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Paired domination in graphs with minimum degree four

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A B S T R A C T

A set S of vertices in a graph G is a paired dominating set if every vertex of G is adjacent to a vertex in S and the subgraph induced by S admits a perfect matching. The minimum cardinality of a paired dominating set of G is the paired domination number $\gamma_{\rm pr}(G)$ of G. We show that if G is a graph of order n and $\delta(G) \ge 4$, then $\gamma_{\rm pr}(G) \le \frac{10}{17}n < 0.5883n$.

1. Introduction

A set S of vertices in a graph G is a *dominating set* if every vertex in $V(G) \setminus S$ is adjacent to a vertex in S. The *domination number* $\gamma(G)$ of G is the minimum cardinality among all dominating sets of G. The study of domination in graphs is a vibrant and growing area of research in graph theory. A thorough treatment of this topic can be found in the recent so-called "domination books" [10–12,18]. A *paired dominating set*, abbreviated PD-set, of G is a dominating set S of G such that the induced subgraph $G[S]$ contains a perfect matching M (not necessarily induced). With respect to the matching M, two vertices joined by an edge of M are *paired* and are called *partners* in S. The *paired domination number*, $\gamma_{\rm pr}(G)$, of G is the minimum cardinality among all PD-sets of G. A $\gamma_{\rm pr}$*-set* of G is a PD-set of G of minimum cardinality. Necessarily, the paired domination number of a graph is an even integer. Paired domination in graphs is well studied in the literature, and was first studied by Haynes and Slater [13,14] in 1995. Surveys on paired domination in graphs can be found in [7,8]. In this paper, we study bounds on the paired domination number of a graph with minimum degree at least 4.

1.1. Graph theory notation and terminology

For notation and graph theory terminology, we generally follow [12]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. If S is a set of vertices in G, the graph $G - S$ denotes the graph obtained from G by removing the vertices (and their incident edges) from S. If $S = \{v\}$, then we simply write $G - v$ rather than $G - \{v\}$. The subgraph induced by the set S is given by $G[S]$. The *open* (resp., *closed*) *neighborhood* of the set $S \subseteq V(G)$ is the union of the open (resp., closed) neighborhoods of vertices in S, denoted by $N_G(S)$ (resp., $N_G[S]$).

We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$. The minimum and maximum degree in G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. A graph G is k*-regular* if every vertex has degree k in G. A 3-regular graph is commonly referred

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Table 1
Best known upper bounds on $\gamma_{\rm pr}(G)$ with small minimum degree $\delta \in [3]$.

1998:	$\delta(G) \ge 1$	$\Rightarrow$	$\gamma_{\rm pr}(G)$	$\le$	$n-1$	([14])
1998:	$\delta(G) \ge 2$	$\Rightarrow$	$\gamma_{\rm pr}(G)$	$\le$	$\dfrac{2}{3}\,n$	([14,19])
2022:	$\delta(G) \ge 3$	$\Rightarrow$	$\gamma_{\rm pr}(G)$	$<$	$\dfrac{127}{200}\,n = 0.635\,n$ [a]	([17])

[a] The precise bound given in [17] is $\gamma_{\rm pr}(G) \le \frac{19037}{30000}n < 0.634567\,n$.

Table 2
Best known upper bounds on $c_{\rm pdom,k}$ for small $k \in [3]$.

		$c_{\rm pdom,1}$	$=$	1	([14])
		$c_{\rm pdom,2}$	$=$	$\frac{2}{3}$	([14,15])
$\frac{3}{5}$	$\le$	$c_{\rm pdom,3}$	$<$	$\frac{3}{5} + \frac{13}{375}$	([6,9,17])

to a *cubic graph* in the literature. An *isolated vertex* is a vertex of degree 0. If X is a set of vertices of G, then $\deg_X(v)$ is the number of neighbors of v in G that belong to the set X. In the special case when $X = V(G)$, we note that $\deg_X(v) = \deg_G(v)$. An F*-component* of G is a component of G isomorphic to F. We denote a path and cycle on n vertices by P_n and C_n, respectively.

2. Motivation and known results

In 1998 Haynes and Slater [14] proved that if G is a connected graph of order $n \ge 3$, then $\gamma_{\rm pr}(G) \le n-1$ and they characterized the extremal family of graphs achieving equality in this bound. Moreover, Haynes and Slater [14] proved that if G is a connected graph of order n and $\delta(G) \ge 2$, then $\gamma_{\rm pr}(G) \le \frac{2}{3}n$. Their proof contained an error, which was subsequently corrected by Huang and Shan [19]. The graphs achieving equality in this $\frac{2}{3}n$-bound were characterized in [15]. In 2007 Chen, Sun, and Xing [6] proved that if G is a cubic graph of order n, then $\gamma_{\rm pr}(G) \le \frac{3}{5}n$. Goddard and Henning [9] in 2009 showed that the only connected cubic graph achieving equality in this $\frac{3}{5}n$-bound is the Petersen graph, and conjectured that if we exclude this exceptional graph, then the bound can be improved to $\gamma_{\rm pr}(G) \le \frac{4}{7}n$. Lu, Wang, Wang, and Wu [21] in 2019 proved the conjecture in the special case of claw-free graphs. Kosari, Shao, Shi, Sheikholeslami, Chellali, Khoeilar, and Karami [20] in 2022 proved the full conjecture. In 2022 Henning, Pilśniak, and Tumidajewicz [17] proved that if G is a graph of order n and $\delta(G) \ge 3$, then $\gamma_{\rm pr}(G) < \frac{127}{200}n = 0.635n$. We summarize the best known upper bounds to date on the paired domination number for graphs with small minimum degree in Table 1.

For $k \ge 1$, let $\mathcal{G}_k$ denote the class of all connected graphs with minimum degree at least k containing no isolated edge. We note that $\mathcal{G}_1$ is the class of all connected graphs of order at least 3. The following problem is posed in [16]

Problem 1. ([16]) Determine or estimate the best possible constants $c_{\rm pdom,k}$ (which depends only on k) such that $\gamma_{\rm pr}(G) \le c_{\rm pdom,k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\rm pdom,k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma_{\rm pr}(G)}{n(G)}.$$

The constants $c_{\rm pdom,k}$ are surprisingly only known for $k=1$ and $k=2$. We summarize the best known results to date on the constants $c_{\rm pdom,k}$ for small $k \in [3]$ in Table 2.

3. Main result

It remains an open problem to determine the best possible upper bound on the paired domination number of a connected graph with minimum degree at least 4 in terms of its order n. By the result in [17], we have $\gamma_{\rm pr}(G) < 0.634567\,n$. In this paper, we present an improvement of this best known upper bound to date on the paired domination number of a connected graph with minimum degree at least 4. We shall prove the following result, a proof of which is given in Section 4.

Theorem 1. *If G is a graph of order n with $\delta(G) \ge 4$, then $\gamma_{\rm pr}(G) \le \frac{10}{17}n < 0.5883n$.*

In the 4-regular graph $G = H_8$ of order $n = 8$ illustrated in Fig. 1, every edge is incident with a triangle. Therefore, $|N_G[u] \cup N_G[v]| \le 7$ holds for any two adjacent vertices u and v. Thus, two adjacent vertices cannot form a dominating set

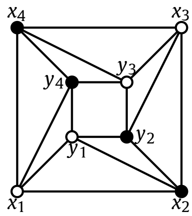

Fig. 1. The graph H_8.

of G, and so $\gamma_{\rm pr}(G) > 2$. Since the paired domination number of a graph is an even integer, this shows that $\gamma_{\rm pr}(G) \ge 4$. The set $\{x_2, x_4, y_2, y_4\}$ (indicated by the shaded vertices in Fig. 1) is an example of a PD-set of G where x_2 and y_2 are paired and x_4 and y_4 are paired, and so $\gamma_{\rm pr}(G) \le 4$. Consequently, $\gamma_{\rm pr}(G) = 4 = \frac{1}{2}n$. As a consequence of Theorem 1, this yields the following bounds on $c_{\rm pdom,4}$.

Corollary 1. $\frac{1}{2} \le c_{\rm pdom,4} \le \frac{10}{17} = \frac{1}{2} + \frac{3}{34}$.

4. Proof of Theorem 1

We define the *boundary* $\partial_G(S)$ of a set $S \subseteq V(G)$ in a graph G as all neighbors of vertices of S that belong outside the set S, that is, $\partial_G(S) = N_G[S] \setminus S$. We present the definition of a colored graph according to [17] and remark that this notion has a similar flavor to that of a residual graph defined in [1] (and also in other papers, such as in [2–5]).

Definition 1. ([17]) Let G be a graph and let S be a set of vertices such that $G[S]$ contains a perfect matching. The *colored graph* G_S of G associated with the set S is the graph obtained from G as follows:

(a) A vertex is colored **amber** if it has no neighbor in S.
(b) A vertex is colored **beige** if it has a neighbor in S and a neighbor not dominated by S.
(c) A vertex is colored **cyan** if it and all its neighbors are dominated by S.
(d) All edges not incident to amber vertices are removed from G_S. Thus every edge in G_S joins two amber vertices or a beige and an amber vertex.

By Definition 1, each vertex in the colored graph G_S is colored amber, beige or cyan. In particular, a vertex in S is colored cyan. We let A, B, and C be the set of amber, beige, and cyan vertices, respectively, in G_S, and so (A, B, C) is a partition of $V(G)$. The *amber graph* is defined in [17] as the graph $G[A]$ induced by the set A of amber vertices. The number of amber and beige vertices adjacent to a vertex v in G_S is the *amber-degree* and *beige-degree*, respectively, of v, and is denoted by $\deg_A(v)$ and $\deg_B(v)$, respectively. The maximum amber-degree of a vertex in A (resp., B) is denoted by $\Delta_A(A)$ (resp., $\Delta_A(B)$). If v is an amber vertex, then its amber and beige neighbors are given by $N_A(v)$ and $N_B(v)$, respectively. We let $N_A[v] = N_A(v) \cup \{v\}$.

As observed in [17], an amber vertex has no cyan neighbor, and therefore its degree in G is the sum of its amber-degree and beige-degree in the colored graph G_S. Hence, the number of amber and beige neighbors of an amber vertex v in G_S is precisely its degree in G, namely $\deg_G(v) \ge \delta(G) \ge 4$. By construction of the colored graph, a beige vertex has at least one amber neighbor and possibly beige neighbors but has no cyan neighbor in G_S. Moreover, if v is a beige vertex in G_S, then at least one of its neighbors in G is colored cyan in G_S.

We are now in a position to prove Theorem 1. Recall its statement.

Theorem 1 *If G is a graph of order n with $\delta(G) \ge 4$, then $\gamma_{\rm pr}(G) \le \frac{10}{17}n < 0.5883n$.*

Proof. If $\delta(G) \ge 4$ and G contains an edge $e = uv$ such that $\deg_G(u) \ge \deg_G(v) \ge 5$, we delete this edge and obtain the graph $G' = G - e$. Then, the condition $\delta(G') \ge 4$ remains valid and $\gamma_{\rm pr}(G') \ge \gamma_{\rm pr}(G)$ also holds. Sequentially deleting edges between vertices of degree higher than 4, we obtain a graph G'' with $\delta(G'') = 4$ and $\gamma_{\rm pr}(G'') \ge \gamma_{\rm pr}(G)$ such that $\{v \in V(G'') : \deg_{G''}(v) \ge 5\}$ is an independent vertex set in G''. We may assume in the proof that G already satisfies these properties. Since the paired domination number of a disconnected graph is the sum of the paired domination numbers of its components, we will also assume that G is connected.

Let G be a connected graph of order n such that $\delta(G) = 4$ and every edge is incident with at least one vertex of degree 4. Let D be a subset of $V(G)$ such that $G[D]$ contains a perfect matching. We define $B_{\ge 4}$ as the set of beige vertices of degree at least 4 in G_D, and we define B_i as the set of beige vertices of degree exactly i in G_D for $i \in [4]$. Thus, every (beige) vertex in B_i has exactly i amber neighbors for $i \in [4]$. The weight $\mathrm{w}(v)$ of a vertex v is defined according to Table 3.

Table 3
The weight $w(v)$ of a vertex v

Set containing v	A	$B_{\geq 4}$	B_3	B_2	B_1	C
$w(v)$	45	33	31	29	27	0

Table 4
Change of weight from an amber vertex A to a beige vertex B_i

$A \to B_{\geq 4}$	$A \to B_3$	$A \to B_2$	$A \to B_1$
12	14	16	18

Table 5
Change of weight from a beige vertex in B_{i+1} to a beige vertex in B_i

$B_4 \to B_3$	$B_3 \to B_2$	$B_2 \to B_1$	$B_1 \to C$
2	2	2	27

We define the weight of the colored graph G_D as the sum of the weights of its vertices, and so

$$w(G_D) = \sum_{v \in V(G)} w(v) = 45|A| + 33|B_{\geq 4}| + 31|B_3| + 29|B_2| + 27|B_1|.$$

The set D is a PD-set in G if and only if all vertices are colored cyan in G_D, in which case $w(G_D) = 0$. Given the set $D \subseteq V(G)$ such that $G[D]$ contains a perfect matching and a set $S \subseteq V(G) \setminus D$ where $G[S]$ contains a perfect matching (and so, $|S| \geq 2$ is even), we define

$$\xi(S) = w(G_D) - w(G_{D \cup S})$$

as the decrease of the weight when extending D to $D \cup S$. Such a set S is a D-*desirable set* if

$$\xi(S) \geq 76.5|S|.$$

If D and S are clear from the context, we set $D' = D \cup S$ and denote by A', B' and C' the set of amber, beige, and cyan vertices, respectively, in $G_{D'}$. We define B'_i as the set of beige vertices of degree exactly i in $G_{D'}$ for $i \in [4]$ and where $B'_{\geq 4}$ is the set of beige vertices of degree at least 4 in $G_{D'}$. The change of weight from an amber vertex $v \in A$ in G_D to a beige vertex in $B'_{\geq 4}$ or to beige vertex in B'_i in $G_{D'}$ where $i \in [3]$ is given in Table 4.

We note that the weight of an amber vertex in G decreases by at least 12 when recolored beige in $G_{D'}$. Next, we prove our key claim that if the weight of the colored graph G_D is positive, then there exists a D-desirable set.

Fact 1. If $w(G_D) > 0$, then the graph G contains a D-desirable set.

Proof. Suppose, to the contrary, that $w(G_D) > 0$, but the graph G does not contain a D-desirable set with the given property. Suppose that e is an edge incident with $y \in B_i$ in G_D for some $i \in [4]$ and the edge e joins y to an amber vertex. If the edge e is removed from G_D, then y has exactly $i-1$ amber neighbors, implying that either y is recolored cyan or y belongs to the set B_{i-1} in the colored graph $G_D - e$. By definition of the weights of the vertices given in Table 3, this in turn implies that the removal of the edge e decreases the weight of y by 2 if i equals 4, 3, or 2 and by 27 if $i = 1$. The change of weight from a beige vertex in B_i to a beige vertex in B_{i-1} where $i \in [4] \setminus \{1\}$, and from a beige vertex in B_1 to a cyan vertex is given in Table 5. We will frequently refer to these observations in the proof.

In the proof, we present a series of subclaims stating structural properties of G_D which culminate in the implication of its nonexistence. Throughout the proof of the claim, we let v be an amber vertex of maximum amber-degree, and we let w be a beige vertex of maximum amber-degree. Thus, $\deg_A(v) = \Delta_A(A)$ and $\deg_A(w) = \Delta_A(B)$.

Let $X = \partial(N_A[v])$ be the boundary of the set $N_A[v]$ in the amber graph $G[A]$, and so X is the set of amber vertices that do not belong to $N_A[v]$ but have a neighbor in $N_A(v)$. Recall that if $x \in V(G)$, then $\deg_X(x)$ is the number of neighbors of x that belong to the set X. Among all amber neighbors of v, let v' be chosen so that $\deg_X(v')$ is a maximum.

Let $Y = \partial(N_A[w])$ be the boundary of the set $N_A[w]$ in the amber graph $G[A]$, and so Y is the set of amber vertices that do not belong to $N_A[w]$ but have a neighbor in $N_A(w)$. Among all amber neighbors of w, let w' be chosen so that $\deg_Y(w')$ is a maximum.

Claim 1. $\Delta_A(A) \leq 4$.

Proof. Suppose, to the contrary, that $\Delta_A(A) \ge 5$, and so $\deg_A(v) \ge 5$. By the edge-minimality of G, every neighbor of v has degree of at most 4 in G. Let z be an arbitrary amber neighbor of v.

Suppose first that $\deg_X(z) = 0$, and so $N_A[z] \subset N_A[v]$. Thus, all amber neighbors of z different from v are common amber neighbors of z and v in G. In this case, we let z' be an amber neighbor of v not in $N_A[z]$. We note that z' exists since $\deg_A(v) \ge 5$ and $4 \ge \deg_A(z)$. We now let $S = \{v, z'\}$, and so $D' = D \cup S$. The amber vertices v, z and z' are colored cyan in the colored graph $G_{D'}$, while the further amber neighbors of v are recolored as beige vertices and therefore belong to the set B'_i in $G_{D'}$ for some $i \in [3]$ or are recolored as cyan vertices. We infer that each such neighbor of v has a weight decrease of at least 14 (see Table 4). Therefore the total weight resulting from the set $S = \{v, z'\}$ is at least $3 \times 45 + 3 \times 14 = 177 > 153 = 76.5|S|$. Thus, the set S is a D-desirable set, a contradiction.

Hence, $\deg_X(z) \ge 1$, and so $N_A[z] \not\subseteq N_A[v]$ and z has at least one neighbor that belongs to the set X. We now let $S = \{v, z\}$. The amber vertices v and z are colored cyan in the colored graph $G_{D'}$. As before, every amber neighbor of v different from z is recolored as a beige vertex $z' \in B'_i$ in $G_{D'}$ for some $i \in [3]$ or is recolored cyan, and therefore has a weight decrease of at least 14, while every neighbor of z that belongs to X is recolored in $G_{D'}$ as a beige vertex that belongs to the set $B'_{\ge 4}$ or B'_i for some $i \in [3]$ or is recolored cyan, and therefore has a weight decrease of at least 12 (see Table 4). We therefore infer that the weight decrease in the colored graph $G_{D'}$ is at least $2 \times 45 + 4 \times 14 + 1 \times 12 = 158 > 153 = 76.5|S|$. Thus, the set S is a D-desirable set, a contradiction. □

Claim 2. $\Delta_A(A) \le 3$.

Proof. Suppose, to the contrary, that $\Delta_A(A) = 4$. Let $v \in A$ be a vertex with $\deg_A(v) = 4$ and v' be the amber neighbor of v as defined at the beginning of the proof. We set $S = \{v, v'\}$. Suppose that $\deg_X(v') = 0$, and so $N_A[v'] \subseteq N_A[v]$. Thus by our choice of v', every amber neighbor of v is recolored cyan in $G_{D'}$, resulting in a total weight decrease of at least $5 \times 45 = 225 > 153 = 76.5|S|$, implying that the set S is a D-desirable set, a contradiction. Hence, $\deg_X(v') \ge 1$.

Consider first the case when $\deg_X(v') = 1$ and define $S = \{v, v'\}$ as before. Every neighbor of v' that belongs to X is recolored as a beige vertex $u \in B'_i$ in $G_{D'}$ for some $i \in [3]$ or is recolored cyan, and therefore has a weight decrease of at least 14. By the choice of v', $\deg_X(z) \le 1$ holds for every $z \in N_A(v)$. Consequently, each neighbor of v different from v' belongs to B'_1 or is recolored cyan in $G_{D'}$ and has a weight decrease of at least 18. We therefore infer that the weight decrease in the colored graph $G_{D'}$ is at least $2 \times 45 + 3 \times 18 + 1 \times 14 = 158 > 76.5|S|$. Thus, the set S is a D-desirable set, a contradiction.

Now suppose that $\deg_X(v') \ge 2$. Let $S = \{v, v'\}$. Every neighbor of v' in $N_X(v)$ and every amber neighbor of v different from v' is recolored as a beige vertex $v \in B'_i$ in $G_{D'}$ for some $i \in [3]$ or recolored cyan. Hence, the weight decrease is at least 14 for each of these five (or six) vertices. The vertices in S are cyan vertices in $G_{D'}$. We therefore conclude that the weight decrease in $G_{D'}$ is at least $2 \times 45 + 5 \times 14 = 160 > 76.5|S|$. This gives the desired contradiction and finishes the proof of the claim $\Delta_A(A) \le 3$. □

Claim 3. $\Delta_A(B) \le 4$.

Proof. Suppose, to the contrary, that $\Delta_A(B) \ge 5$, and so $\deg_A(w) = \Delta_A(B) \ge 5$ where recall that w is a beige vertex of maximum amber-degree. By Claim 2, every neighbor of w has amber-degree at most 3. Let w' be as defined earlier so $\deg_Y(w')$ is maximum in $N_A(w)$. We set $S = \{w, w'\}$. The vertices w and w' are recolored cyan in $G_{D'}$, resulting in a weight decrease of $33 + 45$.

If $\deg_Y(w') = 0$, the same is true for every amber neighbor of w. Thus in this case the vertex w and all its amber neighbors in G_D are colored cyan in $G_{D'}$, resulting in a weight decrease of at least $33 + \deg_A(w) \times 45 \ge 33 + 5 \times 45 = 258 > 76.5|S|$. Thus, the set S is a D-desirable set, a contradiction.

If $\deg_Y(w') = 1$, then each amber neighbor of w different from w' in G_D is in B'_1 or is a cyan vertex in $G_{D'}$. Hence, for each such neighbor, the weight decrease is at least 18. An amber neighbor of w' in Y belongs to $B'_1 \cup B'_2$ in $G_{D'}$, or it is a cyan vertex. In either case, the weight of the vertex decreases by at least 16. We infer that the weight decrease in $G_{D'}$ is at least $33 + 45 + 4 \times 18 + 16 = 166 > 76.5|S|$, a contradiction.

If $\deg_Y(w') \ge 2$, then each amber neighbor of w and w' belongs to a set B'_i for some $i \in [3]$ or is a cyan vertex in $G_{D'}$ and consequently, its weight decreases by at least 14. There are at least six such vertices different from w', and so the weight decrease in $G_{D'}$ is at least $33 + 45 + 6 \times 14 = 162 > 76.5|S|$, a contradiction. □

By Claim 2, we have $\Delta_A(A) \le 3$, and by Claim 3, we have $\Delta_A(B) \le 4$. Thus, $B = B_1 \cup B_2 \cup B_3 \cup B_4$, where we recall that if $w \in B_i$ for $i \in [4]$, then $\deg_A(w) = i$.

Claim 4. $\Delta_A(A) \le 2$.

Proof. Suppose, to the contrary, that $\Delta_A(A) = 3$. Let $S = \{v, v'\}$. Suppose first that $\deg_X(v') = 0$. By our choice of v', every amber neighbor of v is recolored cyan in $G_{D'}$, resulting in a total weight decrease of at least $4 \times 45 = 180 > 76.5|S|$, implying that the set S is a D-desirable set, a contradiction.

If $\deg_X(v')=1$, then $\deg_X(z)\le 1$ holds for every amber neighbor z of v that is not in $N_A[v']$. Thus, $z\in B'_1$ or z is a cyan vertex in $G_{D'}$, and its weight decreases by at least 18. The neighbor of v' that belongs to X is recolored as a beige vertex $v\in B'_i$ in $G_{D'}$ for some $i\in[2]$ or is recolored cyan, and therefore has a weight decrease of at least 16. By our assumption $\Delta_A(A)=3$, every amber vertex has at least one beige neighbor in G_D. In particular, every vertex in $N_A[v]\cup N_A[v']$ has at least one beige neighbor. Thus, $\ell\ge 5$ edges join vertices in $N_A[v]\cup N_A[v']$ to beige vertices. As all vertices in $N_A[v]\cup N_A[v']$ are recolored beige or cyan in $G_{D'}$, these ℓ edges are not present in $G_{D'}$. The removal of them decreases the weight of the beige neighbors of vertices in $N_A[v]\cup N_A[v']$ by at least $2\times\ell$ (see Table 5). If $\ell\ge 6$, then

$$\xi(S)\ge 2\times 45+2\times 18+1\times 16+2\times\ell\ge 154>153=76.5|S|,$$

a contradiction. If $\ell=5$, then every vertex $u\in N_A[v]\cup N_A[v']$ has exactly one beige neighbor and consequently, $\deg_A(u)=3$ holds. For a vertex $z\in N_A(v)$ we also know that $\deg_X(z)\le 1$, that is, at most one amber neighbor of z is outside $N_A[v]$. Therefore, every vertex $z\in N_A(v)$ has at least one neighbor from $N_A(v)$. Since $|N_A(v)|=3$, the subgraph induced by $N_A(v)$ in G_D contains at least two edges. Consequently, there is a vertex $z'\in N_A(v)$ with $\deg_X(z)=0$. Such a vertex z' is a cyan vertex in $G_{D'}$, and we now can estimate the weight decrease and get a contradiction by

$$\xi(S)\ge 3\times 45+1\times 18+1\times 16+2\times 5=179>153=76.5|S|.$$

If $\deg_X(v')\ge 2$, then $N_A[v]\cup N_A[v']$ contains at least six amber vertices. In the colored graph $G_{D'}$, vertices v and v' are recolored as cyan vertices; the remaining at least four vertices are either cyan or belong to $B'_1\cup B'_2$. For the latter case, the weight decreases by at least 16 for each such vertex in $B'_1\cup B'_2$. In G_D, there are $\ell\ge 6$ edges joining vertices in $N_A[v]\cup N_A[v']$ to beige vertices. The removal of these edges decreases the weight by at least $2\times\ell\ge 2\times 6=12$ (see Table 5). We therefore infer $\xi(S)\ge 2\times 45+4\times 16+12=166>76.5|S|$. Thus, the set S is a D-desirable set, a contradiction. □

By Claim 4, we have $\Delta_A(A)\le 2$. Recall that by Claim 3, we have $\Delta_A(B)\le 4$.

Claim 5. $\Delta_A(B)\le 3$.

Proof. Suppose, to the contrary, that $\Delta_A(B)=4$, and so $\deg_A(w)=4$. Since $\Delta_A(A)\le 2$, every amber neighbor of w has amber-degree at most 2. Let $S=\{w,w'\}$. The vertices w and w' are recolored cyan in $G_{D'}$, resulting in a weight decrease of $33+45$. If $\deg_Y(w')=0$, then the vertex w and all its amber neighbors in G_D are colored cyan in $G_{D'}$, resulting in a weight decrease of at least $33+4\times 45=213>76.5|S|$. Thus, the set S is a D-desirable set, a contradiction. Hence, $\deg_Y(w')\ge 1$.

Every amber neighbor of w' which belongs to Y is recolored in $G_{D'}$ as a beige vertex that belongs to the set B_1 or is recolored cyan, and therefore has a weight decrease of at least 18 (see Table 4). Furthermore, every amber neighbor of w different from w' is recolored in $G_{D'}$ as a beige vertex that belongs to the set B_i for some $i\in[2]$ or is recolored cyan, and therefore has a weight decrease of at least 16 (see Table 4). We note that $\ell\ge 6$ edges join vertices in $N_A[w']\cup N_A(w)$ to beige vertices different from w. As all vertices in $N_A[w']\cup N_A(w)$ are recolored beige or cyan in $G_{D'}$, these ℓ edges are not present in $G_{D'}$. The removal of them decreases the weight of the beige neighbors of vertices in $N_A[w']\cup N_A(w)$ by at least $2\times\ell\ge 2\times 6=12$ (see Table 5). We therefore infer that the weight decrease in the colored graph $G_{D'}$ is at least $33+45+3\times 16+18+12=156>76.5|S|$. Thus, the set S is a D-desirable set, a contradiction. □

By Claim 5, we have $\Delta_A(B)\le 3$. As Claim 4 states $\Delta_A(A)\le 2$, and so the graph $G[A]$ induced by the amber vertices in G_D consists of path and cycle components.

Claim 6. *$G[A]$ contains no component that is a path P_k of order $k\ge 3$.*

Proof. Suppose, to the contrary, that $G[A]$ contains a component that is a path $P\colon v_1v_2\dots v_k$ for some $k\ge 3$. If $k=3$, then let u be a beige neighbor of v_2 and set $S=\{v_2,u\}$. In $G_{D'}$, the four vertices v_1,v_2,v_3,u are recolored cyan. Without counting the further weight decreases, we get $\xi(S)\ge 3\times 45+27=162>76.5|S|$, a contradiction.

If P is a path component on $k\ge 4$ vertices, choose $S=\{v_2,v_3\}$. Vertices v_1,v_2,v_3 are colored cyan in $G_{D'}$, while v_4 is either cyan or belongs to B'_1. Without counting the decreases in the weights of beige vertices, we obtain $\xi(S)\ge 3\times 45+18=153=76.5|S|$. Thus, the set S is a D-desirable set, a contradiction. □

Claim 7. *$G[A]$ contains no cycle component.*

Proof. Suppose, to the contrary, that $G[A]$ contains a component that is a cycle $C\colon v_1v_2\dots v_kv_1$ for some $k\ge 3$.

If $k\ge 8$, let $S=\{v_2,v_3,v_6,v_7\}$. In $G_{D'}$, each of the vertices $v_2,\dots v_7$ is cyan. Further, v_1 and v_8 are either cyan or belong to B'_1. Without counting the decrease in the weights of the adjacent beige vertices, we get $\xi(S)\ge 6\times 45+2\times 18=306=76.5|S|$, a contradiction.

If $k = 7$, let $S = \{v_2, v_3, v_6, v_7\}$. In $G_{D'}$, all vertices of C are recolored as cyan vertices. Without counting the further weight decreases, we obtain $\xi \geq 7 \times 45 = 315 > 306 = 76.5|S|$ in $G_{D'}$. Thus, the set S is a D-desirable set, a contradiction.

If $k = 6$, we distinguish two cases. If there is a beige vertex z that has a neighbor, say v_1, from $V(C)$ and a neighbor z' outside $V(C)$, then we consider the set $S = \{z, v_1, v_4, v_5\}$. In this case the six vertices from $V(C)$ together with the vertex z are recolored cyan in $G_{D'}$. The (amber) neighbor z' of z belongs to $B'_1 \cup B'_2$ or is colored cyan in $G_{D'}$. The weight of z' therefore decreases by at least 16. Without counting further decreases in weights, we get that $\xi(S) \geq 6 \times 45 + 29 + 16 = 315 > 306 = 76.5|S|$, once again yielding a contradiction.

In the other cases, all beige neighbors of the vertices on C have neighbors only from $V(C)$. Let $S = \{v_1, v_2, v_4, v_5\}$ and assume that the neighborhood of $V(C)$ contains b_3, b_2, and b_1 vertices from B_3, B_2, and B_1, respectively. There are $\ell \geq 12$ edges between $V(C)$ and these beige vertices, and $\ell = 3b_3 + 2b_2 + b_1$. In $G_{D'}$, all these beige vertices are recolored as cyan, which means the following decrease in the weight of the beige vertices:

$$31b_3 + 29b_2 + 27b_1 > 10(3b_3 + 2b_2 + b_1) = 10\ell \geq 120.$$

Together with the decrease in the weight of vertices from the cycle, we obtain that the weight of G_D decreases by at least $6 \times 45 + 120 = 390 > 306 = 76.5|S|$, which is a contradiction.

If $k = 5$, similarly to the case of a 6-cycle, we consider two possibilities. Suppose firstly that there is a beige vertex z that has a neighbor, say v_1, from $V(C)$ and a neighbor z' outside $V(C)$, consider the set $S = \{z, z', v_3, v_4\}$. In $G_{D'}$, the vertices in $V(C) \cup \{z, z'\}$ are all cyan. There are $\ell \geq 9$ edges in G_D between $V(C) \cup \{z'\}$ and beige vertices different from z. The removal of these edges decreases the weights by at least $2 \times \ell \geq 18$. The total decrease in the weight of the colored graph is at least $6 \times 45 + 29 + 8 = 317 > 306 = 76.5|S|$, which is again a contradiction.

Suppose now that all beige neighbors of the vertices on C have neighbors only from $V(C)$. Let $S = \{v_1, v_2, v_4, v_5\}$ and use the notation introduced for $k = 6$. Now, we have $\ell \geq 10$ edges between $V(C)$ and the beige vertices and $\ell = 3b_3 + 2b_2 + b_1$. In $G_{D'}$, all these vertices are cyan, and the decrease in the weight of the beige vertices is

$$31b_3 + 29b_2 + 27b_1 > 10(3b_3 + 2b_2 + b_1) = 10\ell \geq 100.$$

As all vertices from $V(C)$ are recolored as cyan in $G_{D'}$, the decrease in the weight of the colored graph is at least $5 \times 45 + 100 = 325 > 306 = 76.5|S|$, a contradiction.

If $k = 4$, we simply set $S = \{v_1, v_2\}$. Then, all vertices in $V(C)$ are cyan vertices in $G_{D'}$. It shows that the weight decrease in the colored graph $G_{D'}$ is larger than $4 \times 45 = 180 > 76.5|S|$. This gives the desired contradiction.

Finally, if $k = 3$, we choose $S = \{v_1, z\}$, where z is an arbitrary beige neighbor of v_1. In the colored graph $G_{D'}$, vertices z, v_1, v_2, and v_3 are colored cyan. This fact itself proves that the weight decrease in the colored graph $G_{D'}$ is larger than $3 \times 45 + 27 = 162 > 76.5|S|$. This contradiction finishes the proof of the claim. □

By Claims 6 and 7, the graph $G[A]$ consists of isolated vertices and P_2-components. In particular, $\Delta_A(A) \leq 1$. We show next that beige vertex has amber-degree at most 2.

Claim 8. $\Delta_A(B) \leq 2$.

Proof. Suppose, to the contrary, that $\Delta_A(B) = 3$. Thus, w denotes a beige vertex of amber-degree 3 and w' is an amber neighbor of w with a maximum $\deg_Y(w')$. Let $S = \{w, w'\}$. If $\deg_Y(w') = 0$, then the four vertices in $N[w]$ are recolored cyan in $G_{D'}$ and $\xi(S) \geq 31 + 3 \times 45 = 166 > 76.5|S|$, a contradiction.

If $\deg_Y(w') > 0$, it equals 1 and w' belongs to a P_2-component $w'w''$ in $G[A]$. In $G_{D'}$, vertices w, w', and w'' are all cyan. The other two (amber) neighbors of w are either cyan vertices in $G_{D'}$ or belong to B'_1. This shows that the total decrease in the weight of the colored graph is greater than $31 + 2 \times 45 + 2 \times 18 = 157 > 76.5|S|$, a contradiction. □

By Claim 8, we have $\Delta_A(B) \leq 2$. Thus, $B = B_1 \cup B_2$.

Claim 9. *No beige vertex has a neighbor from a P_1-component and from a P_2-component of $G[A]$.*

Proof. Suppose, to the contrary, that $w \in B_2$ with two neighbors w_1 and w_2 such that w_1 belongs to a P_1-component and $w_2w'_2$ is a path in a P_2-component in $G[A]$. Let $S = \{w, w_2\}$. Since w, w_1, w_2, w'_2 are all colored as cyan vertices in $G_{D'}$, the decrease in the weight of the colored graph is greater than $3 \times 45 + 29 = 164 > 76.5|S|$, a contradiction. □

Claim 10. $B_1 = \emptyset$.

Proof. Suppose, to the contrary, that $B_1 \neq \emptyset$. Suppose first that a vertex $z \in B_1$ has an amber neighbor z' from a P_1-component of $G[A]$. If every beige neighbor of z' belongs to B_1, we simply take $S = \{z, z'\}$ and observe that the entire $N[z']$ is recolored cyan in $G_{D'}$. Consequently, $\xi(S) \geq 45 + 4 \times 27 = 153 = 76.5|S|$, a contradiction. Hence at least one neighbor of z' belongs to B_2.

If $z_2 \in B_2$ is a neighbor of z', let z'_2 be the other neighbor of z_2. By Claim 9, vertex z_2 belongs to a P_1-component in $G[A]$. Let $S = \{z', z_2\}$. In $G_{D'}$, the amber vertices z', z'_2 and the beige vertices z, z_2 are recolored as cyan vertices. There are at least five edges between $\{z', z'_2\}$ and beige vertices different from z and z_2. Therefore, $\xi(S) \geq 2 \times 45 + 27 + 29 + 5 \times 2 = 156 > 76.5|S|$. This contradiction shows that no vertex from B_1 is adjacent to a vertex that belongs to a P_1-component of $G[A]$.

Now, suppose that a vertex $z \in B_1$ has an amber neighbor from a P_2-component z_1z_2 of $G[A]$. If there is a beige vertex z' that has a neighbor z_i from $\{z_1, z_2\}$ and an amber neighbor $u \notin \{z_1, z_2\}$, we set $S = \{z', z_i\}$. In $G_{D'}$, vertices z_1, z_2, z, z' are all cyan, and u belongs to B'_1. Therefore, $\xi(S) \geq 2 \times 45 + 27 + 29 + 18 = 164 > 76.5|S|$, a contradiction. If there is no beige vertex in $N_B[\{z_1, z_2\}]$ with a neighbor different from z_1 or z_2, then let $b_1 = |B_1 \cap N_B[\{z_1, z_2\}]|$ and $b_2 = |B_2 \cap N_B[\{z_1, z_2\}]|$. Since there are at least six edges between $\{z_1, z_2\}$ and $B_1 \cup B_2$, we have $b_1 + 2b_2 \geq 6$. Choosing $S = \{z_1, z_2\}$, all vertices in $\{z_1, z_2\} \cup N_B[\{z_1, z_2\}]$ are recolored cyan in $G_{D'}$ and we have

$$\xi(S) = 2 \times 45 + 27b_1 + 29b_2 \geq 90 + 14.5 \times (b_1 + 2b_2) \geq 90 + 14.5 \times 6 = 177 > 76.5|S|.$$

This contradiction finishes the proof of the claim. □

By Claims 8 and 10, all beige vertices belong to B_2, that is, $B = B_2$. By Claim 9, no beige vertex connects P_1- and P_2-components from $G[A]$. It follows then that $G[A \cup B]$ consists of two types of components. In a *type*-1 *component*, every amber vertex x has $\deg_A(x) = 0$ and they are connected via B_2-vertices. In a *type*-2 *component*, every amber vertex x has $\deg_A(x) = 1$ and the beige neighbors belong to B_2. In the continuation of the proof, we point out that neither type-1 nor type-2 components exist in G_D.

Claim 11. *There is no type*-1 *component in* G_D.

Proof. Suppose, to the contrary, that there is a type-1 component Q^1 in G_D. By our earlier observations, every beige vertex in Q^1 has both its neighbors in P_1-component in $G[A]$. Since every vertex of Q^1 is incident to at least two edges, there is a cycle C^1 in Q^1. Observe that amber and beige vertices alternate along the cycle. Let $C^1 : v_1u_1v_2u_2 \dots v_ku_kv_1$ where $k \geq 2$ and where $v_i \in A$ with $\deg_A(v_i) = 0$ and $u_i \in B_2$ for every $i \in [k]$.

Suppose firstly that k is even. In this case, define

$$S = \bigcup_{i=1}^{\frac{k}{2}} \{v_{2i-1}, u_{2i-1}\},$$

and so, $|S| = k$ and $G[S]$ contains a perfect matching. In $G_{D'}$, all vertices of C^1 are colored cyan decreasing the weight by $45k + 29k = 74k$. There are at least $2k$ edges between the amber vertices on the cycle C^1 and beige vertices that are not included in the cycle. Their removal further decreases the weight by at least $2 \times 2k = 4k$. Therefore, the weight of the colored graph decreases by at least $74k + 4k = 78k > 76.5k = 76.5|S|$. Thus, the set S is a D-desirable set, a contradiction.

Suppose next that $k \geq 3$ is odd. We consider two cases. Suppose firstly that there is a beige vertex z having a neighbor from C^1, say it is v_k, and another (amber) neighbor $z' \notin V(C^1)$. In this case, we let

$$S = \{z, z'\} \cup \left(\bigcup_{i=1}^{\frac{k-1}{2}} \{v_{2i-1}, u_{2i-1}\} \right)$$

and so, $|S| = k + 1$ and $G[S]$ contains a perfect matching. In $G_{D'}$ the vertices from $Z = V(C^1) \cup \{z, z'\}$ are recolored cyan, resulting in a decrease in the weight by $45(k+1) + 29(k+1) = 74(k+1)$. There are at least $2k+2$ edges between $Z \cap A$ and $B \setminus Z$, the removal of which decreases the weights of beige vertices by at least $2(2k+2)$. The total decrease in the weight of G_D is then at least $74(k+1) + 2(2k+2) = 78(k+1) > 76.5(k+1) = 76.5|S|$, a contradiction.

Suppose next that there is no beige vertex that connects $V(C^1) \cap A$ to other amber vertices. In this case the component Q^1 consists of $v_1, \dots, v_k$ and their beige neighbors. Since each v_i has at least four beige neighbors, and every beige vertex has exactly two amber neighbors, we have $|V(Q^1) \cap B| \geq 2k$. We now let

$$S = \bigcup_{i=1}^{\frac{k+1}{2}} \{v_{2i-1}, u_{2i-1}\}$$

and so, $|S| = k + 1$ and $G[S]$ contains a perfect matching. In $G_{D'}$, all vertices of Q^1 are colored cyan, resulting in

$$\xi(S) \geq 45k + 29|V(Q^1) \cap B| \geq 45k + 58k = 103k > 76.5(k+1) = 76.5|S|,$$

as $k \geq 3$. Hence, S is a D-desirable set, and this contradiction proves the claim. □

Claim 12. *There is no type-2 component in G_D.*

Proof. Suppose that Q^2 is a type-2 component which consists of $2k$ amber vertices and their beige neighbors. Since each amber vertex has at least three beige neighbors, and every beige vertex has exactly two amber neighbors, we have $|V(Q^2) \cap B| \geq \frac{3}{2} \times 2k = 3k$. Now, let $S = V(Q^2) \cap A$. Since Q^2 is of type-2, we note that $|S| = 2k$ and $G[S]$ contains a perfect matching. In $G_{D'}$, all vertices of Q^2 are colored cyan and we obtain

$$\xi(S) \geq 45 \times 2k + 29|V(Q^2) \cap B| \geq 90k + 29 \times 3k = 177k > 76.5 \times 2k = 76.5|S|.$$

Consequently, S is a D-desirable set and this contradiction proves the claim. □

Claims 11 and 12 complete the proof of Fact 1. □

We now return to the proof of Theorem 1. By Fact 1, if $\mathrm{w}(G_D) > 0$, then there is a D-desirable set in the graph G. Let $D_0 = \emptyset$ and let $G_0 = G_{D_0}$, and so G_0 is the graph G with all vertices colored amber. We note that $V(G_0) = A$ and $\mathrm{w}(G_0) = 45n$. By Fact 1, there exists a D_0-desirable set S_1, and so letting $D_1 = D_0 \cup S_1 = S_1$ and $G_1 = G_{D_1}$, we have $\mathrm{w}(G_0) - \mathrm{w}(G_1) \geq 76.5|S_1|$, that is,

$$\mathrm{w}(G_1) \leq \mathrm{w}(G_0) - 76.5|S_1|.$$

If $\mathrm{w}(G_1) > 0$, then there is a D_1-desirable set S_2 by Fact 1, and so letting $D_2 = S_1 \cup S_2$ and $G_2 = G_{D_2}$, we have $\mathrm{w}(G_1) - \mathrm{w}(G_2) \geq 76.5|S_2|$, that is,

$$\mathrm{w}(G_2) \leq \mathrm{w}(G_1) - 76.5|S_2|.$$

If $\mathrm{w}(G_2) > 0$, then we continue the process, thereby obtaining a sequence of colored graphs $G_0, G_1, \ldots, G_k$ and a PD-set $D = S_1 \cup \cdots \cup S_k$ of G such that

$$\begin{aligned} 0 = \mathrm{w}(G_k) &\leq \mathrm{w}(G_{k-1}) - 76.5|S_k| \\ &\leq \mathrm{w}(G_0) - 76.5\sum_{i=1}^{k}|S_i| \\ &= 45n - 76.5|D|. \end{aligned}$$

Consequently,

$$\gamma_{\mathrm{pr}}(G) \leq |D| \leq \frac{45}{76.5}n = \frac{10}{17}n < 0.5883\,n,$$

completing the proof of Theorem 1. □

5. Closing comment and conjectures

In this paper we continue the study of upper bounds on the paired domination of a graph with given minimum degree in terms of its order. We establish a best known upper bound to date on the paired domination of a graph with minimum degree at least 4. However, several open problems and conjectures remain.

In 2007 Chen, Sun, and Xing [6] conjectured that if G is a connected graph of order $n \geq 11$ with $\delta(G) \geq 3$, then $\gamma_{\mathrm{pr}}(G) \leq \frac{4}{7}n$. A slightly stronger conjecture was posed in 2009 by Goddard and Henning [9], namely that if G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\mathrm{pr}}(G) \leq \frac{4}{7}n$, unless G is the Petersen graph, in which case $\gamma_{\mathrm{pr}}(G) = \frac{3}{5}n$. These two conjectures have yet to be settled. In this paper, we have shown (see Theorem 1) that if G is a graph of order n and $\delta(G) \geq 4$, then $\gamma_{\mathrm{pr}}(G) \leq \frac{10}{17}n < \frac{3}{5}n$. We do not know if this $\frac{10}{17}n$-upper bound is achievable. Motivated by the $\frac{4}{7}n$-conjectures stated above in [6] and [9], we pose the conjecture that the $\frac{4}{7}n$-upper bound for the paired domination number holds if the minimum degree is at least 4. We state the conjecture formally as follows.

Conjecture 1. *If G is a graph of order n with $\delta(G) \geq 4$, then $\gamma_{\mathrm{pr}}(G) \leq \frac{4}{7}n = \left(\frac{10}{17} - \frac{2}{119}\right)n$.*

In 2014 Desormeaux and Henning in [7] posed the conjecture that if G is a bipartite cubic graph of order n, then $\gamma_{\mathrm{pr}}(G) \leq \frac{1}{2}n$. This conjecture has yet to be settled. Motivated by this $\frac{1}{2}n$-conjecture in [7], we close with the following conjecture.

Conjecture 2. ([7]) *If G is a bipartite 4-regular graph of order n, then $\gamma_{\mathrm{pr}}(G) \leq \frac{1}{2}n$.*

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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