

CONCENTRATING SOLUTIONS OF THE FRACTIONAL (p, q)-CHOQUARD EQUATION WITH EXPONENTIAL GROWTH

YUEQIANG SONG¹, XUEQI SUN¹, DUŠAN D. REPOVŠ^{2,3,4}

ABSTRACT. This article deals with the following fractional (p, q)-Choquard equation with exponential growth of the form:

$$\varepsilon^{ps}(-\Delta)_p^s u + \varepsilon^{qs}(-\Delta)_q^s u + Z(x)(|u|^{p-2}u + |u|^{q-2}u) = \varepsilon^{\mu-N}[|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^N,$$

where $s \in (0, 1)$, $\varepsilon > 0$ is a parameter, $2 \leq p = \frac{N}{s} < q$, and $0 < \mu < N$. The nonlinear function f has an exponential growth at infinity and the continuous potential function Z satisfies suitable natural conditions. With the help of the Ljusternik-Schnirelmann category theory and variational methods, the multiplicity and concentration of positive solutions are obtained for $\varepsilon > 0$ small enough. In a certain sense, we generalize some previously known results.

1. INTRODUCTION

In this paper, we consider the multiplicity and concentration of solutions for the following fractional (p, q)-Choquard problem in \mathbb{R}^N :

$$\varepsilon^{ps}(-\Delta)_p^s u + \varepsilon^{qs}(-\Delta)_q^s u + Z(x)(|u|^{p-2}u + |u|^{q-2}u) = \varepsilon^{\mu-N}[|x|^{-\mu} * F(u)]f(u), \quad (\mathcal{Q})$$

where ε is small positive parameter, $0 < \mu < N$, $0 < s < 1$, $2 \leq p = \frac{N}{s} < q$, the continuous potential Z is bounded from below by $Z_0 > 0$, the nonlinearity f has an exponential critical growth at infinity, and $(-\Delta)_\varphi^s$ ($\varphi \in \{p, q\}$) is the fractional φ -Laplace operator defined by

$$(-\Delta)_\varphi^s u(x) = 2 \lim_{r \rightarrow 0^+} \int_{\mathbb{R}^N \setminus B_r(x)} \frac{|u(x) - u(y)|^{\varphi-2}(u(x) - u(y))}{|x - y|^{N+\varphi s}} dy \text{ for every } x \in \mathbb{R}^N,$$

up to a normalization constant in the integral, where $u \in C_0^\infty(\mathbb{R}^N)$ and $B_r(x)$ denotes the ball with center x of radius $r > 0$.

Many scholars have studied fractional and nonlocal operators because of their applications in various contexts, for example, in optimization, finance, crystal dislocations, phase transitions, etc. For more on these topics, we refer to Ambrosio [4] and di Nezza et al. [21].

We shall assume that the potential function Z and the nonlinearity f satisfy the following conditions:

- (\mathcal{Z}_1) There exists $Z_0 > 0$ such that $Z(x) \geq Z_0$, for every $x \in \mathbb{R}^N$;
- (\mathcal{Z}_2) There exists an open bounded set $\Omega \subset \mathbb{R}^N$ such that

$$Z_0 = \inf_{x \in \Omega} Z(x) < \min_{x \in \partial\Omega} Z(x).$$

* Corresponding author: Dušan D. Repovš.

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(f₁) f is a continuous function such that $f(t) = 0$ and for every $t \leq 0$ and every q_1, q_2 , such that

$$q_1 \geq q, \quad q_2 \geq \frac{N}{s},$$

there exist real numbers $a_1 > 0$, $a_2 > 0$, and β_0 , with $0 < \beta_0 < \alpha_*(s, N)$, such that

$$f'(t) \leq a_1 |t|^{q_1-2} + a_2 \mathcal{H}_{N,s}(\beta_0 |t|^{N/(N-s)}) |t|^{q_2-2} \quad \text{for every } t \geq 0,$$

where (see Parini and Ruf [32] and Zhang [40])

$$\mathcal{H}_{N,s}(t) = e^t - \sum_{j=0}^{j_p-2} \frac{t^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \geq p\}, \quad \text{and}$$

$$\alpha_* = \alpha_*(s, N) = N \left[\frac{2(N\omega_N)^2 \Gamma(p+1)}{N!} \sum_{k=0}^{\infty} \frac{(N+k-1)!}{k!} \frac{1}{(N+2k)^p} \right]^{s/(N-s)}, \quad (1.1)$$

$$\omega_N = \frac{\pi^{N/2}}{\Gamma(1+N/2)}.$$

(f₂) $\lim_{t \rightarrow 0^+} \frac{f'(t)}{t^{q-2}} = 0$.

(f₃) There exists $\theta > q$ such that $f(t)t \geq \theta F(t) > 0$, for every $t > 0$, where $F(t) = \int_0^t f(\tau) d\tau$.

(f₄) There exists $\gamma_1 > 0$ large enough such that $F(t) \geq \gamma_1 |t|^\theta$, for every $t \geq 0$, where θ is as given in (f₃).

(f₅) The function $t \mapsto f(t)t^{1-q}$ is strictly increasing on $\mathbb{R}^+ = (0, +\infty)$.

Once $s = 1$, problem (Q) reduces to a typical (p, q) -elliptic equation:

$$-\Delta_p u - \Delta_q u + Z(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = H(u) \quad \text{in } \mathbb{R}^N, \quad (1.2)$$

where H is nonlinear reaction, $\Delta_\varphi u = \operatorname{div}(|\nabla u|^{\varphi-2} \nabla u)$, and $\varphi \in \{p, q\}$. It has been widely studied in physics, biophysics, plasma physics, chemical reaction design and elsewhere. For more physical examples, we refer to Antontsev and Shmarev [8], Benci et al. [9], and Cherfil and Il'yasov [16], and the references therein. The multiple phases equation was proposed in the study of the Born-Infeld equation (see Bonheure et al. [10], Born and Infeld [11], and Brézis and Lieb [12]), which models electromagnetic fields, electrostatics and electrodynamics, and was a model based on the Maxwell-Lagrangian density

$$-\operatorname{div} \left(\frac{\nabla u}{(1 - 2|\nabla u|^2)^{1/2}} \right) = h(u) \quad \text{in } \mathbb{R}^N.$$

When $p = q$, problem (Q) becomes the fractional p -Laplace Choquard equation of the form:

$$\varepsilon^{ps} (-\Delta)_p^s u + Z(x) |u|^{p-2} u = \varepsilon^{\mu-N} \left[|x|^{-\mu} * F(u) \right] f(u(x)) \quad \text{in } \mathbb{R}^N, \quad (1.3)$$

where $\varepsilon > 0$ is a sufficiently small parameter, typically the Planck constant and $F(t) = \int_0^t f(\tau) d\tau$. We say that a solution of problem (1.3) is *semi-classical* if $\varepsilon \rightarrow 0^+$. From the physics point of view, the semi-classical solution is also a solutions of problem (1.3), when $\varepsilon \rightarrow 0^+$. Floer and Weinstein [22] established the existence of semi-classical solutions of problem (1.3). A special form of problem (1.3) is

$$-\varepsilon^2 \Delta u + Z(x)u = \varepsilon^{\mu-N} [|x|^{-\mu} * F(u)] f(u) \quad \text{in } \mathbb{R}^N \quad \text{for every } 0 < \mu < N, \quad (1.4)$$

where f is a nonlinear reaction. It is worth noting that problem (1.4) was introduced in the theory of the Bose-Einstein condensation and used to describe the finite-range many body interactions between particles. There are already many works on this topic. By variational methods, Alves et al. [1] considered the concentration solutions of problem (1.4) in \mathbb{R}^2 , where f has exponential critical growth and Z satisfies some appropriate conditions. Once $F(u) = |u|^p$ in problem (1.4), one obtains the following Choquard equation

$$-\Delta u + Z(x)u = (I_\alpha * |u|^p)|u|^{p-2}u \text{ in } \mathbb{R}^N, \quad (1.5)$$

where I_α is the Riesz potential, Γ is the Gamma function, and Z is a potential function.

When $p = \alpha = 2$, $N = 3$, and $Z(x) = \nu$, problem (1.5) reduces to the Choquard-Pekar type equation

$$-\Delta u + \nu u = (I_2 * u^2)u, \text{ for every } x \in \mathbb{R}^3, \quad (1.6)$$

which was proposed in 1976 by Lieb [26], in order to describe an electron trapped in its own hole. Problem (1.6) is called the Schrödinger-Newton equation. Inspired by the work of Lieb [26] and Lions [28], many researchers have studied the Choquard equation by variational methods.

Recently, these methods have become more useful for establishing the existence of weak solutions of the Choquard equations. For example, Chen and Yang [15] studied the following Choquard equation with upper critical exponent on a bounded domain

$$-\Delta u = \mu f(x)|u|^{p-2}u + g(x)(I_\alpha^*(g|u|^{2^*}))|u|^{2^*-2}u \text{ for every } x \in \Omega,$$

where $\mu > 0$ is a parameter, $N > 4$, $0 < \alpha < N$, I_α is the Riesz potential, $\frac{N}{N-2} < p < 2$, Ω is a bounded domain with smooth boundary, and f and g are continuous functions. For μ small enough, with the help of variational methods, they established the relationship between the number of solutions and the profile of potential g .

Yang and Zhao [38] studied the singularly perturbed fractional Choquard equation

$$\varepsilon^{2s}(-\Delta)^s u + V(x)u = \varepsilon^{\mu-3} \left(\int_{\mathbb{R}^3} \frac{|u(y)|^{2_{\mu,s}^*} + F(u(y))}{|x-y|^\mu} dy \right) (|u|^{2_{\mu,s}^*-2}u + \frac{1}{2_{\mu,s}^*}f(u)) \text{ in } \mathbb{R}^N, \quad (1.7)$$

where $2_{\mu,s}^* = \frac{6-\mu}{3-2s}$ is the critical exponent in the sense of the Hardy-Littlewood-Sobolev inequality and the continuous function f satisfies subcritical growth conditions. By variational methods, penalization techniques and the Ljusternik-Schnirelmann theory, the authors established the multiplicity and concentration behaviour of solutions for problem (1.7).

We need to point out some recent results: Zuo et al. [44] developed a variational approach, based on the scaling function method to solve optimization problems. Here, the authors dealt with the mass subcritical case, and referred to the fractional framework setting. Zhang et al. [41] considered a class of fractional parabolic equation with general nonlinearities. The authors established monotone increasing property of the positive solutions in one direction. Based on this, nonexistence of the solutions was demonstrated, via a contradiction argument. For more information, we refer to Chen et al. [13, 14], Cingolani and Tanaka [17], Clemente et al. [18], and Böer and Miyagaki [19], and the references therein.

For fractional (p, q) -Laplace problems, some interesting existence and multiplicity results have emerged in recent years. Zhang et al. [42] studied multiplicity and concentration solution for the double phase equation in \mathbb{R}^N , especially, they assumed the nonlinearity of $f \in C^1(\mathbb{R}^N)$ and that the continuous potential function satisfies the global condition. Later, using the penalization method, the Ljusternik-Schnirelmann theory, and variational methods, Ambrosio [5] first studied existence of multiple solutions and concentration of the (p, q) -fractional Choquard equation

$$(-\Delta)_p^s u + (-\Delta)_q^s u + Z(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = (|x|^{-\mu} * F(u))f(u) \text{ in } \mathbb{R}^N, \quad (1.8)$$

where $\mu \in [0, ps)$, $0 < s < 1$, $1 < p < q < \frac{N}{s}$, f has subcritical growth and the potential function Z satisfies the local conditions. Molica Bisci et al. [29] extended the results of Zhang et al. [42] to the fractional Choquard problem (1.8), where the potential function $Z(x)$ satisfies the global condition. With the help of the Ljusternik-Schnirelmann category theory and variational methods, Liang et al. [24] explored the multiplicity and concentration behaviors of solutions for the (p, q) fractional Choquard equation with exponential growth.

To the best of our knowledge, there are no known results concerning problem (Q), when the continuous function f has the exponential growth behavior at infinity in the sense of Trudinger-Moser. Inspired by the results of Liang et al. [24], we show in this paper the existence and concentration behavior of solutions of problem (Q) involving exponential growth. In comparison to Liang et al. [24], we assume the nonlinearity and function f is supposed to be only continuous, which makes the corresponding Nehari manifold possibly nondifferentiable. Therefore, we cannot directly use the differentiability of the Nehari manifold. Furthermore, it is not possible to apply the Ljusternik-Schnirelmann category theory on the Nehari manifold in order to obtain the multiplicity of solutions for problem (Q). Whereas Liang et al. [24] have $f \in C^1(\mathbb{R})$, we need to apply some other techniques to overcome this difficulty.

In addition, this is the first time that problem (Q) with the Trudinger-Moser nonlinearities has been studied in both cases: $s \in (0, 1)$ and $s \rightarrow 1^-$. Comparing with Ambrosio [5], he studied the subcritical growth $p < \frac{N}{s}$ and the local case. We obtain the Sobolev embedding from $W^{s,p}(\mathbb{R}^N)$ into $L^t(\mathbb{R}^N)$, for every $t \in [p, p_s^*]$. However, in this paper we consider the case $N = ps$, so the embedding $W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$ may not exist. To overcome this obstacle, it is essential to apply the fractional Trudinger-Moser inequality, which is the main difference with Ambrosio [5], Molica Bisci et al. [29], and Zhang et al. [42]. The other major challenge which we encountered, is the loss of compactness of the Palais-Smale sequences associated with the underlying functionals, corresponding to problems (Q) and (Q_ε) , so we use certain analytical techniques to overcome this obstacle.

Definition 1.1. We denote the category of a set A with respect to a set B by $cat_B(A)$ as the least integer k such that $A \subset A_1 \cup \dots \cup A_k$, where each A_i , $i = 1, \dots, k$, is a closed and contractible subset of B . We set $cat_B(\emptyset) = 0$ and $cat_B(A) = \infty$ if there is no integer with the above property.

Let

$$\mathcal{M} = \{x \in \mathbb{R}^N : Z(x) = Z_0\} \quad (1.9)$$

and for every $\delta > 0$ define

$$\mathcal{M}_\delta = \{x \in \mathbb{R}^N : \text{dist}(x, \mathcal{M}) \leq \delta\}.$$

We are now ready to state the main results of this paper.

Theorem 1.1. Suppose that conditions (Z_1) , (Z_2) , and $(f_1) - (f_5)$ are satisfied. Then for every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (Q) has at least $cat_{\mathcal{M}_\delta}(\mathcal{M})$ positive (weak) solutions for every $\varepsilon > 0$ satisfying $\varepsilon < \varepsilon_\delta$. Furthermore, let w_ε be a solution of problem (Q) and ζ_ε its global maximum. Then, up to a subsequence, $\zeta_\varepsilon \rightarrow y \in \mathcal{M}$ and $\lim_{\varepsilon \rightarrow 0^+} Z(\zeta_\varepsilon) = Z_0$.

Theorem 1.2. Suppose that conditions (Z_1) , (Z_2) , and $(f_1) - (f_5)$ are satisfied. Let w_ε be a solution of problem (Q), which exists by Theorem 1.1, and let ζ_ε be its global maximum. Then $u_\varepsilon(x) = w_\varepsilon(\varepsilon x + \zeta_\varepsilon)$ converges strongly in $W^{s,p}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ to a ground state solution u of the following problem

$$(-\Delta)_p^s u + (-\Delta)_q^s u + Z_0(|u|^{p-2}u + |u|^{q-2}u) = [|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^N.$$

Remark 1.1. As $s \rightarrow 1^{-1}$, condition (f_1) reduces to the following condition

$(f_1)'$ Continuous function $f(t)$ vanishes for every $t \in (-\infty, 0)$ and for every q_1, q_2 , with

$$q_1 \geq q, \quad q_2 > N,$$

there exist constants $a_1 > 0, a_2 > 0$ and α_0 , with $0 < \beta_0 < \alpha_*$, such that

$$f(t) \leq a_1 |t|^{q_1-1} + a_2 \mathcal{H}_N(\beta_0 |t|^{N/(N-1)}) |t|^{q_2-1} \quad \text{for every } t \in \mathbb{R}_0^+,$$

where

$$\mathcal{H}_N(t) = e^t - \sum_{j=0}^{N-2} \frac{t^j}{j!}, \quad 0 < \alpha_* \leq \alpha_*(1, N) = \lim_{s \rightarrow 1^-} \alpha_*(s, N)$$

and $\alpha_*(s, N)$ is given in (1.1).

Invoking Theorem 1.1 and Theorem 1.2 for the case when $s \rightarrow 1^{-1}$, we get the following results for problem (\mathcal{Q}) , respectively.

Corollary 1.3. *Suppose that conditions $(\mathcal{Z}_1), (\mathcal{Z}_2), (f_1)'$, and $(f_2) - (f_5)$ hold. Then for every $\delta > 0$, there exists $\varepsilon_\delta > 0$ such that problem (\mathcal{Q}) has at least $\text{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ positive (weak) solutions for every $\varepsilon \in (0, \varepsilon_\delta)$. Furthermore, let w_ε be a solution of problem (\mathcal{Q}) and ζ_ε its global maximum. Then, up to a subsequence, $\zeta_\varepsilon \rightarrow y \in \mathcal{M}$ and $\lim_{\varepsilon \rightarrow 0^+} Z(\zeta_\varepsilon) = Z_0$.*

Corollary 1.4. *Suppose that conditions $(\mathcal{Z}_1), (\mathcal{Z}_2), (f_1)'$, and $(f_2) - (f_5)$ hold. If w_ε is a solution of problem (\mathcal{Q}) , which exists by Corollary 1.3, and ζ_ε is its global maximum, then $u_\varepsilon(x) = w_\varepsilon(\varepsilon x + \zeta_\varepsilon)$ converges strongly in $W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$ to a ground state solution u of*

$$-\Delta_N u - \Delta_q u + Z_0(|u|^{N-2}u + |u|^{q-2}u) = [|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^N$$

and there exist $c > 0, C > 0$ such that $|w_\varepsilon(x)| \leq C e^{-c|x-\zeta_\varepsilon|/\varepsilon}$, for every $x \in \mathbb{R}^N$.

The organization of this paper is as follows. In Section 2, we introduce some notations and recall certain technical results which will be needed in the paper. In Section 3, we study the autonomous problem (\mathcal{Q}_ϑ) associated to problem (\mathcal{Q}) . In Section 4, we deal with the auxiliary problem $(\mathcal{Q}_\varepsilon)$. In addition, we also verify that the Palais-Smale condition holds for its energy functional and apply some new tools to obtain a multiplicity result. In Section 5, we establish the multiplicity of solutions for the modified problem and complete the proof of the main results.

2. PRELIMINARIES

In this section, we state some results and notions which will be used later. For all other background material we refer to the comprehensive monograph by Papageorgiou et al. [31].

Let us recall that $p = \frac{N}{s}$ in problem (\mathcal{Q}) . For $1 < p < \infty$, we define the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ as

$$W^{s,p}(\mathbb{R}^N) = \left\{ u \in L^p(\mathbb{R}^N) : [u]_{s,p} = \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{2N}} dx dy \right)^{1/p} < \infty \right\}$$

and we endow it with the norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} = \left(\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p \right)^{1/p}.$$

By Pucci et al. [33, Lemma 10], the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ is a uniformly convex Banach space.

Now, fix $\vartheta > 0$ and endow $W^{s,p}(\mathbb{R}^N)$ with the norm

$$\|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)} = \left(\vartheta \|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p \right)^{1/p}.$$

Obviously, the norm $\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}$ is equivalent to $\|\cdot\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}$ on $W^{s,p}(\mathbb{R}^N)$.

Let conditions (\mathcal{Z}_1) and (\mathcal{Z}_2) be satisfied. We denote by $W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)$ the completion of $C_0^\infty(\mathbb{R}^N)$, with norm

$$\|u\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)} = \left([u]_{s,p}^p + \|u\|_{p,Z,\varepsilon}^p \right)^{1/p}, \quad \|u\|_{p,Z,\varepsilon}^p = \int_{\mathbb{R}^N} Z(\varepsilon x) |u(x)|^p dx \quad \text{for every } \varepsilon > 0.$$

Pucci et al. [33, Lemma 10] showed that $W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)$ is also a uniformly convex Banach space for $1 < p < \infty$. Moreover, $W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)$ is a reflexive Banach space. Invoking conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$, and Di Nezza et al. [21, Theorem 6.9], we obtain the continuous embedding $W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$ for arbitrary $\nu \in [N/s, \infty)$.

Let $s \in (0, 1)$ and $p, q \in (1, \infty)$. The natural solution space of problem (\mathcal{Q}) is defined as

$$\mathcal{W}_\varepsilon = W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N) \cap W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)$$

and is equipped with the norm

$$\|u\|_{\mathcal{W}_\varepsilon} = \|u\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)} + \|u\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}.$$

With the aid of above definitions, assumptions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$, and the fact that $p = \frac{N}{s}$, it is easy to get the continuous embeddings

$$\mathcal{W}_\varepsilon \hookrightarrow W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N) \hookrightarrow W^{s,p}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N) \quad \text{for every } \nu \in [N/s, \infty).$$

Therefore, there exist the best constants

$$S_{\nu,\varepsilon} = \inf_{\substack{u \in \mathcal{W}_\varepsilon \\ u \neq 0}} \frac{\|u\|_{\mathcal{W}_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}} \quad \text{for every } \nu \in [N/s, \infty).$$

By a change of variable $x \mapsto \varepsilon x$, problem (\mathcal{Q}) becomes equivalent to the following equation

$$(-\Delta)_{N/s}^s u + (-\Delta)_q^s u + Z(\varepsilon x) (|u|^{\frac{N}{s}-2} u + |u|^{q-2} u) = [|x|^{-\mu} * F(u)] f(u) \quad \text{in } \mathbb{R}^N \quad (\mathcal{Q}_\varepsilon)$$

which is variational and the (weak) solutions of problem $(\mathcal{Q}_\varepsilon)$ satisfy the following definition.

Definition 2.1. Let $u \in \mathcal{W}_\varepsilon$. If for every $\varphi \in \mathcal{W}_\varepsilon$, we have

$$\begin{aligned} & \sum_{\varphi \in \{p,q\}} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\varphi-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+\varphi s}} dx dy \\ & + \int_{\mathbb{R}^N} Z(\varepsilon x) (|u|^{p-2} u + |u|^{q-2} u) \varphi dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(y))}{|x - y|^\mu} f(u(x)) \varphi(x) dx dy, \end{aligned}$$

then u is called a weak *solution* of problem $(\mathcal{Q}_\varepsilon)$.

3. THE AUTONOMOUS PROBLEM (\mathcal{Q}_ϑ)

Fix $\vartheta > 0$. In this section, we shall consider the autonomous problem (\mathcal{Q}_ϑ) , associated with problem (\mathcal{Q}) , that is

$$(-\Delta)_{N/s}^s u + (-\Delta)_q^s u + \vartheta(|u|^{\frac{N}{s}-2}u + |u|^{q-2}u) = [|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^N. \quad (\mathcal{Q}_\vartheta)$$

We consider the Euler-Lagrange functional $\mathcal{E}_\vartheta : W^{s,N/s}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N) \rightarrow \mathbb{R}$ corresponding to problem (\mathcal{Q}_ϑ) as follows

$$\mathcal{E}_\vartheta(u) = \frac{1}{p} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \mathcal{L}_\mu(x) F(u(x)) dx, \quad (3.1)$$

where

$$\mathcal{L}_\mu(u)(x) = \int_{\mathbb{R}^N} \frac{F(u(y))}{|x-y|^\mu} dy.$$

Here, $\mathcal{W} = W^{s,N/s}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$ is the Banach space, with the norm

$$\|u\| = \|u\|_{W^{s,p}(\mathbb{R}^N)} + \|u\|_{W^{s,q}(\mathbb{R}^N)}.$$

We also endow \mathcal{W} with the equivalent norm

$$\|u\|_{\vartheta, \mathcal{W}} = \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)} + \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}.$$

Thus, \mathcal{W} is a uniformly convex Banach space and so \mathcal{W} is also a reflexive Banach space. By Di Nezza et al. [21, Theorem 6.9], we obtain the continuous embeddings

$$\mathcal{W} \hookrightarrow W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N) \text{ for every } \nu \in [N/s, \infty).$$

Hence, there exists the best constant $A_{\nu, \vartheta} > 0$ given by

$$A_{\nu, \vartheta} = \inf_{\substack{u \in \mathcal{W} \\ u \neq 0}} \frac{\|u\|_{\vartheta, \mathcal{W}}}{\|u\|_{L^\nu(\mathbb{R}^N)}} \text{ for every } \nu \in [N/s, \infty).$$

Lemma 3.1. (see Zhang [40, Theorem 1.1]) *Let $s \in (0, 1)$ and $N = sp$. Then for every α , with $0 < \alpha < \alpha_* \leq \alpha_*(s, N)$,*

$$\sup_{\substack{v \in W^{s,p}(\mathbb{R}^N) \\ \|v\|_{W^{s,p}(\mathbb{R}^N)} \leq 1}} \int_{\mathbb{R}^N} \mathcal{H}_{N,s}(\alpha |v|^{N/(N-s)}) dx < \infty,$$

$$\text{where } \mathcal{H}_{N,s}(t) = e^t - \sum_{j=0}^{j_p-2} \frac{t^j}{j!}, \quad j_p = \min\{j \in \mathbb{N} : j \geq p\}.$$

Moreover, for $\alpha > \alpha_*(s, N)$,

$$\sup_{\substack{v \in W^{s,p}(\mathbb{R}^N) \\ \|v\|_{W^{s,p}(\mathbb{R}^N)} \leq 1}} \int_{\mathbb{R}^N} \mathcal{H}_{N,s}(\alpha |v|^{N/(N-s)}) dx = \infty,$$

where

$$\alpha_*(s, N) = N \left(\frac{2(N\omega_N)^2 \Gamma(p+1)}{N!} \sum_{k=0}^{+\infty} \frac{(N+k-1)!}{k!} \frac{1}{(N+2k)^p} \right)^{s/(N-s)} = N(\gamma_{s,N})^{s/(N-s)}.$$

Remark 3.1. In Lemma 3.1, if we take the norm $\|\cdot\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}$ in $W^{s,N/s}(\mathbb{R}^N)$, then

$$(\max\{1, \vartheta\})^{-1/p} \|v\|_{\vartheta, W^{s,p}(\mathbb{R}^N)} \leq \|v\|_{W^{s,p}(\mathbb{R}^N)} \leq (\min\{1, \vartheta\})^{-1/p} \|v\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}, v \in W^{s,N/s}(\mathbb{R}^N).$$

Moreover,

$$\sup_{\substack{v \in W^{s,p}(\mathbb{R}^N) \\ \|v\|_{\vartheta, W^{s,p}(\mathbb{R}^N)} \leq (\min\{1, \vartheta\})^{s/N} \mathbb{R}^N}} \int_{\mathbb{R}^N} \mathcal{H}_{N,s}(\alpha |v|^{N/(N-s)}) dx < \infty \text{ for every } \alpha, 0 < \alpha < \alpha_* \leq \alpha_*(s, N).$$

Lemma 3.2 (The Hardy-Littlewood-Sobolev inequality, see Lieb [26, 27]). *Let $r, t > 1$, and $0 < \mu < N$ such that*

$$\frac{1}{r} + \frac{\mu}{N} + \frac{1}{t} = 2.$$

Then there exists a sharp constant $C(r, N, \mu, t) > 0$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x)v(y)|}{|x-y|^\mu} dx dy \leq C(r, N, \mu, t) \|u\|_{L^r(\mathbb{R}^N)} \|v\|_{L^t(\mathbb{R}^N)} \text{ for every } u \in L^r(\mathbb{R}^N), v \in L^t(\mathbb{R}^N).$$

Since $r = t$, we can use Lemma 3.2 and the following equality

$$\frac{2}{t} + \frac{\mu}{N} = 2,$$

that is, when $t = \frac{2N}{2N-\mu}$, then the integral

$$\int_{\mathbb{R}^N} [|x|^{-\mu} * F(u)] F(u) dx \text{ for every } F(u) = |u|^q$$

is well-defined on $L^t(\mathbb{R}^N)$, with $t = 2N/(2N - \mu)$, along every $u \in W^{s,N/s}(\mathbb{R}^N)$, provided that $qt \geq \frac{N}{s}$, due to the continuous embedding $W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^\nu(\mathbb{R}^N)$, for every $\nu \in [N/s, \infty)$. Hence,

$$q \geq \frac{N}{st} = \frac{2N - \mu}{2s}.$$

Consequently, Lemma 3.1, assumption (f_1) and the fact that $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$, imply that \mathcal{E}_ϑ is well-defined on \mathcal{W} and of class $C^1(\mathcal{W})$. Furthermore, for every $u \in \mathcal{W}$,

$$\begin{aligned} \langle \mathcal{E}'_\vartheta(u), \varphi \rangle &= \sum_{\varphi \in \{p, q\}} \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\varphi-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+\varphi s}} dx dy \\ &\quad + \vartheta \int_{\mathbb{R}^N} (|u|^{\frac{N}{s}-2} u + |u|^{q-2} u) \varphi dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(u(y)) f(u(x)) \varphi(x)}{|x - y|^\mu} dx dy \text{ for every } \varphi \in \mathcal{W}. \end{aligned}$$

Here $\langle \cdot, \cdot \rangle$ is the dual pairing between \mathcal{W} and its dual space \mathcal{W}' . Consequently, the (weak) solutions of problem (3.1) are also the critical points of \mathcal{J}_ϑ in \mathcal{W} .

Lemma 3.3. *Suppose that conditions (\mathcal{Z}_1) – (\mathcal{Z}_2) , (f_1) , and (f_5) hold. Then there exist constants $t_0, \rho_0 > 0$ such that $\mathcal{E}_\vartheta(u) \geq \rho_0$, for every $u \in \mathcal{W}$, with $\|u\|_{\vartheta, \mathcal{W}} = t_0$.*

Proof. By condition (f_1) , with

$$q_1 \geq q > \frac{N}{s} > \frac{2N - \mu}{2s}, \quad q_2 \geq \frac{N}{s},$$

there exist $a_1 > 0, a_2 > 0$ such that

$$f'(t) \leq a_1 |t|^{q_1-2} + a_2 \mathcal{H}_{N,s}(\beta_0 |t|^{N/(N-s)}) |t|^{q_2-2} \text{ for every } t \in \mathbb{R}.$$

This implies that

$$f(t) \leq a_1 |t|^{q_1-1} + a_2 \mathcal{H}_{N,s}(\beta_0 |t|^{N/(N-s)}) |t|^{q_2-1} \quad \text{for every } t \in \mathbb{R}.$$

Therefore, we get

$$|F(t)| \leq a_1 |t|^{q_1} + a_2 |t|^{q_2} \mathcal{H}_{N,s}(\beta_0 |t|^{N/(N-s)}) \quad \text{for every } t \in \mathbb{R}. \quad (3.2)$$

By Lemma 3.2, we obtain that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))F(u(x))|}{|x-y|^\mu} dx dy \leq C(r, N, \mu) \|F(u)\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}^2 \quad \text{for every } u \in \mathcal{W}. \quad (3.3)$$

Moreover, (3.2) yields that

$$\|F(u)\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)} \leq a_1 \|u^{q_1}\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)} + a_2 \| |u|^{q_2} \mathcal{H}_{N,s}(\beta_0 |u|^{N/(N-s)}) \|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}. \quad (3.4)$$

Applying the Hölder inequality with $\kappa > 1$ and $\kappa' > 1$ close to 1, $1/\kappa + 1/\kappa' = 1$, and using the arguments from the proof of Li and Yang [23, Lemma 2.3], we can show that for every

$$\mathfrak{l} > \frac{2N\kappa'}{2N-\mu},$$

there exists a constant $C(\mathfrak{l}) > 0$ such that

$$\left(\mathcal{H}_{N,s}(\beta_0 |t|^{N/(N-s)}) \right)^{\frac{2N\kappa'}{2N-\mu}} \leq C(\mathfrak{l}) \mathcal{H}_{N,s}(\mathfrak{l} \beta_0 |t|^{N/(N-s)}) \quad \text{for every } t \in \mathbb{R}. \quad (3.5)$$

Hence, (3.5) implies that for every $u \in \mathcal{W}$,

$$\begin{aligned} \| |u|^{q_2} \mathcal{H}_{N,s}(\beta_0 |u|^{N/(N-s)}) \|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)} &= \left(\int_{\mathbb{R}^N} (|u|^{q_2} \mathcal{H}_{N,s}(\beta_0 |u|^{N/(N-s)}))^{\frac{2N}{2N-\mu}} dx \right)^{\frac{2N-\mu}{2N}} \\ &\leq \|u\|_{L^{\frac{2N\kappa q_2}{2N-\mu}}(\mathbb{R}^N)}^{q_2} \left(\int_{\mathbb{R}^N} C(\mathfrak{l}) \mathcal{H}_{N,s}(\mathfrak{l} \beta_0 |u|^{N/(N-s)}) dx \right)^{\frac{2N-\mu}{2N}}. \end{aligned} \quad (3.6)$$

We apply Lemma 3.1, taking $\|u\|_{\vartheta, \mathcal{W}}$ small enough, and get

$$\mathfrak{l} \beta_0 \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} \leq \mathfrak{l} \beta_0 \|u\|_{\vartheta, \mathcal{W}}^{N/(N-s)} < \alpha_*, \quad (3.7)$$

hence

$$\int_{\mathbb{R}^N} \mathcal{H}_{N,s}(\mathfrak{l} \beta_0 |u|^{N/(N-s)}) dx = \int_{\mathbb{R}^N} \mathcal{H}_{N,s} \left(\mathfrak{l} \beta_0 \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} \left(\frac{|u|}{\|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}} \right)^{N/(N-s)} \right) dx < \infty. \quad (3.8)$$

Together with (3.3)-(3.8), assuming $\|u\|_{\vartheta, \mathcal{W}}$ to be small enough, we conclude that there exist appropriate constants $\mathfrak{h}_1 > 0$ and $\mathfrak{h}_2 > 0$ such that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))F(u(x))|}{|x-y|^\mu} dx dy \leq \mathfrak{h}_1 \|u\|_{L^{\frac{2Nq_1}{2N-\mu}}(\mathbb{R}^N)}^{2q_1} + \mathfrak{h}_2 \|u\|_{L^{\frac{2N\kappa q_2}{2N-\mu}}(\mathbb{R}^N)}^{2q_2}. \quad (3.9)$$

Thus, by (3.1) and the continuity of the embeddings $\mathcal{W} \hookrightarrow W^{s, N/s}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$, for every $t \geq N/s$, we obtain that

$$\begin{aligned}
\mathcal{E}_\vartheta(u) &\geq \frac{s}{N} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \frac{1}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \mathfrak{h}_1 A_{\frac{2Nq_1}{2N-\mu}, \vartheta}^{-2q_1} \|u\|_{\vartheta, \mathcal{W}}^{2q_1} - \mathfrak{h}_2 A_{\frac{2N\kappa q_2}{2N-\mu}, \vartheta}^{-2q_2} \|u\|_{\vartheta, \mathcal{W}}^{2q_2} \\
&\geq \frac{2^{1-q}}{q} \|u\|_{\vartheta, \mathcal{W}}^q - \mathfrak{h}_1 A_{\frac{2Nq_1}{2N-\mu}, \vartheta}^{-2q_1} \|u\|_{\vartheta, \mathcal{W}}^{2q_1} - \mathfrak{h}_2 A_{\frac{2N\kappa q_2}{2N-\mu}, \vartheta}^{-2q_2} \|u\|_{\vartheta, \mathcal{W}}^{2q_2}
\end{aligned} \tag{3.10}$$

for $\|u\|_{\vartheta, \mathcal{W}}$ small enough. Let

$$\mathcal{C}(t) = \frac{2^{1-q}}{q} - \mathfrak{a}_1 A_{\frac{2Nq_1}{2N-\mu}, \vartheta}^{-2q_1} t^{2q_1 - \frac{N}{s}} - \mathfrak{h}_2 A_{\frac{2N\kappa q_2}{2N-\mu}, \vartheta}^{-2q_2} t^{2q_2 - \frac{N}{s}}, \quad t \geq 0.$$

We claim that there exists $t_0 > 0$ so small that

$$\mathcal{C}(t_0) \geq \frac{2^{1-q}}{2q} = \mathcal{C}_0.$$

Clearly, \mathcal{C} is continuous in \mathbb{R}_0^+ and $\lim_{t \rightarrow 0^+} \mathcal{C}(t) = 2\mathcal{C}_0$, so there exists t_0 such that $\mathcal{C}(t) \geq \mathcal{C}_0$, for every $t \in [0, t_0]$. We take t_0 even smaller, if necessary, so that $\|u\|_{\vartheta, \mathcal{W}} = t_0$ satisfies (3.7). This proves the claim. Hence $\mathcal{E}_\vartheta(u) \geq \mathcal{C}_0 t_0^q = \rho_0$, for every $u \in \mathcal{W}$, with $\|u\|_{\vartheta, \mathcal{W}} = t_0$. This completes the proof of Lemma 3.3. \square

In the sequel, $\mathcal{A}_\mu : \mathcal{W} \rightarrow \mathbb{R}$ will denote the functional

$$\mathcal{A}_\mu(u) = \frac{1}{2} \int_{\mathbb{R}^N} \mathcal{L}_\mu(u)(x) F(u(x)) dx, \tag{3.11}$$

where \mathcal{L}_μ is given in (3.1).

Lemma 3.4. *Suppose that conditions (\mathcal{Z}_1) , (\mathcal{Z}_2) , and (f_4) hold. Then there exists a nonnegative function $v \in C_0^\infty(\mathbb{R}^N)$, with $\|v\|_{\vartheta, \mathcal{W}} > t_0$, such that $\mathcal{E}_\vartheta(v) < 0$, where $t_0 > 0$ is the number given by Lemma 3.3.*

Proof. Fix $u_0 \in C_0^\infty(\mathbb{R}^N) \setminus \{0\}$, with $u_0 \geq 0$ in \mathbb{R}^N . Set

$$\mathcal{H}_\mu(t) = \mathcal{A}_\mu(tu_0/\|u_0\|_{\vartheta, \mathcal{W}}) \quad \text{for every } t > 0,$$

where \mathcal{A}_μ is defined in (3.11). Condition (f_4) gives

$$\begin{aligned}
\mathcal{H}'_\mu(t) &= \mathcal{H}'_\mu(tu_0/\|u_0\|_{\vartheta, \mathcal{W}}) \frac{u_0}{\|u_0\|_{\vartheta, \mathcal{W}}} \\
&= \int_{\mathbb{R}^N} \left[|x|^{-\mu} * F(tu_0/\|u_0\|_{\vartheta, \mathcal{W}}) \right] f(tu_0/\|u_0\|_{\vartheta, \mathcal{W}}) \frac{u_0}{\|u_0\|_{\vartheta, \mathcal{W}}} dx > \frac{\theta}{t} \mathcal{H}_\mu(t).
\end{aligned}$$

Integrating the above inequality on $[1, t\|u_0\|_{\vartheta, \mathcal{W}}]$, with $t > 1/\|u_0\|_{\vartheta, \mathcal{W}}$, we get

$$\mathcal{H}_\mu(t\|u_0\|_{\vartheta, \mathcal{W}}) \geq \mathcal{H}_\mu(1)(t\|u_0\|_{\vartheta, \mathcal{W}})^\theta$$

which implies that

$$\mathcal{A}_\mu(tu_0) \geq \mathcal{A}_\mu\left(\frac{u_0}{\|u_0\|_{\vartheta, \mathcal{W}}}\right) \|u_0\|_{\vartheta, \mathcal{W}}^\theta t^\theta.$$

Therefore, we have

$$\mathcal{E}_\vartheta(tu_0) = \frac{st^{N/s}}{N} \|u_0\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \frac{t^q}{q} \|u_0\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \int_{\mathbb{R}^N} \mathcal{L}_\mu(tu_0)(x) F(tu_0) dx$$

$$\leq \frac{st^{N/s}}{N} \|u_0\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \frac{t^q}{q} \|u_0\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \mathcal{L}_\mu \left(\frac{u_0}{\|u_0\|_{\vartheta, \mathcal{W}}} \right) \|u_0\|_{\vartheta}^\theta t^\theta$$

for every $t > 1/\|u_0\|_{\vartheta, \mathcal{W}}$, choosing $v = tu_0$ and t large enough. This completes the proof of Lemma 3.4. \square

Lemmas 3.3 and 3.4 show that \mathcal{E}_ϑ satisfies the geometric conditions of the Mountain Pass Theorem, therefore there exists a Palais-Smale sequence $\{u_n\}_n \subset \mathcal{W}$ for \mathcal{E}_ϑ at level c_ϑ , briefly $(PS)_{c_\vartheta}$, that is,

$$\begin{aligned} \mathcal{E}_\vartheta(u_n) &\rightarrow c_\vartheta \quad \text{and} \quad \mathcal{E}'_\vartheta(u_n) \rightarrow 0 \text{ in } \mathcal{W}', \text{ as } n \rightarrow \infty, \\ c_\vartheta &= \inf_{\zeta \in \Pi} \max_{t \in [0,1]} \mathcal{E}_\vartheta(\zeta(t)), \end{aligned} \quad (3.12)$$

where $\Pi = \{\zeta \in C([0,1], \mathcal{W}) : \zeta(0) = 0, \mathcal{E}_\vartheta(\zeta(1)) < 0\}$. We denote Nehari manifold \mathcal{M}_ϑ related to \mathcal{E}_ϑ by

$$\mathcal{M}_\vartheta = \{u \in \mathcal{W} \setminus \{0\} : \langle \mathcal{E}'_\vartheta(u), u \rangle = 0\}.$$

Let us define

$$\mathcal{T}_\vartheta^+ := \{u \in \mathcal{W} : |\text{supp}(u^+)| > 0\} \quad (3.13)$$

and $\mathbb{S}_{Z_0}^+ = \mathbb{S}_{Z_0} \cap \mathcal{T}_\vartheta^+$, where \mathbb{S}_ϑ is the unit sphere in \mathcal{W} . We know that \mathcal{T}_ϑ^+ is an open subset of \mathcal{W} .

Invoking the fact above and the definition of \mathbb{S}_ϑ^+ , we obtain that \mathbb{S}_ϑ^+ is an incomplete $C^{1,1}$ -manifold of codimension 1 modelled on \mathcal{W} and contained in \mathcal{T}_ϑ^+ . Therefore, $\mathcal{W} = T_u \mathbb{S}_{Z_0}^+ \oplus \mathbb{R}u$ for every $u \in \mathbb{S}_{Z_0}^+$, where

$$\begin{aligned} T_u \mathbb{S}_\vartheta^+ &= \{v \in \mathcal{W} : \sum_{\vartheta \in \{p,q\}} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\vartheta-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+\vartheta s}} dx dy \\ &\quad + \int_{\mathbb{R}^N} \vartheta (|u|^{p-2}u + |u|^{q-2}u) \varphi dx = 0\}. \end{aligned}$$

Since f is only continuous, the following result plays an important role in overcoming the non-differentiability of \mathcal{M}_ϑ and the incompleteness of \mathbb{S}_ϑ .

Lemma 3.5. *Suppose that conditions $(Z_1) - (Z_2)$ and $(f_1) - (f_5)$ hold. Then*

- (i) *For every $u \in \mathcal{T}_\vartheta^+$ and $\mathcal{B}_u : [0, \infty) \rightarrow \mathbb{R}$, defined as $\mathcal{B}_u(t) := \mathcal{E}_\vartheta(tu)$, there exists a unique $t_u > 0$ such that $\mathcal{B}'_u(t) > 0$ in $(0, t_u)$ and $\mathcal{B}'_u(t) < 0$ on $(t_u, +\infty)$.*
- (ii) *\mathcal{M}_ϑ is bounded away from 0 and \mathcal{M}_ϑ is closed in \mathcal{W} . There exists $\tau > 0$ independent on u , such that $t_u \geq \tau$, for every $u \in \mathbb{S}_\vartheta$. Moreover, for each compact set $\mathcal{K} \subset \mathbb{S}_\vartheta$, there exists $C_{\mathcal{K}}$ such that $t_u \leq C_{\mathcal{K}}$, for every $u \in \mathcal{K}$.*
- (iii) *The map $\tilde{m}_\vartheta : \mathcal{T}_\vartheta^+ \rightarrow \mathcal{M}_\vartheta$, given by $\tilde{m}_\vartheta(u) = t_u u$, is continuous and $m_\vartheta := \tilde{m}_\vartheta|_{\mathbb{S}_\vartheta^+}$ is a homeomorphism between \mathbb{S}_ϑ^+ and \mathcal{M}_ϑ , and $m_\vartheta^{-1}(u) = \frac{u}{\|u\|_{\vartheta, \mathcal{W}}}$.*

Proof. For every $u \in \mathcal{T}_\vartheta^+$ and $t \geq 0$,

$$\mathcal{B}_u(t) = \mathcal{E}_\vartheta(tu) = \frac{t^p}{p} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{t^q}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(tu(y))F(tu(x))|}{|x - y|^\mu} dx dy.$$

By (3.9), (3.10) and the continuity of the embeddings $\mathcal{W} \hookrightarrow W^{s,N/s}(\mathbb{R}^N) \hookrightarrow L^t(\mathbb{R}^N)$ for every $t \in [\frac{N}{s}, +\infty)$, we get

$$\begin{aligned} \mathcal{B}_u(t) &= \mathcal{E}_\vartheta(tu) \geq \frac{t^p}{p} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{t^q}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \mathbf{a}_1 \|u\|_{L^{\frac{2Nq_1}{2N-\mu}}(\mathbb{R}^N)}^{2q_1} - D \|u\|_{L^{\frac{2N\kappa q_2}{2N-\mu}}(\mathbb{R}^N)}^{2q_2} \\ &\geq \frac{t^p}{p} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{t^q}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - t^{2q_1} \mathbf{a}_1 A^{\frac{-2q_1}{2N-\mu}, \vartheta} \|u\|_{\vartheta, \mathcal{W}}^{2q_1} - t^{2q_2} D A^{\frac{-2q_2}{2N-\mu}, \vartheta} \|u\|_{\vartheta, \mathcal{W}}^{2q_2} \end{aligned} \quad (3.14)$$

which yields $\mathcal{B}_u(t) \rightarrow 0^+$, as $t \rightarrow 0^+$. Moreover, we obtain that $\mathcal{B}_u(t) \rightarrow -\infty$, as $t \rightarrow \infty$. Therefore, there exists $t_u \in (0, \infty)$ such that $\mathcal{B}_u(t_u) = \max_{t \geq 0} \mathcal{B}_u(t)$. Furthermore, $\mathcal{B}'_u(t_u) = 0$. Now we shall verify that t_u is a unique critical point of \mathcal{B}_u in $(0, \infty)$. Arguing by contradiction, suppose that there exist $0 < t_1 < t_2 < \infty$ such that $\mathcal{B}'_u(t_1) = \mathcal{B}'_u(t_2) = 0$. Consequently, we have

$$t_1^p \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + t_1^q \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(t_1 u(y)) f(t_1 u(x)) t_1 u(x)|}{|x-y|^\mu} dx dy$$

and

$$t_2^p \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + t_2^q \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(t_2 u(y)) f(t_2 u(x)) t_2 u(x)|}{|x-y|^\mu} dx dy.$$

By two equalities above and (f₄), we obtain

$$\begin{aligned} 0 &< \left(\frac{1}{t_1^{q-p}} - \frac{1}{t_2^{q-p}} \right) \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p \\ &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^\mu} \left| \frac{F(t_1 u(y))}{(t_1 u(y))^{\frac{q}{2}}} \frac{f(t_1 u(x))}{(t_1 u(x))^{\frac{q}{2}-1}} (u(y))^{\frac{q}{2}} (u(x))^{\frac{q}{2}} \right| dy dx \\ &\quad - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^\mu} \left| \frac{F(t_2 u(y))}{(t_2 u(y))^{\frac{q}{2}}} \frac{f(t_2 u(x))}{(t_2 u(x))^{\frac{q}{2}-1}} (u(y))^{\frac{q}{2}} (u(x))^{\frac{q}{2}} \right| dy dx \\ &< 0 \end{aligned}$$

which is impossible. Therefore, we have completed the proof of (i).

(ii) Let $u \in \mathcal{M}_\vartheta$. We shall prove that the first part of conclusion is true in the following two cases.

Case 1. $\|u_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} > \frac{\alpha_*}{l\beta_0} \sigma^{s/(N-s)}$.

In this case we are done.

Case 2. $\|u_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\alpha_*}{l\beta_0} \sigma^{s/(N-s)}$.

Applying the Trudinger-Moser inequality, we obtain

$$\sup_{u \in W^{s,p}(\mathbb{R}^N)} \int_{\mathbb{R}^N} \Phi_{N,s}(\alpha_0 |u|^{N/(N-s)}) dx < +\infty \quad \text{for every } 0 \leq \alpha < \alpha_*. \quad (3.15)$$

Using the Hardy-Littlewood-Sobolev inequality again and (f₃), it follows that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y)) f(u(x)) u(x)|}{|x-y|^\mu} dy dx \leq C \|F(u)\|_{L^{\frac{2N}{2N-\mu}}} \|f(u)u\|_{L^{\frac{2N}{2N-\mu}}} \leq C \|f(u)u\|_{L^{\frac{2N}{2N-\mu}}}^2.$$

By (f_1) and (f_2) , for every $\varepsilon_* > 0$ and $q_1 \geq q > \frac{N}{s} > \frac{2N-\mu}{2s}$ and $q_2 \geq \frac{N}{s}$, there exists $C_{q,\varepsilon_*} > 0$ such that

$$\|f(u)u\|_{L^{\frac{2N}{2N-\mu}}} \leq \varepsilon_* \| |u|^{q_1} \|_{L^{\frac{2N}{2N-\mu}}} + C_{q,\varepsilon_*} \| |u|^{q_2} \mathcal{H}_{N,s}(\alpha_0 |t|^{N/(N-s)}) \|_{L^{\frac{2N}{2N-\mu}}} \quad (3.16)$$

for every $t \geq 0$. Using inequality (3.16) and the definition of $A_{\nu,\vartheta}$, there exists a constant $C(\varepsilon_*)$ such that

$$\|f(u)u\|_{L^{\frac{2N}{2N-\mu}}}^2 \leq \varepsilon_* A_{\frac{2Nq_1}{2N-\mu},\vartheta}^{-2q_1} \|u\|_{\vartheta,\mathcal{W}}^{2q_1} + C(q,\varepsilon_*) \|u\|_{\vartheta,\mathcal{W}}^{2q_2} \quad \text{for some } q_1 \geq q, q_2 \geq \frac{N}{s}. \quad (3.17)$$

In view of $\langle \mathcal{E}'_{\vartheta}(u), u \rangle = 0$, we get

$$\begin{aligned} \|u\|_{\vartheta,W^{s,p}(\mathbb{R}^N)}^p + \|u\|_{\vartheta,W^{s,q}(\mathbb{R}^N)}^q &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} dy dx \\ &\leq \varepsilon_* A_{\frac{2Nq_1}{2N-\mu},\vartheta}^{-2q_1} \|u\|_{\vartheta,\mathcal{W}}^{2q_1} + C(q,\varepsilon_*) \|u\|_{\vartheta,\mathcal{W}}^{2q_2}. \end{aligned}$$

Therefore

$$\|u\|_{\vartheta,\mathcal{W}} \geq \alpha \quad \text{for every } u \in \mathcal{M}_{\vartheta}. \quad (3.18)$$

For any sequence $\{\mathbf{u}_n\}_n \subset \mathcal{M}_{\vartheta}$, such that $\mathbf{u}_n \rightarrow u$ in \mathcal{W} , we have to prove that $u \in \mathcal{M}_{\vartheta}$. Indeed, by the fact that $\mathbf{u}_n \rightarrow u$ in \mathcal{W} , we obtain $\|\mathbf{u}_n - u\|_{\vartheta,\mathcal{W}} \rightarrow 0$, as $n \rightarrow \infty$. Using the discussion as in Willem [37, Lemma A.1], there exists a subsequence $\{\mathbf{u}_n\}_n$ of $\{u_n\}_n$ satisfying $\|\mathbf{u}_{i+1} - \mathbf{u}_i\|_{\vartheta,\mathcal{W}} \leq 2^{-i}$ for every $i \geq 1$. We denote $\mathcal{U}(x) := |\mathbf{u}_1(x)| + \sum_{i=1}^{\infty} |\mathbf{u}_{i+1}(x) - \mathbf{u}_i(x)|$. Together with the fact that $\mathbf{u}_n \rightarrow u$ in \mathcal{W} , we obtain $|\mathbf{u}_n(x)| \leq \mathcal{U}(x)$ for every $x \in \mathbb{R}^N$ and $|u(x)| \leq \mathcal{U}(x)$ in \mathbb{R}^N . Clearly, $\|\mathcal{U}\|_{\vartheta,\mathcal{W}} \leq \|\mathbf{u}_1\|_{\vartheta,\mathcal{W}} + \sum_{i=1}^{\infty} 2^{-i} < +\infty$, hence $\mathcal{U} \in \mathcal{W}$.

Since $\{\mathbf{u}_n\}_n$ is a subsequence of $\{\mathbf{u}_n\}_n$, we have

$$\|u\|_{\vartheta,W^{s,p}(\mathbb{R}^N)}^p + \|u\|_{\vartheta,W^{s,q}(\mathbb{R}^N)}^q = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(\mathbf{u}_n(y))f(\mathbf{u}_n(x))\mathbf{u}_n(x)|}{|x-y|^\mu} dy dx. \quad (3.19)$$

We shall now prove

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(\mathbf{u}_n(y))f(\mathbf{u}_n(x))\mathbf{u}_n(x)|}{|x-y|^\mu} dy dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} dy dx. \quad (3.20)$$

We have

$$\left| \frac{|F(\mathbf{u}_n(y))f(\mathbf{u}_n(x))\mathbf{u}_n(x)|}{|x-y|^\mu} - \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} \right| \leq 2 \frac{|F(\mathcal{U}(y))f(\mathcal{U}(x))\mathcal{U}(x)|}{|x-y|^\mu}.$$

Now we shall show that

$$\frac{|F(\mathcal{U}(y))f(\mathcal{U}(x))\mathcal{U}(x)|}{|x-y|^\mu} \in L^1(\mathbb{R}^N). \quad (3.21)$$

By Zhang et al. [39, Lemma 2.4], we have

$$\int_{\mathbb{R}^N} \mathcal{H}_{N,s}(\alpha_0 |\mathcal{U}|^{N/(N-s)}) dx < +\infty. \quad (3.22)$$

Together with (3.16)-(3.17), we deduce (3.21). Therefore,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(\mathbf{u}_n(y))f(\mathbf{u}_n(x))\mathbf{u}_n(x)|}{|x-y|^\mu} dy dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} dy dx \in L^1(\mathbb{R}^N).$$

Furthermore,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(\mathbf{u}_n(y))f(\mathbf{u}_n(x))\mathbf{u}_n(x)|}{|x-y|^\mu} dy dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} dy dx \rightarrow 0,$$

pointwisely on \mathbb{R}^N outside a set of measure zero. By the Dominated Convergence Theorem, we obtain that (3.20) is true. By $\mathbf{u}_n \rightarrow u$ in $\mathcal{W}(\mathbb{R}^N)$, we have

$$\|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|F(u(y))f(u(x))u(x)|}{|x-y|^\mu} dy dx$$

which yields $u \in \mathcal{M}_\vartheta$.

In the sequel, we shall verify that the second part of the conclusion is also true.

Applying (i), there exists $t_u > 0$ such that $t_u u \in \mathcal{M}_\vartheta$ for every $u \in \mathcal{S}_\vartheta$. Therefore, it follows from (3.18) that $t_u \geq \alpha$. We shall argue by contradiction that $\mathbf{u}_n \in \mathcal{K}$ satisfies $t_n := t_{\mathbf{u}_n} \rightarrow \infty$. Due to the compactness of \mathcal{K} , we may suppose that $\mathbf{u}_n \rightarrow u$ in \mathcal{W} . Then $u \in \mathcal{K} \subset \mathcal{S}_\vartheta$. By (f₄), we obtain

$$\begin{aligned} \mathcal{E}_\vartheta(t_n \mathbf{u}_n) &= \frac{1}{p} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_n \mathbf{u}_n(y))}{|x-y|^\mu} F(t_n \mathbf{u}_n(x)) dx dy \\ &\geq \frac{1}{p} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \gamma_1^2 t_n^{2\theta} \|\mathbf{u}_n\|_{L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)}^\theta \rightarrow -\infty \end{aligned}$$

due to $\theta > q$. However, since $t_n \mathbf{u}_n \in \mathcal{M}_\vartheta$, we have

$$\begin{aligned} \mathcal{E}_\vartheta|_{\mathcal{M}_\vartheta}(t_n \mathbf{u}_n) &= \frac{1}{p} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} t_n^p \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(\mathbf{u}_n(y))F(\mathbf{u}_n(x))}{|x-y|^\mu} dy dx \\ &\leq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_n \mathbf{u}_n(y))}{|x-y|^\mu} \left[\frac{1}{p} f(t_n \mathbf{u}_n(x)) t_n \mathbf{u}_n(x) - \frac{1}{2} F(t_n \mathbf{u}_n(x)) \right] dy dx \geq 0 \end{aligned}$$

which is impossible.

(iii) By (i) – (ii) and the arguments from the proof of Szulkin and Weth [34, Proposition 3.1], we obtain (iii). This completes the proof of Lemma 3.5. \square

Remark 3.2. By Lemma 3.5, the least energy c_{Z_0} satisfies the following equality:

$$c_\vartheta = \inf_{u \in \mathcal{M}_\vartheta} \mathcal{E}_\vartheta(u) = \inf_{u \in \mathcal{W} \setminus \{0\}} \max_{t > 0} \mathcal{E}_\vartheta(tu) = \inf_{u \in \mathcal{S}_\vartheta} \max_{t > 0} \mathcal{E}_\vartheta(tu). \quad (3.23)$$

Considering the functional $\Phi_\vartheta : \mathcal{S}_\vartheta \rightarrow \mathbb{R}$ given by

$$\Phi_\vartheta(\omega) := \mathcal{E}_\vartheta(m_\vartheta(\omega)) \quad (3.24)$$

similarly to Szulkin and Weth [34, Corollary 3.3], we have:

Lemma 3.6. Suppose that conditions $(Z_1) - (Z_2)$ and $(f_1) - (f_5)$ hold. Then the following statements are true:

- (i) If $\{\mathbf{u}_n\}_n$ is a $(PS)_{c_\vartheta}$ sequence for Φ_ϑ , then $\{m_\vartheta(\mathbf{u}_n)\}$ is a $(PS)_{c_\vartheta}$ sequence for \mathcal{E}_ϑ . If $\{\mathbf{u}_n\}_n \subset \mathcal{M}_\vartheta$ is a bounded $(PS)_{c_\vartheta}$ sequence for \mathcal{E}_ϑ , then $\{m_\vartheta^{-1}(\mathbf{u}_n)\}_n$ is a $(PS)_{c_\vartheta}$ sequence for Φ_ϑ .
- (ii) u is a critical point of Φ_ϑ if and only if $m_\vartheta(u)$ is a nontrivial critical point of \mathcal{E}_ϑ . Moreover, $\inf_{\mathcal{M}_\vartheta} \mathcal{E}_\vartheta = \inf_{\mathcal{S}_\vartheta} \Phi_\vartheta$.

Lemma 3.7. (see Liang et al. [25, Lemma 5]) Let (f_1) be satisfied. We suppose that $\{\mathbf{u}_n\}_n$ is a sequence in $W^{s,p}(\mathbb{R}^N)$ such that

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\alpha_*}{\mathfrak{l}\beta_0} \sigma^{s/(N-s)} \quad \text{for some } \mathfrak{l} > 1,$$

where $\sigma = \min\{1, \vartheta\}$. Then there exists $C_0 > 0$ such that

$$\left| |x|^{-\mu} * F(\mathbf{u}_n) \right| \leq C_0 \quad \text{for every } n.$$

Lemma 3.8. (see Molica Bisci et al. [30, Lemma 4]) Let $\varsigma \in [N/s, \infty)$. If $\{\mathbf{u}_n\}_n$ is a bounded sequence in $W^{s,p}(\mathbb{R}^N)$ and

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |\mathbf{u}_n(x)|^\varsigma dx = 0$$

for some $R > 0$, then $\mathbf{u}_n \rightarrow 0$ in $L^\nu(\mathbb{R}^N)$, for every $\nu \in (\varsigma, \infty)$.

As in the proof of Molica Bisci et al. [30, Theorem 7], we get

Lemma 3.9. Suppose that conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$ and (f_1) hold. Then

$$\limsup_{t \rightarrow 0^+} f(t)t^{1-\varsigma} = 0$$

for some $\varsigma \geq N/s$. Let $\{\mathbf{u}_n\}_n \subset \mathcal{W}$ be weakly convergent to 0 and such that

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\alpha_*}{\mathfrak{l}\beta_0} \sigma^{s/(N-s)},$$

where β_0, α_* are given in (f_1) and (1.1), $\sigma = \min\{1, \vartheta\}$, and $\mathfrak{l} > 1$ is a suitable constant. If there exists $R > 0$ such that

$$\liminf_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |\mathbf{u}_n|^\varsigma dx = 0,$$

then

$$\int_{\mathbb{R}^N} [|x|^{-\mu} * F(\mathbf{u}_n)] f(\mathbf{u}_n) \mathbf{u}_n \rightarrow 0 \quad \text{and} \quad \int_{\mathbb{R}^N} [|x|^{-\mu} * F(\mathbf{u}_n)] F(\mathbf{u}_n) \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Lemma 3.10. (see Liang et al. [24, Lemma 2.6]) Suppose that conditions (f_3) and (f_4) hold. Then there exists a constant C_{γ_1} such that $\rho_0 \leq c_\vartheta \leq C_{\gamma_1}$, where ρ_0 is the number determined by Lemma 3.3, γ_1 is the constant given in (f_4) ,

$$C_{\gamma_1} = a \left(1 - \frac{N}{2\theta s} \right) \left(\frac{aN}{2\theta sb} \right)^{N/(2\theta s - N)},$$

and

$$a = \frac{s}{N} \|u\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{\frac{N}{s}} + \frac{1}{q} \|u\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q.$$

Proposition 3.1. Suppose that conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$ and $(f_1) - (f_5)$ hold. Then problem (\mathcal{Q}_ϑ) has a nontrivial nonnegative (weak) solution.

Proof. By Lemmas 3.3 and 3.4, there exists a $(PS)_{c_\vartheta}$ sequence $\{\mathbf{u}_n\}_n \subset \mathcal{W}$, satisfying (3.12). We shall divide the proof into two steps.

Step 1. $\{\mathbf{u}_n\}_n$ is a bounded sequence in \mathcal{W} .

Up to a subsequence, we may suppose that $\{\mathbf{u}_n\}_n$ is strongly convergent in \mathcal{W} . From (3.12), we have

$$\mathcal{E}_\vartheta(\mathbf{u}_n) - \frac{1}{\theta} \langle \mathcal{E}'_\vartheta(\mathbf{u}_n), \mathbf{u}_n \rangle = c_\vartheta + o_n(1) + o_n(1) \|\mathbf{u}_n\|_{\vartheta, \mathcal{W}} \quad \text{as } n \rightarrow \infty, \quad (3.25)$$

where θ is given in (f₃). Moreover,

$$\begin{aligned} \mathcal{E}_\vartheta(\mathbf{u}_n) - \frac{1}{\theta} \langle \mathcal{E}'_\vartheta(\mathbf{u}_n), \mathbf{u}_n \rangle &= \left(\frac{s}{N} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \left(\frac{1}{q} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q \\ &\quad + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(\mathbf{u}_n)}{|x-y|^\mu} \left[\frac{1}{\theta} f(\mathbf{u}_n) \mathbf{u}_n - \frac{1}{2} F(\mathbf{u}_n) \right] dx dy. \end{aligned}$$

Condition (f₂) implies that

$$\mathcal{E}_\vartheta(\mathbf{u}_n) - \frac{1}{\theta} \langle \mathcal{E}'_\vartheta(\mathbf{u}_n), \mathbf{u}_n \rangle \geq \left(\frac{s}{N} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \left(\frac{1}{q} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q. \quad (3.26)$$

Combining (3.25) and (3.26), we get as $n \rightarrow \infty$,

$$\left(\frac{s}{N} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} + \left(\frac{1}{q} - \frac{1}{\theta} \right) \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^q \leq c_\vartheta + o_n(1) + o_n(1) \|\mathbf{u}_n\|_{\vartheta, \mathcal{W}}. \quad (3.27)$$

Observing that $\lim_{x \rightarrow \infty, y \rightarrow \infty} \frac{cx^{N/s} + \mathfrak{k}y^q}{x+y} = \infty$, for fixed numbers $\mathfrak{c} > 0$, $\mathfrak{k} > 0$, we conclude that $\{\mathbf{u}_n\}_n$ is a bounded sequence in \mathcal{W} . Since $\mathcal{E}_\vartheta(\mathbf{u}_n) - \frac{1}{\theta} \langle \mathcal{E}'_\vartheta(\mathbf{u}_n), \mathbf{u}_n \rangle \rightarrow c_\vartheta$ as $n \rightarrow \infty$, it follows that

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/s} \leq \frac{c_\vartheta}{\frac{1}{p} - \frac{1}{\theta}} \leq \frac{C\gamma_1}{\frac{1}{p} - \frac{1}{\theta}} \quad (3.28)$$

and

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q \leq \frac{c_\vartheta}{\frac{1}{q} - \frac{1}{\theta}} \leq \frac{C\gamma_1}{\frac{1}{q} - \frac{1}{\theta}}. \quad (3.29)$$

From (3.28), we have

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\alpha_*}{\mathfrak{l}\beta_0} \sigma^{s/(N-s)} \quad (3.30)$$

for some $\mathfrak{l} > 0$ and γ_1 large enough such that

$$\frac{C\gamma_1}{\frac{1}{p} - \frac{1}{\theta}} < \left(\frac{\alpha_*}{\mathfrak{l}\beta_0} \right)^{(N-s)/N} \sigma,$$

when $\gamma_1 \geq \gamma_*$, where

$$\gamma_* = \frac{1}{|NB_1(0)|} \sqrt{\frac{2a(N-\mu)}{\mathfrak{B}(N, N-\mu+1)}} \quad (3.31)$$

and $\mathfrak{B} = \int_0^1 t^{x-1} (1-t)^{y-1} dt$, $x > 0$, $y > 0$ is Beta function. This implies that

$$\gamma_1 \geq \frac{1}{|NB_1(0)|} \sqrt{\frac{aN(N-\mu)}{\theta a \mathfrak{B}(N, N-\mu+1)}} \left[\frac{a(1 - \frac{N}{2\theta s})}{(\frac{s}{N} - \frac{1}{\theta}) \sigma (\frac{\alpha_*}{\mathfrak{b}\beta_0})^{(N-s)/N}} \right]^{\frac{2\theta s - N}{2N}} = \gamma_{**}.$$

Hence, (3.30) holds for every $\gamma_1 \geq \max\{\gamma_*, \gamma_{**}\}$.

Step 2. We shall show that there exist $R > 0$, $\delta > 0$ and a sequence $\{y_n\}_n \subset \mathbb{R}^N$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(y_n)} |\mathbf{u}_n(x)|^\vartheta dx \geq \delta > 0, \quad \vartheta \in \{p, q\}. \quad (3.32)$$

Arguing by contradiction, we assume that for some $R > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |\mathbf{u}_n(x)|^\varphi dx = 0, \quad \varphi \in \{p, q\}. \quad (3.33)$$

Then Lemma 3.8 yields that $\mathbf{u}_n \rightarrow 0$ in $L^\nu(\mathbb{R}^N)$, for every $\nu > \varphi$. Condition (3.33), Lemma 3.9 and the Trudinger-Moser inequality imply that

$$\int_{\mathbb{R}^N} \left[|x|^{-\mu} * F(\mathbf{u}_n) \right] f(\mathbf{u}_n) \mathbf{u}_n dx \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Therefore,

$$\begin{aligned} o_n(1) &= \langle \mathcal{E}'_\vartheta(\mathbf{u}_n), \mathbf{u}_n \rangle = \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q - \int_{\mathbb{R}^N} \left[|x|^{-\mu} * F(\mathbf{u}_n) \right] f(\mathbf{u}_n) \mathbf{u}_n dx \\ &= \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p + \|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q + o_n(1) \text{ as } n \rightarrow \infty. \end{aligned}$$

So $\mathbf{u}_n \rightarrow 0$ in $W^{s,N/s}(\mathbb{R}^N) \cap W^{s,q}(\mathbb{R}^N)$. Passing to the limit as $n \rightarrow \infty$, we have

$$\mathcal{E}_\vartheta(\mathbf{u}_n) = \frac{\|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^p}{p} + \frac{\|\mathbf{u}_n\|_{\vartheta, W^{s,q}(\mathbb{R}^N)}^q}{q} - \int_{\mathbb{R}^N} \left[|x|^{-\mu} * F(\mathbf{u}_n) \right] F(\mathbf{u}_n) dx \rightarrow 0$$

which contradicts with the fact that $\mathcal{E}_\vartheta(\mathbf{u}_n) \rightarrow c_\vartheta > 0$, as $n \rightarrow \infty$. Thus (3.32) holds.

Put $\mathbf{v}_n = \mathbf{u}_n(\cdot + y_n)$, then from (3.32) we have

$$\int_{B_R(0)} |\mathbf{v}_n|^\varphi dx \geq \delta/2 > 0, \quad \varphi \in \{p, q\} \text{ for some } \delta > 0. \quad (3.34)$$

Because \mathcal{E}_ϑ and \mathcal{E}'_ϑ are invariant under translation, we have

$$\mathcal{E}_\vartheta(\mathbf{v}_n) \rightarrow c_\vartheta \quad \text{and} \quad \mathcal{E}'_\vartheta(\mathbf{v}_n) \rightarrow 0 \quad \text{in } \mathcal{W}'.$$

Since $\|\mathbf{v}_n\|_{\vartheta, \mathcal{W}} = \|\mathbf{u}_n\|_{\vartheta, \mathcal{W}}$ for every n , then $\{\mathbf{v}_n\}_n$ is also bounded in \mathcal{W} and

$$\limsup_{n \rightarrow \infty} \|\mathbf{v}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} = \limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\vartheta, W^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\alpha_*}{\beta_0} \sigma^{s/(N-s)}. \quad (3.35)$$

Hence, choosing a subsequence if necessary, we may assume that there exists $\mathbf{v} \in \mathcal{W}$ such that $\mathbf{v}_n \rightharpoonup \mathbf{v}$ in \mathcal{W} , $\mathbf{v}_n \rightarrow \mathbf{v}$ in $L^\vartheta(B_R(0))$, for every $\vartheta \in [N/s, \infty)$ and $R > 0$, and $\mathbf{v}_n \rightarrow \mathbf{v}$ a.e. in \mathbb{R}^N . Clearly, (3.34) implies that

$$\int_{B_R(0)} |\mathbf{v}|^\varphi dx \geq \delta/2 > 0, \quad \varphi \in \{p, q\},$$

hence, $\mathbf{v} \neq 0$.

Arguing similarly to the proof of Thin et al. [36, Lemma 13], we get that $\mathcal{E}'_\vartheta(\mathbf{v}) = 0$ and \mathbf{v} is indeed a ground state solution of problem (\mathcal{Q}_ϑ) . This completes the proof of Proposition 3.1. \square

4. THE AUXILIARY PROBLEM $(\mathcal{Q}_\varepsilon)$

Using the transformation $x \mapsto \varepsilon x$, problem (\mathcal{Q}) can be rewritten as follows

$$(-\Delta)_p^s u + (-\Delta)_q^s u + Z(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u) = [|x|^{-\mu} * F(u)]f(u). \quad (\mathcal{Q}_\varepsilon)$$

Inspired by the work of del Pino and Felmer [20], we introduce a penalized function which will play an essential role to obtain our main results. In general, we assume that $0 \in \Omega$ and $Z(0) = Z_0$. Fix $h_0 > 0$. We define

$$\hat{f}(t) := \begin{cases} f(t) & \text{if } t \leq a, \\ \frac{Z_0}{h_0} t^{q-1} & \text{if } t > a, \end{cases}$$

and

$$\hat{g}(t) := \begin{cases} g(x, t) = \chi_\Omega(x)f(t) + (1 - \chi_\Omega(x))\hat{f}(t) & \text{if } t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

To study problem $(\mathcal{Q}_\varepsilon)$, we introduce the Euler-Lagrange functional $\mathcal{I}_\varepsilon : \mathcal{W}_\varepsilon \rightarrow \mathbb{R}$ by

$$\mathcal{I}_\varepsilon(u) = \sum_{\wp \in \{p, q\}} \frac{1}{\wp} \|u\|_{W_{Z, \varepsilon}^{s, \wp}(\mathbb{R}^N)}^\wp - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u(y))G(u(x))}{|x - y|^\mu} dy dx.$$

By conditions (Z_1) , (Z_2) and (f_1) , the functional \mathcal{I}_ε is well defined on \mathcal{W}_ε and of class $C^2(\mathcal{W}_\varepsilon)$. Moreover, the critical points of \mathcal{I}_ε are exactly the (weak) solutions of problem $(\mathcal{Q}_\varepsilon)$. Associated to the energy functional \mathcal{I}_ε , we denote the Nehari manifold \mathcal{N}_ε by

$$\mathcal{N}_\varepsilon = \{v \in \mathcal{W}_\varepsilon \setminus \{0\} : \langle \mathcal{I}'_\varepsilon(v), v \rangle = 0\},$$

where

$$\begin{aligned} \langle \mathcal{I}'_\varepsilon(v), \varphi \rangle &= \sum_{\wp \in \{p, q\}} \int_{\mathbb{R}^{2N}} \frac{|v(x) - v(y)|^{\wp-2} (v(x) - v(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+\wp s}} dx dy \\ &+ \int_{\mathbb{R}^N} Z(\varepsilon x) (|v|^{p-2}v + |v|^{q-2}v) \varphi dx - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(v(y))g(v(x))\varphi(x)}{|x - y|^\mu} dy dx \end{aligned}$$

for every $v, \varphi \in \mathcal{W}_\varepsilon$.

Lemma 4.1. *Suppose that conditions $(f_1) - (f_5)$ hold. The the following statements hold:*

- (g₁) $\lim_{t \rightarrow 0^+} \frac{g(x, t)}{t^{q-1}} = 0$ uniformly with respect to $x \in \mathbb{R}^N$;
- (g₂) (i) $0 < g(x, t) \leq f(t)$ for every $t > 0$, for every $x \in \mathbb{R}^N$;
(ii) $g(x, t) = 0$ for every $t \leq 0$, for every $x \in \mathbb{R}^N$;
- (g₃) (i) $0 < \theta G(x, t) \leq g(x, t)t$, for every $x \in \Omega$ and $t > 0$;
(ii) $0 \leq qG(x, t) \leq g(x, t)t \leq \frac{Z_0}{h_0} (t^p + t^q)$ for every $x \in \mathbb{R}^N \setminus \Omega$ and $t > 0$;
- (g₄) for every $x \in \Omega$, the function $t \mapsto \frac{g(x, t)}{t^{\frac{q}{2}-1}}$ is strictly increasing on $(0, +\infty)$;
- (g₅) for every $x \in \mathbb{R}^N \setminus \Omega$, function $t \mapsto \frac{g(x, t)}{t^{\frac{q}{2}-1}}$ is strictly increasing on $(0, a)$.

Proof. We shall only give the proof of $(g_3) - (ii)$. The rest of the properties can be verified by the definition of g . Using (f_5) , we obtain that

$$\frac{f(t)}{t^q} \leq \frac{f(a)}{a^q} = \frac{Z_0}{h_0} \quad \text{for every } t \in [0, a].$$

Consequently,

$$g(x, t)t = f(t)t \leq \frac{Z_0}{h_0} t^q \leq \frac{Z_0}{h_0} (t^p + t^q) \text{ for every } t \in [0, a].$$

If $t \in (a, +\infty)$, we have

$$g(x, t) = \frac{Z_0}{h_0} t^q < \frac{Z_0}{h_0} (t^p + t^q).$$

From (f_2) and (g_2) , we obtain that

$$g(x, t)t = f(t)t \geq \theta F(t) > qF(t) \geq qG(x, t) > 0 \text{ for every } t \in [0, a].$$

In addition, if $t \in (a, +\infty)$, we have

$$g(x, t) = \hat{f}(t) = \frac{Z_0}{h_0} t^{q-1}.$$

Hence

$$G(x, t) = \frac{Z_0}{qh_0} t^q$$

and

$$g(x, t)t = \frac{Z_0}{h_0} t^q = qG(x, t).$$

This completes the proof of Lemma 4.1. \square

Lemma 4.2. (see [24, Proposition 3.1]) *Suppose that conditions $(\mathcal{Z}_1) - \mathcal{Z}_2$ and $(f_1) - (f_3)$ hold. Then there is a real number $\mathfrak{r}_* > 0$ such that*

$$\|u\|_{\mathcal{W}_\varepsilon} \geq \mathfrak{r}_* > 0 \text{ for every } u \in \mathcal{N}_\varepsilon.$$

Lemma 4.3. *Suppose that conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$ and $(f_1) - (f_5)$ hold. Then \mathcal{I}_ε satisfies the following the geometric conditions:*

- (i) *There exist real numbers $\alpha_* > 0, \rho_* > 0$ such that for every $u \in \mathcal{W}_\varepsilon : \|u\|_{\mathcal{W}_\varepsilon} = \rho_*$, we have $\mathcal{I}_\varepsilon(u) \geq \alpha_* > 0$;*
- (ii) *There exists $u \in \mathcal{W}_\varepsilon$ such that $\|u\|_{\mathcal{W}_\varepsilon} > \rho_*$ and $\mathcal{I}_\varepsilon(u) < 0$.*

Proof. One can apply a similar discussion as in Lemmas 3.3 and 3.4, combined with the Trudinger-Moser inequality, so we shall omit the details here. \square

By virtue of Lemma 4.2 and the Mountain Pass Theorem, there exists a $(PS)_{c_\varepsilon}$ sequence $\{\mathbf{u}_n\}_n \subset \mathcal{W}_\varepsilon$, that is,

$$\mathcal{I}_\varepsilon(\mathbf{u}_n) \rightarrow c_\varepsilon \text{ and } \mathcal{I}'_\varepsilon(\mathbf{u}_n) \rightarrow 0,$$

where

$$c_\varepsilon := \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} \mathcal{I}_\varepsilon(\gamma(t))$$

and $\Gamma = \{\gamma \in (C^0[0, 1], \mathcal{W}_\varepsilon(\mathbb{R}^N)) : \gamma(0) = 0, \mathcal{I}_\varepsilon(\gamma(1))\} < 0$.

We give the definition of $\mathcal{T}_\varepsilon^+$ as follows:

$$\mathcal{T}_\varepsilon^+ := \{u \in \mathcal{W}_\varepsilon(\mathbb{R}^N) : |\text{supp}(u^+) \cap \Omega_\varepsilon| > 0\} \subset \mathcal{W}_\varepsilon(\mathbb{R}^N),$$

$\Omega_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in \Omega\}$. Let \mathbb{S}_ε be a the unit sphere in $\mathcal{W}_\varepsilon(\mathbb{R}^N)$ and denote by $\mathbb{S}_\varepsilon^+ = \mathbb{S}_\varepsilon \cap \mathcal{T}_\varepsilon^+$. Note that \mathbb{S}_ε^+ is an incomplete $C^{1,1}$ -manifold of codimension 1, modelled on $\mathcal{W}_\varepsilon(\mathbb{R}^N)$ and contained in the open $\mathcal{T}_\varepsilon^+$. Thus, $\mathcal{W}_\varepsilon = T_u \mathbb{S}_\varepsilon^+ \oplus \mathbb{R}u$, for every $u \in \mathbb{S}_\varepsilon^+$, where

$$T_u \mathbb{S}_\varepsilon^+ = \{v \in \mathcal{W} : \sum_{\varnothing \in \{p, q\}} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\varnothing-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{N+\varnothing s}} dx dy\}$$

$$+ \int_{\mathbb{R}^N} Z(\varepsilon x)(|u|^{p-2}u + |u|^{q-2}u)\varphi dx = 0\}.$$

As in Lemmas 3.5 and 3.6, the following results can be obtained.

Lemma 4.4. *Suppose that conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$ and $(f_1) - (f_5)$ hold. Then the following statements are true:*

- (i) *There exists a unique $t_u u \in \mathcal{N}_\varepsilon$ and $\mathcal{I}_\varepsilon(t_u u) = \max_{t>0} \mathcal{I}_\varepsilon(tu)$ for every $u \in \mathcal{T}_\varepsilon^+$. Moreover, we have $c_\varepsilon \geq \zeta > 0$ and*

$$c_\varepsilon = \inf_{u \in \mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon(u) = \inf_{u \in \mathcal{T}_\varepsilon^+} \max_{t>0} \mathcal{I}_\varepsilon(tu) = \inf_{u \in \mathbb{S}_\varepsilon^+} \max_{t>0} \mathcal{I}_\varepsilon(tu).$$

- (ii) *\mathcal{N}_ε is bounded away from 0, and there exists $\alpha > 0$ such that $t_u \geq \alpha$, for every $u \in \mathbb{S}_\varepsilon$. Moreover, for every compact subset $\mathcal{K} \subset \mathbb{S}_\varepsilon^+$, there exists $C_{\mathcal{K}} > 0$ such that $t_u \leq C_{\mathcal{K}}$, for every $u \in \mathcal{K}$.*
- (iii) *The continuous map $\hat{m}_\varepsilon : \mathcal{W}_\varepsilon \rightarrow \mathcal{N}_\varepsilon$ is given by $\hat{m}_\varepsilon(u) = t_u u$ and $m_\varepsilon := \hat{m}_\varepsilon|_{\mathbb{S}_\varepsilon^+}$ is a homeomorphism between \mathbb{S}_ε^+ and \mathcal{N}_ε , and $m_\varepsilon^{-1}(u) = \frac{u}{\|u\|_{\mathcal{W}_\varepsilon}}$.*

Exploring the functional $\Theta_\varepsilon(u) := \mathcal{I}_\varepsilon(m_\varepsilon(u))$, together with argument similar to the proof of Szulkin and Weth [34, Corollary 3.3], we obtain the following results.

Lemma 4.5. *Suppose that conditions $(\mathcal{Z}_1) - (\mathcal{Z}_2)$ and $(f_1) - (f_5)$ hold. Then the following statements are true:*

- (i) *If $\{\mathbf{u}_n\}_n$ is a $(PS)_{c_\varepsilon}$ sequence for Θ_ε , then $\{m_\varepsilon(\mathbf{u}_n)\}_n$ is a $(PS)_{c_\varepsilon}$ sequence for \mathcal{I}_ε . Moreover, if $\{\mathbf{u}_n\}_n \subset \mathcal{N}_\varepsilon$ is a bounded $(PS)_{c_\varepsilon}$ sequence for \mathcal{I}_ε , then $\{m_\varepsilon^{-1}(\mathbf{u}_n)\}_n$ is a $(PS)_{c_\varepsilon}$ sequence for Θ_ε .*
- (ii) *u is a critical point of Θ_ε if and only if $m_\varepsilon(u)$ is a nontrivial critical point of \mathcal{I}_ε . Furthermore, $\inf_{\mathcal{N}_\varepsilon} \mathcal{I}_\varepsilon = \inf_{\mathbb{S}_\varepsilon^+} \Theta_\varepsilon$.*

Lemma 4.6. *c_ε and c_ϑ satisfy the following inequalities*

$$\limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon \leq c_\vartheta \leq C_{\gamma_1}. \quad (4.1)$$

Proof. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $\varphi \equiv 1$ on $B_{\delta/2}(0)$, $\text{supp}(\varphi) \subset B_\delta(0) \subset \Omega$ for some $\delta > 0$ and $\varphi \equiv 0$ on $B_\delta(0)^c$. We define

$$u_\varepsilon(x) := \varphi(\varepsilon x)u(x) \quad \text{for every } \varepsilon > 0,$$

where u is the ground state solution of problem (\mathcal{Q}_ϑ) obtained in Proposition 3.1. We know that $\text{supp}(u_\varepsilon) \subset \Omega_\varepsilon$ and $u_\varepsilon \rightarrow u$ in \mathcal{W} (see Ambrosio and Isernia [6, Lemma 2.4]). If we assume that $t_\varepsilon > 0$ with $t_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$, then

$$\begin{aligned} c_\varepsilon \leq \mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon) &= \frac{t_\varepsilon^p}{p} \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{t_\varepsilon^q}{q} \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(t_\varepsilon u_\varepsilon(y))G(t_\varepsilon u_\varepsilon(x))}{|x-y|^\mu} dy dx \\ &= \frac{t_\varepsilon^p}{p} \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{t_\varepsilon^q}{q} \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))F(t_\varepsilon u_\varepsilon(x))}{|x-y|^\mu} dy dx. \end{aligned}$$

Since $t_\varepsilon u_\varepsilon \in \mathcal{N}_\varepsilon$, we have

$$t_\varepsilon^p \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + t_\varepsilon^q \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))f(t_\varepsilon u_\varepsilon(x))t_\varepsilon u_\varepsilon(x)}{|x-y|^\mu} dy dx. \quad (4.2)$$

Moreover, it follows from (f₃) that

$$\begin{aligned} \mathcal{I}_\varepsilon(t_\varepsilon v_\varepsilon) &= \frac{t_\varepsilon^p}{p} \|u_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{t_\varepsilon^q}{q} \|u_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))F(t_\varepsilon u_\varepsilon(x))}{|x-y|^\mu} dy dx \\ &\geq \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))}{|x-y|^\mu} \left[\frac{1}{q} f(t_\varepsilon u_\varepsilon) t_\varepsilon u_\varepsilon dx - \frac{1}{2} F(t_\varepsilon u_\varepsilon) \right] dx \geq 0. \end{aligned} \quad (4.3)$$

This fact implies that $\{t_\varepsilon\}$ is bounded, as $\varepsilon \rightarrow 0^+$. Indeed, suppose to the contrary, that $\{t_\varepsilon\}$ is unbounded when $\varepsilon \rightarrow 0^+$. Consequently, together with (f₄), we would have

$$\mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon) \geq \frac{t_\varepsilon^p}{p} \|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{t_\varepsilon^q}{q} \|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q - \gamma_1^2 t_\varepsilon^{2\theta} \|u_\varepsilon\|_{L^{\frac{2N}{2N-\mu}}}^2 \rightarrow -\infty$$

which is a contradiction with (4.3). Thus, we may suppose that $t_\varepsilon \rightarrow t_0$, as $\varepsilon \rightarrow 0^+$. Using the Vitali's Theorem, we obtain

$$\limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon \leq \frac{t_0^p}{p} \|u\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{t_0^q}{q} \|u\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q - \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_0 u(y))F(t_0 u(x))}{|x-y|^\mu} dy dx = \mathcal{E}_\vartheta(t_0 u).$$

We shall now verify that $t_0 = 1$. For every $\varepsilon > 0$, there exists $t_\varepsilon > 0$ such that

$$\mathcal{I}_\varepsilon(t_\varepsilon u_\varepsilon) = \max_{t \geq 0} \mathcal{I}_\varepsilon(t u_\varepsilon).$$

Therefore, $\langle \mathcal{I}'_\varepsilon(t_\varepsilon u_\varepsilon), u_\varepsilon \rangle = 0$ and we have

$$t_\varepsilon^p \|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + t_\varepsilon^q \|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))f(t_\varepsilon u_\varepsilon(x))t_\varepsilon u_\varepsilon(x)}{|x-y|^\mu} dy dx$$

which means that

$$\|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + t_\varepsilon^{q-p} \|u_\varepsilon\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_\varepsilon u_\varepsilon(y))f(t_\varepsilon u_\varepsilon(x))u_\varepsilon(x)}{t_\varepsilon^{p-1}|x-y|^\mu} dy dx. \quad (4.4)$$

Passing to the limit as $\varepsilon \rightarrow 0$ in (4.4) and using the fact that $u_\varepsilon \rightarrow u$ in $W^{s,\frac{N}{s}}(\mathbb{R}^N)$ (see Ambrosio and Isernia [6, Lemma 2.4]), we obtain

$$\|u\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + t_0^{q-p} \|u\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(t_0 u(y))f(t_0 u(x))u(x)}{t_0^{p-1}|x-y|^\mu} dy dx.$$

Together with this fact, $u \in \mathcal{M}_\vartheta$ and (f₅), we deduce that $t_0 = 1$. Therefore,

$$\limsup_{\varepsilon \rightarrow 0^+} c_\varepsilon \leq \mathcal{E}_\vartheta(v) = c_\vartheta.$$

Combining with Lemma 3.10, we can obtain that the last inequality hold in (4.1). This completes the proof of Lemma 4.6. \square

Lemma 4.7. *The functional \mathcal{I}_ε satisfies the Palais-Smale condition at level c_ε , for every $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let $\{u_n\}_n \subset \mathcal{W}_\varepsilon(\mathbb{R}^N)$ be a $(PS)_{c_\varepsilon}$ sequence for \mathcal{I}_ε , i.e.,

$$\mathcal{I}_\varepsilon(u_n) \rightarrow c_\varepsilon \text{ and } \mathcal{I}'_\varepsilon(u_n) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We shall complete the proof of this lemma by the following two claims.

Claim 1. $\{u_n\}_n$ is bounded in $\mathcal{W}_\varepsilon(\mathbb{R}^N)$.

Indeed, by (f₃), we have

$$\begin{aligned} c_\varepsilon + o_n(1) &= \mathcal{I}_\varepsilon(\mathbf{u}_n) - \frac{1}{2q} \langle \mathcal{I}'_\varepsilon(\mathbf{u}_n), \mathbf{u}_n \rangle = \left(\frac{1}{p} - \frac{1}{2q} \right) \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \frac{1}{2q} \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{F(\mathbf{u}_n(y))}{|x-y|^\mu} \left[\frac{1}{q} f(\mathbf{u}_n(x)) \mathbf{u}_n(x) - F(\mathbf{u}_n(x)) \right] dy dx \\ &\geq \frac{1}{2q} (\|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}(\mathbb{R}^N)}^p + \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}(\mathbb{R}^N)}^q) = \frac{1}{2q} \|\mathbf{u}_n\|_{\mathcal{W}_\varepsilon} \end{aligned}$$

which implies that $\{\mathbf{u}_n\}_n$ is bounded in $\mathcal{W}_\varepsilon(\mathbb{R}^N)$. Moreover, we obtain

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\mathcal{W}_\varepsilon}^p \leq 2qc_\varepsilon. \quad (4.5)$$

Together with this fact and Lemma 3.10, we obtain that

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\mathcal{W}_\varepsilon} \leq 2qc_\varepsilon \leq 2qC_{\gamma_1} = 2qa \left(1 - \frac{N}{2\theta_s} \right) \left(\frac{aN}{2\theta_sb} \right)^{N/(2\theta_s-N)} := \mathcal{G} \quad (4.6)$$

for $\gamma_1 \geq \max\{\gamma^*, a\}$, where γ^*, a are given in (3.31) and Lemma 3.10, respectively. Therefore, going to a subsequence if necessary, we may assume that $\mathbf{u}_n \rightharpoonup u$ in \mathcal{W}_ε , $\mathbf{u}_n \rightarrow u$ in $L_{loc}^q(\mathbb{R}^N)$ for every $q \in [\frac{N}{s}, +\infty)$ and $\mathbf{u}_n \rightarrow u$ a.e. in \mathbb{R}^N .

Claim 2. $(PS)_{c_\varepsilon}$ condition holds in \mathcal{W}_ε .

We shall divide the proof into three steps. We first verify that u is a critical point of \mathcal{I}_ε . For every $\varphi \in C_c^\infty(\mathbb{R}^N)$, we have

$$\begin{aligned} &\sum_{\wp \in \{p,q\}} \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{\wp-2} (\mathbf{u}_n(x) - \mathbf{u}_n(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+\wp s}} dx dy \\ &\rightarrow \sum_{\wp \in \{p,q\}} \iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{\wp-2} (u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x-y|^{N+\wp s}} dx dy \end{aligned}$$

and

$$\int_{\mathbb{R}^N} Z(\varepsilon x) |\mathbf{u}_n|^{\wp-2} \mathbf{u}_n \varphi dx \rightarrow \int_{\mathbb{R}^N} Z(\varepsilon x) |u|^{\wp-2} u \varphi dx \quad \text{for every } \wp \in \{p,q\}.$$

Step 1. We prove that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(\mathbf{u}_n(y)) g(\mathbf{u}_n(x)) \varphi(x)}{|x-y|^\mu} dy dx \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u(y)) g(u(x)) \varphi(x)}{|x-y|^\mu} dy dx. \quad (4.7)$$

Since the boundedness of $\{G(\varepsilon x, \mathbf{u}_n)\}_n$ in $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$, $\mathbf{u}_n \rightarrow u$ a.e. in \mathbb{R}^N , and $t \mapsto G(\cdot, t)$ is continuous, hence

$$G(\varepsilon x, \mathbf{u}_n) \rightharpoonup G(\varepsilon x, u) \text{ in } L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N).$$

From Lemma 3.2, we obtain the linear bounded operator

$$\frac{1}{|x|^\mu} * F \in L^{\frac{2N}{\mu}}(\mathbb{R}^N) \quad \text{for every } F \in L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N),$$

from $L^{\frac{2N}{2N-\mu}}(\mathbb{R}^N)$ to $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Therefore, $\frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n) \rightharpoonup \frac{1}{|x|^\mu} * G(\varepsilon y, u)$ in $L^{\frac{2N}{\mu}}(\mathbb{R}^N)$. Since $g(\mathbf{u}_n) \rightarrow g(u)$ in $L_{loc}^\nu(\mathbb{R}^N)$, for every $\nu \in [p, +\infty)$, we deduce that (4.7) is true. Consequently, in view of $\langle \mathcal{I}'_\varepsilon(\mathbf{u}_n), \phi \rangle = o_n(1)$, for every $\phi \in C_c^\infty(\mathbb{R}^N)$, we obtain $\langle \mathcal{I}'_\varepsilon(u), \phi \rangle = 0$, for every $\phi \in C_c^\infty(\mathbb{R}^N)$. Since $C_c^\infty(\mathbb{R}^N)$ is dense in \mathcal{W}_ε , we obtain that u is a critical point of \mathcal{I}_ε .

Step 2. We shall prove that for every ξ , there exists $R = R(\xi) > 0$ such that

$$\limsup_{n \rightarrow \infty} \int_{B_R^c} \left(\sum_{\varphi \in \{p, q\}} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^\varphi}{|x - y|^{N+\varphi s}} dy + Z(\varepsilon x)|\mathbf{u}_n|^p + Z(\varepsilon x)|\mathbf{u}_n|^q \right) dx < \xi. \quad (4.8)$$

For every $R > 0$, let $\phi_R \in C_c^\infty(\mathbb{R}^N)$ such that $0 \leq \phi_R \leq 1$, $\phi_R = 0$ in $B_R(0)$, $\phi_R = 1$ in $B_{2R}^c(0)$, and $|\nabla \phi_R| \leq \frac{C}{R}$ for some constant $C > 0$ independent of R . Since $\{\phi_R \mathbf{u}_n\}_n$ is bounded in $\mathcal{W}_\varepsilon(\mathbb{R}^N)$, it follows that $\langle \mathcal{I}'_\varepsilon(\mathbf{u}_n), \phi_R \mathbf{u}_n \rangle = o_n(1)$, i.e.,

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^p}{|x - y|^{N+sp}} \phi_R(x) dy dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^q}{|x - y|^{N+sq}} \phi_R(x) dy dx \\ & + \int_{\mathbb{R}^N} Z(\varepsilon x) |\mathbf{u}_n(x)|^p dx + \int_{\mathbb{R}^N} Z(\varepsilon x) |\mathbf{u}_n(x)|^q dx \\ & = o_n(1) + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(\mathbf{u}_n(y))g(\mathbf{u}_n(x))\phi_R(x)\mathbf{u}_n(x)}{|x - y|^\mu} dy dx \\ & - \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{p-2}(\mathbf{u}_n(x) - \mathbf{u}_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+sp}} \mathbf{u}_n(y) dx dy \\ & - \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{q-2}(\mathbf{u}_n(x) - \mathbf{u}_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+sq}} \mathbf{u}_n(y) dx dy. \end{aligned} \quad (4.9)$$

By Lemma 3.7, then there exists $\hbar_0 > 0$ such that

$$\frac{\sup_{\mathbf{u}_n \in W^{s,p}(\mathbb{R}^N)} \left| |x|^{-\mu} * G(\mathbf{u}_n) \right|}{\hbar_0} < \frac{1}{2}. \quad (4.10)$$

Let $R > 0$ be such that $\Omega_\varepsilon \subset B_R$. By the definition of ϕ_R , $(g_3) - (ii)$ and (4.10), we get

$$\begin{aligned} & \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^p}{|x - y|^{N+sp}} \phi_R(x) dy dx + \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^q}{|x - y|^{N+sq}} \phi_R(x) dy dx \\ & + \frac{1}{2} \int_{\mathbb{R}^N} Z(\varepsilon x) (|\mathbf{u}_n(x)|^p + |\mathbf{u}_n(x)|^q) dx \\ & \leq o_n - \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{p-2}(\mathbf{u}_n(x) - \mathbf{u}_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+sp}} \mathbf{u}_n(y) dx dy \\ & - \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{q-2}(\mathbf{u}_n(x) - \mathbf{u}_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+sq}} \mathbf{u}_n(y) dx dy. \end{aligned} \quad (4.11)$$

For $\varphi \in \{p, q\}$, by virtue of the Hölder inequality and the boundedness of $\{\mathbf{u}_n\}_n$ in \mathcal{W}_ε , we have

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^{\varphi-2}(\mathbf{u}_n(x) - \mathbf{u}_n(y))(\phi_R(x) - \phi_R(y))}{|x - y|^{N+\varphi s}} \mathbf{u}_n(y) dx dy \\ & \leq C \left(\iint_{\mathbb{R}^{2N}} \frac{|\phi_R(x) - \phi_R(y)|^\varphi}{|x - y|^{N+\varphi s}} |\mathbf{u}_n(y)|^\varphi dy \right)^{\frac{1}{\varphi}}. \end{aligned} \quad (4.12)$$

Next, by the definition of ϕ_R , polar coordinates and the boundedness of $\{\mathbf{u}_n\}_n$ in \mathcal{W}_ε , we obtain

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{|\phi_R(x) - \phi_R(y)|^\varphi}{|x - y|^{N+\varphi s}} |\mathbf{u}_n(x)|^\varphi dx dy \\ & = \int_{\mathbb{R}^N} \int_{|y-x|>R} \frac{|\phi_R(x) - \phi_R(y)|^\varphi}{|x - y|^{N+\varphi s}} |\mathbf{u}_n(x)|^\varphi dx dy + \int_{\mathbb{R}^N} \int_{|y-x|\leq R} \frac{|\phi_R(x) - \phi_R(y)|^\varphi}{|x - y|^{N+\varphi s}} |\mathbf{u}_n(x)|^\varphi dx dy \end{aligned}$$

$$\begin{aligned}
&\leq C \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi \left(\int_{|y-x|>R} \frac{dy}{|x-y|^{N+\varphi s}} \right) dx + \frac{C}{R^\varphi} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi \left(\int_{|y-x|\leq R} \frac{dy}{|x-y|^{N+\varphi s-\varphi}} \right) dx \\
&\leq C \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi \left(\int_{|z|>R} \frac{dz}{|z|^{N+\varphi s}} \right) dx + \frac{C}{R^\varphi} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi \left(\int_{|z|\leq R} \frac{dz}{|z|^{N+\varphi s-\varphi}} \right) dx \\
&\leq C \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi dx \left(\int_R^\infty \frac{d\rho}{\rho^{s\varphi+1}} \right) + \frac{C}{R^\varphi} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi dx \left(\int_0^R \frac{d\rho}{\rho^{s\varphi-\varphi+1}} \right) \\
&\leq \frac{C}{R^{s\varphi}} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi dx + \frac{C}{R^\varphi} R^{-s\varphi+\varphi} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi dx \leq \frac{C}{R^{s\varphi}} \int_{\mathbb{R}^N} |\mathbf{u}_n(x)|^\varphi dx \\
&\leq \frac{C}{R^{s\varphi}} \rightarrow 0 \text{ as } R \rightarrow \infty, \text{ where } \varphi \in \{p, q\}. \tag{4.13}
\end{aligned}$$

Gathering (4.11)-(4.13), we infer that (4.8) is satisfied.

Step 3. We verify that $\mathbf{u}_n \rightarrow u$ in \mathcal{W}_ε as $n \rightarrow \infty$.

In view of (4.8), we obtain $\mathbf{u}_n \rightarrow u$ in $L^\nu(\mathbb{R}^N)$, for every $\nu \in [p, +\infty)$. Indeed, fixed $\xi > 0$, there exists $R = R(\xi) > 0$ such that (4.8) holds. Using the compactness embedding $\mathcal{W}_\varepsilon \hookrightarrow L^{\nu}_{loc}(\mathbb{R}^N)$ and (\mathcal{Z}_1) , we see

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \|\mathbf{u}_n - u\|_{L^p(\mathbb{R}^N)}^p &= \limsup_{n \rightarrow \infty} \left[\|\mathbf{u}_n - u\|_{L^p(B_R(0))}^p + \|\mathbf{u}_n - u\|_{L^p(B_R^c(0))}^p \right] \\
&\leq 2^{p-1} \limsup_{n \rightarrow \infty} \left(\|\mathbf{u}_n\|_{L^p(B_R^c(0))}^p + \|u\|_{L^p(B_R^c(0))}^p \right) \\
&\leq \frac{2^{p-1}}{Z_0} \limsup_{n \rightarrow \infty} \int_{B_R^c(0)} \left(\int_{\mathbb{R}^N} \frac{|\mathbf{u}_n(x) - \mathbf{u}_n(y)|^p}{|x-y|^{2N}} dy + Z(\varepsilon x) |\mathbf{u}_n|^p \right) dx \\
&\quad + \frac{2^{p-1}}{Z_0} \int_{B_R^c(0)} \left(\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x-y|^{2N}} dy + Z(\varepsilon x) |u|^p \right) dx < \frac{2^p}{Z_0} \xi.
\end{aligned}$$

Due to the arbitrariness of ξ , $\mathbf{u}_n \rightarrow u$ in $L^p(\mathbb{R}^N)$. By interpolation, $\mathbf{u}_n \rightarrow u$ in $L^\nu(\mathbb{R}^N)$ for every $\nu \in [p, +\infty)$, as desired. Arguing similarly as in the proof of [43, Lemma 22], we can obtain that

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(\mathbf{u}_n(y))g(\mathbf{u}_n(x))}{|x-y|^\mu} dy dx \rightarrow \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{G(u(y))g(u(x))}{|x-y|^\mu} dy dx. \tag{4.14}$$

Therefore, it follows from $\langle \mathcal{I}'_\varepsilon(\mathbf{u}_n), \mathbf{u}_n \rangle = o_n(1)$ and $\langle \mathcal{I}'_\varepsilon(u), u \rangle = o_n(1)$ that

$$\|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,p}}^p + \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,q}}^q = \|u\|_{W_{Z,\varepsilon}^{s,p}}^p + \|u\|_{W_{Z,\varepsilon}^{s,q}}^q + o_n(1).$$

By the Brézis-Lieb lemma, we get that

$$\|\mathbf{u}_n - u\|_{\mathcal{W}_\varepsilon}^\varphi = \|\mathbf{u}_n\|_{W_{Z,\varepsilon}^{s,\varphi}}^\varphi - \|u\|_{W_{Z,\varepsilon}^{s,\varphi}}^\varphi + o_n(1) \text{ for every } \varphi \in \{p, q\}.$$

Therefore

$$\|\mathbf{u}_n - u\|_{W_{Z,\varepsilon}^{s,p}}^p + \|\mathbf{u}_n - u\|_{W_{Z,\varepsilon}^{s,q}}^q = o_n(1).$$

Moreover, we obtain $\mathbf{u}_n \rightarrow u$ in \mathcal{W}_ε . This completes the proof of Lemma 4.7. \square

Lemma 4.8. (see Ambrosio [5, Corollary 3.1]) *The functional Φ_ε satisfies the $(PS)_{c_\varepsilon}$ condition at level c_ε , for every $\varepsilon \in (0, \varepsilon_0)$ on \mathbb{S}_ε .*

5. MULTIPLICITY AND CONCENTRATION OF POSITIVE SOLUTIONS OF PROBLEM (\mathcal{Q})

In this section, we shall prove the main results. To this end, we shall give some notations and useful results which will be used later. Fix $\delta > 0$. Let \mathbf{w} be a ground state solution of equation (\mathcal{Q}_{Z_0}) , so that $\mathcal{E}_{Z_0}(\mathbf{w}) = c_{Z_0}$ and $\mathcal{E}'_{Z_0}(\mathbf{w}) = 0$. Let η be a smooth nonincreasing cut-off function in \mathbb{R}_0^+ such that $\eta(t) = 1$ if $0 \leq t \leq \delta/2$ and $\eta(t) = 0$ if $t \geq \delta$. For $\varepsilon > 0$ and any $y \in \mathcal{M}$, we define

$$\psi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|) \mathbf{w} \left(\frac{\varepsilon x - y}{\varepsilon} \right), \quad x \in \mathbb{R}^N$$

and $\Phi_\varepsilon : \mathcal{M} \rightarrow \mathcal{N}_\varepsilon$ is given by $\Phi_\varepsilon(y) = t_\varepsilon \psi_{\varepsilon,y}$, when $t_\varepsilon > 0$ satisfies

$$\max_{t \geq 0} \mathcal{I}_\varepsilon(t\psi_{\varepsilon,y}) = \mathcal{I}_\varepsilon(t_\varepsilon \psi_{\varepsilon,y}).$$

We obtain that $\Phi_\varepsilon(y)$ has compact support in \mathbb{R}^N for every $y \in \mathcal{M}$.

Lemma 5.1. (see Liang et al. [24, Lemma 5.1]) *The function Φ_ε has the following property*

$$\lim_{\varepsilon \rightarrow 0^+} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = c_{Z_0}, \quad \text{uniformly in } y \in \mathcal{M}.$$

For any $\delta > 0$, let $\varrho = \varrho(\delta) > 0$ be such that $\mathcal{M}_\delta \subset B_\varrho(0)$. We define the function $\mathcal{X} : \mathbb{R}^N \rightarrow \mathbb{R}^N$ by

$$\mathcal{X}(x) = \begin{cases} x, & \text{if } |x| < \varrho, \\ \frac{\varrho x}{|x|}, & \text{if } |x| \geq \varrho. \end{cases}$$

In what follows, let the barycenter map $\beta_\varepsilon : \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^N$ be defined by

$$\beta_\varepsilon(u) = \frac{\int_{\mathbb{R}^N} \mathcal{X}(\varepsilon x) (|u(x)|^p + |u(x)|^q) dx}{\int_{\mathbb{R}^N} (|u|^p + |u|^q) dx}, \quad u \in \mathcal{N}_\varepsilon.$$

Arguing as in the similar discussion of [35, Lemma 14], we obtain the following result.

Lemma 5.2. *The map $\beta_\varepsilon \circ \Phi_\varepsilon$ satisfies the following limit*

$$\lim_{\varepsilon \rightarrow 0^+} \beta_\varepsilon(\Phi_\varepsilon(y)) = y, \quad \text{uniformly in } y \in \mathcal{M}. \quad (5.1)$$

Lemma 5.3. *Let $\varepsilon_n \rightarrow 0^+$ and $\{\mathbf{u}_n\}_n \subset \mathcal{N}_{\varepsilon_n}$ satisfy $\mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) \rightarrow c_{V_0}$, as $n \rightarrow \infty$. Then there exists a sequence $\{\tilde{y}_n\}_n \subset \mathbb{R}^N$ such that the sequence $\mathbf{v}_n = \mathbf{u}_n(\cdot + \tilde{y}_n)$ has a subsequence which strongly converges in \mathcal{W} . Furthermore, up to a subsequence, $y_n = \varepsilon \tilde{y}_n \rightarrow y \in \mathcal{M}$.*

Proof. Since $\langle \mathcal{I}'_{\varepsilon_n}(\mathbf{u}_n), \mathbf{u}_n \rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) \rightarrow c_{Z_0}$, we can see that $\{\mathbf{u}_n\}_n$ is a bounded sequence in \mathcal{W} . Indeed, by $(g_3) - (ii)$, we have

$$\begin{aligned} \mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) &= \mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) - \frac{1}{2q} \langle \mathcal{I}'_{\varepsilon_n}(\mathbf{u}_n), \mathbf{u}_n \rangle = \left(\frac{1}{p} - \frac{1}{2q} \right) \|\mathbf{u}_n\|_{Z, W_\varepsilon^{s,p}(\mathbb{R}^N)}^p + \frac{1}{2q} \|\mathbf{u}_n\|_{Z, W_\varepsilon^{s,q}(\mathbb{R}^N)}^q \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n(y)) \right] \left(\frac{1}{q} g(\varepsilon x, \mathbf{u}_n(x)) \mathbf{u}_n(x) - G(\varepsilon x, \mathbf{u}_n(x)) \right) dx \geq \frac{1}{2q} \|\mathbf{u}_n\|_{\mathcal{W}_{\varepsilon_n}}. \end{aligned}$$

Therefore

$$\limsup_{n \rightarrow \infty} \|\mathbf{u}_n\|_{\mathcal{W}_\varepsilon} \leq 2qc_{Z_0}. \quad (5.2)$$

By conditions (\mathcal{Z}_1) and (\mathcal{Z}_2) , we obtain that

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} \leq \min\{1, V_0\}^{\frac{1}{p}} \|u\|_{\mathcal{W}_\varepsilon}. \quad (5.3)$$

Together with the continuous embedding $\mathcal{W}_{\varepsilon_n} \hookrightarrow W^{s,p}(\mathbb{R}^N)$, we get that $\{\mathbf{u}_n\}_n$ is bounded in \mathcal{W} .

Next, we claim that there exist a sequence $\{\tilde{y}_n\}_n \subset \mathbb{R}^N$ and constants $R, \delta > 0$ such that

$$\liminf_{n \rightarrow +\infty} \int_{B_R(\tilde{y}_n)} |u_n|^q dx \geq \delta > 0. \quad (5.4)$$

Suppose to the contrary, that for every $R > 0$, we deduce that

$$\lim_{n \rightarrow +\infty} \sup_{y \in \mathbb{R}^N} \int_{B_R(y)} |u_n|^q dx = 0.$$

Together with Lemma 3.8, we obtain that $\mathbf{u}_n \rightarrow 0$ in $L^\nu(\mathbb{R}^N)$, for every $\nu \in (p, +\infty)$. Using Lemma 3.1 and (5.2), we get

$$\lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n(y)) \right] G(\varepsilon x, \mathbf{u}_n(x)) dx = 0.$$

Together with $\mathbf{u}_n \in \mathcal{N}_{\varepsilon_n}$, we have $\mathbf{u}_n \rightarrow 0$ in \mathcal{W}_ε . Hence $\mathcal{I}_\varepsilon(\mathbf{u}_n) \rightarrow 0$, which is impossible due to $c_{Z_0} > 0$. We now suppose $v_n(x) = \mathbf{u}_n(x + \tilde{y}_n)$, so $\{v_n\}_n$ is bounded in \mathcal{W}_ε . Therefore, we may assume that $v_n \rightharpoonup v$ in \mathcal{W} , as $n \rightarrow \infty$. It follows from (5.4) that $v \neq 0$.

Let $t_n > 0$ be such that $\tilde{v}_n := t_n v_n \in \mathcal{N}_{Z_0}$ and $y_n := \varepsilon_n \tilde{y}_n$. Applying Lemma 4.4, for every n there exists a unique $t_{\mathbf{u}_n} > 0$ such that $\mathcal{I}_{\varepsilon_n}(t_{\mathbf{u}_n} \mathbf{u}_n) = \sup_{t \geq 0} \mathcal{I}_{\varepsilon_n}(t \mathbf{u}_n)$. Then $t_{\mathbf{u}_n} \mathbf{u}_n \in \mathcal{N}_{\varepsilon_n}$, which yields $t_{\mathbf{u}_n} = 1$, due to $\mathbf{u}_n \in \mathcal{N}_{\varepsilon_n}$. Therefore, $\sup_{t \geq 0} \mathcal{I}_{\varepsilon_n}(t \mathbf{u}_n) = \mathcal{I}_{\varepsilon_n}(\mathbf{u}_n)$. By the change of variable $z = x + \tilde{y}_n$, we deduce that

$$\begin{aligned} c_{Z_0} \leq \mathcal{E}_{Z_0}(\tilde{v}_n) &= \frac{1}{p} \|\tilde{v}_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q - \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\tilde{v}_n(y)) \right] F(\tilde{v}_n(x)) dx \\ &\leq \frac{1}{p} \|\tilde{v}_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \\ &\quad - \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon_n y + y_n, \tilde{v}_n(y)) \right] G(\varepsilon_n x + y_n, v_n(x)) dx \\ &= \mathcal{I}_{\varepsilon_n}(t_n \mathbf{u}_n) \leq \mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) \leq c_{Z_0} + o_n(1) \end{aligned}$$

which yields $\mathcal{E}_{Z_0}(\tilde{v}_n) \rightarrow c_{Z_0}$, as $n \rightarrow +\infty$. By the fact that $\{\tilde{v}_n\}_n \subset \mathcal{N}_{Z_0}$ and (f_3) , we can pick $C_1 > 0$ such that $\|\tilde{v}_n\|_{Z_0} \leq C_1$, for every $n \in \mathbb{N}$. In addition, since $v_n \not\rightarrow 0$ in \mathcal{W} , there exists $\tilde{C}_1 > 0$ such that $\|v_n\|_{Z_0} \geq \tilde{C}_1 > 0$, for every $n \in \mathbb{N}$. Therefore,

$$\tilde{C}_1 t_n \leq \|t_n v_n\|_{Z_0, \mathcal{W}} = \|\tilde{v}_n\|_{Z_0, \mathcal{W}} \leq C_1$$

which yields $t_n \leq \frac{C_1}{\tilde{C}_1}$, for every $n \in \mathbb{N}$. Consequently, going to a subsequence if necessary, we suppose that $t_n \rightarrow t_0 \geq 0$ and $\tilde{v}_n \rightharpoonup \tilde{v} := t_0 v \neq 0$ in \mathcal{W} and $\tilde{v}_n \rightarrow \tilde{v}$ a.e. in \mathbb{R}^N . If $t_0 = 0$, then $\tilde{v}_n \rightarrow 0$ in \mathcal{W} . Therefore $\mathcal{E}_{Z_0}(\tilde{v}) \rightarrow 0$, which is impossible since $c_{Z_0} > 0$, so we get that $t_0 > 0$. Arguing as Proposition 3.1, we obtain $\mathcal{E}'_{Z_0}(\tilde{v}) = 0$.

In the sequel, we shall prove that

$$\lim_{n \rightarrow +\infty} \|\tilde{v}_n\|_{Z_0, \mathcal{W}} = \|\tilde{v}\|_{Z_0, \mathcal{W}}. \quad (5.5)$$

Invoking the Fatou lemma, we can deduce

$$\|\tilde{v}\|_{Z_0, \mathcal{W}} \leq \liminf_{n \rightarrow +\infty} \|\tilde{v}_n\|_{Z_0, \mathcal{W}}. \quad (5.6)$$

Assume to the contrary, that

$$\|\tilde{v}\|_{Z_0, \mathcal{W}} < \limsup_{n \rightarrow \infty} \|\tilde{v}_n\|_{Z_0, \mathcal{W}}.$$

In such a case we would get

$$\begin{aligned} c_{Z_0} + o_n(1) &= \mathcal{E}_{Z_0}(\tilde{v}_n) - \frac{1}{2q} \langle \mathcal{E}'_{Z_0}(\tilde{v}_n), \tilde{v}_n \rangle \\ &= \left(\frac{1}{p} - \frac{1}{2q}\right) \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q + \frac{1}{2q} \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\tilde{v}_n(y)) \right] \left(\frac{1}{q} f(\tilde{v}_n) \tilde{v}_n - F(\tilde{v}_n) \right) dx \end{aligned}$$

and, by (f_3) and the Fatou lemma, we would have

$$\begin{aligned} c_{Z_0} &\geq \limsup_{n \rightarrow +\infty} \left[\left(\frac{1}{p} - \frac{1}{2q}\right) \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q + \frac{1}{2q} \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \right] \\ &\quad + \frac{1}{2} \liminf_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\tilde{v}_n(y)) \right] \left(\frac{1}{q} f(\tilde{v}_n) \tilde{v}_n - F(\tilde{v}_n) \right) dx \\ &> \left(\frac{1}{p} - \frac{1}{2q}\right) \|\tilde{v}\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q + \frac{1}{2q} \|\tilde{v}\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * F(\tilde{v}(y)) \right] \left(\frac{1}{q} f(\tilde{v}) \tilde{v} - F(\tilde{v}) \right) dx \\ &= \mathcal{E}_{Z_0}(\tilde{v}) - \frac{1}{2q} \mathcal{E}'_{Z_0}(\tilde{v})(\tilde{v}) = \mathcal{E}_{Z_0}(\tilde{v}) \geq c_{Z_0} \end{aligned}$$

which is a contradiction. Therefore, $w_n \rightharpoonup w$ in \mathcal{W} and (5.5) implies $\tilde{v}_n \rightarrow \tilde{v}$ in \mathcal{W} . Moreover, $v_n \rightarrow v$ in \mathcal{W} , as $n \rightarrow +\infty$.

In order to complete the proof of this lemma, we explore $y_n = \varepsilon_n y_n$. We claim that $\{y_n\}_n$ allows a subsequence, still denoted the same, such that $y_n \rightarrow y_0$, for some $y_0 \in \mathcal{M}$. In the sequel, we have to verify that the following two claims hold.

Claim 1. $\{y_n\}_n$ is bounded.

We shall argue by contradiction. Assume that, up to a subsequence, $|y_n| \rightarrow \infty$, as $n \rightarrow \infty$. Since $\langle \mathcal{I}'_{\varepsilon_n}(\mathbf{u}_n), \mathbf{u}_n \rangle = 0$ and $\mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) \rightarrow c_{V_0}$, by Lemma 3.7, we can infer that there exists $\hat{C}_0 \in (0, \frac{h_0}{2})$ such that

$$\left| \frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n) \right| < \hat{C}_0.$$

Fixed $R > 0$ such that $\Lambda \subset B_R(0)$, and assume that $|y_n| > 2R$. Therefore,

$$|\varepsilon_n x + y_n| \geq |y_n| - |\varepsilon_n x| > R \text{ for every } x \in B_{\frac{R}{\varepsilon_n}}(0). \quad (5.7)$$

Note that $\mathbf{u}_n \in \mathcal{N}_{\varepsilon_n}$, so we have

$$\|\mathbf{u}_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \|\mathbf{u}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \leq \|\mathbf{u}_n\|_{W_{Z, \varepsilon_n}^{s,p}}^p + \|\mathbf{u}_n\|_{W_{Z, \varepsilon_n}^{s,q}}^q = \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n) \right] g(\varepsilon_n x, \mathbf{u}_n) \mathbf{u}_n dx.$$

Using the change of variables $x \mapsto x + \tilde{y}_n$ and $y \mapsto y + \tilde{y}_n$, we get

$$\begin{aligned} \|v_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \|v_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q &= \|\mathbf{u}_n\|_{W_{Z, \varepsilon_n}^{s,p}(\mathbb{R}^N)}^p + \|\mathbf{u}_n\|_{W_{Z, \varepsilon_n}^{s,q}(\mathbb{R}^N)}^q \\ &\leq \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon y, \mathbf{u}_n) \right] g(\varepsilon_n x, \mathbf{u}_n) \mathbf{u}_n dx \end{aligned}$$

$$= \int_{\mathbb{R}^N} \left[\frac{1}{|x|^\mu} * G(\varepsilon y + y_n, v_n) \right] g(\varepsilon_n x + y_n, v_n) v_n dx.$$

By (5.7), the definition of g , $f(t) \leq \frac{Z_0}{h_0} t^{p-1}$, $\hat{C}_0 \in (0, \frac{h_0}{2})$, $v_n \rightarrow v$ in \mathcal{W}_ε , the Dominated Convergence Theorem, we have that

$$\begin{aligned} \|v_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \|v_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q &\leq \hat{C}_0 \int_{\mathbb{R}^N} g(\varepsilon_n x + y_n, v_n) v_n dx \\ &\leq \hat{C}_0 \int_{B_{\frac{R}{\varepsilon_n}}(0)} \hat{f}(v_n) v_n dx + \hat{C}_0 \int_{B_{\frac{R}{\varepsilon_n}}^c(0)} f(v_n) v_n dx \\ &\leq \frac{1}{2} \int_{B_{\frac{R}{\varepsilon_n}}(0)} Z_0 (|v_n|^p + |v_n|^q) dx + o_n(1) \end{aligned}$$

which gives

$$\|v_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \|v_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q = o_n(1).$$

Therefore, we have that $\{y_n\}_n$ is bounded in \mathbb{R}^N .

Claim 2. $y_0 \in \mathcal{M}$.

By Claim 1, up to a subsequence, we can suppose that $y_n \rightarrow y_0$. Once $y_0 \notin \bar{\Omega}$, which implies the closure of Ω , we can argue as above to get $v_n \rightarrow 0$ in \mathcal{W}_ε , which is impossible. Therefore, we obtain $y_0 \in \bar{\Omega}$. Now, suppose by contradiction that $Z(y_0) > Z_0$, then by using $\tilde{v}_n \rightarrow v$ in \mathcal{W} and the Fatou lemma, we can deduce that

$$\begin{aligned} c_{Z_0} = \mathcal{E}_{Z_0}(\tilde{v}) &< \liminf \left[\frac{1}{p} \|\tilde{v}_n\|_{Z_0, W^{s,p}(\mathbb{R}^N)}^p + \frac{1}{q} \|\tilde{v}_n\|_{Z_0, W^{s,q}(\mathbb{R}^N)}^q \right. \\ &\quad \left. - \frac{1}{2} \int_{\mathbb{R}^N} \left(\frac{1}{|x|^\mu} * F(\tilde{v}_n) \right) F(\tilde{v}_n) dx \right] \\ &\leq \liminf_{n \rightarrow \infty} \mathcal{I}_{\varepsilon_n}(t_n \mathbf{u}_n) \leq \liminf_{n \rightarrow \infty} \mathcal{I}_{\varepsilon_n}(\mathbf{u}_n) = c_{Z_0} \end{aligned}$$

which is impossible. Therefore, $Z(y_0) = Z_0$ and $y_0 \in \bar{\Omega}$. Thanks to (Z_2) , $y_0 \notin \partial\Omega$, and thus $y_0 \in \mathcal{M}$. This completes the proof of Lemma 5.3. \square

Let $h(\varepsilon)$ be any positive function satisfying $h(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$. Define

$$\tilde{\mathcal{N}}_\varepsilon = \{u \in \mathcal{N}_\varepsilon : \mathcal{I}_\varepsilon(u) \leq c_{Z_0} + h(\varepsilon)\}.$$

For any $y \in \mathcal{M}$, we deduce from Lemma 5.1 that $h(\varepsilon) = |\mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) - c_{Z_0}| \rightarrow 0$, as $\varepsilon \rightarrow 0$. Thus, $\Phi_\varepsilon(y) \in \tilde{\mathcal{N}}_\varepsilon$ and $\tilde{\mathcal{N}}_\varepsilon \neq \emptyset$ for every $\varepsilon > 0$.

Lemma 5.4. (see Thin [35, Lemma 16]) For every $\delta > 0$,

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{u \in \tilde{\mathcal{N}}_\varepsilon} \text{dist}(\beta_\varepsilon(u), \mathcal{M}_\delta) = 0.$$

Lemma 5.5. Suppose that conditions $(Z_1) - (Z_2)$ and $(f_1) - (f_5)$ hold and denote by \mathbf{v}_n a nontrivial nonnegative solution in \mathbb{R}^N of

$$(-\Delta)_{N/s}^s \mathbf{v}_n + (-\Delta)_q^s \mathbf{v}_n + Z_n(x) (|\mathbf{v}_n|^{\frac{N}{s}-2} \mathbf{v}_n + |\mathbf{v}_n|^{q-2} \mathbf{v}_n) = [|x|^{-\mu} * F(\mathbf{v}_n)] g(\varepsilon_n x + \varepsilon_n \tilde{y}_n, \mathbf{v}_n), \quad (5.8)$$

where $Z_n(x) = Z(\varepsilon_n x + \varepsilon_n \tilde{y}_n)$ and $\varepsilon_n \tilde{y}_n \rightarrow y \in \mathcal{M}$. Then, if $(\mathbf{v}_n)_n$ is a bounded sequence in \mathcal{W} satisfying

$$\limsup_{n \rightarrow \infty} \|u_n\|_{\vartheta, \mathcal{W}^{s,p}(\mathbb{R}^N)}^{N/(N-s)} < \frac{\beta_* \mathfrak{d}_*^{s/(N-s)}}{\mathfrak{c} \alpha_0}, \quad \text{with } 0 < \beta_* < \alpha_*$$

for a suitable constant $\mathfrak{c} > 1$ and if $\mathbf{v}_n \rightarrow \mathbf{v}$ in \mathcal{W} , then each $\mathbf{v}_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|\mathbf{v}_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for every n . Moreover,

$$\lim_{|x| \rightarrow \infty} \mathbf{v}_n(x) = 0, \quad \text{uniformly in } n.$$

Proof. In view of $\mathcal{I}_{\varepsilon_n}(u_n) \leq c_{Z_0} + h(\varepsilon)$, with $h(\varepsilon) \rightarrow 0$, as $n \rightarrow \infty$. We argue as in the proof of Lemma 5.4 to show that $\mathcal{I}(\varepsilon_n)(u_n) \rightarrow c_{Z_0}$. Then by Lemma 5.3, there exists $\{\tilde{y}_n\} \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + \tilde{y}_n)$ strongly converges in \mathcal{W} and $\varepsilon_n \tilde{y}_n \rightarrow y_0 \in \mathcal{M}$. By the boundedness of $\{v_n\}_n$ in \mathcal{W} , we can proceed as in the proof of Lemma 3.7 to obtain that there exists $\hat{C}_0 > 0$ such that

$$\frac{1}{|x|^\mu} * G(\varepsilon_n x + \varepsilon_n \tilde{y}_n, v_n) \leq \hat{C}_0.$$

Repeating the same Moser iteration argument developed in the proof of Liang et al. [24, Lemma 5.5], we have that $\|\mathbf{v}_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for every $n \in \mathbb{N}$. Now, we note that v_n satisfies problem (5.8).

Using Ambrosio and Rădulescu [7, Corollary 2.1] and the fact that \mathbf{v}_n is uniformly bounded in $L^\infty(\mathbb{R}^N) \cap \mathcal{W}$, we can conclude that $\mathbf{v}_n(x) \rightarrow 0$ as $|x| \rightarrow \infty$ uniformly in $n \in \mathbb{N}$. This completes the proof of Lemma 5.5. \square

Proof of Theorem 1.1. Using the similar arguments to the proof of Ambrosio [3, Theorem 5.2] and [2, Theorem 1.1]. We define $\alpha_\varepsilon : \mathcal{M} \rightarrow \mathbb{S}_\varepsilon$ by setting $\alpha_\varepsilon(y) = m_\varepsilon^{-1}(\Phi_\varepsilon(y))$ for every $\varepsilon > 0$. Applying Lemma 5.1 and the definition of Φ_ε , we obtain that

$$\lim_{\varepsilon \rightarrow 0} \psi_\varepsilon(\alpha_\varepsilon(y)) = \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon(\Phi_\varepsilon(y)) = c_{Z_0}, \quad \text{uniformly in } y \in \mathcal{M}.$$

Therefore, there exists $\tilde{\varepsilon} > 0$ such that $\tilde{\mathbb{S}}_\varepsilon := \{w \in \mathbb{S}_\varepsilon : \psi_\varepsilon(w) \leq c_{Z_0} + h(\varepsilon)\} \neq \emptyset$, for every $\varepsilon \in (0, \tilde{\varepsilon})$. With the aid of Lemma 4.4-(iii), Lemma 5.1 and Lemma 5.4, for every $\delta > 0$, there exists $\tilde{\varepsilon} = \tilde{\varepsilon}_\delta > 0$ such that the diagram of continuous mappings

$$\mathcal{M} \xrightarrow{\Phi_{\tilde{\varepsilon}}} \tilde{\mathcal{N}}_{\tilde{\varepsilon}} \xrightarrow{m_{\tilde{\varepsilon}}^{-1}} \mathbb{S}_{\tilde{\varepsilon}} \xrightarrow{m_{\tilde{\varepsilon}}} \tilde{\mathcal{N}}_{\tilde{\varepsilon}} \xrightarrow{\beta_{\tilde{\varepsilon}}} \mathcal{M}_\delta = \tilde{\varepsilon}_\delta > 0$$

is well-defined, for every $\varepsilon \in (0, \tilde{\varepsilon})$. Invoke Lemma 5.2 and take a function $\varpi(\varepsilon, y)$ with $|\varpi(\varepsilon, y)| < \frac{\delta}{2}$ uniformly in $y \in \mathcal{M}$, for every $\varepsilon \in (0, \tilde{\varepsilon})$ such that $\beta_\varepsilon(\Phi_\varepsilon(y)) = y + \varpi(\varepsilon, y)$, for every $y \in \mathcal{M}$. Therefore, we obtain that $\mathcal{F}(t, y) = y + (1-t)\varpi(\varepsilon, y)$ with $(t, y) \in [0, 1] \times \mathcal{M}$ is a homotopy between $\beta_\varepsilon \circ \Phi_\varepsilon = (\beta_\varepsilon \circ m_\varepsilon) \circ \alpha_\varepsilon$ and the inclusion map $\text{id}: \mathcal{M} \rightarrow \mathcal{M}_\delta$. Together with [4, Lemma 6.3.21], we obtain that

$$\text{cat}_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon) \geq \text{cat}_{\mathcal{M}_\delta}(\mathcal{M}). \quad (5.9)$$

In what follows, we choose a function $h(\varepsilon) > 0$ such that $h(\varepsilon) \rightarrow 0$, as $\varepsilon \rightarrow 0$ and such that $c_{Z_0} + h(\varepsilon)$ is not a critical level for \mathcal{I}_ε . From Lemma 4.8, we see that \mathcal{I}_ε satisfies the Palais-Smale condition in $\tilde{\mathbb{S}}_\varepsilon$ as $\varepsilon > 0$. Invoking Ambrosio [4, Theorem 6.3.20], we get that ψ_ε has at least $\text{cat}_{\tilde{\mathbb{S}}_\varepsilon}(\tilde{\mathbb{S}}_\varepsilon)$ critical points on $\tilde{\mathbb{S}}_\varepsilon$. Consequently, by Lemma 4.5 and (5.9), we deduce that \mathcal{I}_ε has at least $\text{cat}_{\mathcal{M}_\delta}(\mathcal{M})$ critical points.

Let u_{ε_n} be a solution of problem $(\mathcal{Q}_{\varepsilon_n})$, then $\mathbf{v}_n(x) = u_{\varepsilon_n}(x + \tilde{y}_n)$ is also a solution of problem (5.8). Moreover, there exists $\mathbf{v} \in \mathcal{W}$, such that, up to a subsequence, $\mathbf{v}_n \rightarrow \mathbf{v}$ in \mathcal{W} and $y_n = \varepsilon_n \tilde{y}_n \rightarrow y \in \mathcal{M}$ by Lemma 5.3.

We claim that there exists $\bar{\delta} > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geq \bar{\delta}$, for every n large enough. In fact, (5.4) in the proof of Lemma 5.3 implies

$$0 < \frac{\beta}{2} \leq \int_{B_r(0)} |\mathbf{v}_n|^{N/s} dx \leq |B_r(0)| \|\mathbf{v}_n\|_{L^\infty(\mathbb{R}^N)}^{N/s} \quad (5.10)$$

for every n large enough. Therefore, we choose $\bar{\delta} = \left(\frac{\beta}{2|B_r(0)|}\right)^{s/N}$. Applying the fact that $\mathbf{v}_n \rightarrow \mathbf{v}$ in \mathcal{W} , we have $\lim_{|x| \rightarrow \infty} \mathbf{v}_n(x) = 0$, uniformly in n by Lemma 5.5.

Let \mathbf{q}_n be a global maximum point of \mathbf{v}_n . Invoking Lemma 5.5, we see that there exists $R > 0$ such that $|\mathbf{q}_n| \leq R$, for every n . Consequently, the maximum point of u_{ε_n} is denoted by $\mathfrak{z}_{\varepsilon_n} = \mathbf{q}_n + \tilde{y}_n$. Furthermore, problem (Q) possesses a nontrivial nonnegative solution $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$. Therefore, the maximum points ζ_ε of w_ε and \mathfrak{z}_ε of u_ε satisfy $\zeta_\varepsilon = \varepsilon \mathfrak{z}_\varepsilon$. We see that

$$\lim_{\varepsilon \rightarrow 0^+} Z(\zeta_\varepsilon) = \lim_{n \rightarrow \infty} Z(\varepsilon_n \mathfrak{z}_{\varepsilon_n}) = Z_0.$$

This completes the proof of Theorem 1.1. \square

Proof of Theorem 1.2. We know that $w_\varepsilon(x) = u_\varepsilon(x/\varepsilon)$ is a nontrivial nonnegative solution of problem (Q). Set

$$\mathbf{v}_{\varepsilon_n} = w_{\varepsilon_n}(\varepsilon_n \cdot + \eta_{\varepsilon_n}) = u_{\varepsilon_n}(\cdot + \mathfrak{z}_{\varepsilon_n}).$$

Therefore, Lemma 5.3 yields that $(\mathbf{v}_{\varepsilon_n})_n \rightarrow \mathbf{v}$ in \mathcal{W} and \mathbf{v} is a ground state solution of the following equation

$$(-\Delta)_p^s u + (-\Delta)_q^s u + Z_0(|u|^{p-2}v + |u|^{q-2}u) = \left[|x|^{-\mu} * F(u(y))\right] f(u) \text{ in } \mathbb{R}^N.$$

This completes the proof of Theorem 1.2. \square

Proofs of Corollaries 1.3 and 1.4. Apply a similar discussion as in Liang et al. [24]. \square

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¹ COLLEGE OF MATHEMATICS, CHANGCHUN NORMAL UNIVERSITY, CHANGCHUN, 130032, P.R. CHINA
Email address: Yueqiang Song: `songyq16@mails.jlu.edu.cn`
Email address: Xueqi Sun: `sunxueqi1@126.com`

² FACULTY OF EDUCATION, UNIVERSITY OF LJUBLJANA, LJUBLJANA, 1000, SLOVENIA

³ FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, LJUBLJANA, 1000, SLOVENIA

⁴ INSTITUTE OF MATHEMATICS, PHYSICS AND MECHANICS, LJUBLJANA, 1000, SLOVENIA
Email address: Dušan D. Repovš: `dusan.repovs@guest.arnes.si`