcal{B}(K_n) $	$ \mathcal{A}(K_n) $
4	4	1	0	0
5	12	0	0	0
6	56	3	0	0
7	456	9	0	0
8	6880	119	0	0
9	191536	2373	10	0
10	9733056	90782	567	0
11	90375348	5918592	26629	0

Since $G \cong G^R$, statements (i) and (ii) are equivalent. Moreover, (iii) implies (i) and (ii). Hence, it suffices to show that (i) implies (iv) and that (iv) implies (iii).

Let $\mathbf{x} \in \ker G$. From the local condition (3) we get that

$$\mathbf{x}(i, j) = -\mathbf{x}(i + 1, j - 1). \tag{5}$$

Let C be the subgroup of $\mathbb{Z}_n \times \mathbb{Z}_m$ generated by $(1, -1)$ and let D be the subgroup generated by $(2, -2)$. Let $a = \mathbf{x}(0, 0)$. From (5) it follows that $\mathbf{x}(v) = a$ for $v \in D$ and $\mathbf{x}(v) = -a$ for $v \in C \setminus D$.

Suppose now that (i) holds. If $D = C$ then $a = 0$, which implies that G is not a dextro-nut digraph, a contradiction. Hence, D is a subgroup of C of index 2. Let $\mathbf{y}: V(G) \rightarrow \mathbb{R}$ such that $\mathbf{y}(v) = 1$ for $v \in D$, $\mathbf{y}(v) = -1$ for $v \in C \setminus D$ and $\mathbf{y}(v) = 0$ for $v \in V(G) \setminus C$. Then $\mathbf{y} \in \ker G$. Since G is a dextro-nut digraph it follows that $V(G) = C$. In particular, $\mathbb{Z}_n \times \mathbb{Z}_m$ is a cyclic group generated by the element $(1, -1)$. But then $\gcd(m, n) = 1$, and since $V(G)$ contains a subgroup of index 2, mn is even. This proves that (i) implies (iv).

Suppose that (iv) holds. Then $C = V(G)$ and D is a subgroup of $V(G)$ of index 2 and thus \mathbf{y} , as defined above, is a full vector such that $\mathbf{y} \in \ker G$ and $\mathbf{y} \in \text{coker } G$. Since $(1, -1)$ generates the whole $V(G)$, (5) implies \mathbf{y} is a unique vector from $\ker G$ up to scalar multiplication. Since $\eta(G) = 1$, G is an ambi-nut digraph, as required. \square

We have found several families of ambi-nut digraphs whose underlying graphs are nut graphs. However, we have also encountered ambi-nut digraphs, whose underlying graphs are not nut graphs. As the next proposition will show, in this case the underlying graphs are necessarily *core graphs*. In fact, this holds more generally, for the inter-nut digraphs.

Proposition 10. *If G is an inter-nut digraph then its underlying graph Γ is a core graph.*

Proof. Let G be an inter-nut digraph. Then there exists a full vector $\mathbf{x} \in \ker G \cap \text{coker } G$. Then (3) and (4) hold. These imply that $\sum_{u \in G^+(v)} \mathbf{x}(u) + \sum_{u \in G^-(v)} \mathbf{x}(u) = 0$. The latter is precisely the local condition for the underlying graph Γ . Therefore $\mathbf{x} \in \ker \Gamma$, which implies that Γ is a core graph. \square

In this survey of computational results, we also enumerated nut tournaments, i.e., nut digraphs whose underlying graph is K_n ; see Table 3. As one can see, there are no ambi-nut tournaments listed in the table. This is a direct consequence of Proposition 10, as K_n is not a core graph, and therefore $\mathcal{A}(K_n) = \emptyset$ for all n .

Proposition 10 also suggests an improved strategy of enumerating ambi-nut digraphs. For a given order n , we first find all core graphs on order n . Each core graph Γ gives rise to a collection of oriented graphs (for which Γ is the underlying graph). We can further restrict consideration to digraphs G with $\delta^-(G) \geq 2$ and $\delta^+(G) \geq 2$ (see Lemma 24 in Section 4). Consequently, it is enough to generate core graphs Γ with $\delta(\Gamma) \geq 4$. Tables 4 and 5 show the results of such a search for ambi-nut digraphs based on general oriented cores and 4-regular oriented cores, respectively. The search on 4-regular cores of odd order was performed only to illustrate the fact that 4-regular cores of odd order may exist, even though ambi-nut digraphs of odd order do not.

Note that many core graphs do not produce ambi-nut digraphs. See, for example, the graph Γ^\dagger in Fig. 6. As Γ^\dagger is a nut graph, the null space eigenvector is uniquely determined (up to scalar multiplication). The orientation of edges in any ambi-nut digraph that has Γ^\dagger as underlying graph must be consistent with this null space eigenvector. This observation could be used to filter core graphs, and also to carry out substantial pruning of branches in the process of generation of digraphs.

Furthermore, the notion of a core graph extends naturally to digraphs as follows. A digraph G is dextro-core (resp. laevo-core) if $\ker G$ (resp. $\text{coker } G$) contains a full vector. A digraph G is bi-core if it is both dextro-core and laevo-core. A digraph G is ambi-core if $\ker G = \text{coker } G$ and $\ker G$ contains a full vector. A digraph G is inter-core if $\ker G \cap \text{coker } G$ contains a full vector. In Fig. 7 we show some small examples of ambi-core digraphs with nullity greater than 1.

The searches carried out to produce Tables 1–5 were deliberately limited to oriented graphs. If this restriction is lifted, ambi-nut digraphs that contain pairs of oppositely oriented arcs appear from small order. See Fig. 8 for examples. On 4 vertices, there exists one ambi-nut digraph that is not an oriented graph. On 6 vertices, there are 14 ambi-nut digraphs that are not oriented graphs. Three of them are shown in the figure. Of course, every (undirected) graph is in fact a digraph

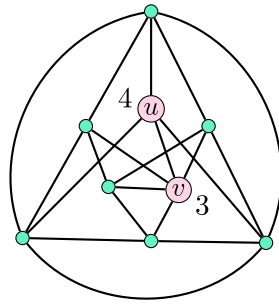


Fig. 6. A ‘Bad’ core graph, i.e., a nut graph Γ^+ that is not the underlying graph of any ambi-nut digraph. The vertices labelled u and v carry respective entries 4 and 3 in the kernel eigenvector, while the rest carry entry -1 . In a putative ambi-nut digraph obtained by orienting edges of Γ^+ , vertex u has $d^-(u) = d^+(u) = 2$. From the entries of the kernel eigenvector, it is clear that this condition cannot be satisfied.

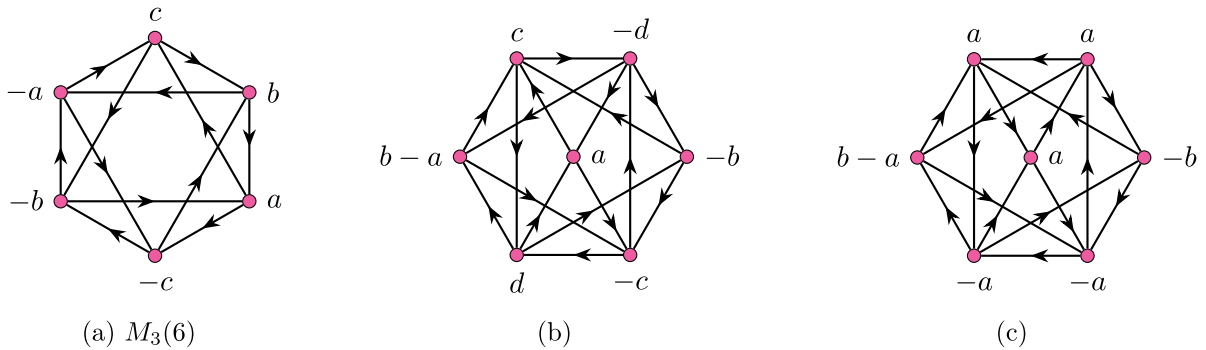


Fig. 7. Small examples of ambi-core digraphs with nullity strictly greater than 1. Letters a, b, c, \dots are used to indicate a choice of independent parameters spanning the nullspace of each digraph. The digraph $M_3(6)$ has nullity 3 and is the unique such digraph on 6 vertices. There are four such digraphs on 7 vertices. They all have the same underlying graph. One (b) has nullity 4, and the three others, one of which is shown in (c), have nullity 2. Note that the ambi-nut digraph in Fig. 3(d) has the same underlying graph.

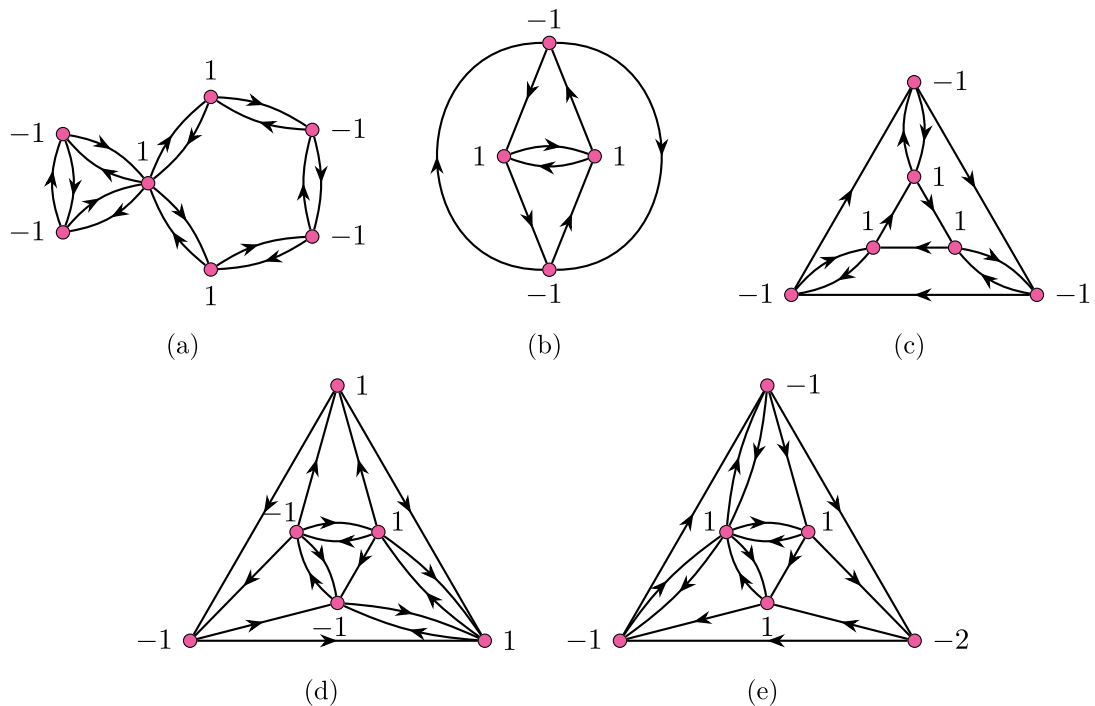


Fig. 8. Small ambi-nut digraphs with pairs of oppositely oriented arcs. Panel (a) shows the Sciriha graph S_1 as we view it in the theory of digraphs. Panel (b) shows the unique example on 4 vertices, while (c)–(e) show three out of 14 examples on 6 vertices.

Table 4

Enumeration of ambi-nut digraphs on $n \leq 10$ vertices by the method based on Proposition 10. The column labelled ‘Core’ gives the number of core graphs Γ with $\delta(\Gamma) \geq 4$. The ‘Oriented’ column gives the number of oriented graphs G that were obtained from the subset satisfying the condition $\delta^-(G) \geq 2$ and $\delta^+(G) \geq 2$. The column ‘Good’ counts the graphs from the ‘Core’ set that produce at least one ambi-nut digraph of the set \mathcal{A}_n .

n	Core	Oriented	$ \mathcal{A}_n $	Good
6	1	4	2	1
7	1	26	1	1
8	13	20958	104	10
9	117	16 677 343	3371	68
10	5299	657 400.1126	1 404 682	2544

Table 5

Enumeration of 4-regular ambi-nut digraphs on $n \leq 15$ vertices by the method based on Proposition 10. The column ‘Core’ gives the number of 4-regular core graphs. The column ‘Oriented’ gives the number of oriented graphs G that were obtained from the subset that satisfy $\delta^-(G) = 2$ and $\delta^+(G) = 2$. The column ‘Good’ counts the graphs from ‘Core’ that produce at least one ambi-nut digraph of the set \mathcal{A}_n .

n	Core	Oriented	$ \mathcal{A}_n $	Good
5	0	0	0	0
6	1	4	2	1
7	0	0	0	0
8	5	47	5	3
9	0	0	0	0
10	21	1645	32	16
11	0	0	0	0
12	446	146 371	860	225
13	0	0	0	0
14	20 794	24 196 518	35 219	6263
15	4	4945	0	0

where each edge is now viewed as a pair of oppositely oriented arcs. Thus, every (undirected) nut graph is an ambi-nut digraph whose arcs all appear in pairs.

Finally, in this survey of examples, we consider inter-nut digraphs G that are not ambi-nut digraphs, i.e., $\dim \ker(G) > 1$. There are 2 such inter-nut digraphs on 6 vertices, and 27 on 7 vertices. These 27 inter-nut digraphs can all be obtained from just 6 different underlying graphs by choosing a suitable orientation. It happens that those 6 underlying graphs include the three nut graphs on 7 vertices, i.e., the three Sciriha graphs [6].

4. Basic properties of nut digraphs

In the undirected universe, nut graphs are connected, non-bipartite and leafless. All these properties have their counterparts in the directed universe. We establish some useful machinery which will be used in the present section and also invoked for some constructions in Section 5. For a digraph G , a function $\mathbf{x}: V(G) \rightarrow \mathbb{R}$ and a vertex v of G , let

$$S_{\mathbf{x}}^+(v) = \sum_{u \in G^+(v)} \mathbf{x}(u) \quad \text{and} \quad S_{\mathbf{x}}^-(v) = \sum_{u \in G^-(v)} \mathbf{x}(u) \tag{6}$$

and recall from (3) and (4) that $\mathbf{x} \in \ker G$ provided $S_{\mathbf{x}}^+(v) = 0$ for every $v \in V(G)$ and that $\mathbf{x} \in \text{coker } G$ provided $S_{\mathbf{x}}^-(v) = 0$ for every $v \in V(G)$.

In the undirected universe, vertices of singular graphs may be partitioned into *core* and *core-forbidden* vertices. Namely, a vertex v of a singular graph G is a core vertex if there exists some $\mathbf{x} \in \ker G$ such that $\mathbf{x}(v) \neq 0$, otherwise v is a core-forbidden vertex (see [53, Definition 1]). In a further refinement, core-forbidden vertices are partitioned into *middle* and *upper*, accordingly as $\eta(G - v) = \eta(G)$ or $\eta(G - v) = \eta(G) + 1$. As $\ker G$ and $\text{coker } G$ are not necessarily the same in the digraph universe, we have to take this fact into consideration in framing corresponding definitions for digraphs.

Definition 11. Let G be a singular digraph. A vertex $u \in V(G)$ is a *dextro-core* vertex if there exists some $\mathbf{x} \in \ker G$ such that $\mathbf{x}(u) \neq 0$, otherwise v is a *dextro-core-forbidden* vertex. A vertex $u \in V(G)$ is a *laevo-core* vertex if there exists some $\mathbf{y} \in \text{coker } G$ such that $\mathbf{y}(u) \neq 0$, otherwise v is a *laevo-core-forbidden* vertex.

The following lemma shows that if G is a laevo-nut digraph, then a function \mathbf{y} happens to be an element of $\ker G$ under slightly weaker assumptions.

Lemma 12. Let G be a digraph and let $w \in V(G)$. If a function $\mathbf{y} \in \mathbb{R}^V$ satisfies the condition $S_{\mathbf{y}}^+(u) = 0$ for all $u \in V(G) \setminus \{w\}$ and if w is a laevo-core vertex then $S_{\mathbf{y}}^+(w) = 0$, and thus $\mathbf{y} \in \ker G$. Similarly, if a function $\mathbf{y} \in \mathbb{R}^V$ satisfies the condition $S_{\mathbf{y}}^-(u) = 0$ for all $u \in V(G) \setminus \{w\}$ and w is a dextro-core vertex then $S_{\mathbf{y}}^-(w) = 0$, and thus $\mathbf{y} \in \text{coker } G$.

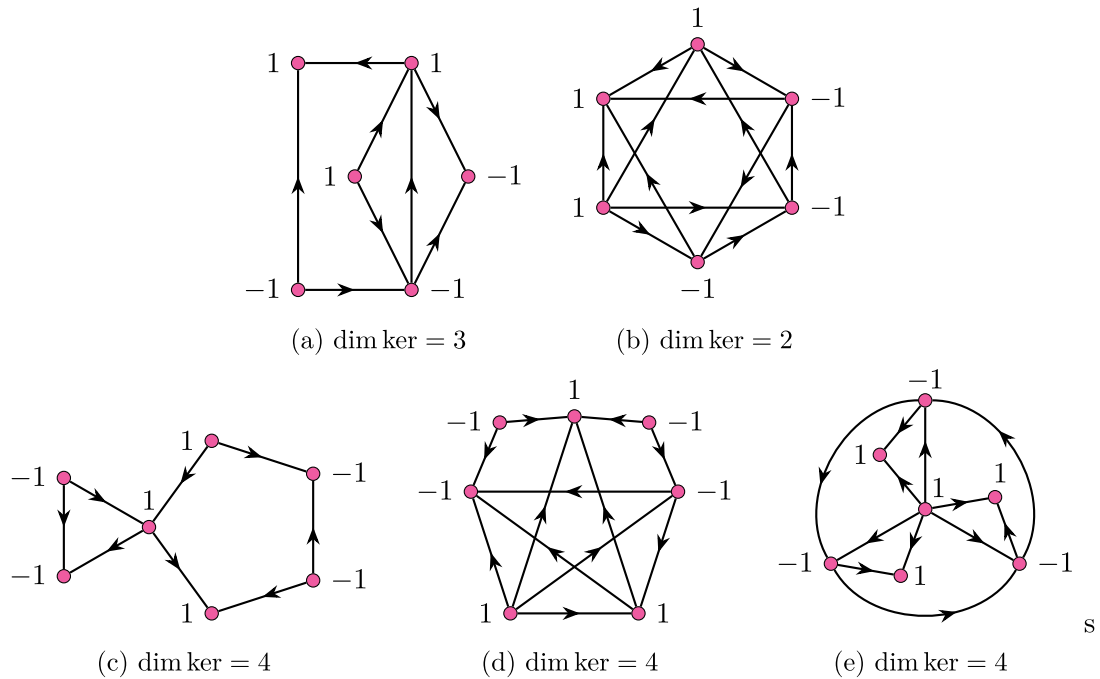


Fig. 9. Examples of inter-nut digraphs that are not ambi-nut digraphs. Panels (a) and (b) show both such inter-nut digraphs on 6 vertices. Panels (c) to (e) show three of 27 such inter-nut digraphs on 7 vertices. Note that their underlying graphs are the Sciriha graphs S_1 , S_2 and S_3 , respectively [6].

Proof. For two functions $\mathbf{w}, \mathbf{z} \in \mathbb{R}^V$, let

$$\langle \mathbf{w}, \mathbf{z} \rangle = \sum_{\substack{u, v \in V(G) \\ v \rightarrow u}} \mathbf{w}(v)\mathbf{z}(u). \tag{7}$$

As w is a laevo-core vertex, there exists some $\mathbf{x} \in \text{coker } G$ such that $\mathbf{x}(w) \neq 0$. Observe that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{u \in V(G)} \sum_{v \in G^-(u)} \mathbf{x}(v)\mathbf{y}(u) = \sum_{u \in V(G)} \mathbf{y}(u)S_x^-(u) = 0. \tag{8}$$

On the other hand,

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{v \in V(G)} \sum_{u \in G^+(v)} \mathbf{x}(v)\mathbf{y}(u) = \sum_{v \in V(G) \setminus \{w\}} \mathbf{x}(v)S_y^+(v) + \mathbf{x}(w)S_y^+(w) = \mathbf{x}(w)S_y^+(w). \tag{9}$$

Therefore, $\mathbf{x}(w)S_y^+(w) = 0$, and since $\mathbf{x}(w) \neq 0$, we see that $S_y^+(w) = 0$, as claimed. The second claim of the lemma can be proved by applying the first claim to the reverse digraph G^R . \square

4.1. Connectedness and vertex deletion

A digraph G is said to be *connected* if its underlying graph is connected.

Lemma 13. Every dextro-nut digraph, every laevo-nut digraph and every inter-nut digraph is connected.

Proof. If G is a dextro-nut digraph then there exists a full vector $\mathbf{x} \in \ker G$. Let $V(G) = V_1 \cup V_2$, $V_1 \neq \emptyset$ and $V_2 \neq \emptyset$, such that $G[V_1]$ is a connected component. (Then $G[V_2]$ is a disjoint union of one or more connected components.) Let us define a function $\mathbf{y}: V(G) \rightarrow \mathbb{R}$ by $\mathbf{y}(u) = \mathbf{x}(u)$ for $u \in V_1$ and $\mathbf{y}(u) = 0$ for $u \in V_2$. As (3) holds for \mathbf{y} , we have that $\mathbf{y} \in \ker G$. This contradicts the antecedent that G is a dextro-nut digraph. An analogous argument applies to laevo-nut digraphs.

For inter-nut digraphs, the above arguments also apply with minor adjustments. Namely, one has to take $\mathbf{x} \in \ker G \cap \text{coker } G$ and observe that then the function $\mathbf{y} \in \ker G \cap \text{coker } G$ as it satisfies both (3) and (4). \square

All examples of ambi-nut digraphs which we have encountered happen to be *strongly connected* (i.e., there exists a directed path between every pair of vertices). We introduce notions needed to prove that this is a general property of ambi-nut digraphs. First, we prove an elementary fact from linear algebra [55].

Lemma 14. Let $A \in \mathbb{C}^{n \times m}$ (i.e., A is an $n \times m$ matrix over the field \mathbb{C}). Let B be a submatrix obtained from A by deleting any row and any column. Then

$$\eta(A) - 1 \leq \eta(B) \leq \eta(A) + 1 \quad \text{and} \quad \eta(A^T) - 1 \leq \eta(B^T) \leq \eta(A^T) + 1. \tag{10}$$

Proof. Without loss of generality, assume that we delete the first row and the first column. We can write

$$A = \begin{bmatrix} z & \mathbf{c}^T \\ \mathbf{d} & B \end{bmatrix}, \tag{11}$$

where $z \in \mathbb{C}$ and $\mathbf{c}, \mathbf{d} \in \mathbb{C}^{(n-1) \times 1}$. Let $D := \begin{bmatrix} \mathbf{c}^T \\ B \end{bmatrix}$. When \mathbf{c}^T is added to the row-space of B , the rank is either unchanged or increases by 1, i.e., $\text{rank}(B) \leq \text{rank}(D) \leq \text{rank}(B) + 1$. When the vector $\begin{bmatrix} z \\ \mathbf{d} \end{bmatrix}$ is added to the column-space of D , again, the rank is either unchanged or increases by 1. Therefore,

$$\text{rank}(B) \leq \text{rank}(A) \leq \text{rank}(B) + 2. \tag{12}$$

Recall that $\text{rank}(A) + \eta(A) = m$ and $\text{rank}(A^T) + \eta(A^T) = n$. Similarly, $\text{rank}(B) + \eta(B) = m - 1$ and $\text{rank}(B^T) + \eta(B^T) = n - 1$. Moreover, $\text{rank}(A) = \text{rank}(A^T)$ and $\text{rank}(B) = \text{rank}(B^T)$. From (12) we obtain $\eta(A) - 1 \leq \eta(B) \leq \eta(A) + 1$ and, similarly, $\eta(A^T) - 1 \leq \eta(B^T) \leq \eta(A^T) + 1$. \square

Lemma 14 has an immediate consequence.

Corollary 15. Let G be a digraph of order $n \geq 2$ and $v \in V(G)$ an arbitrary vertex. Then

$$\eta(G) - 1 \leq \eta(G - v) \leq \eta(G) + 1. \tag{13}$$

It is known that the above corollary holds for undirected graphs (see, e.g., [1, Theorem 1.2]), where the usual proof relies on the Cauchy Interlacing Theorem for symmetric matrices, which is not applicable in the directed universe.

In the undirected universe, if a core-vertex v is deleted from a graph G , then $\eta(G - v) = \eta(G) - 1$ (see, e.g., [47, Corollary 13]), whereas if a core-forbidden vertex v is deleted, then $\eta(G - v) = \eta(G)$ or $\eta(G - v) = \eta(G) + 1$ (see, e.g., [25]). We will now generalise these notions to digraphs.

Lemma 16. Let G be a singular digraph with at least 2 vertices and let $v \in V(G)$ be any vertex. If v is laevo-core then $\eta(G - v) \leq \eta(G)$. If, in addition, v is dextro-core then $\eta(G - v) = \eta(G) - 1$.

Proof. Let $\eta = \eta(G)$ and $\eta' = \eta(G - v)$. Let $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{\eta'}\}$ be a basis for $\ker(G - v)$, and let $\tilde{\mathbf{x}}_i: V(G) \rightarrow \mathbb{R}$, $1 \leq i \leq \eta'$, be defined by $\tilde{\mathbf{x}}_i(u) = \mathbf{x}_i(u)$ for $u \neq v$ and $\tilde{\mathbf{x}}_i(v) = 0$. It is easy to see that $\tilde{\mathbf{x}}_1, \tilde{\mathbf{x}}_2, \dots, \tilde{\mathbf{x}}_{\eta'}$ are linearly independent and each of them satisfies the assumptions of Lemma 12. Since v is a laevo-core vertex there exists $\tilde{\mathbf{y}} \in \text{coker } G$ such that $\tilde{\mathbf{y}}(v) \neq 0$. This allows us to use Lemma 12 to conclude that $\langle \tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{\eta'} \rangle \subseteq \ker G$ and hence $\eta' \leq \eta$.

If v is also dextro-core, then there exists $\tilde{\mathbf{x}}_0 \in \ker G$ such that $\tilde{\mathbf{x}}_0(v) \neq 0$. Then $\tilde{\mathbf{x}}_0 \notin \langle \tilde{\mathbf{x}}_1, \dots, \tilde{\mathbf{x}}_{\eta'} \rangle$, implying that $\eta' < \eta$. From Corollary 15 it follows that $\eta(G - v) = \eta(G) - 1$, as claimed. \square

Lemma 17. Let G be a singular digraph with at least 2 vertices and let $v \in V(G)$ be any vertex. If v is laevo-core-forbidden or dextro-core-forbidden then $\eta(G - v) \in \{\eta(G), \eta(G) + 1\}$.

Proof. Let $\eta = \eta(G)$. Suppose first that v is dextro-core-forbidden. Let $\tilde{\mathbf{z}}_1, \tilde{\mathbf{z}}_2, \dots, \tilde{\mathbf{z}}_\eta$ be a basis of $\ker G$ and let $\mathbf{z}_i: V(G - v) \rightarrow \mathbb{R}$, $1 \leq i \leq \eta$, be defined by $\mathbf{z}_i(u) = \tilde{\mathbf{z}}_i(u)$ for $u \in V(G - v)$. Since v is dextro-core-forbidden, $\tilde{\mathbf{z}}_1(v) = \dots = \tilde{\mathbf{z}}_\eta(v) = 0$, implying that $\mathbf{z}_1, \dots, \mathbf{z}_\eta$ are linearly independent and $\langle \mathbf{z}_1, \dots, \mathbf{z}_\eta \rangle \subseteq \ker(G - v)$. Therefore, $\eta(G - v) \geq \eta$. By Corollary 15 it follows that $\eta(G - v) = \eta$ or $\eta(G - v) = \eta + 1$.

If v is laevo-core-forbidden, then v is dextro-core-forbidden in the reverse digraph G^R . Since $\eta(G) = \eta(G^R)$, the results follows by the previous paragraph. \square

Theorem 18. Let G be a singular digraph of order $n \geq 2$ and let $v \in V(G)$ be any vertex.

- (i) If v is dextro-core and laevo-core then $\eta(G - v) = \eta(G) - 1$.
- (ii) If v is dextro-core-forbidden and laevo-core then $\eta(G - v) = \eta(G)$.
- (iii) If v is dextro-core and laevo-core-forbidden then $\eta(G - v) = \eta(G)$.
- (iv) If v is dextro-core-forbidden and laevo-core-forbidden then $\eta(G - v) \in \{\eta(G), \eta(G) + 1\}$. Moreover, if G is bipartite then $\eta(G - v) = \eta(G) + 1$.

Proof. Statement (i) follows directly from Lemma 16 and the first part of statement (iv) follows directly from Lemma 17. Statement (ii) is obtained by combining Lemmas 16 and 17. Finally, statement (iii) follows by application of (ii) to the reverse digraph G^R .

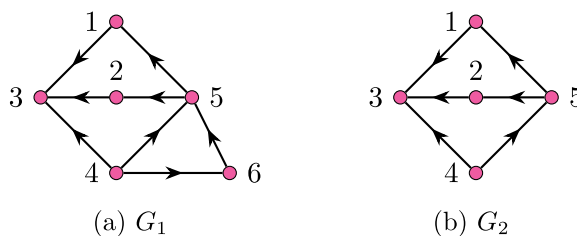


Fig. 10. Examples of digraphs covering all scenarios in Theorem 18.

Now we prove the second part of statement (iv). Without loss of generality, for G bipartite the adjacency matrices of G and $G - v$ can be written as

$$A(G) = \begin{bmatrix} 0 & \mathbf{0}^T & \mathbf{u}^T \\ \mathbf{0} & O_{p-1} & B \\ \mathbf{w} & C & O_q \end{bmatrix}, \quad A(G - v) = \begin{bmatrix} O_{p-1} & B \\ C & O_q \end{bmatrix}, \tag{14}$$

where O_t denotes the all-zero $t \times t$ matrix. If there exists a column vector \mathbf{a} such that $\mathbf{w} = C\mathbf{a}$, then $[-1 \ \mathbf{a} \ 0]^T \in \ker A(G)$, contradicting the assumption that v is a dextro-core-forbidden vertex. Therefore, $\mathbf{w} \notin \text{im } C$. Recall that $\text{im } C$ denotes the image of C , i.e., $\text{im } C = \{C\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^{p-1}\}$. Similarly, since v is also laevo-core-forbidden, it is dextro-core-forbidden in G^R , implying that $\mathbf{u} \notin \text{im } B^T$. In particular, this implies that \mathbf{w} is not in the column-space of C and \mathbf{u}^T is not in the row-space of B . But then $\text{rank } A(G) = \text{rank } A(G - v) + 2$. As in the proof of Lemma 14, by the Rank-Nullity Theorem it then follows that $\eta(G - v) = \eta(G) + 1$. \square

Theorem 18 gives us a way of retrieving the middle/upper stratification of core-forbidden vertices.

Definition 19. Let G be a singular digraph. A vertex $v \in V(G)$ that is both dextro-core-forbidden and laevo-core-forbidden is called *upper* if $\eta(G - v) = \eta(G) + 1$, and *middle* if $\eta(G - v) = \eta(G)$.

For a bipartite (undirected) graph G the parity of $\eta(G)$ is the same as the parity of $|V(G)|$ by the Pairing Theorem, and hence middle vertices are absent. However, for bipartite digraphs, as Theorem 18 shows, it is also possible that deletion of a vertex leaves the nullity unchanged.

Example 20. The digraphs in Fig. 10 show that all scenarios in Theorem 18 may occur.

Consider the digraph G_1 in Fig. 10(a) with $\eta(G_1) = 2$. Vertices 1 and 2 belong to case (i), i.e., they are dextro-core and laevo-core, and thus $\eta(G_1 - 1) = \eta(G_1 - 2) = 1$. Vertices 3 and 4 belong to cases (ii) and (iii), respectively, and thus $\eta(G_1 - 3) = \eta(G_1 - 4) = 2$. Vertices 5 and 6 belong to case (iv), where vertex 5 is upper and vertex 6 is middle, i.e., $\eta(G_1 - 5) = 3$ and $\eta(G_1 - 6) = 2$. These nullities are easily verified by direct calculation of the kernel and co-kernel of G_1 and its vertex-deleted subgraphs.

The digraph G_2 in Fig. 10(b) is bipartite with $\eta(G_2) = 2$. Vertices 1 and 2 belong to case (i) and thus $\eta(G_2 - 1) = \eta(G_2 - 2) = 1$. Vertices 3 and 4 belong to cases (ii) and (iii), respectively, and thus $\eta(G_2 - 3) = \eta(G_2 - 4) = 2$. The vertex 5 belongs to case (iv), but since G_2 is bipartite, it can only be upper, with $\eta(G_2 - 5) = 3$.

Since every vertex of a bi-nut digraph is dextro-core and laevo-core, Theorem 18 implies the following corollary, which generalises [47, Corollary 15] from nut graphs to bi-nut digraphs.

Corollary 21. Let G be a bi-nut digraph and let $w \in V(G)$ be an arbitrary vertex. Then $G - w$ is a non-singular digraph.

We will now make use of another concept from linear algebra. An $n \times n$ matrix A is *reducible* (in the sense of [35, Definition 6.2.21]) if it is permutation-equivalent to a matrix

$$\begin{bmatrix} B & C \\ O & D \end{bmatrix}, \tag{15}$$

where neither B nor D is an empty matrix (i.e., a matrix with no rows or no columns). A matrix that is not reducible is called *irreducible*. A digraph G is strongly connected if and only if $A(G)$ is irreducible (see, e.g., [35, Theorem 6.2.24] or [13, Theorem 3.2.1]). Now, we have all the tools required to prove the next theorem.

Theorem 22. Every bi-nut digraph is strongly connected.

Proof. For contradiction, suppose that a bi-nut digraph G is not strongly connected. Then its adjacency matrix is reducible and can be written, without loss of generality, as

$$A(G) = \begin{bmatrix} B & C \\ O & D \end{bmatrix}, \tag{16}$$

where O is an all-zero matrix and B and D are both square matrices. Since G is a bi-nut digraph, it is singular and thus

$$\det A(G) = (\det B)(\det D) = 0. \tag{17}$$

Now, let G' be obtained from G by deleting the first vertex, i.e., the vertex that corresponds to the first row and first column in $A(G)$. The adjacency matrix of G' is of the form

$$A(G') = \begin{bmatrix} B' & C' \\ O' & D \end{bmatrix}, \tag{18}$$

where O' is an all-zero matrix. By Corollary 21, G' is a non-singular digraph, and thus $\det A(G') = (\det B')(\det D) \neq 0$. In particular, $\det D \neq 0$. Let G'' be obtained from G by deleting the last vertex. By similar reasoning we get that $\det B \neq 0$. But this contradicts (17). \square

Note that every ambi-nut digraph is a bi-nut digraph, and hence ambi-nut digraphs are strongly connected.

4.2. Bipartiteness

Nut graphs are non-bipartite. The same is true in the universe of digraphs, as we now show. Note that a digraph G is bipartite if and only if its underlying graph is bipartite.

Lemma 23. *Every dextro-nut digraph, every laevo-nut digraph and every inter-nut digraph is non-bipartite.*

Proof. Suppose that G is a bipartite digraph and U and W are the parts of the bipartition of its underlying graph. If $\mathbf{x} \in \ker G$, then the function $\mathbf{x}' : V(G) \rightarrow \mathbb{R}$ coinciding with \mathbf{x} on U and mapping each vertex of W to 0 is also in $\ker G$, implying that G is not a dextro-nut digraph. Since the reverse digraph of a bipartite digraph is also bipartite, the same argument applied to the reverse digraph G^R shows that it is not a dextro-nut digraph, and thus G is not a laevo-digraph nut.

For inter-nut digraphs, the above arguments also apply with minor adjustments. Namely, one has to take $\mathbf{x} \in \ker G \cap \text{coker } G$ and observe that then the function $\mathbf{x}' \in \ker G \cap \text{coker } G$, implying that G is not an inter-nut digraph. \square

Clearly, the above lemma applies to ambi-nut digraphs and bi-nut digraphs, as they are dextro-nut digraphs.

4.3. Vertex degrees and leaflessness

Nut graphs have no vertices of degree 1. In the universe of digraphs, the situation is more complicated. Leaflessness generalises as follows.

Lemma 24. *If G is a dextro-nut digraph, then $\delta^-(G) \geq 1$ and $\delta^+(G) \neq 1$. Similarly, if G is a laevo-nut digraph, then $\delta^+(G) \geq 1$ and $\delta^-(G) \neq 1$. Consequently, if G is a bi-nut digraph, then $\delta^-(G) \geq 2$ and $\delta^+(G) \geq 2$.*

Proof. Let G be a dextro-nut digraph with its kernel spanned by \mathbf{x} . If there exists a vertex $v \in V(G)$ with no in-neighbours, then a function $\mathbf{x}' : V(G) \rightarrow \mathbb{R}$ coinciding with \mathbf{x} in each vertex except in v while mapping v to 0 is also in $\ker G$, contradicting the fact that $\ker G$ is one-dimensional. Hence, $\delta^-(G) \geq 1$. Furthermore, if there is vertex $v \in V(G)$ with exactly one out-neighbour, say w , then $\sum_{u \in G^+(v)} \mathbf{x}(u) = \mathbf{x}(w) \neq 0$, contradicting the assumption that $\mathbf{x} \in \ker G$. The claim about laevo-nut digraphs follows, by applying the preceding reasoning to the reverse digraph G^R . \square

In other words, Lemma 24 says that dextro-nut digraphs may not possess sources. However, dextro-nut digraphs can have sinks; see Fig. 3(a). Similarly, by Lemma 24, laevo-nut digraphs may not possess sinks, but can have sources. Note that the underlying graph of a dextro- or laevo-nut digraph may have leaves; see, e.g., Fig. 3(b). Also note that a given inter-nut digraph G may have both sinks and sources, as seen from examples in Fig. 9.

Lemma 24 implies that if G is a bi-nut digraph then the minimum degree of its underlying graph is at least 4. Table 2 gives us some information about the simplest case, where the underlying graph is quartic. From Table 2 it appears that bi-nut digraphs (and therefore ambi-nut digraphs) do not exist if the underlying graph is quartic and of odd order. The next proposition generalises this observation.

Proposition 25. *Let G be a bi-nut digraph, such that its underlying graph is 4-regular. Then G is of even order $2n$. Moreover, $\ker G$ is spanned by a vector \mathbf{x} that consists of entries from $\{+1, -1\}$, n of which are $+1$, and $\text{coker } G$ is spanned by a vector \mathbf{y} that consists of entries from $\{+1, -1\}$, n of which are $+1$.*

Proof. By Lemma 24, $d^+(v) = d^-(v) = 2$ for every $v \in V(G)$. Let $\ell : V(G) \rightarrow \{+a, -a, \star\}$ denote a mapping that assigns to each v vertex of G value $+a$ or value $-a$ or leaves $\ell(v)$ undefined, which is indicated by the \star symbol. We describe an algorithm which finds a non-trivial vector from $\ker G$.

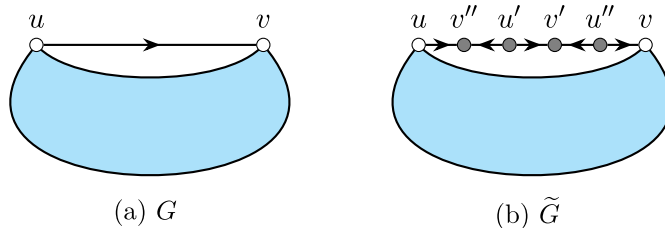


Fig. 11. A possible subdivision construction for digraphs.

Initially, set $\ell(u) = \star$ for every $u \in V(G)$. Choose an arbitrary vertex $v_0 \in V(G)$ and set $\ell(v_0) = +a$. While there still exists a vertex $v \in V(G)$ with out-neighbours v' and v'' , such that $\ell(v') \neq \star$ and $\ell(v'') = \star$, update $\ell(v'')$ by setting $\ell(v'') = -\ell(v')$. When this algorithm stops, for each vertex $v \in V(G)$ with out-neighbours v' and v'' either (i) $\ell(v') = \ell(v'') = \star$ or (ii) $\ell(v') \neq \star$ and $\ell(v'') \neq \star$ holds.

If there exists a vertex $v \in V(G)$ with out-neighbours v' and v'' such that $\ell(v') = \ell(v'') = +a$ or $\ell(v') = \ell(v'') = -a$, then it follows that $a = 0$ and thus $\ker G$ is trivial or contains a non-full vector. This contradicts the assumption that G is a bi-nut digraph. Therefore, if (ii) holds it must be that $\ell(v') \neq \ell(v'')$.

Let us now define the vector $\mathbf{x}: V(G) \rightarrow \mathbb{R}$ by

$$\mathbf{x}(v) = \begin{cases} 0 & \text{if } \ell(v) = \star, \\ +1 & \text{if } \ell(v) = +a, \\ -1 & \text{if } \ell(v) = -a. \end{cases} \tag{19}$$

The vector \mathbf{x} satisfies (3), so $\mathbf{x} \in \ker G$. The existence of a vertex $v \in V(G)$ such that $\mathbf{x}(v) = 0$ would contradict the assumption that G is a bi-nut digraph. It follows that $\mathbf{x}(v) \in \{+1, -1\}$ for every $v \in V(G)$ and vector \mathbf{x} spans $\ker G$. The fact that half of the vertices carry entry $+1$ and the other half -1 is straightforward to prove by standard combinatorial ‘double counting’.

The same approach is used to prove the existence of the vector \mathbf{y} from the proposition. The only difference is that we need to consider in-neighbours instead of out-neighbours. \square

Note that the algorithm in the above proof emulates the standard pencil-and-paper approach [39,62] to determining the null space of a chemical graph.

5. Constructions

In the realm of undirected graphs, several constructions have been described for producing larger nut graphs from smaller. Examples include the *bridge construction* (insertion of two vertices on a bridge) [52], the *subdivision construction* (insertion of four vertices on an edge) [52], and the *coalescence construction* (identification of a vertex in one nut graph with a vertex of another) [47]. Multiplier constructions, so called because they produce a nut graph from a $(2t)$ -regular parent graph, in which the order of the new graph is a fixed multiple of the order of the parent, were also introduced previously [6]. Analogues can be defined for nut digraphs.

5.1. Subdivision construction

One possible construction that in some sense mimics subdivision is illustrated in Fig. 11.

Theorem 26. *Let G be a digraph and let $u \rightarrow v$ be an arc in G . Let \tilde{G} be the digraph obtained from G by removing the arc $u \rightarrow v$, and adding four new vertices u', u'', v', v'' and five new arcs $u \rightarrow v'', u' \rightarrow v'', u' \rightarrow v', u'' \rightarrow v', u'' \rightarrow v$. If G is an inter-nut digraph, then \tilde{G} is an inter-nut digraph that is not an ambi-nut digraph.*

Proof. If G is an inter-nut digraph, then there exists a full vector $\mathbf{x} \in \ker G \cap \text{coker } G$. The vector \mathbf{x} can be extended to $\bar{\mathbf{x}} \in \mathbb{R}^{V(\tilde{G})}$ by defining $\bar{\mathbf{x}}(w) = \mathbf{x}(w)$ for $w \in V(G)$ and $\bar{\mathbf{x}}(v') = -\bar{\mathbf{x}}(v'') = \mathbf{x}(v)$ and $\bar{\mathbf{x}}(u') = -\bar{\mathbf{x}}(u'') = \mathbf{x}(u)$. Clearly, $\bar{\mathbf{x}}$ is full and $\bar{\mathbf{x}} \in \ker \tilde{G} \cap \text{coker } \tilde{G}$. This implies that \tilde{G} is an inter-core digraph.

Take any non-trivial vector $\bar{\mathbf{q}} \in \ker \tilde{G} \cap \text{coker } \tilde{G}$. Eqs. (3) and (4) imply that $\bar{\mathbf{q}}(v) = -\bar{\mathbf{q}}(v') = \bar{\mathbf{q}}(v'')$ and $\bar{\mathbf{q}}(u) = -\bar{\mathbf{q}}(u') = \bar{\mathbf{q}}(u'')$. Let \mathbf{q} be the restriction of $\bar{\mathbf{q}}$ to G . More precisely, $\mathbf{q}(w) = \bar{\mathbf{q}}(w)$ for $w \in V(G)$. It is easy to see that $\mathbf{q} \in \ker G \cap \text{coker } G$. As G is an inter-nut digraph, $\mathbf{q} = \lambda \mathbf{x}$ for some $\lambda \neq 0$. But then $\bar{\mathbf{q}} = \lambda \bar{\mathbf{x}}$. This means that $\bar{\mathbf{x}}$ spans $\ker \tilde{G} \cap \text{coker } \tilde{G}$ and \tilde{G} is thus an inter-nut digraph.

Moreover, let $\bar{\mathbf{y}}, \bar{\mathbf{z}} \in \mathbb{R}^{V(\tilde{G})}$ be vectors defined as

$$\bar{\mathbf{y}}(w) = \begin{cases} 1 & w = u' \text{ or } w = u'', \\ 0 & \text{otherwise;} \end{cases} \quad \text{and} \quad \bar{\mathbf{z}}(w) = \begin{cases} 1 & w = v' \text{ or } w = v'', \\ 0 & \text{otherwise.} \end{cases} \tag{20}$$

It is clear that $\bar{\mathbf{y}} \in \ker \tilde{G}$ and $\bar{\mathbf{y}} \notin \text{coker } \tilde{G}$, while $\bar{\mathbf{z}} \in \text{coker } \tilde{G}$ and $\bar{\mathbf{z}} \notin \ker \tilde{G}$. Hence, \tilde{G} is not an ambi-nut digraph. \square

Note that even if G in [Theorem 26](#) is an ambi-nut digraph, \tilde{G} is not an ambi-nut digraph. The described construction can only be used to produce larger inter-nut digraphs from existing inter-nut digraphs.

As we prove next, inter-nut and bi-nut digraphs have no cut-arcs (i.e., analogues of bridges).

Proposition 27. *No inter-nut digraph contains a cut-arc.*

Proof. Suppose that there exists an inter-nut digraph \widehat{G} with a cut-arc $u \rightarrow w$. When the cut-arc is removed, we obtain two connected components, G_1 and G_2 , such that $u \in V(G_1)$ and $w \in V(G_2)$. Since \widehat{G} is an inter-nut digraph, there exists a full vector $\widehat{\mathbf{x}} \in \ker \widehat{G} \cap \text{coker } \widehat{G}$. Let \mathbf{x} denote the restriction of $\widehat{\mathbf{x}}$ to G_2 , i.e., $\mathbf{x}(v) = \widehat{\mathbf{x}}(v)$ for all $v \in V(G_2)$. Observe that G_2 is a dextro-nut digraph and that \mathbf{x} is full and $\mathbf{x} \in \ker G_2$. Further observe that $S_{\mathbf{x}}^-(v) = 0$ for all $v \in V(G_2) \setminus \{w\}$. By [Lemma 12](#), $S_{\mathbf{x}}^-(w) = 0$ and $\mathbf{x} \in \text{coker } G_2$. Observe that $G^-(w) = G_2^-(w) \cup \{u\}$. It follows that $\widehat{\mathbf{x}}(u) = 0$. But this contradicts the fact that $\widehat{\mathbf{x}}$ is full. Therefore \widehat{G} cannot be an inter-nut digraph. \square

The following is a direct consequence of [Theorem 22](#).

Corollary 28. *No bi-nut digraph contains a cut-arc.*

Note that every ambi-nut digraph is a bi-nut digraph, and hence ambi-nut digraphs do not contain cut-arcs. Dextro-nut digraphs and thus laevo-nut digraphs may, however, possess cut-arcs; see [Fig. 12\(a\)](#).

5.2. Coalescence construction

Let G and H be two digraphs and v a vertex of G and u a vertex of H . The *coalescence* $(G, v) \odot (H, u)$ is defined as the digraph obtained from the disjoint union of G and H by identifying the vertices v and u , where the ‘merged’ vertex retains all the out-neighbours and all the in-neighbours of both v and u . Note that when the adjacency relations of G and H are both symmetric and irreflexive (i.e., when G and H are simple, undirected graphs), the coalescence as defined here coincides with the coalescence of graphs defined in [\[47\]](#).

Theorem 29. *Let G and H be bi-nut digraphs, where $v^* \in V(G)$ and $u^* \in V(H)$. Then $(G, v^*) \odot (H, u^*)$ is a bi-nut digraph. Moreover, if both G and H are ambi-nut digraphs, then so is $(G, v^*) \odot (H, u^*)$.*

Proof. Let $\mathbf{x}, \mathbf{y}, \bar{\mathbf{x}}$, and $\bar{\mathbf{y}}$ be such that $\ker G = \langle \mathbf{x} \rangle$, $\text{coker } G = \langle \bar{\mathbf{x}} \rangle$, $\ker H = \langle \mathbf{y} \rangle$, and $\text{coker } H = \langle \bar{\mathbf{y}} \rangle$. Without loss of generality we may assume that $\mathbf{x}(v^*) = \bar{\mathbf{x}}(v^*) = \mathbf{y}(u^*) = \bar{\mathbf{y}}(u^*)$. Further, let V denote the vertex-set of $C := (G, v^*) \odot (H, u^*)$. We will abuse the notation slightly and denote the vertex of C obtained by identifying u^* and v^* by u^* or v^* , or, when we want to avoid ambiguity, by $\bar{u}\bar{v}$, whichever is most convenient.

Define the functions $\mathbf{z}, \bar{\mathbf{z}} \in \mathbb{R}^V$ by letting them coincide with the functions \mathbf{x} and $\bar{\mathbf{x}}$ on $V(G) \setminus \{v^*\}$ and with \mathbf{y} and $\bar{\mathbf{y}}$ on $V(H) \setminus \{u^*\}$, respectively. Further, let $\mathbf{z}(\bar{u}\bar{v}) = \mathbf{x}(v^*) = \mathbf{y}(u^*)$ and $\bar{\mathbf{z}}(\bar{u}\bar{v}) = \bar{\mathbf{x}}(v^*) = \bar{\mathbf{y}}(u^*)$. Observe that $\mathbf{z} \in \ker C$ and $\bar{\mathbf{z}} \in \text{coker } C$, showing that both $\ker C$ and $\text{coker } C$ contain a full vector. We now need to show that they are both one-dimensional.

Suppose that $\mathbf{w} \in \ker C$. Since the out-neighbourhood in C of each vertex in $V(G) \setminus \{v^*\}$ coincides with its out-neighbourhood in G , it follows that

$$\sum_{u \in C^+(v)} \mathbf{w}(u) = \sum_{u \in G^+(v)} \mathbf{w}(u) = 0 \quad \text{for all } v \in V(G) \setminus \{v^*\}.$$

Now, since $\mathbf{y} \in \text{coker } G$ and $\mathbf{y}(v^*) \neq 0$, [Lemma 12](#) implies that the restriction \mathbf{w}_G of \mathbf{w} to $V(G)$ lies in $\ker G$. Since G is a dextro-nut digraph, it follows that $\mathbf{w}_G = \lambda \mathbf{x}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$. By an analogous argument, one can show that the restriction \mathbf{w}_H of \mathbf{w} to $V(H)$ satisfies $\mathbf{w}_H = \mu \mathbf{y}$. Since $\mathbf{x}(v^*) = \mathbf{y}(u^*) = \mathbf{w}(\bar{u}\bar{v})$, it follows that $\lambda = \mu$ and thus $\mathbf{w} = \lambda \mathbf{z}$. This shows that $\ker C$ is indeed one-dimensional and spanned by \mathbf{z} . The proof that $\text{coker } C$ is also one-dimensional can be obtained by applying the above to the reverse digraphs G^R and H^R . This shows that C is indeed a bi-nut digraph, the co-kernel of which is spanned by the function $\bar{\mathbf{z}}: V \rightarrow \mathbb{R}$, coinciding with $\bar{\mathbf{x}}$ on $V(G)$ and with $\bar{\mathbf{y}}$ on $V(H)$.

If G and H are both ambi-nut digraphs, then $\bar{\mathbf{x}} = \mathbf{x}$ and $\bar{\mathbf{y}} = \mathbf{y}$, and thus $\bar{\mathbf{z}} = \mathbf{z}$, showing that C is an ambi-nut digraph. \square

Note that the coalescence of two dextro-nut digraphs is not necessarily a dextro-nut digraph. [Fig. 12\(a\)](#) shows an example where coalescence of dextro-nut digraphs from [Figs. 3\(a\)](#) and [3\(b\)](#) produces a dextro-nut digraph (which contains a cut-arc but no leaves). On the other hand, [Fig. 12\(b\)](#) shows an example where a coalescence of two copies of the dextro-nut digraph from [Fig. 3\(a\)](#) is not a dextro-nut digraph, as confirmed by the existence of a non-full kernel eigenvector of the resulting digraph.

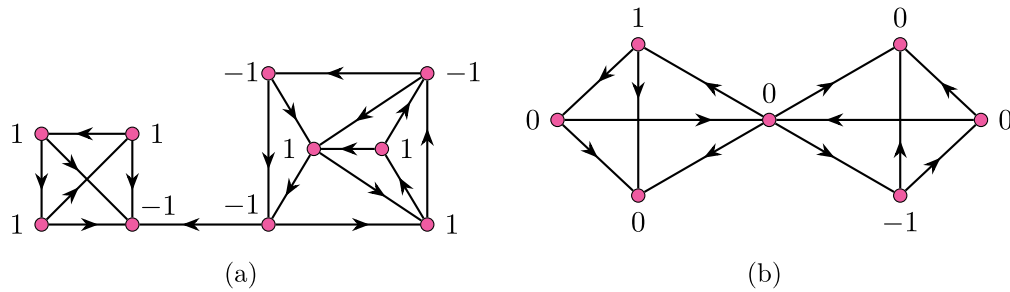


Fig. 12. Coalescence of two dextro-nut digraphs may or may not result in a dextro-nut digraph. (a) Dextro-nut digraphs from Figs 3(a) and 3(b) yield a dextro-nut digraph; but (b) two copies of the dextro-nut digraph from Fig. 3(a) (at one of the vertices bearing 1 in the displayed kernel eigenvector) do not.

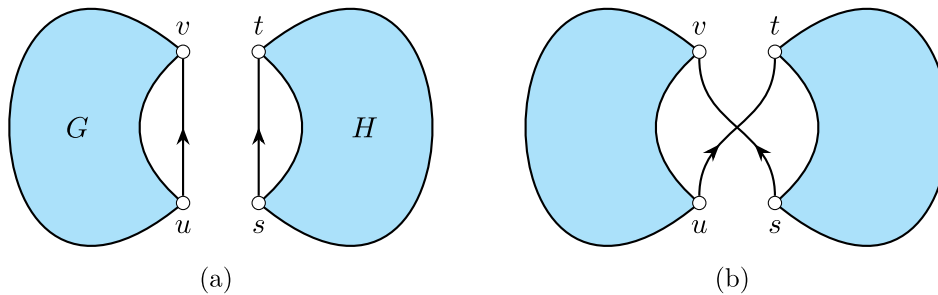


Fig. 13. Schematic of the crossover construction. Panel (a) shows the disjoint union of digraph G and H that contain arcs $u \rightarrow v$ and $s \rightarrow t$, respectively. Panel (b) shows the digraph $(G, u, v) \bowtie (H, s, t)$.

5.3. Crossover construction

Let $u \rightarrow v$ be an arc of a digraph G and let $s \rightarrow t$ be an arc of a digraph H . The *crossover* of G and H with respect to $u \rightarrow v$ and $s \rightarrow t$, denoted by $(G, u, v) \bowtie (H, s, t)$, is defined as the digraph obtained from the disjoint union of G and H by deleting the arcs $u \rightarrow v$ and $s \rightarrow t$ and adding the arcs $u \rightarrow t$ and $s \rightarrow v$; see Fig. 13.

Theorem 30. Let G and H be two ambi-nut digraphs with $\ker G = \langle \mathbf{x} \rangle = \text{coker } G$ and $\ker H = \langle \mathbf{y} \rangle = \text{coker } H$. If $u^* \rightarrow v^*$ is an arc of G and $s^* \rightarrow t^*$ an arc of H such that

$$\mathbf{x}(u^*) = \mathbf{y}(s^*), \quad \mathbf{x}(v^*) = \mathbf{y}(t^*), \tag{21}$$

then the digraph $(G, u^*, v^*) \bowtie (H, s^*, t^*)$ is an ambi-nut digraph.

Proof. Let $C := (G, u^*, v^*) \bowtie (H, s^*, t^*)$. Without loss of generality, we shall assume that $V(G)$ and $V(H)$ are disjoint, so that $V(C)$ is simply the union of $V(G)$ and $V(H)$. Further, let $\mathbf{z}: V(C) \rightarrow \mathbb{R}$ be defined by letting $\mathbf{z}(u)$ to be equal to $\mathbf{x}(u)$ for all $u \in V(G)$ and equal to $\mathbf{y}(u)$ for all $u \in V(H)$. By the definition of the crossover and by the assumption (21), it follows that \mathbf{z} is a full vector contained both in $\ker C$ and in $\text{coker } C$. What remains to show is that $\ker C$ and $\text{coker } C$ are both 1-dimensional.

Let $\mathbf{w}: V(C) \rightarrow \mathbb{R}$ be an element of $\ker C$ such that $\mathbf{w}(u^*) = \mathbf{x}(u^*)$ and let \mathbf{w}_G be its restriction to $V(G)$. Since the out-neighbourhood $C^+(v)$ is contained in $V(G)$ for every $v \in V(G) \setminus \{u^*\}$ (and is in fact equal to $G^+(v)$), we see that \mathbf{w}_G satisfies the local condition $S_{\mathbf{w}_G}^+(v) = 0$ for all $v \in V(G)$ except possibly for $v = u^*$. However, the existence of a full vector in $\text{coker } G$, together with Lemma 12, implies that this condition is satisfied also for $v = u^*$, and thus that $\mathbf{w}_G \in \ker G$. Since $\ker G$ is a one-dimensional space spanned by \mathbf{x} and since \mathbf{w}_G coincides with \mathbf{x} in one vertex, we see that $\mathbf{w}_G = \mathbf{x}$. Using an analogous argument, one can also show that the restriction of \mathbf{w} to $V(H)$ equals \mathbf{y} , implying that $\mathbf{w} = \mathbf{z}$, as required.

Moreover, applying the above to the reverse digraphs G^R and H^R , one can show that $\text{coker } C$ is one-dimensional (and thus spanned by \mathbf{z}), proving that C is an ambi-nut. \square

Note that a minor refinement of the above proof (by separately investigating the kernel and co-kernel) yields the slightly stronger claim that the crossover of two bi-nut digraphs G and H with their kernel and co-kernel vectors being compatible on the arcs $u^* \rightarrow v^*$ and $s^* \rightarrow t^*$ is also a bi-nut. However, as the example in Fig. 14 shows, the crossover of dextro-nut digraphs is not necessarily a dextro-nut digraph.

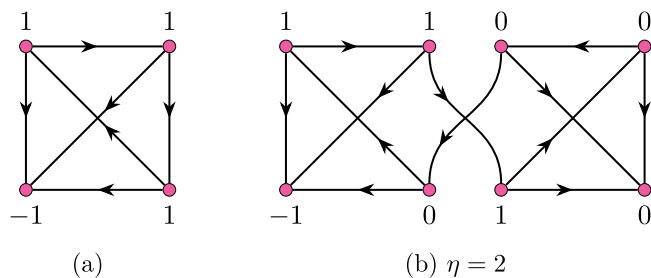


Fig. 14. Crossover of two copies of the dextro-nut digraph in (a) results in the digraph in (b), which is not a dextro-nut as witnessed by the presence of zero entries in the displayed non-full kernel vector.

5.4. Multiplier constructions

In [6, Section 4.2] triangle- and pentagon-multiplier constructions for nut graphs were studied. These constructions proved useful for obtaining nut graphs from any regular graph of even degree, and were crucial for proving that any finite group can be realised by a nut graph [4]. Here, we generalise this theory to ambi-nut digraphs.

We begin by defining the notion of a *gadget*. Let $M_{(i)}$ denote the matrix obtained from M by deleting the i th row. For the present purpose, then:

Definition 31. A gadget is a pair (G, r) , where G is a digraph and $r \in V(G)$, with the property that there exists a full vector \mathbf{x} that spans the respective (1-dimensional) kernels of both $A(G)_{(r)}$ and $A(G^R)_{(r)}$. The vertex r will be called the *root* of the gadget.

Note that the vector \mathbf{x} in the above definition satisfies $S_{\mathbf{x}}^+(v) = S_{\mathbf{x}}^-(v) = 0$ for all $v \in V(G) \setminus \{r\}$. Recall that $S_{\mathbf{x}}^+(v)$ and $S_{\mathbf{x}}^-(v)$ are defined in (6).

Proposition 32. Let (G, r) be a gadget. Then $S_{\mathbf{x}}^+(r) = S_{\mathbf{x}}^-(r)$, where \mathbf{x} is the full vector from Definition 31.

Proof. We shall use a double-counting approach, as in the proof of Lemma 12. Let

$$M := \sum_{\substack{u, v \in V(G) \\ v \rightarrow u}} \mathbf{x}(v)\mathbf{x}(u). \tag{22}$$

On one hand, we see that

$$M = \sum_{u \in V(G)} \sum_{v \in G^-(u)} \mathbf{x}(v)\mathbf{x}(u) = \sum_{u \in V(G)} \mathbf{x}(u)S_{\mathbf{x}}^-(u) = \mathbf{x}(r)S_{\mathbf{x}}^-(r). \tag{23}$$

On the other hand

$$M = \sum_{v \in V(G)} \sum_{u \in G^+(v)} \mathbf{x}(v)\mathbf{x}(u) = \sum_{v \in V(G)} \mathbf{x}(v)S_{\mathbf{x}}^+(v) = \mathbf{x}(r)S_{\mathbf{x}}^+(r). \tag{24}$$

Since, by assumption, \mathbf{x} is a full vector, it follows that $S_{\mathbf{x}}^-(r) = S_{\mathbf{x}}^+(r)$, as claimed. \square

Definition 33. The *demand* of a gadget (G, r) , denoted $\text{dem}(G, r)$, is the value $-S_{\mathbf{x}}^+(r)/\mathbf{x}(r)$, where \mathbf{x} is the full vector from Definition 31.

The demand is well defined, since $\mathbf{x}(r) \neq 0$ and \mathbf{x} is unique up to scalar multiplication. By Proposition 32, $\text{dem}(G, r) = -S_{\mathbf{x}}^-(r)/\mathbf{x}(r)$.

Proposition 34. Let (G, r) be a gadget where r is an arbitrary vertex of G . Then G is an ambi-nut digraph if and only if $\text{dem}(G, r) = 0$.

Proof. If G is an ambi-nut digraph then (3) and (4) immediately imply that $\text{dem}(G, r) = 0$. Conversely, suppose that $\text{dem}(G, r) = 0$. Let \mathbf{x} be the full vector from Definition 31. We already noted that $S_{\mathbf{x}}^+(v) = S_{\mathbf{x}}^-(v) = 0$ for all $v \in V(G) \setminus \{r\}$. From the definition of demand and Proposition 32 it follows that $S_{\mathbf{x}}^+(r) = S_{\mathbf{x}}^-(r) = 0$. Hence, (3) and (4) hold, so $\mathbf{x} \in \ker G$ and $\mathbf{x} \in \text{coker } G$. Suppose that $\eta(G) > 1$. Then there exists $\mathbf{y} \in \ker G$ which is linearly independent from \mathbf{x} . But then \mathbf{y} is also in the kernel of $A(G)_{(r)}$, contradicting the requirement of Definition 31 that $\ker A(G)_{(r)}$ is 1-dimensional. Thus, $\eta(G) = 1$ and G is therefore an ambi-nut digraph. \square

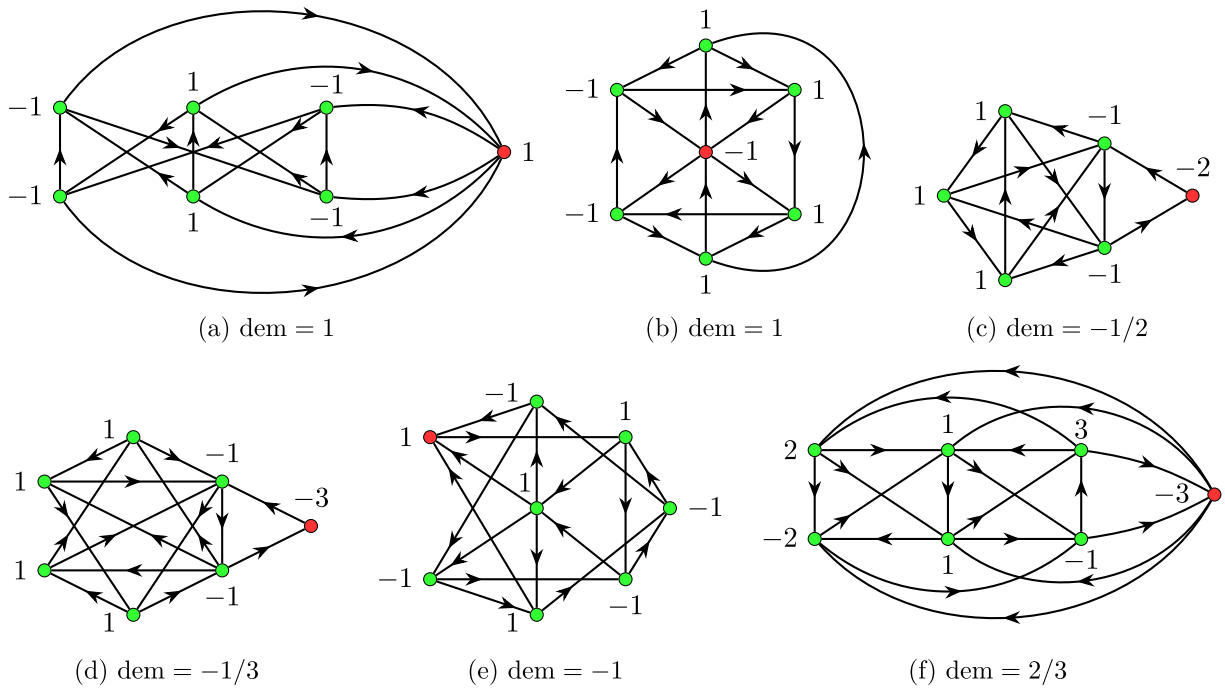


Fig. 15. Examples of gadgets with various demands. The root vertex is coloured red. Labels show the vector \mathbf{x} from Definition 31. (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Fig. 15 shows examples of gadgets with demands from the set $\{1, -1/2, -1/3, -1, 2/3\}$. Note that the demand of a gadget is always a rational number.

Lemma 35 ([42, Lemma 1.2]). *Let G be a digraph. If λ is a rational eigenvalue of G then λ is an integer.*

Proof. The characteristic polynomial φ of a digraph is a monic polynomial with integer coefficients. By the Rational Root Theorem, the rational roots of φ are integers. \square

Lemma 35 will be significant for the formulation of the main theorem of the present section (Theorem 38), for which we will require gadgets that have integer demand. The next theorem shows that ambi-nut digraphs provide a ready source of gadgets with demand -1 .

Theorem 36. *Let H be an ambi-nut digraph and let $\tilde{\mathbf{x}} \in \ker H$ be a non-trivial vector. Let $(u_1, u_2) \in E(H)$ such that $\tilde{\mathbf{x}}(u_1) = \tilde{\mathbf{x}}(u_2)$. Let G be a digraph with the vertex set $V(G) = V(H) \cup \{u_0\}$ and the arc-set $E(G) = E(H) \setminus \{(u_1, u_2)\} \cup \{(u_1, u_0), (u_0, u_2)\}$. Then (G, u_0) is a gadget with $\text{dem}(G, u_0) = -1$.*

Proof. Let us define $\mathbf{x}: V(G) \rightarrow \mathbb{R}$ by $\mathbf{x}(v) = \tilde{\mathbf{x}}(v)$ for $v \in V(H)$ and $\mathbf{x}(u_0) = \tilde{\mathbf{x}}(u_1)$. It is clear that \mathbf{x} is full, $\mathbf{x} \in \ker A(G)_{(u_0)}$ and $\mathbf{x} \in \ker A(G^R)_{(u_0)}$. Suppose that $\mathbf{y} \in \ker A(G)_{(u_0)}$. Let us define $\tilde{\mathbf{y}}: V(H) \rightarrow \mathbb{R}$ by $\tilde{\mathbf{y}}(v) = \mathbf{y}(v)$ for $v \in V(H)$. By Lemma 12, $\tilde{\mathbf{y}}(v) \in \ker H$. But H is an ambi-nut digraph, so $\tilde{\mathbf{y}} = \mu \tilde{\mathbf{x}}$ for some $\mu \in \mathbb{R}$. Now, $\mu \tilde{\mathbf{x}}$ satisfies the out-local condition at vertex u_1 in the graph H , i.e.,

$$S_{\mu \tilde{\mathbf{x}}}^+(u_1) = \sum_{v \in H^+(u_1)} \mu \tilde{\mathbf{x}}(v) = \sum_{v \in H^+(u_1) \setminus \{u_2\}} \mu \tilde{\mathbf{x}}(v) + \mu \tilde{\mathbf{x}}(u_2) = 0. \tag{25}$$

Similarly, \mathbf{y} satisfies the out-local condition at vertex u_1 in the graph G , i.e.,

$$S_{\mathbf{y}}^+(u_1) = \sum_{v \in G^+(u_1)} \mathbf{y}(v) = \sum_{v \in G^+(u_1) \setminus \{u_0\}} \mathbf{y}(v) + \mathbf{y}(u_0) = 0. \tag{26}$$

From (25) and (26) we obtain $\mathbf{y}(u_0) = \mu \tilde{\mathbf{x}}(u_2)$. This implies that $\mathbf{y} = \mu \mathbf{x}$ and thus $\ker A(G)_{(u_0)}$ is 1-dimensional. By analogy we can show that $\ker A(G^R)_{(u_0)}$ is also 1-dimensional, so \mathbf{x} satisfies the requirements of Definition 31, and thus (G, u_0) is a gadget. As u_0 has out-degree one, connecting only to u_2 , it is straightforward to see that $\text{dem}(G, u_0) = -1$. \square

Note that the condition in Theorem 36, namely the existence of an arc $(u_1, u_2) \in E(H)$ such that $\tilde{\mathbf{x}}(u_1) = \tilde{\mathbf{x}}(u_2)$, is not obeyed by all ambi-nut digraphs. The smallest examples among oriented graphs that are unsuitable for the theorem

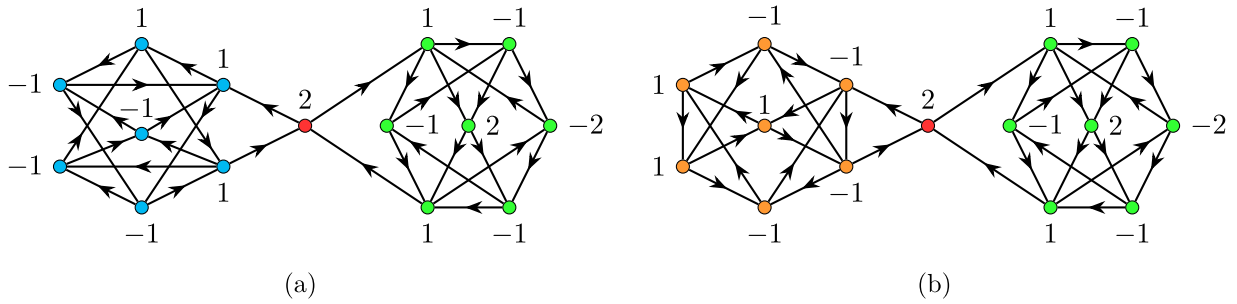


Fig. 16. (a): A composite gadget of demand -1 obtained from two non-isomorphic gadgets with demands $-1/2$. (b): A composite gadget of demand 0 , which is an ambi-nut digraph, obtained from two non-isomorphic gadgets with respective demands $1/2$ and $-1/2$.

have 10 vertices. There are exactly 6 such ambi-nut digraphs among the oriented graphs on 10 vertices [5]. All ambi-nut digraphs among oriented graphs up to order 9 contain at least one ‘suitable’ arc.

In light of the following lemma, gadgets with fractional demand can also be useful.

Lemma 37. *Let (G_1, r_1) and (G_2, r_2) be gadgets. Then the digraph $(G_1, r_1) \odot (G_2, r_2)$ with the root being the merged vertex is a gadget with demand $\text{dem}(G_1, r_1) + \text{dem}(G_2, r_2)$.*

Proof. Let $H := (G_1, r_1) \odot (G_2, r_2)$. Let $\mathbf{x}^{(1)} \in \ker A(G_1)_{(r_1)} \cap \ker A(G_1^R)_{(r_1)}$ and $\mathbf{x}^{(2)} \in \ker A(G_2)_{(r_2)} \cap \ker A(G_2^R)_{(r_2)}$, which exist by Definition 31. We can assume that $\mathbf{x}^{(1)}(r_1) = \mathbf{x}^{(2)}(r_2) \neq 0$. Define $\mathbf{y}: V(H) \rightarrow \mathbb{R}$ by $\mathbf{y}(u) = \mathbf{x}^{(1)}(u)$ if $u \in V(G_1)$ and $\mathbf{y}(u) = \mathbf{x}^{(2)}(u)$ if $u \in V(G_2)$. Vector \mathbf{y} is full and $\mathbf{y} \in \ker A(H)_{(r_1)} \cap \ker A(H^R)_{(r_1)}$.

Let $\mathbf{z} \in \ker A(H)_{(r_1)}$. The restriction of \mathbf{z} to G_1 (resp. G_2) is in $\ker A(G_1)_{(r_1)}$ (resp. $\ker A(G_2)_{(r_2)}$), which is 1-dimensional by Definition 31, hence the restriction of \mathbf{z} to G_1 (resp. G_2) is a multiple of $\mathbf{x}^{(1)}$ (resp. $\mathbf{x}^{(2)}$). This implies that \mathbf{z} is a multiple of \mathbf{y} and hence, $\ker A(H)_{(r_1)}$ is 1-dimensional. By similar reasoning, we can show that $\ker A(H^R)_{(r_1)}$ is also 1-dimensional.

Moreover,

$$\begin{aligned} \text{dem}(H, r_1) &= -\frac{\sum_{u \in H^+(r_1)} \mathbf{y}(u)}{\mathbf{y}(r_1)} \\ &= -\frac{\sum_{u \in G_1^+(r_1)} \mathbf{y}(u)}{\mathbf{y}(r_1)} - \frac{\sum_{u \in G_2^+(r_2)} \mathbf{y}(u)}{\mathbf{y}(r_2)} = \text{dem}(G_1, r_1) + \text{dem}(G_2, r_2). \end{aligned} \quad \square \quad (27)$$

A gadget that can be obtained by (possibly repeated) use of Lemma 37 will be called a *composite gadget*. Note that this allows gadgets with integer demand to be obtained by coalescing gadgets of fractional demand; the gadget in Fig. 16(a) is an example. Note that if we combine Lemma 37 with Proposition 34, we have yet another avenue for obtaining large ambi-nut digraphs. If we coalesce, at their common root, a set of gadgets for which the demands add to 0, the result is an ambi-nut digraph; the ambi-nut digraph in Fig. 16(b) is an example.

We now give a theorem that can be used to produce ambi-nut digraphs by attaching gadgets to a *base digraph*.

Theorem 38. *Let G be the base digraph, i.e., a digraph of order n with $V(G) = \{v_1, v_2, \dots, v_n\}$ and an integer eigenvalue λ such that the λ -eigenspaces of G and G^R are both 1-dimensional and spanned by the same full vector \mathbf{x} . Let $(\Gamma_1, r_1), (\Gamma_2, r_2), \dots, (\Gamma_n, r_n)$ be gadgets, such that $\text{dem}(\Gamma_i, r_i) = \lambda$ for all $i = 1, \dots, n$. Then the digraph M obtained from the disjoint union $G \sqcup \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_n$ by identifying v_i with r_i for all $i = 1, \dots, n$ is ambi-nut.*

Proof. For each i , let $\mathbf{x}^{(i)}$ be the vector that spans the kernel of $A(\Gamma_i)_{(r_i)}$. We can assume that $\mathbf{x}^{(i)}(r_i) = \mathbf{x}(v_i)$. Let $\mathbf{y}: V(M) \rightarrow \mathbb{R}$, such that for each i and for each $v \in V(\Gamma_i)$, $\mathbf{y}(v) = \mathbf{x}^{(i)}(v)$. From Definition 31 it follows that every vertex $v \in V(M) \setminus V(G)$ satisfies $S_{\mathbf{y}}^+(v) = S_{\mathbf{y}}^-(v) = 0$. Moreover, $S_{\mathbf{y}}^+(v_i) = \sum_{u \in M^+(v_i)} \mathbf{y}(u) = \sum_{u \in G^+(v_i)} \mathbf{y}(u) + \sum_{u \in \Gamma_i^+(v_i)} \mathbf{y}(u) = \lambda \mathbf{x}(v_i) - \text{dem}(\Gamma_i, r_i) \mathbf{x}(v_i) = 0$. Similarly, $S_{\mathbf{y}}^-(v_i) = 0$. It follows that $\mathbf{y} \in \ker M$ and $\mathbf{y} \in \text{coker } M$. Since \mathbf{x} and all $\mathbf{x}^{(i)}$ are full, \mathbf{y} is also full.

Let $\mathbf{z} \in \ker M$. The vector \mathbf{z} restricted to Γ_i is in $\ker A(\Gamma_i)_{(r_i)}$ and therefore it is a multiple of $\mathbf{x}^{(i)}$. The local condition at vertex r_i is

$$\sum_{u \in M^+(r_i)} \mathbf{z}(u) = \sum_{u \in G^+(r_i)} \mathbf{z}(u) + \sum_{u \in \Gamma_i^+(r_i)} \mathbf{z}(u) = 0. \quad (28)$$

Recall that $\sum_{u \in \Gamma_i^+(r_i)} \mathbf{z}(u) = -\mathbf{z}(r_i) \text{dem}(\Gamma_i, r_i)$. Eq. (28) yields

$$\sum_{u \in G^+(r_i)} \mathbf{z}(u) - \mathbf{z}(r_i) \text{dem}(\Gamma_i, r_i) = 0. \quad (29)$$

Since $\text{dem}(\Gamma_i, r_i) = \lambda$, we further obtain

$$\sum_{u \in G^+(r_i)} \mathbf{z}(u) = \lambda \mathbf{z}(r_i). \tag{30}$$

This implies that \mathbf{z} restricted to G is an eigenvector for λ , so it is a multiple of \mathbf{x} . In summary, \mathbf{z} is a multiple of \mathbf{y} . We conclude that $\ker M$ is 1-dimensional. By similar reasoning, we can see that $\text{coker } M$ is also 1-dimensional. \square

Using [Theorem 36](#), [Lemma 37](#) and the examples in [Fig. 15](#), it is easy to produce an abundance of gadgets with integer demands. Following the definition used in [\[56\]](#), a digraph G is *diregular of degree k* (or *k -diregular*) if $d^+(v) = d^-(v) = k$ for $v \in V(G)$. (Some authors, e.g. [\[11\]](#), use the term *k -regular digraph* to mean what we are calling a *k -diregular digraph*. However, in the present paper, we have already used *k -regular digraph* to denote a digraph whose underlying graph is *k -regular*.) Further examples of gadgets can be found in [\[5\]](#). For the base digraph G in [Theorem 38](#), there exist whole classes of examples:

- (i) Every connected k -diregular digraph is a base digraph for $\lambda = k$.
- (ii) Every connected bipartite k -diregular digraph is a base digraph for $\lambda = -k$ (in addition to $\lambda = k$).
- (iii) [Fig. 17](#) shows examples of base digraphs for $\lambda \in \{-1, 1, 2\}$ that are not diregular. More can be found in [\[5\]](#).

It is not difficult to see the following.

Proposition 39. *Every connected k -diregular digraph ($k \geq 0$) is strongly connected.*

Proof. Let G be a connected k -diregular digraph and let \tilde{G} be the condensation of G . Recall that the vertices of \tilde{G} are the strongly connected components S_i of G with $(S_i, S_j) \in E(\tilde{G})$ if there exist vertices $u \in S_i$ and $v \in S_j$ such that $(u, v) \in E(G)$ [\[29\]](#). If \tilde{G} contains exactly one vertex, there is nothing to prove, as the whole graph G is a strongly connected component. Suppose that \tilde{G} contains at least two vertices. As \tilde{G} is a DAG (directed acyclic graph), there exists a vertex $S^* \in V(\tilde{G})$ with $d^-(S^*) = 0$, which implies

$$\sum_{\substack{v \rightarrow u \\ u \in S^*, v \notin S^*}} 1 = 0.$$

Since G is connected, $d^+(S^*) > 0$, which implies

$$\sum_{\substack{u \rightarrow v \\ u \in S^*, v \notin S^*}} 1 > 0.$$

For $U \subseteq V(G)$, let us define

$$\Delta(U) = \sum_{u \in U} d^+(u) - \sum_{u \in U} d^-(u). \tag{31}$$

Since G is diregular, we obtain $\Delta(S^*) = 0$. On the other hand, we can write

$$\Delta(U) = \sum_{\substack{u \rightarrow v \\ u \in U}} 1 - \sum_{\substack{v \rightarrow u \\ u \in U}} 1 = \sum_{\substack{u \rightarrow v \\ u \in U}} 1 + \sum_{\substack{u \rightarrow v \\ u \in U, v \notin U}} 1 - \sum_{\substack{v \rightarrow u \\ u \in U, v \in U}} 1 - \sum_{\substack{v \rightarrow u \\ u \in U, v \notin U}} 1 = \sum_{\substack{u \rightarrow v \\ u \in U}} 1 - \sum_{\substack{v \rightarrow u \\ u \in U, v \notin U}} 1. \tag{32}$$

Choosing $U = S^*$ in [\(32\)](#) gives that $\Delta(S^*) > 0$, a contradiction. Therefore, \tilde{G} cannot have two or more vertices. \square

By [Proposition 39](#), digraphs from (i) and (ii) above are strongly connected, i.e., have irreducible adjacency matrices, and the Perron–Frobenius theory for non-symmetric matrices applies. If G is a k -diregular digraph, then the spectral radius $\rho(A(G)) = k$ by [\[12, Theorem 8.3.6\]](#). Moreover, by [\[12, Theorem 8.3.4\]](#), G has the Perron eigenvalue k with (algebraic and geometric) multiplicity 1. The corresponding eigenspace is, in fact, spanned by the all-ones vector. Finally, by [\[11, Theorem 3.1\]](#), the spectrum of a bipartite digraph is invariant under multiplication by -1 .

[Theorem 38](#) also applies to undirected graphs (viewed as digraphs where for every arc the opposite arc is also present) and hence generalises [Propositions 17–19](#) from [\[6\]](#). For convenience, we give the undirected versions of [Definition 31](#) and [Theorem 38](#):

Definition 40. An *undirected gadget* is a pair (G, r) , where G is a graph and $r \in V(G)$, with the property that there exists a full vector \mathbf{x} that spans the (1-dimensional) kernel of $A(G)_{(r)}$. The vertex r will be called the *root*.

Note that the triangle and pentagon graphs used in [Section 4.2](#) of [\[6\]](#) are, in fact, undirected gadgets with respective demands 2 and -2 . The various bouquets used in that section correspond to composite undirected gadgets.

Theorem 41. *Let G be the base graph, i.e., a graph of order n with $V(G) = \{v_1, v_2, \dots, v_n\}$ and an integer eigenvalue λ such that the (1-dimensional) λ -eigenspace of G is spanned by a full vector \mathbf{x} . Let $(\Gamma_1, r_1), (\Gamma_2, r_2), \dots, (\Gamma_n, r_n)$ be undirected gadgets, such that $\text{dem}(\Gamma_i, r_i) = \lambda$ for all $i = 1, \dots, n$. Then the graph M obtained from the disjoint union $G \sqcup \Gamma_1 \sqcup \Gamma_2 \sqcup \dots \sqcup \Gamma_n$ by identifying v_i with r_i for all $i = 1, \dots, n$ is a nut graph.*

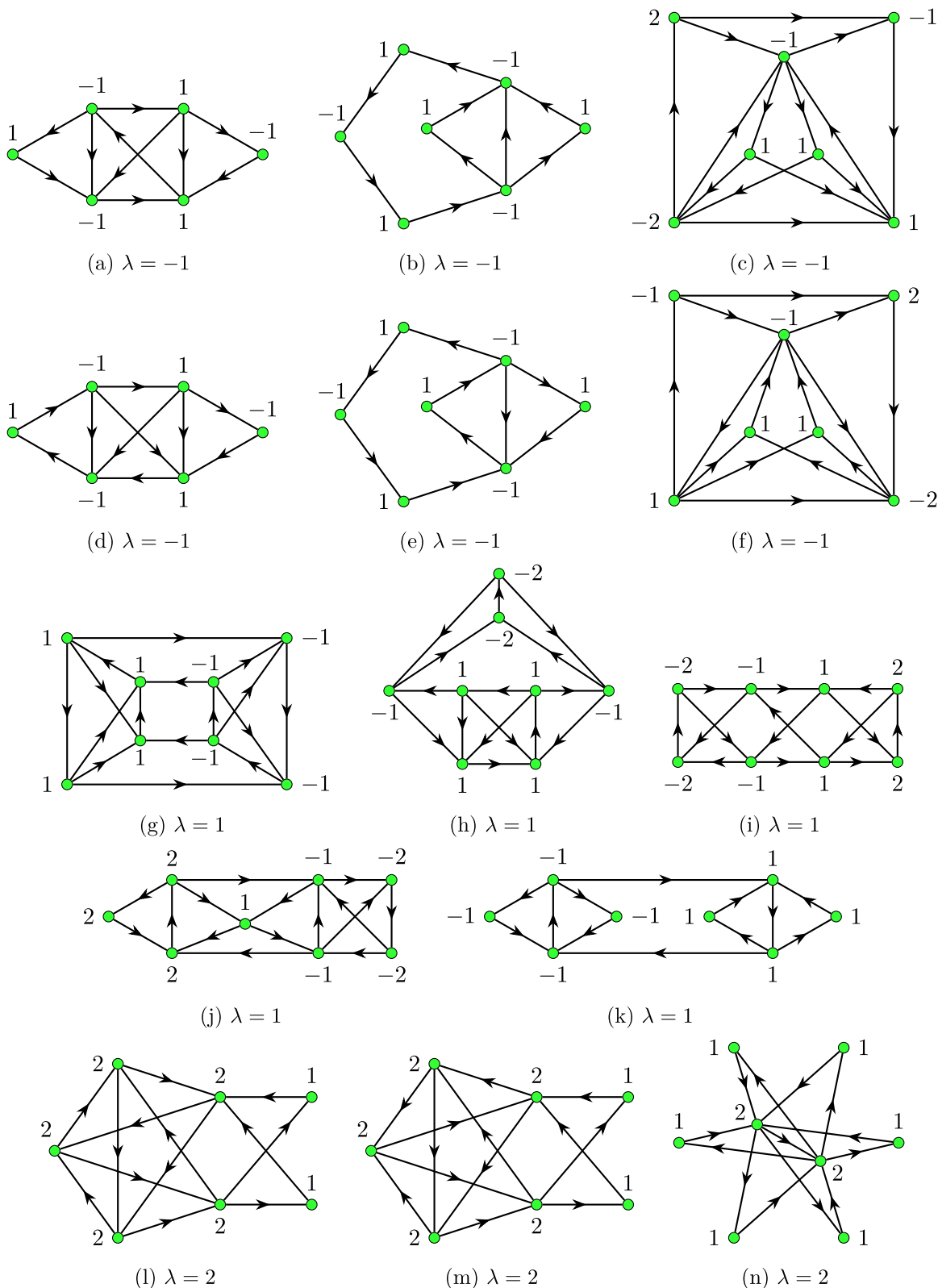


Fig. 17. Non-diregular base digraphs for [Theorem 38](#) with $\lambda \in \{-1, 1, 2\}$. The labels show entries of the unique (up to scalar multiplication) eigenvector for the chosen eigenvalue λ .

6. Perspective

To give an indication of how digraphs and hence nut digraphs may have applications to physics and chemistry, we first note some applications of graphs. Adjacency matrices have many applications to physics and chemistry, where it is often appropriate to use a system of linear equations. For example, in the Hückel (tight-binding) model of π systems, linear combinations of basis functions on atomic sites (vertices of the molecular graph) are used to form molecular orbitals, which are then used to construct a many-electron wavefunction for the physical state of the system. Interactions are limited to adjacent sites, and in the simplest cases the one-electron Hamiltonian determining the molecular orbitals reduces to a scaled and shifted adjacency matrix of the molecular graph, and π electron distribution and π energy follow from the eigenvectors and eigenvalues of this matrix.

Adjacency matrices also arise in approximate descriptions of photonic, optomechanical, or phononic devices. In these systems a set of oscillatory modes (for example, a photon at resonant frequency in a cavity) defines a basis. Modes may be coupled to one another (for instance, due to the spatial proximity of cavities) with couplings defining edges. Eigenvalues then correspond to the resonant frequencies of the entire coupled system, and eigenvectors to the spatial profile of the many-mode wavefunctions at that frequency.

Typically H for an isolated system is Hermitian to ensure real energy eigenvalues (a requirement of the postulates of quantum mechanics), but connecting the system to a larger environment in a controlled non-conservative way allows an effective non-Hermitian matrix (including directional arc weights [9,41]) to model the system [33,34,38,40] and is readily realisable experimentally [22,37,54,57,60]. This generalisation implies that many chemical and physical applications may be usefully formulated in terms of digraphs.

In topological physics, non-Hermitian model Hamiltonians may undergo so-called *topological phase transitions*, where Hamiltonians are split into distinct equivalence classes, and may be mapped from one class to another by some continuous adjustment of system parameters (for example, altering arc weights of a digraph to change the parity of the number of negative-eigenvalue states [61]). In some approaches to the classification of such phases, ambi- and bi- nut digraphs may describe systems that have two distinct topological phases [30,36,61] and (with unit arc weights) may lie on a topological phase boundary. This connection gives a strong hint that nut digraphs may also be applied to the classification of topological phases.

Ambi- and bi-nut digraphs also have potential applications in chemical physics, for example in models of molecular conduction. In the eponymous source-and-sink model, where a device in a circuit is modelled by connection of vertices of the molecular graph to special source and sink vertices, graph nullity is a crucial factor. In particular, a molecular graph can be a *strong omni-conductor*, such that connection to any vertex or pair of vertices leads to a prediction of transmission at the Fermi level [25,31]. The nut graphs are exactly the strong omni-conductors of nullity 1, as the property depends on the fact that all first minors of the adjacency matrix are non-zero, a requirement fulfilled by all ambi- and bi-nut digraphs. (By [Corollary 21](#), all principal minors are non-zero, but this result is easily extended to all first minors.)

Laevo- and dextro-nut digraphs could be useful in the description of novel physical phenomena. Left and right null-vectors are not simultaneously full in such digraphs, and the Green's function of the adjacency matrix [10,14] indicates that the digraph will display (asymmetric) directional transport. Furthermore, such digraphs are prime candidates for exhibiting a form of *topological protection* (which has been experimentally observed for undirected graphs in [58] and will be subjected to further theoretical exploration).

Our discussion indicates that applications of laevo- and dextro-nut digraphs will be distinct from those of ambi- and bi-nut digraphs, but we expect that the construction of larger digraphs using the various forms of nut digraphs as building blocks will result in systems with novel physics.

We suspect that there are many more applications of nut digraphs, beyond the few outlined above.

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Data availability

The data are in a public GitHub repository and are referred to from within the paper.

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