ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.)
ARS MATHEMATICA CONTEMPORANEA 25 (2025) #P3.07
https://doi.org/10.26493/1855-3974.3344.4c8
(Also available at http://amc-journal.eu)

Mutual-visibility problems in Kneser and Johnson graphs*

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Received 22 March 2024, accepted 16 July 2024, published online 6 June 2025

Abstract

Let G be a connected graph and $\mathcal{X} \subseteq V(G)$. By definition, two vertices u and v are $\mathcal{X}$-visible in G if there exists a shortest u,v-path with all internal vertices being outside of the set $\mathcal{X}$. The largest size of $\mathcal{X}$ such that any two vertices of G (resp. any two vertices from $\mathcal{X}$) are $\mathcal{X}$-visible is the total mutual-visibility number (resp. the mutual-visibility number) of G.

In this paper, we determine the total mutual-visibility number of Kneser graphs, bipartite Kneser graphs, and Johnson graphs. The formulas proved for Kneser, and bipartite Kneser graphs are related to the size of transversal-critical uniform hypergraphs, while the total mutual-visibility number of Johnson graphs is equal to a hypergraph Turán number. Exact values or estimations for the mutual-visibility number over these graph classes are also established.

Keywords: Mutual-visibility set, total mutual-visibility set, Kneser graph, bipartite Kneser graph, Johnson graph, Turán-type problem, covering design.

Math. Subj. Class. (2020): 05C12, 05C38, 05C65, 05C76

*This research was initiated with the visit of Csilla Bujtás to Ege University in Izmir, Turkey; the authors thank the financial support of TÜBITAK under the grant BIDEB 2221 (1059B212300041).

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‡The research of Csilla Bujtás was partially supported by the Slovenian Research and Innovation Agency (ARIS) under the grants N1-0355 and P1-0297.

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1 Introduction

Several studies have been conducted to address network visibility problems, both from theoretical and practical points of view. From the practical perspective, many papers focus on robot navigation problems (see, e.g., [1, 5, 12, 24]). From the theoretical perspective, the mutual-visibility notion in graphs was proposed by Di Stefano [13] and then investigated in several graph classes [2, 6, 7, 9, 10, 19]. As a natural extension, Cicerone *et al.* [5] proposed the notion of total mutual-visibility to provide a "visibility" property to all the network nodes instead of only the nodes where robots are located. Total mutual-visibility sets have been investigated recently in product graphs [20, 26] and Hamming graphs [4]. It is also shown that both the mutual-visibility number and the total mutual-visibility number are NP-complete to compute in general [8, 13]. Very recently, some new variants of the mutual-visibility notion were defined and discussed for different graph classes [3, 8, 11].

For a connected graph G with vertex set $V(G)$ and edge set $E(G)$, let $\mathcal{X}$ be a subset of $V(G)$. Two vertices u and v from $V(G)$ are *$\mathcal{X}$-visible* if there exists a shortest u,v-path such that no internal vertex belongs to $\mathcal{X}$. That is, u and v are $\mathcal{X}$-visible if there exists a shortest u,v-path P in G such that $V(P) \cap \mathcal{X} \subseteq \{u, v\}$. If any two vertices from $\mathcal{X}$ are $\mathcal{X}$-visible then the set $\mathcal{X}$ is a *mutual-visibility set*, while if any two vertices from $V(G)$ are $\mathcal{X}$-visible, then the set $\mathcal{X}$ is a *total mutual-visibility set* in G. The cardinality of a largest mutual-visibility (resp. total mutual-visibility) set in G is defined to be the *mutual-visibility number* (resp. *total mutual-visibility number*), $\mu(G)$ (resp. $\mu_{\mathrm{t}}(G)$), of G. By definition, $\mu(G) \geq \mu_{\mathrm{t}}(G)$ holds for every (connected) graph G.

It has been proved that the mutual-visibility problems are connected to several classical combinatorial problems which are extremely difficult to solve. To determine the mutual-visibility numbers for Cartesian products of two complete graphs is equivalent to the famous problem of Zarankievicz [9]. The total mutual-visibility number of Hamming graphs and line graphs of complete graphs can be expressed as Turán numbers for graphs or hypergraphs [4, 11].

In this manuscript, we prove exact formulas for the total mutual-visibility numbers of Kneser graphs, bipartite Kneser graphs, and Johnson graphs. For Kneser and bipartite Kneser graphs, these formulas show a direct connection to the minimum size of covering designs. For Johnson graphs, we prove that $\mu_{\mathrm{t}}(G)$ equals a hypergraph Turán number. Along the way, we also obtain exact values or estimations for the mutual-visibility number over these graph classes.

1.1 Terminology

This paper considers only simple undirected graphs that are connected. The *distance* between two vertices $u, v \in V(G)$, denoted by $\mathrm{dist}(u, v)$, is the length of a shortest path between u and v. The maximum distance between any two vertices in G is the *diameter* of G and is denoted by $\mathrm{diam}(G)$. Throughout the paper, $[n]$ stands for the set $\{1, \ldots, n\}$ for every positive integer n. For graph theory concepts not defined here, we refer the reader to the book [29].

A *hypergraph* $\mathcal{H}$ is a set system over the underlying vertex set $V(\mathcal{H})$. We assume, as usual, that the edge set $E(\mathcal{H})$ does not contain the empty set that is $E(\mathcal{H}) \subseteq 2^{V(\mathcal{H})} \setminus \{\emptyset\}$. A hypergraph is *$k$-uniform*, if every (hyper)edge $e \in E(\mathcal{H})$ consists of exactly k vertices. Then 2-uniform hypergraphs correspond to graphs without loops. A k-uniform hypergraph is often called a *k-graph*, while an edge $e \in E(\mathcal{H})$ is a *k-edge* if $|e| = k$. The *degree* of a vertex $v \in V(\mathcal{H})$ is the number of edges incident to v, and v is called *an isolated vertex* (isolate) if its degree is 0. For a set $Y \subseteq V(\mathcal{H})$, the *induced subhypergraph* contains those edges of $\mathcal{H}$ that are fully contained in Y.

A set $T \subseteq V(\mathcal{H})$ is a *transversal* in $\mathcal{H}$ if $e \cap T \neq \emptyset$ for every edge $e \in E(\mathcal{H})$. The minimum cardinality of a transversal is the *transversal number* $\tau(\mathcal{H})$ of the hypergraph.[1]

Let n and k be integers such that $n \geq k \geq 1$. The *Kneser graph* $KG(n,k)$ is the graph with vertices representing the k-subsets of the *ground set* $[n]$, and where two vertices A and B are adjacent if and only if $A \cap B = \emptyset$ holds for the corresponding sets. Formally, the vertex set is $V(KG(n,k)) = \binom{[n]}{k}$ and the edge set is

$$E(KG(n,k)) = \{\{A,B\} : A, B \in V(KG(n,k)) \text{ and } A \cap B = \emptyset\}.$$

This family of graphs was introduced in 1955 by M. Kneser. If $n < 2k$, then $KG(n,k)$ consists of $\binom{n}{k}$ isolated vertices, and if $n = 2k$, then $KG(n,k)$ is the union of disjoint copies of K_2. Moreover, $KG(n,1)$ is the complete graph on n vertices. In this paper, we consider only connected Kneser graphs. Thus, to avoid some trivial cases, we assume that $n \geq 2k+1$ and $k \geq 2$ for a Kneser graph $KG(n,k)$.

The vertex set of a *bipartite Kneser graph* $H(n,k)$ consists of vertices representing the k-element and $(n-k)$-element subsets of the ground set $[n]$. Two vertices u and v are adjacent if one of the represented sets is a subset of the other. By definition, $H(n,k)$ is bipartite for every $n > 2k \geq 2$. The partite classes correspond to $\binom{[n]}{k}$ and $\binom{[n]}{n-k}$, respectively. As $H(n,k) \cong H(n,n-k)$ and $H(2k,k)$ consists of k copies of K_2-components, we will assume $n \geq 2k+1 \geq 5$ while studying mutual-visibility problems in bipartite Kneser graphs.

The *Johnson graph* $J(n,k)$ is the graph with vertex set consisting of all the k-subsets of the ground set $[n]$ and the edge set is

$$E(J(n,k)) = \{\{A,B\} : A, B \in V(J(n,k)) \text{ and } |A \cap B| = k-1\}.$$

Observe that, by definition, $J(n,2)$ corresponds to the line graph of the complete graph K_n. It is well-known that $J(n,k)$ is isomorphic to $J(n,n-k)$ and that $J(n,1)$ is isomorphic to the complete graph on n vertices. To avoid these cases, in this paper, we deal with Johnson graphs $J(n,k)$ where $n \geq k+2 \geq 4$.

When we consider a graph G that is a Kneser, bipartite Kneser, or a Johnson graph, the vertices will be denoted by italic capital letters, and *the same notation will be used for a vertex of G and the represented subset of* $[n]$. For a subset $\mathcal{S} \subseteq V(G)$, we define the *underlying hypergraph* $\mathcal{F}(\mathcal{S})$ on the ground set $[n]$ that contains edges corresponding to the sets represented by the vertices in $\mathcal{S}$. That is, if G and $\mathcal{S} \subseteq V(G)$ are given,

$$E(\mathcal{F}(\mathcal{S})) = \{S \subseteq [n] : S \in \mathcal{S}\}.$$

If G is a Kneser graph $KG(n,k)$ or a Johnson graph $J(n,k)$, then $\mathcal{F}(\mathcal{S})$ is a k-graph for every nonempty vertex set $\mathcal{S}$.

Turán-type problems form a central area in extremal graph and hypergraph theory. Typically, in a hypergraph Turán problem two integers n, k and a k-graph $\mathcal{F}$ are given. We want to determine the maximum number of edges in a k-graph $\mathcal{H}$ on an n-element vertex

[1] Transversals are often called vertex covers, especially in the context of graphs. However, in this paper, we use the term "cover" in a different sense (as a subset relation between sets) and keep the term "transversal" as defined here.

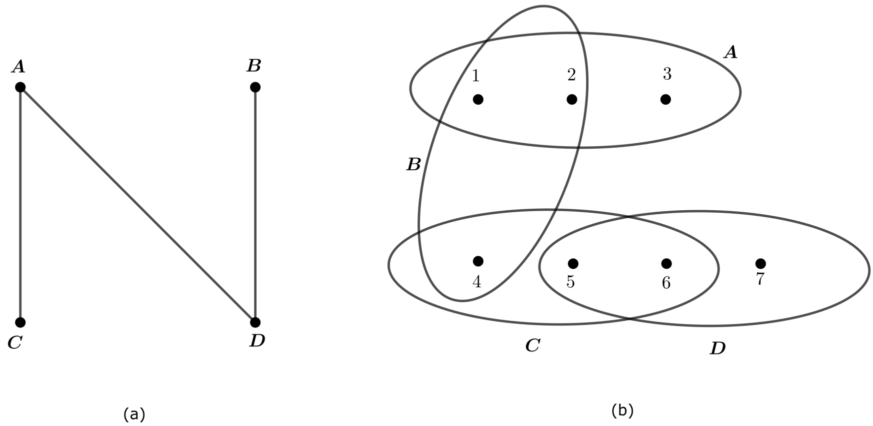

Figure 1: Figure (a) shows the subgraph of $K(7,3)$ induced by $\mathcal{S} = \{A, B, C, D\}$ where vertices A, B, C, D represent the sets $\{1,2,3\}, \{1,2,4\}, \{4,5,6\}, \{5,6,7\}$, respectively. In figure (b) the underlying hypergraph $\mathcal{F}(\mathcal{S})$ is depicted where $E(\mathcal{F}(\mathcal{S})) = \{A, B, C, D\}$.

set such that $\mathcal{H}$ contains no subhypergraph isomorphic to $\mathcal{F}$. This maximum number is denoted by $\mathrm{ex}_k(n, \mathcal{F})$. If $k = 2$ and $\mathcal{F}$ is a graph, we omit the index 2 and write simply $\mathrm{ex}(n, \mathcal{F})$. Most of the hypergraph Turán problems seem extremely difficult. Even the tight asymptotics for $\mathrm{ex}_k(n, \mathcal{F})$ as $n \to \infty$ might be hard to identify. For more results on the subject, we refer the reader to [18] and [14, Chapter 5].

For three positive integers with $n \geq k \geq t$, an (n, k, t) *covering design* is a pair $(X, \mathcal{B})$, where X is a set of n elements and $\mathcal{B}$ is a collection of k-element subsets of X (blocks) such that every t-element subset of X is a subset of at least one block from $\mathcal{B}$. The minimum number of blocks that may occur in an (n, k, t) covering design is the *covering number* $C(n, k, t)$. By double-countation, we obtain that $\binom{k}{t} C(n, k, t) \geq \binom{n}{t}$ is always true. Further, this inequality holds with equality if and only if a Steiner system $S(t, k, n)$ exists (see [16]). This fact shows that determining the values $C(n, k, t)$ is a challenging problem in general. If $(X, \mathcal{B})$ is an (n, k, t) covering design and $\mathcal{H} = (X, \mathcal{B}^c)$ is the $(n-k)$-uniform hypergraph containing the complements of the blocks $B \in \mathcal{B}$, then no t-element subset of X intersects every edge of $\mathcal{H}$. Then, we have $\tau(\mathcal{H}) > t$. As this implication also holds in the other direction, we may conclude that $C(n, n-k, t)$ is the minimum number of edges in a k-uniform hypergraph with n vertices and transversal number $t+1$. This correlation was first remarked by D.B. West (see Chapter 7.1 in [17]).

1.2 Results and structure of the paper

We start our study in Section 2 by proving a lemma on the values of the covering number $C(n, k, t)$ under specific conditions.

In Section 3, the total mutual-visibility numbers of Kneser graphs $KG(n, k)$ are determined. While we prove explicit formulas in terms of n and k for the cases when $n \leq 3k-1$ or $2k^2 \leq n$, the formula for the middle range contains the constant $C(n, n-k, 2k-1)$.

In Section 4, we determine the mutual-visibility number of $KG(n, k)$ for $n \geq 7k-5$ and estimate the value for the remaining cases.

In Section 5, the total mutual-visibility numbers of bipartite Kneser graphs $H(n,k)$ are determined. We also provide a lower bound for the mutual-visibility number $\mu(H(n,k))$.

Section 6 is devoted to the study of Johnson graphs. The main theorem establishes equality between $\mu_{\rm t}(J(n,k))$ and a hypergraph Turán number for all nontrivial cases. The exact mutual-visibility number is proved for $J(n,2)$, and it is shown that $\mu(J(n,k))$ is sandwiched between two Turán numbers in general.

In the last section, we answer a question from the recent manuscript [11] by determining the exact values for the mutual-visibility numbers of Kneser graphs $KG(n,2)$. Further, we give some remarks and formulate open problems related to the topic of this paper.

2 A lemma on the covering number

As a preparation for the proofs in Sections 3 and 4, here we mainly study the covering number. First, we prove a short technical lemma.

Lemma 2.1. *If n and k are two positive integers and $k<n<2k$, then*

$$\binom{n}{k} > 2\binom{n-1}{k}.$$

Proof. The condition $2k>n$ implies $\frac{n}{n-k}>2$ and hence,

$$\binom{n}{k} = \frac{n}{n-k}\binom{n-1}{k} > 2\binom{n-1}{k}$$

holds. □

Now, we concentrate on the covering numbers given in the form $C(n,n-k,2k-1)$. To avoid trivial or non-defined cases, we will assume that $n\geq 3k$ and $k\geq 2$. Under these conditions, to simplify our notation, we set

$$C^*(n,k) = C(n,n-k,2k-1)$$

that is the minimum number of edges in a k-uniform hypergraph of order n with transversal number $2k$. Recall that $\tau(\mathcal{H})$ denotes the transversal number of hypergraph $\mathcal{H}$.

Lemma 2.2. *Let n and k be integers.*

(i) *If $k\geq 2$ and $n\geq 2k^2$, then $C^*(n,k)=2k$.*

(ii) *If $k\geq 3$ and $n\geq 7k-5$, then $C^*(n,k)\leq 2\binom{2k-3}{k}+6$.*

(iii) *If $k\geq 2$ and $n\geq 2k^2+k$, then $C(n,n-k,2k)=2k+1$.*

Proof. To prove (i), we note that having $\tau=2k$ needs at least $2k$ edges. Thus, $C^*(n,k)\geq 2k$. Moreover, the condition $n\geq 2k^2$ allows us to construct a k-uniform hypergraph on n vertices with $2k$ pairwise disjoint edges. As its transversal number equals $2k$, the minimum number of edges in such a hypergraph is at most $2k$. Therefore, $C^*(n,k)=2k$.

To prove (ii) we consider a hypergraph $\mathcal{H}_{n,k}$ built from the following types of components.

- A k-uniform generalized triangle $\mathcal{H}_1^{(k)}$ has $n_1^{(k)} = k + \lceil \frac{k}{2} \rceil$ vertices partitioned into three sets V_1, V_2, V_3 such that $|V_1| = \lfloor \frac{k}{2} \rfloor$ and $|V_2| = |V_3| = \lceil \frac{k}{2} \rceil$. If k is even, we simply put
$$E(\mathcal{H}_1^{(k)}) = \{V_1 \cup V_2, V_1 \cup V_3, V_2 \cup V_3\}.$$
If k is odd, we specify a vertex $x_3 \in V_3$ and replace the edge $V_2 \cup V_3$ with the k-edge $V_2 \cup (V_3 \setminus \{x_3\})$ in the definition of $E(\mathcal{H}_1^{(k)})$. Then, $\mathcal{H}_1^{(k)}$ has $m_1^{(k)} = 3$ edges and $\tau(\mathcal{H}_1^{(k)}) = 2$ for all $k \geq 2$.

- A complete k uniform hypergraph $\mathcal{K}_{2k-3}^{(k)}$ is defined on $2k-3$ vertices such that all k-element subsets of the vertex set are hyperedges in $\mathcal{K}_{2k-3}^{(k)}$. Then, the size of this hypergraph is $m_2^{(k)} = \binom{2k-3}{k}$ and
$$\tau(\mathcal{K}_{2k-3}^{(k)}) = (2k-3) - (k-1) = k-2.$$

Now, to define the hypergraph $\mathcal{H}_{n,k}$, take two disjoint copies of $\mathcal{H}_1^{(k)}$ and two disjoint copies of $\mathcal{K}_{2k-3}^{(k)}$. They together cover
$$2\left(k + \left\lceil \frac{k}{2} \right\rceil\right) + 2(2k-3) \leq 7k-5$$
vertices. The remaining vertices (if any) remain isolates. The transversal number of $\mathcal{H}_{n,k}$ is
$$\tau(\mathcal{H}_{n,k}) = 2 \cdot 2 + 2(k-2) = 2k.$$
By definition of the covering number, we may conclude that
$$C^*(n,k) \leq |E(\mathcal{H}_{n,k})| = 2\binom{2k-3}{k} + 6.$$

The proof for part (iii) is similar to that for (i). Since $C(n, n-k, 2k)$ is the minimum number of edges in a k-uniform hypergraph on n vertices having transversal number at least $2k+1$, at least $2k+1$ edges are needed. On the other hand, $n \geq 2k^2 + k$ vertices are enough to take $2k+1$ pairwise vertex-disjoint k-edges and achieve $\tau \geq 2k+1$. □

3 Total mutual-visibility in Kneser graphs

The main result of this section is Theorem 3.2 that determines the total mutual-visibility number of Kneser graphs for all nontrivial cases. In its proof, we will use the following lemma.

Lemma 3.1. *Let $\mathcal{X}$ be a set of vertices of the Kneser graph $KG(n,k)$ and $\overline{\mathcal{X}} = V(KG(n,k)) \setminus \mathcal{X}$. For every $n \geq 3k-1$, the set $\mathcal{X}$ is a total mutual-visibility set in $KG(n,k)$ if and only if $\tau(\mathcal{F}(\overline{\mathcal{X}})) \geq 2k$.*

Proof. If $n \geq 3k-1$, then $\mathrm{diam}(KG(n,k)) = 2$ (see [28]). Suppose first that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \geq 2k$ and consider two arbitrary vertices A, B from the Kneser graph $G = KG(n,k)$. If A and B are adjacent, they are $\mathcal{X}$-visible. Otherwise $\mathrm{dist}(A,B) = 2$, and the k-sets A and B are intersecting. Thus $|A \cup B| \leq 2k-1 < \tau(\mathcal{F}(\overline{\mathcal{X}}))$, and the set $A \cup B$ cannot intersect all edges of $\mathcal{F}(\overline{\mathcal{X}})$. Let us choose such an edge Y with $Y \cap (A \cup B) = \emptyset$ from $\mathcal{F}(\overline{\mathcal{X}})$. The corresponding vertex Y belongs to $\overline{\mathcal{X}}$ in G. As we have $Y \cap A = Y \cap B = \emptyset$, the vertex Y is a common neighbor of A and B. We conclude that A and B are $\mathcal{X}$-visible, and hence, $\mathcal{X}$ is a total mutual-visibility set in G.

To prove the other direction, we suppose that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \le 2k-1$. Let us choose $2k-1$ elements, say $a_1, \dots, a_{k-1}, b_1, \dots, b_{k-1}, c$, from the ground set $[n]$ such that they together intersect all edges of $\mathcal{F}(\overline{\mathcal{X}})$. Consider the two nonadjacent vertices of G corresponding to the sets $A = \{a_1, \dots, a_{k-1}, c\}$ and $B = \{b_1, \dots, b_{k-1}, c\}$. Since $A \cup B$ intersects every k-edge from $\mathcal{F}(\overline{\mathcal{X}})$, the vertices A and B in G have no common neighbor from $\overline{\mathcal{X}}$. As $\mathrm{dist}(A, B) = 2$ in G, we may conclude that A and B are not $\mathcal{X}$-visible and, therefore $\mathcal{X}$ is not a total mutual-visibility set. $\square$

Theorem 3.2. *If n and k are integers such that $n \ge 2k+1$ and $k \ge 2$, then*

$$\mu_{\mathrm{t}}(KG(n,k)) = \begin{cases} 0 & \text{if } 2k+1 \le n \le 3k-1, \\ \binom{n}{k} - C^*(n,k) & \text{if } 3k \le n \le 2k^2 - 1, \\ \binom{n}{k} - 2k & \text{if } 2k^2 \le n. \end{cases}$$

Proof. First, suppose that $2k+1 \le n \le 3k-1$ and a total mutual-visibility set $\mathcal{X}$ in $G = KG(n,k)$ contains at least one vertex A. Let $\overline{A}$ denote the $(n-k)$-element set $[n] \setminus A$. As $k+1 \le |\overline{A}| \le 2k-1$, we can always find two k-element subsets B and C such that $B \cap C \neq \emptyset$ and $B \cup C = \overline{A}$. The corresponding vertices B and C are nonadjacent in G, and only vertex A is adjacent to both. Consequently, BAC is the unique shortest B,C-path in G. Since $A \in \mathcal{X}$, the vertices B and C are not $\mathcal{X}$-visible, and $\mathcal{X}$ is not a total mutual visibility set in G. This contradiction proves $\mu_{\mathrm{t}}(G) = 0$ for the range $2k+1 \le n \le 3k-1$.

Now, suppose that $n \ge 3k$. By Lemma 3.1, $\tau(\mathcal{F}(\overline{\mathcal{X}})) \ge 2k$ is exactly the property expected from the k-graph $\mathcal{F}(\overline{\mathcal{X}})$ to ensure that $\mathcal{X}$ is a total mutual-visibility set. Thus $C^*(n,k)$ is the minimum number of k-edges in $\mathcal{F}(\overline{\mathcal{X}})$ that is the minimum number of vertices in $\overline{\mathcal{X}}$ when $\mathcal{X}$ is a total mutual-visibility set in G. Therefore $\mu_{\mathrm{t}}(G) = \binom{n}{k} - C^*(n,k)$, if $n \ge 3k$.

As $2k^2 \ge 3k$, the formula $\mu_{\mathrm{t}}(G) = \binom{n}{k} - C^*(n,k)$ remains valid under the condition $n \ge 2k^2$. Moreover, in this case we have $C^*(n,k) = 2k$ according to Lemma 2.2(i). This finishes the proof of the theorem. $\square$

4 Mutual-visibility number of Kneser graphs

Here we study the mutual-visibility number of Kneser graphs. Theorem 4.1 shows that $\mu(KG(n,k))$ equals $\mu_{\mathrm{t}}(KG(n,k))$ if $n \ge 7k-5$, while Proposition 4.2 states a lower bound for the range $2.5k \le n \le 7k-4$.

Theorem 4.1. *If n and k are two integers such that $n \ge 7k-5$ and $k \ge 2$, then*

$$\mu(KG(n,k)) = \binom{n}{k} - C^*(n,k).$$

In particular, if $n \ge 2k^2$ and $k \ge 2$, then

$$\mu(KG(n,k)) = \binom{n}{k} - 2k.$$

Proof. Since $\mu(KG(n,k)) \geq \mu_{\mathrm{t}}(KG(n,k))$ holds by definition, Theorem 3.2 directly implies $\mu(KG(n,k)) \geq \binom{n}{k} - C^*(n,k)$ when $n \geq 7k-5 \geq 3k$. Therefore, it is enough to prove that $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$ when $n \geq 7k-5$.

Let k and n be integers so that $n \geq 7k-5$ and denote the graph $KG(n,k)$ by G. Recall that $\mathrm{diam}(G) = 2$ under the given conditions. Suppose for a contradiction that $\mathcal{X}$ is a mutual-visibility set in G and $|\overline{\mathcal{X}}| \leq C^*(n,k)-1$. It follows that $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2k-1$. By Lemma 2.2(ii), if $k \geq 3$, the number of k-edges in $\mathcal{F}(\overline{\mathcal{X}})$, that is denoted by m and equals $|\overline{\mathcal{X}}|$, satisfies

$$m \leq 2\binom{2k-3}{k} + 5.$$

Our aim is to find two k-element subsets A and B of $[n]$ such that

($\star$) $A, B \notin E(\mathcal{F}(\overline{\mathcal{X}}))$, A intersects B, and $A \cup B$ is a transversal in $\mathcal{F}(\overline{\mathcal{X}})$.

Then, in the graph G, the vertices A and B belong to $\mathcal{X}$ and have no common neighbor in $\overline{\mathcal{X}}$. As A and B are nonadjacent and $\mathrm{diam}(G) = 2$, there is no shortest A,B-path to ensure the mutual-visibility. This will give the contradiction and proves that $|\overline{\mathcal{X}}| \geq C^*(n,k)$ when $\mathcal{X}$ is a mutual-visibility set in G. We consider three cases in the continuation.

Case 1. $k \geq 3$ and $\tau(\mathcal{F}(\overline{\mathcal{X}})) = 2k-1$.
Let T be a minimum transversal in $\mathcal{F}(\overline{\mathcal{X}})$ and denote by $\mathcal{T}$ the subhypergraph induced by T in $\mathcal{F}(\overline{\mathcal{X}})$. Then $|T| = 2k-1$. By the minimality of T, the set $T \setminus \{u\}$ cannot be a transversal if $u \in T$. Equivalently, every vertex $u \in T$ is incident to a 'private' edge e_u from $\mathcal{F}(\overline{\mathcal{X}})$ such that the intersection $e_u \cap T$ contains only the vertex u. For the number of hyperedges in $\mathcal{T}$, which is denoted by $m_{\mathcal{T}}$, we therefore have

$$m_{\mathcal{T}} \leq m - (2k-1) \leq m - 5 \leq 2\binom{2k-3}{k}.$$

To estimate the average vertex degree $\overline{d}_{\mathcal{T}}$ in the induced subhypergraph $\mathcal{T}$, we apply Lemma 2.1:

$$\begin{aligned} \overline{d}_{\mathcal{T}} = \frac{k\, m_{\mathcal{T}}}{2k-1} &\leq 2\binom{2k-3}{k}\frac{k}{2k-1} \\ &< \frac{1}{2}\binom{2k-1}{k}\frac{k}{2k-1} \\ &= \frac{1}{2}\binom{2k-2}{k-1}. \end{aligned} \tag{4.1}$$

The upper bound implies that there exists a vertex $z \in T$ being incident to less than $\frac{1}{2}\binom{2k-2}{k-1}$ hyperedges in $\mathcal{T}$. Consider now the pairs $\{A', B'\}$ where A' and B' are disjoint $(k-1)$-element subsets of $T \setminus \{z\}$. Every $(k-1)$-element subset appears in exactly one such pair and there are $\frac{1}{2}\binom{2k-2}{k-1}$ pairs. By the pigeonhole principle, there exists a pair $\{A', B'\}$ such that neither $A = A' \cup \{z\}$ nor $B = B' \cup \{z\}$ is an edge in $\mathcal{T}$. Consequently, the k-sets A and B are not edges in $\mathcal{F}(\overline{\mathcal{X}})$, they intersect in z, and the union $T = A \cup B$ is a transversal in $\mathcal{F}(\overline{\mathcal{X}})$. We may infer that A and B satisfy ($\star$) and the corresponding vertices $A, B \in \mathcal{X}$ are not $\mathcal{X}$-visible in G. This contradiction proves $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$.

Case 2. $k \geq 3$ and $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2k-2$.
First, suppose that there is an isolated vertex z in $\mathcal{F}(\overline{\mathcal{X}})$. Then there exists a (not necessarily minimum) transversal T in $\mathcal{F}(\overline{\mathcal{X}})$ such that $|T| = 2k-2$ and $z \notin T$. For an arbitrary $(k-1)$-element subset A' of T and for the complement $B' = T \setminus A'$, the sets $A = A' \cup \{z\}$ and $B = B' \cup \{z\}$ satisfy the three properties in $(\star)$ and give the required contradiction.
In the other case, there is no isolated vertex in $\mathcal{F}(\overline{\mathcal{X}})$. Let T be an arbitrary $(2k-1)$-element transversal of this hypergraph and denote by $m_{\mathcal{T}}$ the number of hyperedges in the subhypergraph $\mathcal{T}$ induced by T. For the

$$n - |T| \geq (7k-5) - (2k-1) = 5k-4 > 5(k-1)$$

non-isolated vertices outside T, we need at least 6 hyperedges to cover them all in $\mathcal{F}(\overline{\mathcal{X}})$. It implies $m_{\mathcal{T}} \leq m - 6 < 2\binom{2k-3}{k}$. In the continuation, we can proceed as in Case 1; that is, we estimate the average degree in $\mathcal{T}$ according to (4.1) and choose a vertex $z \in T$ of minimum degree. Considering the pairs of disjoint $(k-1)$-element subsets of $T \setminus \{z\}$, we may conclude that there exist k-sets A and B satisfying $(\star)$. The contradiction proves $\mu(KG(n,k)) \leq \binom{n}{k} - C^*(n,k)$.

Case 3. $k = 2$.
If $k = 2$, the condition $n \geq 7k-5$ gives $n \geq 9$ and implies $n > 2k^2 = 8$. Hence, by Lemma 2.2(i), $C^*(n,2) = 4$ also holds. By our supposition, $\mathcal{X}$ is a mutual visibility set in G and $\mathcal{F}(\overline{\mathcal{X}})$ contains $m \leq C^*(n,2) - 1 = 3$ edges. As $n \geq 9$, there exist isolated vertices. If $\tau(\mathcal{F}(\overline{\mathcal{X}})) = 3$, then we have three vertex-disjoint hyperedges in $\mathcal{F}(\overline{\mathcal{X}})$ and, for a minimum transversal T, it is easy to choose two 2-element sets $A \subset T$ and $B \subset T$ satisfying $(\star)$. If $\tau(\mathcal{F}(\overline{\mathcal{X}})) \leq 2$, we take a 2-element transversal $\{a,b\}$ and a third vertex c that is isolated in $\mathcal{F}(\overline{\mathcal{X}})$. The sets $A = \{a,c\}$, $B = \{b,c\}$ satisfy $(\star)$. This gives the desired contradiction and proves the formula for $k = 2$.

Concerning the second part of the statement, if $k \geq 3$ and $n \geq 2k^2$, then $n \geq 7k-5$ also holds. Thus the formula $\mu(KG(n,k)) = \binom{n}{k} - C^*(n,k)$ and Lemma 2.2(i) directly imply $\mu(KG(n,k)) = \binom{n}{k} - 2k$. The same is true if $k = 2$ and $n \geq 9$. The only remaining case is the graph $KG(8,2)$. Here, a mutual-visibility set $\mathcal{X}$ with more than $\binom{8}{2} - 4$ vertices defines the 2-uniform underlying hypergraph $\mathcal{F}(\overline{\mathcal{X}})$ with $n = 8$ vertices and at most three edges. As in Case 3 of the proof, it is easy to exhibit two non-edges in $\mathcal{F}(\overline{\mathcal{X}})$ that satisfy $(\star)$ and prove that $\mathcal{X}$ is not a mutual visibility set in $KG(8,2)$. □

The mutual-visibility notion is related to the general position problem in graphs. A subset S of vertices in a connected graph G is a *general position set* if no triple of vertices from S lie on a common shortest path in G. Equivalently, for every three different vertices $u, v, z \in S$, the strict inequality $\mathrm{dist}(u,v) + \mathrm{dist}(v,z) > \mathrm{dist}(u,z)$ holds. The general position problem [15, 21] is to find a largest general position set of G and the order of such a set is the *general position number* $\mathrm{gp}(G)$. By definition, every general position set is a mutual visibility set and therefore, $\mu(G) \geq \mathrm{gp}(G)$ holds for every connected graph G.

Proposition 4.2. *If n and k are integers satisfying $2.5k - 0.5 \leq n \leq 7k-5$ and $k \geq 2$, then*

$$\mu(KG(n,k)) \geq \binom{n-1}{k-1}.$$

Proof. The results from [15] and [23] yield that $\mathrm{gp}(KG(n,k)) = \binom{n-1}{k-1}$ holds when $n \geq 2.5k - 0.5$. Then, the relation $\mu(KG(n,k)) \geq \mathrm{gp}(KG(n,k))$ directly implies the statement. □

5 Bipartite Kneser graphs

In this section we study the total mutual-visibility number and mutual-visibility number of bipartite Kneser graphs. Since a k-element subset $S_1 \subseteq [n]$ is a subset of an $(n-k)$-element subset $S_2 \subseteq [n]$ if and only if $[n] \setminus S_2 \subseteq [n] \setminus S_1$, bipartite Kneser graphs satisfy the following property:

Observation 5.1. Let G be the bipartite Kneser graph $H(n,k)$ such that the vertices in the partite classes $\mathcal{P}_1$ and $\mathcal{P}_2$ represent the k-element and $(n-k)$-element subsets of $[n]$, respectively. Let ϕ be the function that assigns to each vertex $S \in V(H(n,k))$ the vertex S^c which represents the complement set $S^c = [n] \setminus S$. Then, ϕ is an automorhism of $H(n,k)$ that maps $\mathcal{P}_1$ into $\mathcal{P}_2$.

It follows from the observation that any property proved for $\mathcal{P}_1$ holds for $\mathcal{P}_2$, and vice versa. Remark that bipartite Kneser graphs satisfy the much stronger property of symmetry i.e., they are vertex- and edge-transitive graphs [25].

In the main result of this section, we will use again the constant $C(n,k,t)$ that is the minimum size of a covering design with the given parameters.

Theorem 5.2. *If n and k are integers such that $n \geq 2k+1$, then*

$$\mu_{\mathrm{t}}(H(n,k)) = \begin{cases} 0 & \text{if } 2k+1 \leq n \leq 3k, \\ 2\binom{n}{k} - 2\,C(n,n-k,2k) & \text{if } 3k+1 \leq n < 2k^2+k, \\ 2\binom{n}{k} - 4k - 2 & \text{if } 2k^2+k \leq n. \end{cases}$$

Proof. Consider a bipartite Kneser graph $H(n,k)$ with parameters satisfying $n \geq 2k+1$, and let $\mathcal{P}_1$ and $\mathcal{P}_2$ denote the two partite classes of $H(n,k)$ such that $\mathcal{P}_1$ consists of the vertices representing the k-element subsets of $[n]$. Let $\mathcal{X}$ be a total mutual visibility set in $H(n,k)$.

We first assume that $n \leq 3k$ and show that $\mathcal{X} \cap \mathcal{P}_1 = \mathcal{X} \cap \mathcal{P}_2 = \emptyset$. Note that $n \leq 3k$ implies $k + 2(n-2k) \leq n$ and then, for every vertex $A \in \mathcal{P}_1$, we can choose two $(n-k)$-element sets B_1 and B_2 such that $B_1 \cap B_2 = A$. Vertex A is clearly the only common neighbor of B_1 and B_2 in $H(n,k)$. The $\mathcal{X}$-visibility of B_1 and B_2 therefore implies $A \notin \mathcal{X}$. By Observation 5.1, we infer that the same is true for the vertices in $\mathcal{P}_2$ and therefore, $\mathcal{X} = \emptyset$. It proves $\mu_{\mathrm{t}}(H(n,k)) = 0$ for the case $n \leq 3k$.

Assume now that $n \geq 3k+1$ and show that the family $\mathcal{P}_2 \cap \overline{\mathcal{X}}$ of $(n-k)$-element sets covers all $(2k)$-element subsets of $[n]$. Suppose to the contrary that no $B \in \mathcal{P}_2 \cap \overline{\mathcal{X}}$ covers the $(2k)$-element set $S \subseteq [n]$. Split S into two disjoint k-sets A_1 and A_2. Since $n-k > 2k$, there exist $(n-k)$-element subsets of $[n]$ that cover both A_1 and A_2. However, by our supposition, none of the vertices corresponding to these covering sets belong to $\mathcal{P}_2 \cap \overline{\mathcal{X}}$. Equivalently, in $H(n,k)$, $\mathrm{dist}(A_1, A_2) = 2$ but none of the common neighbors belong to $\overline{\mathcal{X}}$. This contradicts the $\mathcal{X}$-visibility of A_1 and A_2. We infer that the $(n-k)$-sets in $\mathcal{P}_2 \cap \overline{\mathcal{X}}$ cover all $(2k)$-subsets of $[n]$ and therefore, $|\mathcal{P}_2 \cap \overline{\mathcal{X}}| \geq C(n, n-k, 2k)$. By

Observation 5.1, $|\mathcal{P}_1 \cap \overline{\mathcal{X}}| \geq C(n, n-k, 2k)$ follows and we may conclude

$$\mu_t(H(n,k)) \leq 2\binom{n}{k} - 2\,C(n, n-k, 2k).$$

To prove the other direction, let $\mathcal{B}$ be a family of $(n-k)$-element subsets of $[n]$ that cover all $(2k)$-element subsets. We also suppose that $\mathcal{B}$ is minimum that is, $|\mathcal{B}| = C(n, n-k, 2k)$ and use the same notation $\mathcal{B}$ for the set of corresponding vertices in $\mathcal{P}_2$. Moreover, let

$$\mathcal{A} = \{A \subseteq [n] : [n] \setminus A \in \mathcal{B}\} = \phi(\mathcal{B}),$$

where ϕ is the automorphism from Observation 5.1. We are going to prove that the vertex set $\mathcal{X} = \overline{\mathcal{A} \cup \mathcal{B}}$ forms a total mutual-visibility set in $H(n,k)$.

Take two vertices A_1 and A_2 from the partite class $\mathcal{P}_1$ of $H(n.k)$. Since $|A_1 \cup A_2| \leq 2k$, the union is covered by an $(n-k)$-element set $B \in \mathcal{B}$. The corresponding vertex $B \in \overline{\mathcal{X}}$ is a common neighbor of A_1 and A_2 and consequently A_1 and A_2 are $\mathcal{X}$-visible. By Observation 5.1 and by the symmetry of our definitions for $\mathcal{A}$ and $\mathcal{B}$, the same is true for any two vertices $B_1, B_2 \in \mathcal{P}_2$.

The last case to check is when $A \in \mathcal{P}_1$ and $B \in \mathcal{P}_2$. If they are adjacent, the $\mathcal{X}$-visibility follows. If they are nonadjacent vertices, the distance is at least 3 (in fact, it is exactly 3). We may choose an arbitrary vertex Y from $\mathcal{P}_1$ that is different from A. As we have already seen, there exists a vertex $B_1 \in \mathcal{P}_2 \cap \overline{\mathcal{X}}$ which is a common neighbor of A and Y. Similarly, there is a vertex A_1 in $\mathcal{P}_1 \cap \overline{\mathcal{X}}$ which is a common neighbor of B and B_1. We may infer that AB_1A_1B is a shortest A,B-path and both internal vertices belong to $\overline{\mathcal{X}}$. Therefore, A and B are $\mathcal{X}$-visible. It finishes the proof for $\mathcal{X} = \overline{\mathcal{A} \cup \mathcal{B}}$ being a total mutual-visibility set in $H(n,k)$. We may conclude that

$$\mu_t(H(n,k)) \geq |\mathcal{X}| = 2\binom{n}{k} - |\mathcal{A} \cup \mathcal{B}| = 2\binom{n}{k} - 2\,C(n, n-k, 2k).$$

This proves the desired formula for $n \geq 3k+1$.

If $n \geq 2k^2 + k$, the equality $\mu_t(H(n,k)) = 2\binom{n}{k} - 2\,C(n, n-k, 2k)$ still remains valid, and together with Lemma 2.2(iii) they yield the formula stated for the last case. $\square$

Concerning the mutual-visibility number of bipartite Kneser graphs, we propose two simple lower bounds.

Proposition 5.3. *If n and k are integers such that $n \geq 3k+1$, then*

$$\mu(H(n,k)) \geq \max\left\{\binom{n}{k},\, 2\binom{n}{k} - 2\,C(n, n-k, 2k)\right\}.$$

Proof. By definition, $\mu(G) \geq \mu_t(G)$ holds for every graph G. Theorem 5.2 then implies $\mu(H(n,k)) \geq 2\binom{n}{k} - 2\,C(n, n-k, 2k)$. The inequality $\mu(H(n,k)) \geq \binom{n}{k}$ follows from the fact that $\mathcal{X} = \mathcal{P}_1$ is a mutual-visibility set if $n \geq 3k+1$. Indeed, as $\mathrm{dist}(A_1, A_2) = 2$ holds for every two vertices A_1, A_2 from $\mathcal{P}_1 = \mathcal{X}$, each shortest path between them contains only one internal vertex, and this vertex is from $\mathcal{P}_2 = \overline{\mathcal{X}}$. $\square$

6 Johnson graphs

In this section, we determine the total mutual-visibility number and estimate the mutual visibility number of Johnson graphs in terms of graph and hypergraph Turán numbers. Before stating the main results, we define the notion of *k-uniform suspension*. For a graph G and an integer $k \geq 2$, the k-uniform suspension of G, denoted by $\mathcal{G}^{k+}$, is obtained from G by taking $k-2$ new vertices and adding them to every edge of the graph. Formally, the vertex set is $V(\mathcal{G}^{k+}) = V(G) \cup Y$, where Y is a set of vertices with $|Y| = k-2$ and $V(G) \cap Y = \emptyset$. The edge set is

$$E(\mathcal{G}^{k+}) = \{Y \cup e : e \in E(G)\}.$$

In our results, we will refer to the hypergraphs $\mathcal{C}_4^{k+}$ and $\mathcal{K}_4^{k+}$ which are the k-uniform suspensions of C_4 and K_4, respectively.[2] The $(k+2)$-element vertex set for both of them is given as $Y \cup \{z_1, z_2, z_3, z_4\}$, and the two edge sets are specified as

$$E(\mathcal{C}_4^{k+}) = \{Y \cup \{z_i, z_{i+1}\}\} : i \in [3]\} \cup \{Y \cup \{z_4, z_1\}\}, \tag{6.1}$$

$$E(\mathcal{K}_4^{k+}) = \{Y \cup \{z_i, z_j\}\} : i, j \in [4],\ \ i < j\}. \tag{6.2}$$

The following tight asymptotic result was proved by Mubayi:

Theorem 6.1 ([22]). *For every $k \geq 2$,*

$$\mathrm{ex}_k(n, \mathcal{C}_4^{k+}) = (1 + o(1)) \frac{1}{k!}\, n^{k-0.5}.$$

The main theorem of this section establishes a direct connection between total mutual-visibility numbers of Johnson graphs and hypergraph Turán numbers $\mathrm{ex}_k(n, \mathcal{C}_4^{k+})$. For $k = 2$, the Johnson graph $J(n, 2)$ is the line graph of the complete graph K_n. Therefore, Theorem 3.7 in [11] has already established the equality $\mu_{\mathrm{t}}(J(n, 2)) = \mathrm{ex}(n, C_4)$. Here we use a different method to approach the general problem.

Theorem 6.2. *It holds for every two integers n and k satisfying $n \geq k+2$ and $k \geq 2$ that*

$$\mu_{\mathrm{t}}(J(n,k)) = \mathrm{ex}_k(n, \mathcal{C}_4^{k+})$$

Proof. Assume that $\mathcal{X}$ is a total mutual visibility set of maximum cardinality in $J(n, k)$. We first prove that the k-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free. Suppose to the contrary that $\mathcal{C}_4^{k+}$ is a subhypergraph in $\mathcal{F}(\mathcal{X})$. Naming the vertices in this subhypergraph as in (6.1), let us consider the k-element sets $A_1 = Y \cup \{z_1, z_3\}$ and $A_2 = Y \cup \{z_2, z_4\}$. The corresponding vertices A_1 and A_2 are at distance $k - |Y| = 2$ in $J(n, k)$. Further, A_1 and A_2 have exactly four common neighbors in $J(n, k)$ and these neighbors correspond to the four edges in the subhypergraph $\mathcal{C}_4^{k+}$. By the supposition, the corresponding four vertices all belong to $\mathcal{X}$ and therefore, A_1 and A_2 are not $\mathcal{X}$-visible. This contradiction proves that $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free and, since the size of $\mathcal{X}$ equals the number of edges in $\mathcal{F}(\mathcal{X})$, we have

$$\mu_{\mathrm{t}}(J(n,k)) = |\mathcal{X}| \leq \mathrm{ex}_k(n, \mathcal{C}_4^{k+}).$$

To prove the other direction, we suppose that $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free and prove that every two vertices A and B are $\mathcal{X}$-visible in $J(n, k)$. We proceed by induction on the distance of

[2] Note that $\mathcal{C}_4^{k+}$ corresponds to the complete k-partite k-graph $\mathcal{K}^{(k)}_{1,\ldots,1,2,2}$.

A and B. If $\mathrm{dist}(A,B)=1$, then the statement clearly holds. Suppose that $A\cap B=D$ holds for the corresponding k-sets and then $\mathrm{dist}(A,B)=k-|D|\geq 2$. Let us choose four different vertices $z_1,z_2\in A\setminus D$ and $w_1,w_2\in B\setminus D$. Since $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free, there are two indices i and j such that $A'=A\setminus\{z_i\}\cup\{w_j\}$ is not an edge in $\mathcal{F}(\mathcal{X})$. Then, for the corresponding vertex, $A'\notin\mathcal{X}$. As $\mathrm{dist}(A',B)=\mathrm{dist}(A,B)-1$, we can now apply the hypothesis for A' and B. As there is a shortest path between A' and B such that all internal vertices belong to $\overline{\mathcal{X}}$ and A' itself is from $\overline{\mathcal{X}}$, we can construct a shortest path between A and B that contains no internal vertex from $\mathcal{X}$. This proves that A and B are $\mathcal{X}$-visible and we can conclude that $\mathcal{X}$ is a total mutual-visibility set if $\mathcal{F}(\mathcal{X})$ is $\mathcal{C}_4^{k+}$-free.

By choosing a k-uniform $\mathcal{C}_4^{k+}$-free hypergraph with maximum number of edges, the corresponding vertex set $\mathcal{X}$ will be a total mutual-visibility set in $J(n,k)$. This proves $\mu_{\mathrm{t}}(J(n,k))\geq \mathrm{ex}_k(n,\mathcal{C}_4^{k+})$ and finishes the proof of the theorem. □

By Theorems 6.1 and 6.2, we may conclude the following asymptotically tight result:

Corollary 6.3. *For every fixed $k\geq 2$,*

$$\mu_{\mathrm{t}}(J(n,k))=(1+o(1))\frac{1}{k!}\,n^{k-0.5}.$$

Finally, we turn to the problem of mutual-visibility in Johnson graphs. Here, we can give an exact formula for $k=2$ and $k=n-2$ (that was proved in [11] with a different approach), while establishing lower and upper bounds for the remaining cases.

Theorem 6.4. *Let n and k be integers.*

(i) *If $k\geq 2$ and $n\geq k+2$, then*

$$\mathrm{ex}_k(n,\mathcal{C}_4^{k+})\leq \mu(J(n,k))\leq \mathrm{ex}_k(n,\mathcal{K}_4^{k+}). \tag{6.3}$$

(ii) *If $n\geq 4$, then*

$$\mu(J(n,2))=\mu(J(n,n-2))=\left\lfloor\frac{n^2}{3}\right\rfloor. \tag{6.4}$$

Proof. By definitions of the parameters, $\mu(G)\geq\mu_{\mathrm{t}}(G)$ holds for every graph G. Theorem 6.2 then directly implies $\mathrm{ex}_k(n,\mathcal{C}_4^{k+})\leq\mu(J(n,k))$.

To show the upper bound in (6.3), we prove that the k-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is $\mathcal{K}_4^{k+}$-free whenever $\mathcal{X}$ is a mutual visibility set in $J(n,k)$. Indeed, let us suppose that $\mathcal{F}(\mathcal{X})$ contains a subhypergraph on the vertex set $Y\cup\{z_1,\dots,z_4\}$ that is isomorphic to $\mathcal{K}_4^{k+}$. We name the vertices according to (6.2). Then, by the definition of underlying hypergraph $\mathcal{F}(\mathcal{X})$, the vertices of $J(n,k)$ that correspond to the k-sets $Y\cup\{z_1,z_2\}$ and $Y\cup\{z_3,z_4\}$ belong to $\mathcal{X}$. Their distance equals 2 and all the four common neighbors are from $\mathcal{X}$ as well. Thus, the two vertices $Y\cup\{z_1,z_2\}$ and $Y\cup\{z_3,z_4\}$ from $\mathcal{X}$ are not $\mathcal{X}$-visible, and $\mathcal{X}$ is not a mutual visibility set in $J(n,k)$. We may conclude that $\mathcal{F}(\mathcal{X})$ is $\mathcal{K}_4^{k+}$-free for every mutual visibility set $\mathcal{X}$ of $J(n,k)$ and consequently, $\mu(J(n,k))\leq \mathrm{ex}_k(n,\mathcal{K}_4^{k+})$ is valid.

To prove (6.4), we first recall the well-known facts that $\mathrm{ex}(n,K_4)=\lfloor n^2/3\rfloor$ [27] and that $J(n,2)\cong J(n,n-2)$ holds for every $n\geq 4$. It is enough then to prove the equality $\mu(J(n,2))=\mathrm{ex}(n,K_4)$. As $\mathcal{K}_4^{2+}$ is the graph K_4, part (i) provides the upper bound $\mu(J(n,2))\leq\mathrm{ex}(n,K_4)$.

Consider a set $\mathcal{X} \subseteq V(J(n,2))$ so that the 2-uniform underlying hypergraph $\mathcal{F}(\mathcal{X})$ is K_4-free. For every two vertices A and B from $\mathcal{X}$, they are either adjacent and $\mathcal{X}$-visible or $\mathrm{dist}(A,B) = 2$. In the latter case, the corresponding 2-element sets are disjoint, say $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$. Since $\mathcal{F}(\mathcal{X})$ is K_4-free, there is a 2-element set $\{a_i, b_j\}$ with $i, j \in [2]$ that is not an edge in $\mathcal{F}(\mathcal{X})$. Thus $C = \{a_i, b_j\}$ is in $\overline{\mathcal{X}}$ and A, B are $\mathcal{X}$-visible via the path ACB. Consequently, a mutual-visibility set $\mathcal{X}$ can be chosen in $J(n,2)$ such that $\mathcal{F}(\mathcal{X})$ is an extremal K_4-free graph having $\mathrm{ex}(n, K_4)$ edges. We conclude $\mathrm{ex}(n, K_4) \leq \mu(J(n,2))$, which finishes the proof of (6.4). □

7 Conclusion

In this paper, we determined the total mutual-visibility numbers for all Kneser graphs, bipartite Kneser graphs, and Johnson graphs. Our formulas gave easily computable values for some ranges, but we had to include invariants from extremal combinatorics in other cases. Computing these invariants (namely hypergraph Turán numbers of suspensions and covering numbers from design theory) is considered an extremely challenging problem.

We also studied the mutual-visibility numbers over these three graph classes and obtained exact results for all Kneser graphs $KG(n,k)$ with $n \geq 7k-5$ and $k \geq 2$ (see Theorem 4.1). For the remaining cases, we gave estimations (see Propositions 4.2, 5.3, and Theorem 6.4), the exact determination is posed here as an open problem.

Problem 7.1. Determine the mutual visibility number of the Kneser graph $KG(n,k)$ if $3 \leq k$ and $2k+1 \leq n \leq 7k-6$.

Problem 7.2. Determine the mutual visibility number of the bipartite Kneser graph $H(n,k)$ if $k \geq 2$ and $n \geq 2k+1$.

Problem 7.3. Determine the mutual visibility number of the Johnson graph $H(n,k)$ if $k \geq 3$ and $n \geq k+2$.

For every graph G, the relation $\mu(G) \geq \mu_{\mathrm{t}}(G)$ holds by definition. If G satisfies $\mu(G) = \mu_{\mathrm{t}}(G)$, we call it a (μ, μ_{t})-*graph*. Our Theorems 3.2 and 4.1 imply that all Kneser graphs $KG(n,k)$ with $k \geq 2$ and $n \geq 7k-5$ are (μ, μ_{t})-graphs. Further, by the same theorems, $K(8,2)$ is also a (μ, μ_{t})-graph.

Cicerone *et al.* [8] introduced further parameters on mutual-visibility. Given a graph G and a vertex set $\mathcal{X}$, we say that $\mathcal{X}$ is a *dual mutual-visibility set*, if every two vertices from $\mathcal{X}$ and every two vertices from $\overline{\mathcal{X}}$ are $\mathcal{X}$-visible. The maximum size of a dual mutual-visibility set in G is the *dual mutual-visibility number* $\mu_{\mathrm{d}}(G)$. Similarly, if $\mathcal{X}$-visibility is required for every two vertices u, v when at least one of them is from $\mathcal{X}$, then $\mathcal{X}$ is an *outer mutual-visibility set* and the maximum size of such a set is denoted by $\mu_{\mathrm{o}}(G)$. It holds by definition that

$$\mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{d}}(G) \leq \mu(G) \qquad \text{and} \qquad \mu_{\mathrm{t}}(G) \leq \mu_{\mathrm{o}}(G) \leq \mu(G). \tag{7.1}$$

It follows that the four parameters are equal for each (μ, μ_{t})-graph.

As a consequence of (7.1), Theorems 3.2 and 4.1, it is easy to determine the four mutual-visibility parameters for Kneser graphs $KG(n,2)$ with $n \geq 8$. By doing so, we answer a question of Cicerone *et al.* posed in [11].

Proposition 7.4. *It holds for every integer n with $n \geq 8$ that*

$$\mu(KG(n,2)) = \mu_{\mathrm{d}}(KG(n,2)) = \mu_{\mathrm{o}}(KG(n,2)) = \mu_{\mathrm{t}}(KG(n,2)) = \binom{n}{2} - 4.$$

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