Journal of Algebraic Combinatorics (2026) 63:9
https://doi.org/10.1007/s10801-025-01492-6

Nut graphs with a prescribed number of vertex and edge orbits

Nino Bašić[1,2,3] · **Ivan Damnjanović**[1,4]

Received: 27 February 2025 / Accepted: 3 December 2025 / Published online: 8 January 2026

Abstract
A nut graph is a nontrivial graph whose adjacency matrix has a one-dimensional null space spanned by a vector without zero entries. Recently, it was shown that a nut graph has more edge orbits than vertex orbits. It was also shown that for any even $r \geq 2$ and any $k \geq r + 1$, there exist infinitely many nut graphs with r vertex orbits and k edge orbits. Here, we extend this result by finding all the pairs (r, k) for which there exists a nut graph with r vertex orbits and k edge orbits. In particular, we show that for any $k \geq 2$, there are infinitely many Cayley nut graphs with k edge orbits and k arc orbits.

Keywords Nut graph · Vertex orbit · Edge orbit · Arc orbit · Cayley graph · Automorphism

Mathematics Subject Classification 05C50 · 05C25

1 Introduction

A *nut graph* is a nontrivial graph whose adjacency matrix has a one-dimensional null space spanned by a full vector, i.e., a vector without zero entries. The concept of nut graph was introduced by Sciriha and Gutman [27–30, 36] and then investigated through a series of papers [16, 20, 31, 32]. The chemical justification for studying nut graphs can be found in [8, 17, 18, 33, 34]. In the Hückel molecular orbital (HMO) theory, nut graphs represent idealized molecular structures that theoretically support

---

✉ Nino Bašić
nino.basic@famnit.upr.si

Ivan Damnjanović
ivan.damnjanovic@famnit.upr.si

[1] FAMNIT, University of Primorska, Glagoljaška 8, 6000 Koper, Slovenia

[2] IAM, University of Primorska, Muzejski trg 2, 6000 Koper, Slovenia

[3] Institute of Mathematics, Physics and Mechanics, Jadranska ulica 19, 1000 Ljubljana, Slovenia

[4] Faculty of Electronic Engineering, University of Niš, Aleksandra Medvedeva 4, Niš 18104, Serbia

fully delocalized zero-energy states and omniconducting behavior at the Fermi level. There, the graphs under consideration are typically subcubic. However, the study of nut graphs, which may or may not arise from chemical structures, has raised several questions that are interesting from a purely mathematical viewpoint. For more results on nut graphs, the reader is referred to the monograph [35].

All the orders attainable by a d-regular nut graph were determined for each $d \leq 4$ in [20], and for each $d \in \{5, 6, \dots, 11\}$ in [16]. Afterward, a circulant graph-based construction was used [6] to prove that there exists a 12-regular nut graph of order n if and only if $n \geq 16$. All the pairs (n, d) for which there exists a d-regular circulant nut graph of order n were subsequently determined through a series of papers [9, 10, 15]. Also, all the pairs (n, d) with $4 \mid d$ for which there exists a d-regular Cayley nut graph of order n were determined in [11]. For more results on the realizability problems concerning nut graphs, see [1–3]. The complete classification of quartic circulant nut graphs was given in [12], while the complete classification of cubic tricirculant nut graphs and quartic bicirculant nut graphs was given in [13] and [14], respectively. Moreover, it was shown that cubic tetra- and pentacirculant nut graphs do not exist [1].

For a graph G, let $o_v(G)$, $o_e(G)$ and $o_a(G)$ denote the number of vertex orbits, edge orbits and arc orbits in G, respectively. The properties of vertex and edge orbits in nut graphs were recently investigated [4], yielding the following two results.

Theorem 1 *([4, Theorem 2]) Let G be a nut graph. Then* $o_e(G) \geq o_v(G) + 1$.

Theorem 2 *([4, Theorem 34]) Let* $r \geq 2$ *be even. For every* $k \geq r + 1$ *there exist infinitely many nut graphs G with* $o_v(G) = r$ *and* $o_e(G) = k$.

Theorem 1 compares to the next two results by Buset.

Theorem 3 *([7, Theorem 1]) For any* $r \in \mathbb{N}$ *and* $k \in \mathbb{N}_0$, *there exists a graph G with* $o_v(G) = r$ *and* $o_e(G) = k$ *if and only if* $r \leq 2k + 1$.

Theorem 4 *([7, Theorem 2]) For any* $r \in \mathbb{N}$ *and* $k \in \mathbb{N}_0$, *there exists a connected graph G with* $o_v(G) = r$ *and* $o_e(G) = k$ *if and only if* $r \leq k + 1$.

With Theorem 2 in mind, this leads us to the natural problem of finding all the pairs $(o_v(G), o_e(G))$ attainable by a nut graph G. The main result of our paper is the complete resolution of this Buset-type problem. We first investigate the automorphisms of Cayley nut graphs and obtain the next result.

Theorem 5 *For any* $k \geq 2$, *there exist infinitely many Cayley nut graphs G with* $o_e(G) = o_a(G) = k$.

We subsequently apply subdivisions to the Cayley nut graphs from Theorem 5 to get the following theorem.

Theorem 6 *For any* $r \in \mathbb{N}$ *and* $k \geq r + 1$, *there exist infinitely many nut graphs G with* $o_v(G) = r$ *and* $o_e(G) = k$.

In Sect. 2, we introduce the notation to be used in the next sections and preview some known results that we will need. Afterward, we prove Theorem 5 in Sect. 3 and Theorem 6 in Sect. 4. In Sect. 5, we end the paper with a brief conclusion.

2 Preliminaries

All graphs considered will be undirected, simple and finite. We denote the standard $(0, 1)$-adjacency matrix of a graph G by $A(G)$, and its spectrum, regarded as a multiset, by $\sigma(G)$. All spectral properties considered will correspond to the adjacency matrix. By $\mathrm{Circ}(n, S)$, where $S \subseteq \{1, 2, \dots, \lfloor \frac{n}{2} \rfloor\}$, we denote the circulant graph on the vertex set $\mathbb{Z}_n$ such that any two vertices u and v are adjacent if and only if $u - v \in S$ or $v - u \in S$ (with the subtraction being done in $\mathbb{Z}_n$). We will need the following result on the spectra of circulant matrices.

Lemma 7 *([22, Section 3.1]) For any $n \in \mathbb{N}$, the eigenvalues of the circulant matrix*

$$C = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \dots & c_0 \end{bmatrix}$$

are of the form $P(\zeta)$, as $\zeta \in \mathbb{C}$ ranges over the n-th roots of unity, where

$$P(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}.$$

Moreover, for any n-th root of unity ζ, the vector

$$\begin{bmatrix} 1 & \zeta & \zeta^2 & \cdots & \zeta^{n-1} \end{bmatrix}^{\intercal}$$

is an eigenvector of C for the eigenvalue $P(\zeta)$.

We also need the next two results on nut graphs.

Lemma 8 *([36, Lemma 4.1]) Let G be a nut graph and let G_1 be the graph that arises from G by subdividing an edge of G four times. Then G_1 is also a nut graph.*

Lemma 9 *([6, Theorem 2.4]) For any $k, n \in \mathbb{N}$ such that k and n are both even and $n \geq 2k + 2$, the graph $\mathrm{Circ}(n, \{1, 2, \dots, k\})$ is a nut graph if and only if $\gcd(\frac{n}{2}, \frac{k}{2}) = \gcd(\frac{n}{2}, k + 1) = 1$.*

We will use $G \,\square\, H$ to denote the cartesian product [25] of two graphs G and H. A graph is *prime* (with respect to the cartesian product) if it is not isomorphic to a graph of the form $G \,\square\, H$, where G and H are both nontrivial. It is known that any connected graph G has a unique factorization of the form $G = G_1 \,\square\, G_2 \,\square \cdots \square\, G_k$, where $G_1, G_2, \dots, G_k$ are all prime, up to isomorphisms and the order of the factors [25, Theorem 15.1]. This factorization is called the *prime factorization* of G and we say that two graphs G and H are coprime if they do not have a common factor in their respective prime factorizations. For a comprehensive treatment of various graph products, see [23].

Lemma 10 *([5, Section 1.4.6]) For any two graphs G and H, the spectrum of the cartesian product $G \,\square\, H$ is given by*

$$\sigma(G \,\square\, H) = \{\lambda + \mu \colon \lambda \in \sigma(G),\ \mu \in \sigma(H)\}.$$

Moreover, if $u \in \mathbb{R}^{V(G)}$ is an eigenvector of G for the eigenvalue λ and $v \in \mathbb{R}^{V(H)}$ is an eigenvector of H for the eigenvalue μ, then the vector $w \in \mathbb{R}^{V(G \,\square\, H)}$ defined as

$$w_{(g,h)} = u_g\, v_h \qquad (g \in V(G),\ h \in V(H))$$

is an eigenvector of $G \,\square\, H$ for the eigenvalue $\lambda + \mu$.

We denote the dihedral group of order $2n$ by $\mathrm{Dih}(n)$ and the symmetric group over $\mathbb{Z}_n$ by $\mathrm{Sym}(n)$. For any permutation group Γ acting on V and any point $x \in V$, the stabilizer of x in Γ will be denoted by Γ_x, and the orbit of x in Γ will be denoted by x^Γ.

Lemma 11 *([21, Lemma 2.2.2]) Let Γ be a permutation group acting on V and let x be a point in V. Then*

$$|\Gamma_x|\,|x^\Gamma| = |\Gamma|.$$

We use $\mathrm{Aut}\, G$ to denote the automorphism group of a graph G. We proceed with the following result on the automorphism groups of circulant graphs.

Lemma 12 *([4, Lemma 35]) For every $k \geq 1$ it holds that $\mathrm{Aut}(\mathrm{Circ}(n, \{1, 2, \dots, k\})) \cong \mathrm{Dih}(n)$ for all $n \geq 2k + 3$.*

We end the section by stating a few well-known facts about the cyclotomic polynomials. For any $n \in \mathbb{N}$, the *cyclotomic polynomial* $\Phi_n(x)$ is defined as

$$\Phi_n(x) = \prod_{\xi} (x - \xi),$$

where $\xi \in \mathbb{C}$ ranges over the primitive n-th roots of unity. It is known that all $\Phi_n(x)$ polynomials have integer coefficients and are irreducible in $\mathbb{Q}[x]$ (see, e.g., [19, Chapter 33]). Therefore, any $\mathbb{Q}[x]$ polynomial contains a primitive n-th root of unity among its roots if and only if it is divisible by $\Phi_n(x)$.

3 Edge and arc orbits of Cayley nut graphs

In the present section, we prove Theorem 5 through the following three propositions.

Proposition 13 *For any even $k \geq 2$ and prime $p \geq k+2$, the graph $\mathrm{Circ}(2p, \{1, 2, \dots, k\})$ is a Cayley nut graph with k edge orbits and k arc orbits.*

Proof Let $G = \mathrm{Circ}(n, \{1, 2, \dots, k\})$. The graph G is obviously a Cayley graph. By Lemma 12, we have $\mathrm{Aut}\, G \cong \mathrm{Dih}(2p)$, which means that for each $i \in \{1, 2, \dots, k\}$,

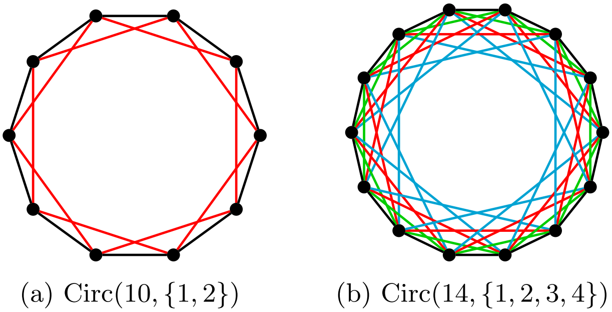

Fig. 1 Cayley nut graphs with two and four edge (arc) orbits from Proposition 13. The edge (arc) orbits are color-coded

the set $\{\{v, v+i\} : v \in \mathbb{Z}_{2p}\}$ forms an edge orbit of G. From here, we obtain $o_e(G) = k$. Moreover, for any two adjacent vertices $v_1, v_2 \in \mathbb{Z}_n$, there is an automorphism of G that swaps v_1 and v_2. Therefore, we also get $o_a(G) = k$. Finally, Lemma 9 implies that G is a nut graph. □

Example 1 Two small examples of graphs from Proposition 13, namely Circ(10, {1, 2}) and Circ(14, {1, 2, 3, 4}), are shown in Fig. 1. ♢

Proposition 14 *For any odd $k \geq 5$ and prime $p \geq 2k+1$, the graph* $\mathrm{Circ}(2p, \{2, 3, \dots, k-1, p\}) \,\square\, K_2$ *is a Cayley nut graph with k edge orbits and k arc orbits.*

Proof Let $G = \mathrm{Circ}(2p, \{2, 3, \dots, k-1, p\}) \,\square\, K_2$ and let $\Gamma = \mathrm{Aut}\, G$. The graph G is obviously a Cayley graph for the group $\mathbb{Z}_{2p} \times \mathbb{Z}_2$. We begin by proving that G has k edge orbits and k arc orbits. By Lemma 11, we have $|\Gamma| = 4p\,|\Gamma_v|$, where Γ_v is the stabilizer of some vertex v in Γ. Now, let $\pi \in \Gamma_v$. Observe that

$$|N_G(v - (2, 0)) \setminus N_G(v)| = |N_G(v + (2, 0)) \setminus N_G(v)| = 6,$$

while $|N_G(u) \setminus N_G(v)| \geq 7$ for any $u \in N_G(v) \setminus \{v - (2, 0), v + (2, 0)\}$. Therefore, π either fixes $v - (2, 0)$ and $v + (2, 0)$ or swaps them. Since there exists an automorphism from Γ_v that swaps $v - (2, 0)$ and $v + (2, 0)$, Lemma 11 implies that $|\Gamma_v| = 2\,|\Gamma_{v,v+(2,0)}|$, where $\Gamma_{v,v+(2,0)}$ is the pointwise stabilizer of v and $v + (2, 0)$ in Γ.

Now, let $\sigma \in \Gamma_{v,v+(2,0)}$. Moreover, let

$$V_1 = \{v + t(2, 0) : t \in \mathbb{Z}\} \qquad \text{and} \qquad V_2 = \{v + (1, 0) + t(2, 0) : t \in \mathbb{Z}\}.$$

Note that σ must fix $v - (2, 0)$. By repeating the same argument from before, we conclude that σ fixes all the vertices from V_1. Observe that any two distinct vertices from V_2 have distinct sets of neighbors among the vertices from V_1. Therefore, σ also fixes all the vertices from V_2, and it is easy to see from here that σ must be the identity permutation. Thus, we have $|\Gamma_{v,v+(2,0)}| = 1$, which implies $|\Gamma| = 8p$.

Observe that Γ contains the automorphisms φ of the form

$$\varphi(v_1, v_2) = (f(v_1), v_2) \qquad \text{and} \qquad \varphi(v_1, v_2) = (f(v_1), v_2 + 1),$$

where $v_1 \in \mathbb{Z}_{2p}$, $v_2 \in \mathbb{Z}_2$ and $f \in \mathrm{Sym}(2p)$ is a dihedral permutation (i.e., it has the form $f(x) = c + x$ or $f(x) = c - x$ for some $c \in \mathbb{Z}_{2p}$). Since there are exactly $8p$ such automorphisms, this means that these are all the automorphisms of Γ. Therefore, for each $i \in \{2, 3, \dots, k-1, p\}$, the set $\{\{v, v + (i, 0)\} \colon v \in \mathbb{Z}_{2p} \times \mathbb{Z}_2\}$ forms an edge orbit of G. In addition, the set $\{\{v, v + (0, 1)\} \colon v \in \mathbb{Z}_{2p} \times \mathbb{Z}_2\}$ is another edge orbit of G. Thus, we have $o_e(G) = k$ and it also trivially follows that $o_a(G) = k$.

Now, we finalize the proof by showing that G is a nut graph. Note that $\sigma(K_2) = \{1, -1\}$, with both corresponding eigenvectors being full. Recall that the adjacency matrix of a circulant graph is a circulant matrix. By combining Lemmas 10 and 7, we have that $\sigma(G)$ consists of the eigenvalues

$$\lambda(\zeta) = \sum_{j=2}^{k-1} \left(\zeta^j + \zeta^{-j}\right) + \zeta^p + 1 \qquad \text{and} \qquad \mu(\zeta) = \sum_{j=2}^{k-1} \left(\zeta^j + \zeta^{-j}\right) + \zeta^p - 1,$$

as ζ ranges over the $2p$-th roots of unity. We trivially observe that

$$\lambda(1) = 2k - 2, \qquad \mu(1) = 2k - 4, \qquad \lambda(-1) = 2 \qquad \text{and} \qquad \mu(-1) = 0.$$

Thus, to complete the proof, it suffices to show that $\lambda(\zeta) \neq 0$ and $\mu(\zeta) \neq 0$ for every $2p$-th root of unity $\zeta \neq 1, -1$.

By way of contradiction, suppose that $\lambda(\zeta) = 0$ or $\mu(\zeta) = 0$ for some $2p$-th root of unity $\zeta \neq 1, -1$. Since $\zeta^p \in \{1, -1\}$, we have that at least one of the expressions

$$\sum_{j=2}^{k-1} \left(\zeta^j + \zeta^{-j}\right) + 2, \qquad \sum_{j=2}^{k-1} \left(\zeta^j + \zeta^{-j}\right) \qquad \text{and} \qquad \sum_{j=2}^{k-1} \left(\zeta^j + \zeta^{-j}\right) - 2$$

equals zero. In other words, ζ is a root of at least one of the nonzero $\mathbb{Z}[x]$-polynomials

$$\sum_{j=0}^{k-3} x^j + \sum_{j=k+1}^{2k-2} x^j + 2x^{k-1}, \qquad \sum_{j=0}^{k-3} x^j + \sum_{j=k+1}^{2k-2} x^j \qquad \text{and} \qquad \sum_{j=0}^{k-3} x^j + \sum_{j=k+1}^{2k-2} x^j - 2x^{k-1},$$

whose degrees are all $2k - 2$. This means that at least one of these three polynomials is divisible by $\Phi_p(x)$ or $\Phi_{2p}(x)$ because the order of ζ is either p or $2p$. The contradiction now follows from $\deg \Phi_p = \deg \Phi_{2p} = p - 1 > 2k - 2$. □

Proposition 15 *For any odd $n \geq 5$, the graph $\mathrm{Circ}(2n, \{1, n\}) \,\square\, K_4$ is a Cayley nut graph with three edge orbits and three arc orbits.*

Proof Let $G' = \mathrm{Circ}(2n, \{1, n\})$ and $\Gamma' = \mathrm{Aut}\, G'$, as well as $G = G' \,\square\, K_4$ and $\Gamma = \mathrm{Aut}\, G$. The graph G is obviously a Cayley graph for the group $\mathbb{Z}_{2n} \times \mathbb{Z}_4$. We first show that G has three edge orbits and three arc orbits. Observe that K_4 is prime with respect to the cartesian product, while G' and K_4 are coprime. For this reason, Γ comprises the automorphisms φ of the form

$$\varphi(v_1, v_2) = (f_1(v_1), f_2(v_2)),$$

where $v_1 \in \mathbb{Z}_{2n}$, $v_2 \in \mathbb{Z}_4$ and $f_1 \in \Gamma'$, while $f_2 \in \mathrm{Sym}(4)$ is any permutation (see, e.g., [25, Theorem 15.5]).

Since G' is vertex-transitive, Lemma 11 gives $|\Gamma'| = 2n\,|\Gamma'_v|$, where Γ'_v is the stabilizer of some vertex v in Γ'. Now, let $\pi \in \Gamma'_v$. Observe that among the three neighbors $v - (1, 0)$, $v + (1, 0)$ and $v + (n, 0)$ of v, only $v - (1, 0)$ and $v + (1, 0)$ do not have an additional common neighbor apart from v. Indeed, $v - (1, 0)$ and $v + (n, 0)$ are both adjacent to $v + (n - 1, 0)$, while $v + (1, 0)$ and $v + (n, 0)$ are both adjacent to $v + (n + 1, 0)$. Therefore, π either fixes $v - (1, 0)$ and $v + (1, 0)$ or swaps them. Since there exists an automorphism from Γ'_v that swaps $v - (1, 0)$ and $v + (1, 0)$, Lemma 11 implies that $|\Gamma'_v| = 2\,|\Gamma'_{v,v+(1,0)}|$, where $\Gamma'_{v,v+(1,0)}$ is the pointwise stabilizer of v and $v + (1, 0)$ in Γ'. By repeating the same argument, we trivially observe that $\Gamma'_{v,v+(1,0)}$ only contains the identity permutation, hence $|\Gamma'| = 4n$ and $\Gamma' \cong \mathrm{Dih}(2n)$.

Having characterized the group Γ, it follows that graph G has exactly three edge orbits: $\{\{v, v + (1, 0)\}\colon v \in \mathbb{Z}_{2n} \times \mathbb{Z}_4\}$ and $\{\{v, v + (n, 0)\}\colon v \in \mathbb{Z}_{2n} \times \mathbb{Z}_4\}$, alongside $\{\{v, v + (0, i)\}\colon v \in \mathbb{Z}_{2n} \times \mathbb{Z}_4, i \in \{1, 2, 3\}\}$. Thus, we obtain $o_e(G) = 3$. Moreover, it is not difficult to see for any two adjacent vertices $v_1, v_2 \in V(G)$, there is an automorphism of G that swaps v_1 and v_2. From here, we also get $o_a(G) = 3$.

Now, we finish the proof by showing that G is a nut graph. Note that $\sigma(K_4) = \{3, -1, -1, -1\}$, with the eigenvector corresponding to 3 being full. By combining Lemmas 10 and 7, we conclude that $\sigma(G)$ consists of the eigenvalues

$$\lambda(\zeta) = \zeta + \zeta^{-1} + \zeta^n + 3 \qquad \text{and} \qquad \mu_i(\zeta) = \zeta + \zeta^{-1} + \zeta^n - 1,$$

as ζ ranges over the $2n$-th roots of unity and i ranges over $\{1, 2, 3\}$. We trivially observe that

$$\lambda(1) = 6, \qquad \mu_i(1) = 2, \qquad \lambda(-1) = 0 \qquad \text{and} \qquad \mu_i(-1) = -4.$$

Thus, to complete the proof, it is enough to show that $\lambda(\zeta) \neq 0$ and $\mu_i(\zeta) \neq 0$ for every $2n$-th root of unity $\zeta \neq 1, -1$.

By way of contradiction, suppose that $\lambda(\zeta) = 0$ or $\mu_i(\zeta) = 0$ for some $2n$-th root of unity $\zeta \neq 1, -1$. Since $\zeta^n \in \{1, -1\}$, we get $\zeta + \zeta^{-1} \in \{-4, -2, 0, 2\}$. The equality $\zeta + \zeta^{-1} = -4$ cannot hold because $|\zeta + \zeta^{-1}| \leq |\zeta| + |\zeta^{-1}| = 2$. Moreover, $\zeta + \zeta^{-1} = -2$ and $\zeta + \zeta^{-1} = 2$ are equivalent to $\zeta = -1$ and $\zeta = 1$, respectively, hence we obtain a contradiction in both of these cases. Finally, $\zeta + \zeta^{-1} = 0$ cannot be satisfied because n is assumed to be odd. □

Example 2 A small example of a graph from Proposition 15, namely $\mathrm{Circ}(10, \{1, 5\}) \,\square\, K_4$, is shown in Fig. 2. ◊

Theorem 5 now follows from Propositions 13, 14 and 15.

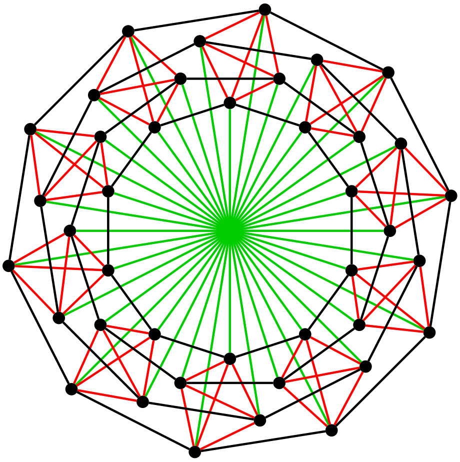

Fig. 2 The Cayley nut graph $\mathrm{Circ}(10, \{1, 5\}) \,\square\, K_4$ with three edge (arc) orbits from Proposition 15. The edge (arc) orbits are color-coded

4 Vertex and edge orbits of nut graphs

In this section, we start from Theorem 5 and derive Theorem 6. We achieve this through the next lemma.

Lemma 16 *Let G be a vertex-transitive nut graph such that $o_e(G) = o_a(G) = k$. Also, let $\mathcal{E}$ be an edge orbit of G and let G_1 be the graph that arises from G by subdividing each edge from $\mathcal{E}$ exactly $4t$ times, where $t \in \mathbb{N}$. Then G_1 is a nut graph such that $o_v(G_1) = 2t + 1$, $o_e(G_1) = 2t + k$ and $o_a(G_1) = 4t + k$.*

Proof By repeated use of Lemma 8, we conclude that G_1 is a nut graph. Let $V_1 = V(G)$ and $V_2 = V(G_1) \setminus V(G)$. The degree of G is at least three, since d-regular nut graphs do not exist for $d < 3$ (see [20]). This means that the vertices from V_1 have a degree of at least three in G_1, while the vertices from V_2 all have the degree two in G_1. Therefore, for any $\pi \in \mathrm{Aut}(G_1)$, the restriction $\pi \restriction V_1$ is a permutation of V_1. For any two vertices $u, v \in V_1$ adjacent in G and such that $uv \notin \mathcal{E}$, we have $u \sim v$ in G_1, hence $\pi(u) \sim \pi(v)$ in G_1, which means that $\pi(u) \sim \pi(v)$ also holds in G. On the other hand, for any two vertices $u, v \in V_1$ adjacent in G and such that $uv \in \mathcal{E}$, the graph G_1 contains a (u, v)-path of length $4t + 1$ whose internal vertices all have the degree two. This means that G_1 also has a $(\pi(u), \pi(v))$-path of length $4t + 1$ whose internal vertices all have the degree two, thus implying $\pi(u) \sim \pi(v)$ in G. With all of this in mind, we obtain that $\pi \restriction V_1 \in \mathrm{Aut}\, G$ holds for every $\pi \in \mathrm{Aut}\, G_1$.

Since G_1 arises from G by subdividing all the edges from the same edge orbit an equal number of times, it is not difficult to see that each automorphism of G has a unique extension to $V(G_1)$ that is an automorphism of G_1. From here, we get $\mathrm{Aut}\, G_1 \cong \mathrm{Aut}\, G$. Also, since $o_e(G) = o_a(G)$, we know that for any edge $uv \in \mathcal{E}$, there exists an automorphism of G that swaps u and v. From here, we trivially observe that $o_v(G_1) = 2t + 1$, $o_e(G_1) = 2t + k$ and $o_a(G_1) = 4t + k$. □

We are now in a position to give the proof of Theorem 6.

Proof of Theorem 6 If r is even, then the result follows from Theorem 2, while for $r = 1$, the result follows from Theorem 5. Now, assume that r is odd and $r \geq 3$, and let $k \geq r + 1$. By Theorem 5, there exist infinitely many vertex-transitive nut graphs G with $o_e(G) = o_a(G) = k - r + 1$. By applying the subdivision transformation from Lemma 16 with $t = \frac{r-1}{2}$ to each of these graphs, we obtain infinitely many nut graphs G_1 with $o_v(G_1) = r$ and $o_e(G_1) = k$.

5 Conclusion

From Theorems 1 and 5, we obtain the following corollary.

Corollary 17 *For any $k \in \mathbb{N}_0$, the following holds:*

(a) there exists a Cayley nut graph with k edge orbits if and only if $k \geq 2$;
(b) there exists a Cayley nut graph with k arc orbits if and only if $k \geq 2$.

Corollary 17 solves the realizability problem for the number of edge orbits and arc orbits, respectively, among the Cayley nut graphs. As shown in Theorem 5, infinite realizability also holds in all the given cases. On the other hand, Theorems 1 and 6 yield the next corollary.

Corollary 18 *For any $r \in \mathbb{N}$ and $k \in \mathbb{N}_0$, there exists a nut graph G with $o_v(G) = r$ and $o_e(G) = k$ if and only if $k \geq r + 1$.*

Corollary 18 solves the realizability problem for the pairs (o_v, o_e) among the nut graphs. Theorem 6 also implies that infinite realizability holds in all the given cases. In other words, there exist infinitely many nut graphs for each of the aforementioned (o_v, o_e) pairs. The related realizability problem for the pairs (o_v, o_a) still remains open.

Problem 19 Determine all the pairs (r, k) for which there exists a nut graph G with r vertex orbits and k arc orbits.

By Proposition 15, there is a Cayley nut graph of order n with three edge orbits for each $n \in \{40, 56, 72, 88, \dots\}$. However, a computer search over the vertex-transitive graphs [24, 26] reveals three vertex-transitive nut graphs G of order 16 with $o_e(G) = o_a(G) = 3$, two of which are Cayley. Moreover, by Proposition 14, there exists a Cayley nut graph of order n with five edge orbits for $n \in \{44, 52, 68, 76, \dots\}$. More precisely, there is such a Cayley nut graph for any $n = 4p$, where $p \geq 11$ is prime. However, there is a unique Cayley nut graph of order 12 with $o_e(G) = o_a(G) = 5$. It is the Cayley graph for the group $\mathbb{Z}_6 \times \mathbb{Z}_2$ with the connection set $\{(i, 0) : i \in \{1, 2, 4, 5\}\} \cup \{(i, 1) : i \in \{0, 1, 3, 5\}\}$; see Fig. 3. With this in mind, it makes sense to pose the following realizability problem.

Problem 20 For any $k \geq 2$, determine all the orders for which:

(a) there exists a Cayley nut graph with k edge orbits;
(b) there exists a Cayley nut graph with k arc orbits.

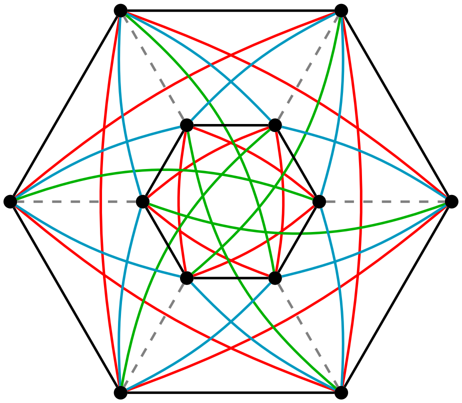

Fig. 3 The Cayley nut graph for the group $\mathbb{Z}_6 \times \mathbb{Z}_2$ with the connection set $\{(i, 0) : i \in \{1, 2, 4, 5\}\} \cup \{(i, 1) : i \in \{0, 1, 3, 5\}\}$. The edge (arc) orbits are color-coded

We conclude the paper with another natural problem.

Problem 21 For any $r \in \mathbb{N}$ and $k \geq r + 1$, determine all the orders for which there exists a nut graph with r vertex orbits and k edge orbits.

Funding N. Bašić is supported in part by the Slovenian Research Agency (research program P1-0294 and research project J5-4596) and in part by the Republic of Slovenia and the European Union — NextGenerationEU as part of the Recovery and Resilience Plan: Development of basic computer science and information technology content and skills in kindergartens and primary schools (B-RIN), NRP code: 3350-23-3505. I. Damnjanović is supported by the Ministry of Science, Technological Development and Innovation of the Republic of Serbia, grant number 451-03-137/2025-03/200102, and the Science Fund of the Republic of Serbia, grant #6767, Lazy walk counts and spectral radius of threshold graphs — LZWK.

Declarations

Conflict of interest The authors declare that they have no Conflict of interest.

References

1. Bašić, N., Damnjanović, I.: On cubic polycirculant nut graphs. Comput. Appl. Math. **44** (2025) https://doi.org/10.1007/s40314-025-03218-7
2. Bašić, N., Damnjanović, I., Fowler, P.W.: On the degrees of regular nut graphs and Cayley nut graphs (2025). arXiv: 2410.14063
3. Bašić, N., Fowler, P.W.: Nut graphs with a given automorphism group. J. Algebr. Comb. **61** (2025) https://doi.org/10.1007/s10801-025-01389-4
4. Bašić, N., Fowler, P.W., Pisanski, T.: Vertex and edge orbits in nut graphs. Electron. J. Comb. **31** (2024) https://doi.org/10.37236/12619

5. Brouwer, A.E., Haemers, W.H.: Spectra of Graphs. Universitext. Springer, New York (2012). https://doi.org/10.1007/978-1-4614-1939-6
6. Bašić, N., Knor, M., Škrekovski, R.: On 12-regular nut graphs. Art Discrete Appl. Math. **5** (2022) https://doi.org/10.26493/2590-9770.1403.1b1
7. Buset, D.: Orbits on vertices and edges of finite graphs. Discrete Math. **57**, 297–299 (1985). https://doi.org/10.1016/0012-365X(85)90181-5
8. Coolsaet, K., Fowler, P.W., Goedgebeur, J.: Generation and properties of nut graphs. MATCH Commun. Math. Comput. Chem. **80**, 423–444 (2018)
9. Damnjanović, I.: Two families of circulant nut graphs. Filomat **37**, 8331–8360 (2023). https://doi.org/10.2298/FIL2324331D
10. Damnjanović, I.: Complete resolution of the circulant nut graph order–degree existence problem. Ars Math. Contemp. **24** (2024) https://doi.org/10.26493/1855-3974.3009.6df
11. Damnjanović, I.: A note on Cayley nut graphs whose degree is divisible by four. Art Discrete Appl. Math. (2025) https://doi.org/10.26493/2590-9770.1662.4e9
12. Damnjanović, I.: On the null spaces of quartic circulant graphs. Discrete Math. Chem. **1** (2025) https://doi.org/10.26493/2820-6657.25_1
13. Damnjanović, I., Bašić, N., Pisanski, T., Žitnik, A.: Classification of cubic tricirculant nut graphs. Electron. J. Comb. **31** (2024) https://doi.org/10.37236/12668
14. Damnjanović, I., Bašić, N., Pisanski, T., Žitnik, A.: Classification of quartic bicirculant nut graphs (2025). arXiv: 2502.06353
15. Damnjanović, I., Stevanović, D.: On circulant nut graphs. Linear Algebra Appl. **633**, 127–151 (2022). https://doi.org/10.1016/j.laa.2021.10.006
16. Fowler, P.W., Gauci, J.B., Goedgebeur, J., Pisanski, T., Sciriha, I.: Existence of regular nut graphs for degree at most 11. Discuss. Math. Graph Theory **40**, 533–557 (2020). https://doi.org/10.7151/dmgt.2283
17. Fowler, P.W., Pisanski, T., Bašić, N.: Charting the space of chemical nut graphs. MATCH Commun. Math. Comput. Chem. **86**, 519–538 (2021)
18. Fowler, P.W., Pickup, B.T., Todorova, T.Z., Borg, M., Sciriha, I.: Omni-conducting and omni-insulating molecules. J. Chem. Phys. **140**, 054115 (2014). https://doi.org/10.1063/1.4863559
19. Gallian, J.A.: Contemporary Abstract Algebra, 10th edn. CRC Press, Boca Raton, FL (2021)
20. Gauci, J.B., Pisanski, T., Sciriha, I.: Existence of regular nut graphs and the Fowler construction. Appl. Anal. Discrete Math. **17**, 321–333 (2023). https://doi.org/10.2298/AADM190517028G
21. Godsil, C., Royle, G.: Algebraic Graph Theory. Graduate Texts in Mathematics, vol. 207. Springer, New York (2001). https://doi.org/10.1007/978-1-4613-0163-9
22. Gray, R.M.: Toeplitz and circulant matrices: A review. Found. Trends Commun. Inf. Theory **2**, 155–239 (2006)
23. Hammack, R., Imrich, W., Klavžar, S.: Handbook of Product Graphs, 2nd edn. Discrete Mathematics and its Applications. CRC Press, Boca Raton, FL (2011). https://doi.org/10.1201/b10959
24. Holt, D., Royle, G.: A census of small transitive groups and vertex-transitive graphs. J. Symb. Comput. **101**, 51–60 (2020). https://doi.org/10.1016/j.jsc.2019.06.006
25. Imrich, W., KlavŽar, S., Rall, D.F.: Topics in Graph Theory: Graphs and Their Cartesian Product, 1st edn. CRC Press, A K Peters, Ltd., Wellesley, MA (2008). pp. xiv+205. https://doi.org/10.1201/b10613
26. Royle, G., Holt, D.: Vertex-transitive graphs on fewer than 48 vertices. Zenodo (2020). https://doi.org/10.5281/zenodo.4010122
27. Sciriha, I.: On the coefficient of λ in the characteristic polynomial of singular graphs. Util. Math. **52**, 97–111 (1997)
28. Sciriha, I.: On singular line graphs of trees. Congr. Numer. **135**, 73–91 (1998)
29. Sciriha, I.: On the construction of graphs of nullity one. Discrete Math. **181**, 193–211 (1998). https://doi.org/10.1016/S0012-365X(97)00036-8
30. Sciriha, I.: The two classes of singular line graphs of trees. Rend. Semin. Mat. Messina, Ser. II **20**, 167–180 (1999)
31. Sciriha, I.: A characterization of singular graphs. Electron. J. Linear Algebra **16**, 451–462 (2007)
32. Sciriha, I.: Coalesced and embedded nut graphs in singular graphs. Ars Math. Contemp. **1**, 20–31 (2008). https://doi.org/10.26493/1855-3974.20.7cc
33. Sciriha, I., Fowler, P.W.: Nonbonding orbitals in fullerenes: Nuts and cores in singular polyhedral graphs. J. Chem. Inf. Model. **47**, 1763–1775 (2007). https://doi.org/10.1021/ci700097j

34. Sciriha, I., Fowler, P.W.: On nut and core singular fullerenes. Discrete Math. **308**, 267–276 (2008). https://doi.org/10.1016/j.disc.2006.11.040
35. Sciriha, I., Farrugia, A.: From Nut Graphs to Molecular Structure and Conductivity. Mathematical Chemistry Monographs, vol. 23. University of Kragujevac, Kragujevac (2021)
36. Sciriha, I., Gutman, I.: Nut graphs: Maximally extending cores. Util. Math. **54**, 257–272 (1998)