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ORIGINAL PAPER

All generalized rose window graphs are hamiltonian

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Abstract

A *bicirculant* is a regular, d-valent graph that admits a semiregular automorphism of order m having two vertex-orbits of size m. The vertices of each orbit induce a circulant graph of order m and the remaining edges span a regular bipartite graph of valence, say s, $1 \leq s \leq d$, connecting the two vertex-orbits. Generalized Petersen graphs constitute a prominent family of bicirculants, with $d = 3$ and $s = 1$. In 1983, Brian Alspach proved that all generalized Petersen graphs are hamiltonian, except for the family $G(m, 2)$ with $m \equiv 5 \pmod 6$. In this paper we conjecture that among all connected bicirculants of valence at least 2, there are no other exceptions. It follows from various sources that the conjecture is true for all cubic bicirculants. In this paper we prove the conjecture for quartic bicirulants with $s = 2$, also known as the generalized rose window graphs.

Keywords Hamilton cycle · Generalized rose window graphs · Bicirculants · Generalized Petersen graphs · Lovász conjecture

Mathematics Subject Classification 05C45 · 05C76 · 05C70 · 05E18

Dedicated to Brian Alspach, who opened this research area.

Communicated by Jack Koolen.

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1 Introduction

Motivated by the inspiring seminal work of Brian Alspach on generalized Petersen graphs [1] and the subsequent papers on the hamiltonian properties of certain families of cubic graphs [3, 4], we address the problem of existence of a Hamilton cycle in a larger class of bicirculant graphs.

A *bicirculant* is a regular, d-valent graph that admits a semiregular automorphism of order m having two vertex-orbits of size m. The vertices of each orbit induce a circulant graph and the remaining edges span a regular bipartite graph of valence, say s, $1 \leq s \leq d$, connecting the two orbits. Formal definitions are given in Section 2.

In general, a regular graph that admits a semiregular automorphism with $k \geq 1$ vertex-orbits is called a *polycirculant* or sometimes a *multicirculant* [6, 16]. Polycirculants with $k = 1$ vertex-orbits are the circulants. Bicirculants therefore constitute the next case where $k = 2$. While it is relatively easy to show that all circulants are hamiltonian, see [24], the problem which bicirculants are hamiltonian is still widely open. In particular, it is not even known whether all Cayley graphs on dihedral groups are hamiltonian [2]. Note that all Cayley graphs on dihedral groups are bicirculants.

The generalized Petersen graphs are clearly bicirculants. The rims determine the orbits and the spokes constitute a matching between them. In [1] Brian Alspach classified the hamiltonian generalized Petersen graphs: he proved that among the generalized Petersen graphs only the graphs $G(m, 2)$ with $m \equiv 5 \pmod 6$ are not hamiltonian. In this paper we pose the following conjecture:

Conjecture 1.1 Every connected bicirculant, except for the K_2 and the generalized Petersen graphs $G(m, 2)$ with $m \equiv 5 \pmod 6$, is hamiltonian.

As bicirculants, generalized Petersen graphs have parameters $d = 3$ and $s = 1$. However, the whole class of bicirculants with parameters $d = 3$, $s = 1$ consists of I-graphs, first introduced in the Foster Census [10]. The classification of hamiltonian generalized Petersen graphs from [1] was extended to I-graphs in 2017 [8]. It has been proven that all proper I-graphs are hamiltonian. The conjecture therefore holds for all bicirculants with parameters $d = 3$, $s = 1$. Cubic bicirculants fall into three classes, depending on s, with $s = 1, 2, 3$ [25]. Alspach and Zhang [4] dealt with the case $d = s = 3$. Note that bicirculants with $d = s$ are known as cyclic Haar graphs; they are a special class of Cayley graphs on dihedral groups [17]. This essentially covered all connected cubic bicirculants.

In this paper, we take the next step in attacking the case $d = 4$ by resolving the subcase $s = 2$. The bicirculants with parameters $d = 4$ and $s = 2$ are called *generalized rose window graphs* [11]. The following is our main result.

Theorem 1.2 *Every connected generalized rose window graph is hamiltonian.*

The rose window graphs, which were introduced by Steve Wilson in 2008 [26], are contained in the family of generalized rose window graphs. Informally, a rose window graph is obtained from a generalized Petersen graph by adding an additional set of spokes to its edge set that preserves the semiregular symmetry. Rose window graphs turned out to be a very interesting family of graphs. As they belong to the class of bicirculant graphs, they have many symmetries. Some are vertex-transitive [13] and some are even edge-transitive [20] or Cayley [12]. In addition, several of their other properties were studied: isomorphsms [14], domination [19], relation to maps [18, 21], etc.

The relationship between generalized rose window graphs and rose window graphs is analogous to the relationship between I-graphs and generalized Petersen graphs. While generalized Petersen graphs and rose window graphs are necessarily connected, the I-graphs and generalized rose window graphs need not be connected. Moreover, the removal of a matching consisting of spokes from a connected generalized rose window graph results in a disconnected graph whose connected components are I-graphs, unless certain arithmetic conditions that will be specified in the next section are satisfied. Because of this, the existence of a Hamilton cycle in a generalized rose window graph cannot easily follow from the existence of a Hamilton cycle in I-graphs. To prove that all generalized rose window graphs are hamiltonian we had to develop several completely novel tools, which are potentially useful for constructing Hamilton cycles in larger families of bicirculants.

From the point of symmetry, the analogy is more intricate. Although some proper I-graphs may possess symetries not present in any generalized Petersen graph, none of them is vertex-transitive [7]. However, there exist generalized rose window graphs that are Cayley graphs and others that are vertex-transitive and non-Cayley [11].

Therefore, the results presented in this paper are also important in the context of the Lovász conjecture, a variant of which can be stated as: Every finite connected vertex-transitive graph, except for the five known exceptions, is hamiltonian [24]. By Theorem 1.2 we confirm this conjecture within the class of vertex-transitive generalized rose window graphs.

The paper is organized as follows. In Section 2 we give a formal definition of bicirculants and review the basic properties of bicirculants, and in particular generalized rose window graphs. We recall the notions of generalized Petersen graphs and I-graphs [7, 10], as these graphs appear as subgraphs, actually as spanning sub-bicirculants, of the generalized rose window graphs. In Section 3 we also review some known results on hamiltonicity of these graphs. We classify Hamilton cycles in I-graphs into three types. Each of the three types is then used in a different construction in Section 4, where we prove our main result that all generalized rose window graphs are hamiltonian.

The proof can be briefly described as follows. By removing a suitable matching from a generalized rose window graph G we obtain an I-graph H, which can be connected or disconnected. As shown in Section 3, every connected component of this I-graph H contains a Hamilton cycle or path of a special type. These structures provide subpaths that can be combined into a Hamilton cycle of the entire graph G by using some of the removed edges.

In the last section we then discuss the hamiltonian problem for more general bicirculant graphs. As a consequence of Theorem 1.2, combined with the results from [4], we obtain that every connected bicirculant with $d \geq 5$ and $s = d - 2$ is hamiltonian if m is a product of at most three prime powers. In particular, this is true for the Tabačjn graphs [5, 23], pentavalent bicirculants with $s = 3$.

2 Bicirculants and their properties

In this section we give a formal definition of bicirculants, generalized rose window graphs and I-graphs, and recall some of their properties.

A *bicirculant* can be described as follows. Given an integer $m \geq 1$ and sets $R, S, T \subseteq \mathbb{Z}_m$ such that $R = -R$, $T = -T$, $0 \notin R \cup T$ and $0 \in S$, the graph $B(m; R, S, T)$ has vertex set $V = V_1 \cup V_2$, where $V_1 = \{u_0, \ldots, u_{m-1}\}$ and $V_2 = \{v_0, \ldots, v_{m-1}\}$, and edge set

$$E = \{u_i u_{i+j} \mid i \in \mathbb{Z}_m, j \in R\} \cup \{v_i v_{i+j} \mid i \in \mathbb{Z}_m, j \in T\} \cup \{u_i v_{i+j} \mid i \in \mathbb{Z}_m, j \in S\}.$$

Obviously, the mapping $\alpha : V \to V$, defined by $\alpha(u_i) = u_{i+1}$, $\alpha(v_i) = v_{i+1}$ is an automorphism of $B(m; R, S, T)$, having two vertex-orbits of the same size.

The circulant graph induced on the set V_1 is called the *outer rim* and the circulant graph induced on the set V_2 is called the *inner rim*. We call the vertices from V_1 the *outer vertices* and the vertices from V_2 the *inner vertices*. There are three types of edges: the edges adjacent to two outer vertices are called *outer edges*, the edges adjacent to two inner vertices are called *inner edges*, and edges connecting an outer vertex to an inner vertex are called *spokes*. Specifically, the edges $u_i u_{i+a}$, $i \in \mathbb{Z}_m$, $a \in R$, are called *outer edges of type a*, the edges $v_i v_{i+b}$, $i \in \mathbb{Z}_m$, $b \in T$, are called *inner edges of type b* and the edges $u_i v_{i+c}$, $i \in \mathbb{Z}_m$, $c \in S$, are called *spokes of type c*. We will also say that a path is *outer* (*inner*) if all of its vertices are outer (inner) vertices.

In accordance with our previous discussion, we have $s = |S|$. The order of a graph $B(m; R, S, T)$ is $n = 2m$, the valence is d and $|R| = |T| = d - s$. In the study of bicirculants, other authors use similar notation, see for instance [22].

We have already mentioned that generalized rose window graphs are bicirculants. For their description, we need four parameters. Let $m \geq 3$ be a positive integer and $a, b, c \in \mathbb{Z}_m \setminus \{0\}$ with $a, b \neq m/2$. If we take $R = \{a, -a\}$, $S = \{0, c\}$ and $T = \{b, -b\}$, the graph $B(m; R, S, T)$ is a generalized rose window graph, which we will denote by $R(m; a, b, c)$. If $a = 1$, an ordinary rose window graph is obtained.

Figure 1 shows two generalized rose window graphs. The generalized rose window graph $R(12; 3, 4, 2)$ that is presented on the right hand side of the figure is not isomorphic to any rose window graph.

An I-graph $I(m; a, b)$ is a bicirculant $B(m; R, S, T)$ with $m \geq 3$, $R = \{a, -a\}$, $T = \{b, -b\}$ and $S = \{0\}$, where $a, b \in \mathbb{Z}_m \setminus \{0, m/2\}$. Generalized Petersen graphs are a subfamily of I-graphs; an I-graph is isomorphic to a generalized Petersen graph if and only if $\gcd(m, a) = 1$ or $\gcd(m, b) = 1$. We denote the generalized Petersen graph $I(m; 1, k)$ by $G(m, k)$.

As we can see, we keep in the description of a specific family of bicirculants for each pair of parameters $x, -x$ only one parameter. Also, we leave out parameters having constant values, such as 0 or 1.

Some properties of bicirculant graphs can be deduced from the general theory of covering graphs [15]. We will use the following notation. Let $A = \{a_1, a_2, \ldots, a_\ell\}$ be a set and let i be an integer. We define $A - i = \{a - i \mid a \in A\}$ and $A/i = \{a/i \mid a \in$

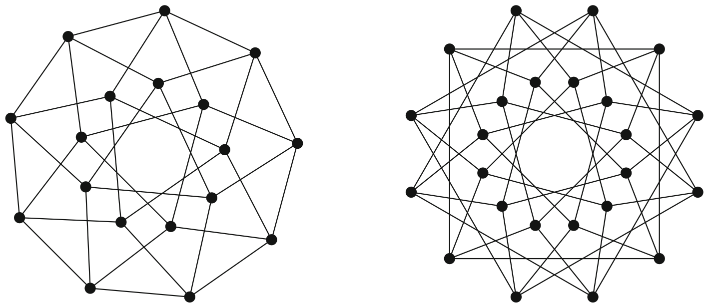

Fig. 1 The generalized rose window graphs $R(9; 1, 3, 2)$ and $R(12; 3, 4, 2)$

$A\}$. Moreover, the notation $\gcd(A)$ should be understood as $\gcd(a_1, a_2, \ldots, a_\ell)$ and $\gcd(i, A)$ as $\gcd(i, \gcd(A))$.

Proposition 2.1 *A bicirculant $B(m; R, S, T)$ is connected if and only if $\gcd(m, R, S, T) = 1$. In particular, the generalized rose window graph $R(m; a, b, c)$ is connected if and only if $\gcd(m, a, b, c) = 1$.*

In the case where a bicirculant graph is disconnected, it is composed of isomorphic connected components.

Proposition 2.2 *Let $G = B(m; R, S, T)$. Suppose $\delta = \gcd(m, R, S, T) > 1$. Then G is a disjoint union of δ isomorphic graphs $G_0, \ldots G_{\delta-1}$ such that $u_i \in G_i$, $i = 0, \ldots, \delta - 1$. Moreover, each G_i is connected and isomorphic to the graph $B(m/\delta; R/\delta, S/\delta, T/\delta)$.*

In many cases there exist isomorphic bicirculants with different parametric descriptions. Two special cases are presented below.

Proposition 2.3 *Graph $B(m; R, S, T)$ is isomorphic to the graph $B(m; R, S - c, T)$ for every $c \in S$.*

Proposition 2.4 *Let $G = B(m; R, S, T)$, let $r \in \mathbb{Z}_m$ be such that $\gcd(m, r) = 1$ and let $G' = B(m; rR, rS, rT)$. Then the graph G' is isomorphic to the graph G.*

For example, this property of bicirculants was applied to I-graphs in the proof that all generalized Petersen graphs are unit-distance graphs [27]. This property also implies that a generalized rose window graph $R(m; a, b, c)$ is isomorphic to a rose window graph if $\gcd(m, a) = 1$ or $\gcd(m, b) = 1$.

3 Hamilton cycles in I-graphs

In this section, we consider Hamilton cycles in I-graphs as they will play an essential role in the construction of Hamilton cycles in rose window graphs. Recall that by removing a set of spokes of the same type from a rose window graph, we obtain an I-graph. We classify Hamilton cycles of I-graphs into three types. For each of these types we define a different construction in Section 4, which shows how to combine Hamilton cycles in connected components of a rose window graph to a Hamilton cycle in the whole graph.

Hamilton cycles and paths in I-graphs are guaranteed by the following results. In [1] Brian Alspach showed that every generalized Petersen graph is hamiltonian, except for the family $G(m, 2)$ with $m \equiv 5 \pmod 6$. However, Alspach and Liu showed that all these exceptional graphs have very many Hamilton paths [3, Theorem 4.2].

Theorem 3.1 *([3]) Every pair of non-adjacent vertices in $G(m, 2)$ with $m \equiv 5 \pmod 6$ is connected by a Hamilton path.*

Later, it was shown by Bonvicini and Pisanski [8] that the non-hamiltonian generalized Petersen graphs are the only non-hamiltonian connected I-graphs. Consequently, we have the following theorem.

Theorem 3.2 *([8]) Every connected I-graph, except for the generalized Petersen graphs $G(m, 2)$ with $m \equiv 5 \pmod 6$, is hamiltonian.*

Clearly, a Hamilton cycle in an I-graph $I(m; a, b)$ alternates between paths in the outer rim and paths in the inner rim, which are connected by the spokes. The paths in each rim cover all the vertices of the rim and there are no paths of length zero. This follows from the fact that every vertex of an I-graph is adjacent to exactly one spoke. If all the rim paths of a Hamilton cycle C have length one, then C contains all the spokes of the I-graph and the spokes alternate with the inner/outer edges; in this case we say that the Hamilton cycle C is *alternating*. Otherwise, it is called *non-alternating*.

Non-alternating Hamilton cycles are further divided into two types, the 4-hooked and the 2-hooked Hamilton cycles. See Section 4 for an explanation of these terms. If there exists a labeling of the vertices of the graph $I(m; a, b)$ such that a non-alternating Hamilton cycle C contains the edges $u_0 u_a$, $u_b u_{a+b}$, $v_0 v_b$, $v_a v_{a+b}$, then C is called 4-*hooked*. If there exists a labeling of the vertices of the graph $I(m; a, b)$ such that a non-alternating Hamilton cycle C provides a Hamilton path connecting the vertices v_0 and v_a, or a Hamilton path connecting the vertices u_0 and u_b, then C is called 2-*hooked*. By saying that the Hamilton cycle provides a certain Hamilton path, we mean that starting from the cycle, one can produce the path by replacing one or more of its edges with edges not in that cycle.

Observe that by symmetry, by adding the same number to the subscripts of the vertices of any given Hamilton cycle, we again obtain a (usually different) Hamilton cycle. This fact will play a key role in Section 4.

Lemma 3.4 gives the classification of Hamilton cycles in an I-graph $I(m; a, b)$ when $a \neq \pm b$. We deal with the case when $a = b$ or $a = -b$ separately.

Lemma 3.3 *Let $G = I(m; a, b)$ be a connected I-graph, with $a = b$ or $a = -b$. Then G contains a 2-hooked Hamilton cycle.*

Proof Observe that $\gcd(m, a) = \gcd(m, b) = 1$ since G is connected. Therefore, the sequence $v_0, u_0, u_a, u_{2a}, \ldots, u_{(m-1)a}, v_{(m-1)a}, \ldots, v_{2a}, v_a, v_0$ defines a non-alternating Hamilton cycle, say C. By removing the edge $v_0 v_a$ from C, we obtain a Hamilton path from v_0 to v_a. That means that the graph G contains a 2-hooked Hamilton cycle. □

Lemma 3.4 *Let $G = I(m; a, b)$ be a connected I-graph, with $a \neq \pm b$. Then every Hamilton cycle of G is alternating or 4-hooked or 2-hooked.*

Proof Let C be a Hamilton cycle in the graph G. Then it is either alternating or non-alternating. We assume that the cycle C is non-alternating and we will show that it is either 4-hooked or 2-hooked.

To this end, we define a special type of non-alternating Hamilton cycles, that we call almost alternating, and then we treat separately the cases in which the Hamilton cycle C is almost alternating, and when it is not. The Hamilton cycle C is said to be *almost alternating*, if all of the outer and inner subpaths of the Hamilton cycle C consist of at most two edges and there exists at least one outer or inner subpath with exactly two edges (so the cycle is not alternating).

Case 1: the Hamilton cycle C is almost alternating. Assume that the Hamilton cycle C is almost alternating. Then there exists at least one outer subpath consisting of two edges. We can label the three vertices of the subpath with (u_{-a}, u_0, u_a) and find the subpath (v_{-b}, v_0, v_b) in C accordingly. We thus have the edges in $u_0\, u_a$, $v_0\, v_b$ in C. If the edges $u_b\, u_{a+b}$, $v_a\, v_{a+b}$ are also in C, then the cycle C is 4-hooked and the assertion follows. Otherwise we consider two cases: (a) none of the edges $u_b\, u_{a+b}$, $v_a\, v_{a+b}$ is in C and (b) exactly one of the edges $u_b\, u_{a+b}$, $v_a\, v_{a+b}$ is in C.

Case (a) Assume that none of the edges $u_b\, u_{a+b}$, $v_a\, v_{a+b}$ is in C. Then C has the edges $u_b\, u_{b-a}$ and $v_a\, v_{a-b}$ that, combined with the fact that the inner and outer subpaths of C consist of at most three vertices, imply the existence of the subpaths $(u_{b-a}, u_b, v_b, v_0, v_{-b})$ and $(v_{a-b}, v_a, u_a, u_0, u_{-a}, v_{-a})$ in C.

If $v_{-a}\, v_{b-a}$ is an edge of C, then by adding a modulo m to the subscripts of the vertices of G we see that the cycle C is 4-hooked, since the edges $v_0\, v_b$, $u_b\, u_{b-a}$, $u_0\, u_{-a}$, $v_{-a}\, v_{b-a}$ are turned into the edges $v_a\, v_{a+b}$, $u_b\, u_{a+b}$, $u_0\, u_a$, $v_0\, v_b$.

If the edge $v_{-a}\, v_{b-a}$ is not in C, then we find the subpaths $(v_{b-a}, u_{b-a}, u_b, v_b, v_0, v_{-b})$ and $(v_{a-b}, v_a, u_a, u_0, u_{-a}, v_{-a}, v_{-a-b})$ in C. Two cases can occur: the edge $u_{-b}\, u_{-b-a}$ is in C or not.

If $u_{-b}\, u_{-b-a}$ is in C, then by adding $(a + b)$ modulo m to the subscripts of the vertices of G we see that the cycle C is 4-hooked, since the edges $u_{-b}\, u_{-b-a}$, $v_0\, v_{-b}$, $u_0\, u_{-a}$, $v_{-a}\, v_{-a-b}$ are turned into the edges $u_0\, u_a$, $v_a\, v_{a+b}$, $u_b\, u_{a+b}$, $v_0\, v_b$.

If the edge $u_{-b}\, u_{-b-a}$ is not in C, then the edge $u_{-b}\, u_{a-b}$ is in C and by adding b modulo m to the subscripts of the vertices of G, the edges $u_{-b}\, u_{a-b}$, $v_0\, v_{-b}$, $u_0\, u_a$, $v_a\, v_{a-b}$ are turned into the edges $u_0\, u_a$, $v_0\, v_b$, $u_b\, u_{a+b}$, $v_a\, v_{a+b}$, therefore C is again a 4-hooked Hamilton cycle.

Case (b) Now assume that exactly one of the edges $u_b\, u_{a+b}$, $v_a\, v_{a+b}$ is in C. For the case where $u_b\, u_{a+b}$ is in C but $v_a\, v_{a+b}$ is not in C, we can repeat the same arguments as above when the edge $u_{-b}\, u_{a-b}$ is in C, and also when both edges $u_{-b}\, u_{-a-b}$, $v_{-a}\, v_{-a-b}$ are in C. In the missing case, that is, when $u_{-b}\, u_{-a-b}$ and $v_{-a}\, v_{b-a}$ are edges of C, we can find a Hamilton path from u_0 to u_b, or from v_0 to v_a in G, so the cycle C is 2-hooked. More specifically, we consider the vertices in clockwise order, and we can always assume that v_b, v_0 occur in that order. The vertices occur in C in one of the following orders: v_{a+b}, u_{a+b}, u_b, v_b, v_0, v_{a-b}, v_a, u_a, u_0, u_{-a}, or v_{a+b}, u_{a+b}, u_b, v_b, v_0, u_{b-a}, v_{b-a}, v_{-a}, u_{-a}, u_0, u_a. In the first case we remove the edges $u_a\, v_a$, $u_{a+b}\, v_{a+b}$, add the chord $v_a\, v_{a+b}$, and find a Hamilton path from u_a to u_{a+b}

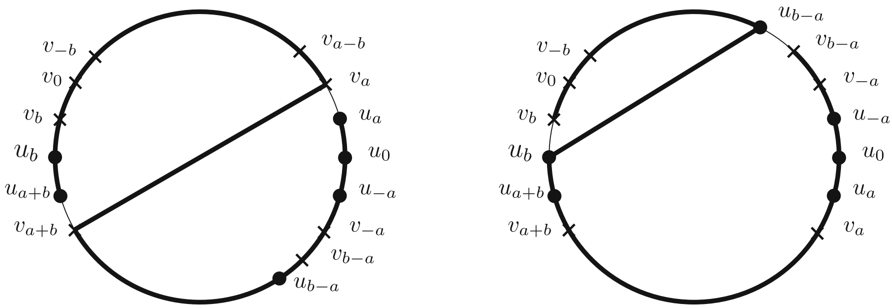

Fig. 2 The almost alternating Hamilton cycle C described in Lemma 3.4 when the edge $u_b\, u_{a+b}$ is in C, but the edge $v_a\, v_{a+b}$ is not. The bold lines define a Hamilton path from u_a to u_{a+b} in the cycle on the left-hand side of the figure and a Hamilton path from v_b to v_{b-a} in the cycle on the right-hand side

that yields a Hamilton path from u_0 to u_b if we add $(-a)$ modulo m to the subscripts of the vertices of G; see the graph on the left-hand side of Figure 2. In the second case we remove the edges $u_b\, v_b$, $u_{b-a}\, v_{b-a}$, add the chord $u_b\, u_{b-a}$, and find a Hamilton path from v_{b-a} to v_b that yields a Hamilton path from v_0 to v_a if we add $(a-b)$ modulo m to the subscripts of the vertices of G; see the graph on the right-hand side of Figure 2.

For the case where $v_a\, v_{a+b}$ is in C but $u_b\, u_{a+b}$ is not in C we can repeat the same arguments as for the case where $u_b\, u_{a+b}$ is in C but $v_a\, v_{a+b}$ is not in C by symmetry. The validity of the lemma is thus proved for the almost alternating cycles of an I-graph.

Case 2: the Hamilton cycle C is not almost alternating. Assume now that the Hamilton cycle C is not almost alternating. Then we find at least one outer or inner subpath in C consisting of at least three edges; by symmetry we may assume that such a path is an outer subpath and we can label the four vertices of the subpath with $(u_{-a}, u_0, u_a, u_{2a})$. We find the inner subpaths (v_{-b}, v_0, v_b), (v_{a-b}, v_a, v_{a+b}) in C accordingly. We therefore find the edges $v_0\, v_b$, $u_0\, u_a$, $v_a\, v_{a+b}$ in C. If the edge $u_b\, u_{a+b}$ is also in C, then the cycle C is 4-hooked and the assertion follows.

We now assume that the edge $u_b\, u_{a+b}$ is not in C. Then the Hamilton cycle C is 2-hooked – and hence the assertion follows – if the vertices occur in C in some prescribed orders. More specifically, in the following we consider the vertices of C in clockwise order; we can always assume that the vertices u_a, u_0 occur in that order; we also set $\{v_x, v_{x'}\} = \{v_{a+b}, v_{a-b}\}$, $\{v_y, v_{y'}\} = \{v_b, v_{-b}\}$.

If the vertices occur in order $u_a, u_0, v_x, v_a, v_{x'}, v_y, v_0, v_{y'}$, we can find a Hamilton path from v_0 to v_a in G. In fact, at least one of the equalities $x - y = a$ or $x - y' = a$ holds. If $x - y = a$ (respectively, $x - y' = a$), then we remove the edges $u_0\, u_a$, $v_a\, v_x$, $v_0\, v_y$ (respectively, $u_0\, u_a$, $v_a\, v_x$, $v_0\, v_{y'}$) in C, add the chords $u_0\, v_0$, $u_a\, v_a$, and find a Hamilton path from v_x to v_y (respectively, from v_x to $v_{y'}$) that yields a Hamilton path from v_0 to v_a, since $x - y = a$ (respectively, $x - y' = a$); see the first two graphs of Figure 3. We can find a Hamilton path from v_0 to v_a even in the case where the vertices occur in the order $u_a, u_0, v_y, v_0, v_{y'}, v_x, v_a, v_{x'}$, with $x' - y = a$. In fact, in this case we remove the edges $u_0\, u_a$, $v_a\, v_{x'}$, $v_0\, v_y$ in C, add the chords $u_0\, v_0$, $u_a\, v_a$,

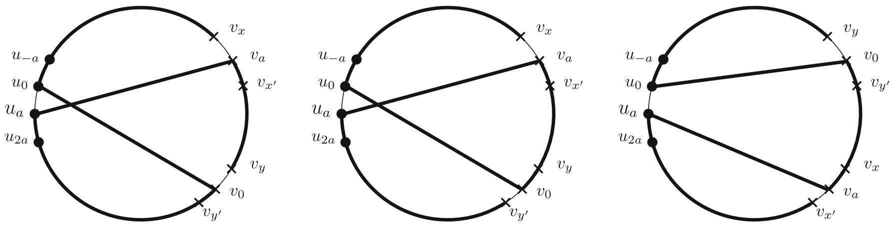

Fig. 3 The Hamilton cycle C described in Lemma 3.4: the cycle is not almost alternating and the vertices occur in the order $u_a, u_0, v_x, v_a, v_{x'}, v_y, v_0, v_{y'}$ with $\{v_x, v_{x'}\} = \{v_{a+b}, v_{a-b}\}$, $\{v_y, v_{y'}\} = \{v_b, v_{-b}\}$ in the first two cycles and in the order $u_a, u_0, v_y, v_0, v_{y'}, v_x, v_a, v_{x'}$ with $x' - y = a$ in the third cycle. The bold lines in the first two cycles define a Hamilton path from v_x to v_y, and a Hamilton path from v_x to $v_{y'}$. The bold lines in the third cycle define a Hamilton path from v_y to $v_{x'}$

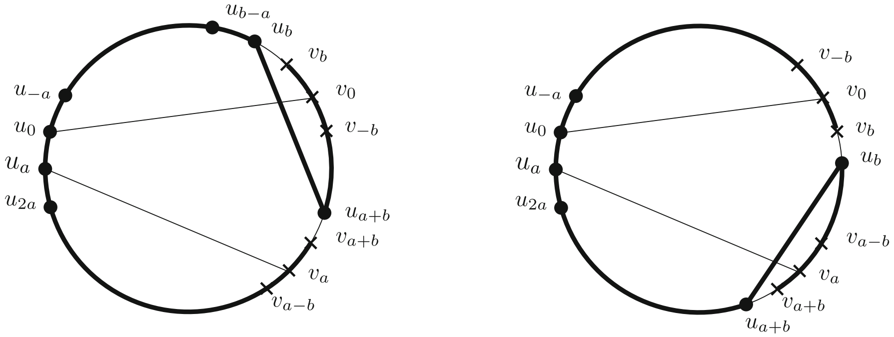

Fig. 4 The Hamilton cycle C described in Lemma 3.4 when the cycle is not almost alternating, does not contain the edge $u_b\, u_{a+b}$, and the vertices occur in the order $u_a, u_0, v_y, v_0, v_{y'}, v_x, v_a, v_{x'}$ with $x' - y \neq a$. Furthermore, $v_y = v_b$ and $v_{x'} = v_{a-b}$ in the cycle on the left-hand side and $v_y = v_{-b}$ and $v_{x'} = v_{a+b}$ in the cycle on the right-hand side. The bold lines define a Hamilton path from v_b to v_{a+b}

and find a Hamilton path from v_y to $v_{x'}$ that yields a Hamilton path from v_0 to v_a, since $x' - y = a$; see the third graph of Figure 3.

It remains to consider the case in which the edge $u_b\, u_{a+b}$ is not in C, and the vertices occur in the order u_a, u_0, v_y, v_0, $v_{y'}$, v_x, v_a, $v_{x'}$, with $x' - y \neq a$, i.e., $(v_y, v_{x'}) = (v_b, v_{a-b})$ or $(v_y, v_{x'}) = (v_{-b}, v_{a+b})$. The nonexistence of the edge $u_b\, u_{b+a}$ in C implies the existence of the subpaths $(u_{b-a}, u_b, v_b, v_0, v_{-b})$ and $(u_{b+a}, v_{b+a}, v_a, v_{a-b})$ in C. In this setting, we remove the edges $u_b\, v_b$, $u_{a+b}\, v_{a+b}$ from C, and add the edge $u_b\, u_{a+b}$; we find a Hamilton path from v_b to v_{a+b} that provides a Hamilton path from v_0 to v_a if we add $-b$ modulo m to the subscripts of the vertices of G; see Figure 4. Therefore, the cycle C is 2-hooked and the assertion follows. □

When constructing a Hamilton cycle in a rose window graph using a 4-hooked Hamilton cycle in its subgraph that is an I-graph, certain orderings of vertices are difficult to deal with. A 4-hooked Hamilton cycle C is called *elusive (of type* 1 *or of type* 2*)*, if the vertices $u_0, u_a, u_b, u_{a+b}, v_0, v_b, v_a, v_{a+b}$ appear on C in one of the orders (1) or (2), starting with vertices u_0, u_a:

$$u_0, u_a, u_b, u_{a+b}, v_{a+b}, v_a, v_b, v_0, \tag{1}$$

$$u_0, u_a, v_a, v_{a+b}, v_0, v_b, u_b, u_{a+b}. \tag{2}$$

Otherwise, a 4-hooked Hamilton cycle is called *standard*. Both types of elusive Hamilton cycles are equivalent in a certain way.

Remark 3.5 Let C be an elusive Hamilton cycle of type 2 in an I-graph $I(m; a, b)$. We relabel the vertices by adding $-a$ modulo m to their indices. Then the sequence of vertices $u_0, u_a, v_a, v_{a+b}, v_0, v_b, u_b, u_{a+b}$ is mapped to the sequence $u_{-a}, u_0, v_0, v_b, v_{-a}, v_{-a+b}, u_{-a+b}, u_b$. By reversing the cycle C we see that it contains the sequence $u_0, u_{-a}, u_b, u_{-a+b}, v_{-a+b}, v_{-a}, v_b, v_0$, so C is an elusive Hamilton cycle of type 1 for the graph $I(m; -a, b)$, which is the same graph as $I(m; a, b)$.

Remark 3.6 One can observe that an elusive Hamilton cycle C that may appear in the proof of Lemma 3.4 is either of type 1 with vertices u_0, v_0 not adjacent in C or it is of type 2 with the property that it contains the subpaths $(v_{a+b}, v_a, u_a, u_0, u_{-a}, v_{-a})$ and $(u_{a+b}, u_b, v_b, v_0, v_{-b}, u_{-b})$ occurring in this order in C; see the first part of the proof regarding the almost alternating Hamilton cycles. By Remark 3.5 we may thus assume that such a cycle is also of type 1 and that it contains the subpaths $(v_b, v_0, u_0, u_a, u_{2a}, v_{2a})$ and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b})$ occurring in this order in C.

The next lemma shows that it is almost always possible to replace elusive Hamilton cycles of type 1 in I-graphs with standard Hamilton cycles or certain Hamilton paths.

Lemma 3.7 *Let $a \neq \pm b$ and let an I-graph $G = I(m; a, b)$ contain an elusive Hamilton cycle C of type 1. Then*

- *the graph G contains a standard 4-hooked Hamilton cycle or a 2-hooked Hamilton cycle or*
- *$b \equiv -2a \pmod{m}$ or $a \equiv -2b \pmod{m}$ and the cycle C contains the subpaths (v_b, v_0, u_0, u_a) and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b})$ occurring in this order in C.*

The proof of Lemma 3.7 requires consideration of different orderings of selected vertices on the Hamilton cycle C to find either a standard 4-hooked cycle or a desired Hamilton path. By Remark 3.6, it is enough to consider three cases: I. the path (u_a, u_0, u_{-a}) is contained in C and u_{-a} is not adjacent to v_{-a} in C, II. the path $(u_a, u_0, u_{-a}, v_{-a})$ is contained in C, III. the cycle C contains the subpaths $(v_b, v_0, u_0, u_a, u_{2a}, v_{2a})$ and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b})$ occurring in this order in C.

For brevity, we will omit the detailed discussion of the three subcases, which is rather long, and similar methods are used as in the proof of Lemma 3.4. Sometimes we need to introduce new vertices on the cycle. In particular, vertices u_0 and u_{2a+b} need to be distinct in case III, and therefore we consider the situation $b \equiv -2a \pmod{m}$ or $a \equiv -2b \pmod{m}$ separately in case III. The full proof of Lemma 3.7 is given in the preprint [9].

4 Hamilton cycles in generalized rose window graphs

In this section, we show how to construct a Hamilton cycle in any given generalized rose window graph. We will use the following notation: $G = R(m; a, b, c)$ will denote a connected generalized rose window graph, so $\gcd(m, a, b, c) = 1$. By H we denote the graph obtained from G by removing the spokes of type c. Note that the graph H is an I-graph. It consists of $\gcd(m, a, b)$ isomorphic connected components, each of which is an I-graph. This follows readily from Proposition 2.2 applied to I-graphs, or more directly from Proposition 1 of [7]. If the graph H is connected, then Theorem 3.2 implies G is hamiltonian in case it is not isomorphic to a generalized Petersen graph $G(n, 2)$, $n \equiv 5 \pmod 6$; this case has to be considered separately.

We now consider the case where H is not connected. Set $\lambda = \gcd(m, a, b) - 1$ and denote by H_i, $0 \le i \le \lambda$, the connected components of H; H_0 will be the component containing the vertex u_0.

The connected component H_i with $i > 0$ can be described as the i-th isomorphic copy of H_0, that is, we leave invariant the adjacencies in H_0 and label the vertices of H_i by adding $i\,c$ modulo m to the subscripts of the vertices in H_0. We will use the notation u_x^i, v_x^i to denote the outer and inner vertices of H_i corresponding to the outer and inner vertices u_x, v_x, respectively, in H_0.That is, $u_x^i = u_{x+i\,c}$ and $v_x^i = v_{x+i\,c}$. The outer vertices u_x^i in H_i are adjacent to the inner vertices v_x^{i+1} in H_{i+1}, since G is connected and H is obtained from G by removing the spokes of type c. Sometimes, for our convenience, we will also use the notation u_x^0, v_x^0 for the vertices in H_0.

Given a generalized rose window graph G whose subgraph H has at least two connected components, we will construct a Hamilton cycle in G by appropriately joining the Hamilton cycles, or paths, in the components H_i. The construction depends on the classification defined in Lemma 3.4. More specifically, for an alternating Hamilton cycle we will define the *alternating construction* (see Proposition 4.1); we will define the 4- and the 2-*hooked construction* for the 4- and the 2-hooked Hamilton cycles, respectively. The terminology follows from the fact that, in the assembly of the cycles in the components of H, the cycle corresponding to C in H_i with $0 \le i \le \lambda - 1$, will be connected to the cycle corresponding to C in H_{i+1} by 4 or 2 spokes, respectively.

The alternating and the hooked constructions can be applied when H_0 is not isomorphic to the generalized Petersen graph $G(n, 2)$ with $n \equiv 5 \pmod 6$. In the latter case, the graph does not have a Hamilton cycle and we will apply the construction described in the proof of Theorem 1.2, and summarized in Figure 8, which could be called the 1-hooked construction in analogy to the previous ones.

We now give the alternating and the hooked constructions; the 2-hooked construction will be also used in Proposition 4.5, which defines the 4-hooked construction. In what follows, we will use the notation $x\,P\,y$ to denote a path P from the vertex x to the vertex y.

Proposition 4.1 **The alternating construction.** *Let $G = R(m; a, b, c)$ be a connected generalized rose window graph. Let H be the graph obtained from G by removing the spokes of type c, and let H_0 be the connected component of H containing the vertex u_0. Assume $\lambda = \gcd(m, a, b) - 1 > 0$. If the graph H_0 has an alternating Hamilton cycle, then the graph G is hamiltonian.*

Proof Let $m_0 = m/\gcd(m, a, b)$. Assume that the graph H_0 has an alternating Hamilton cycle; denote it by C. Note that the existence of an alternating Hamilton cycle in H_0 implies that m_0 is even. We denote the outer and inner vertices of H_0 with u_{x_j}, v_{x_j}, respectively, for $0 \leq j \leq m_0 - 1$, so that the vertices u_{x_j}, v_{x_j} are consecutive in C, as well as $u_{x_j}, u_{x_{j+1}}$ and $v_{x_{j-1}}, v_{x_j}$, with $1 \leq j \leq m_0 - 1$, j odd. The indices x_j are integers modulo m. In H_i with $i > 0$, the vertices corresponding to u_{x_j}, v_{x_j} of H_0 will be denoted with $u^i_{x_j}, v^i_{x_j}$; the vertices in H_0 will be also denoted with $u^0_{x_j}, v^0_{x_j}$.

To construct a Hamilton cycle in the graph G, we keep just the spokes $v^i_{x_j}\, u^i_{x_j}$ (spokes of type 0) in each of the graphs H_i for $1 \leq i \leq \lambda-1$. For every $0 \leq j \leq m_0-1$, we connect the edge $v^i_{x_j}\, u^i_{x_j}$ in H_i to the edge $v^{i+1}_{x_j}\, u^{i+1}_{x_j}$ in H_{i+1} by adding the spoke $u^i_{x_j}\, v^{i+1}_{x_j}$ of type c, for $1 \leq i \leq \lambda - 2$. For every $0 \leq j \leq m_0 - 1$, we get a path from $v^1_{x_j}$ to $u^{\lambda-1}_{x_j}$, to which we add the edges $u^0_{x_j}\, v^1_{x_j}$ and $u^{\lambda-1}_{x_j}\, v^{\lambda}_{x_j}$ in order to obtain a path $u^0_{x_j}\, P\, v^{\lambda}_{x_j}$ from the vertex $u^0_{x_j}$ in H_0 to the vertex $v^{\lambda}_{x_j}$ in H_λ. The union of the paths $u^0_{x_j}\, P\, v^{\lambda}_{x_j}$ is a disconnected graph that covers all the vertices in G, with the exception for the inner vertices in H_0 and the outer vertices in H_λ. Since m_0 is even, we can join the paths $u^0_{x_j}\, P\, v^{\lambda}_{x_j}$ by adding the paths $(u^0_{x_j}, v^0_{x_j}, v^0_{x_{j+1}}, u^0_{x_{j+1}})$ for $0 \leq j \leq m_0 - 1$, j even, and the paths $(v^{\lambda}_{x_j}, u^{\lambda}_{x_j}, u^{\lambda}_{x_{j+1}}, v^{\lambda}_{x_{j+1}})$ for $0 \leq j \leq m_0 - 1$, j odd. We thus obtain a Hamilton cycle in G. We summarize the construction with the diagram in Figure 5. □

Proposition 4.2 The 2-hooked construction. *Let $G = R(m; a, b, c)$ be a connected generalized rose window graph. Let H be the graph obtained from G by removing the spokes of type c, and let H_0 be the connected component of H containing the vertex u_0. Assume $\lambda = \gcd(m, a, b) - 1 > 0$. If the graph H_0 has a 2-hooked Hamilton cycle, then the graph G is hamiltonian.*

Proof Assume that the graph H_0 has a 2-hooked Hamilton cycle; denote it by C. We first assume that the 2-hooked cycle C provides a Hamilton path connecting the vertices v_0 and v_a in H_0, say $v_0\, P\, v_a$. Such a path necessarily contains an outer edge, say $u_x\, u_{x+a}$, since it contains the same number of outer and inner vertices. Without loss of generality, we can assume that u_x precedes u_{x+a} in $v_0\, P\, v_a$, so that the removal of the edge $u_x\, u_{x+a}$ yields the two subpaths $v_0\, P\, u_x$ and $v_a\, P\, u_{x+a}$. We can also find a Hamilton path $u_0\, P\, u_a$ from u_0 to u_a in H_0, since the graph H_0 is hamiltonian and every Hamilton cycle in H_0 has at least one outer edge.

If $\lambda = 1$, so the graph H has two components, we connect the two Hamilton paths $u^0_0\, P\, u^0_a$ in H_0 and $v^1_0\, P\, v^1_a$ in H_1 by adding the spokes $u^0_0\, v^1_0$ and $u^0_a\, v^1_a$ in order to obtain a Hamilton cycle in G.

Now we assume that $\lambda > 1$. In H_i, with $1 \leq i \leq \lambda - 1$, we consider the subpaths $v^i_0\, P\, u^i_x$ and $v^i_a\, P\, u^i_{x+a}$ corresponding to the subpaths $v_0\, P\, u_x$ and $v_a\, P\, u_{x+a}$ of H_0. We turn the subpaths $v^i_0\, P\, u^i_x$ and $v^i_a\, P\, u^i_{x+a}$ into the subpaths $v^i_{(i-1)x}\, P\, u^i_{i\,x}$ and $v^i_{a+(i-1)x}\, P\, u^i_{a+i\,x}$ by adding $(i-1)x$ modulo m to the subscripts of the vertices in H_i. Notice that by adding $(i-1)x$ modulo m to the subscripts of the vertices in H_i, we still get vertices of H_i.

We now construct a Hamilton cycle in G by connecting the above paths as follows. For $1 \leq i \leq \lambda - 2$, we join the path $v^i_{(i-1)x}\, P\, u^i_{i\,x}$ in H_i to the path $v^{i+1}_{i\,x}\, P\, u^{i+1}_{(i+1)x}$ in

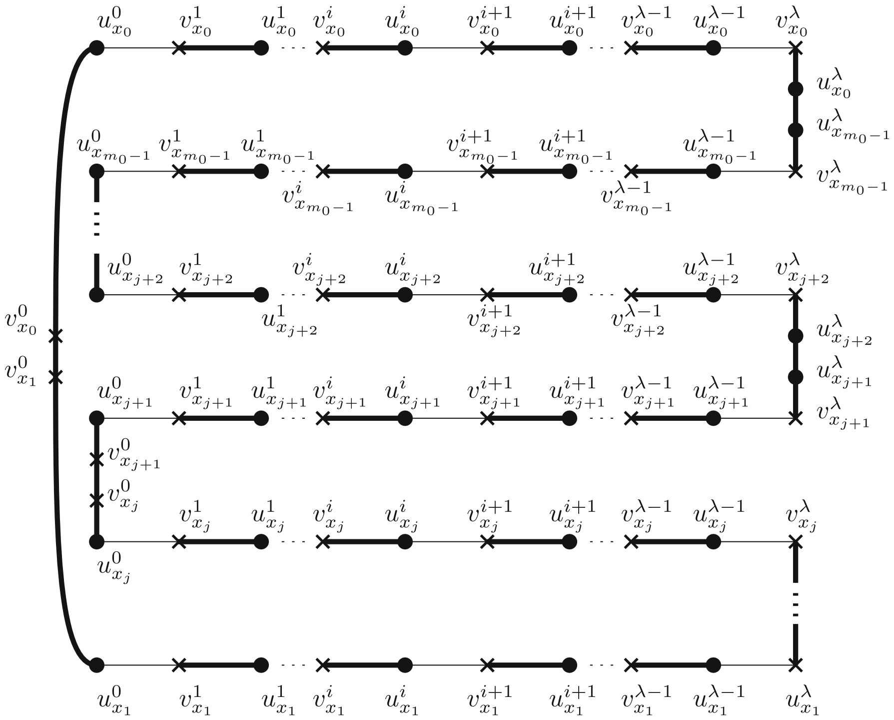

Fig. 5 The alternating construction for the generalized rose window graphs described in Proposition 4.1. The bold lines represent the edges $v^i_{x_j}\ u^i_{x_j}$ in H_i, $1 \le i \le \lambda - 1$, the paths $(u^0_{x_j}, v^0_{x_j}, v^0_{x_{j+1}}, u^0_{x_{j+1}})$ for $0 \le j \le m_0 - 1$, j even, and the paths $(v^{\lambda}_{x_j}, u^{\lambda}_{x_j}, u^{\lambda}_{x_{j+1}}, v^{\lambda}_{x_{j+1}})$ for $0 \le j \le m_0 - 1$, j odd

H_{i+1} by the spoke $u^i_{i\,x}\ v^{i+1}_{i\,x}$, and also join the path $v^i_{a+(i-1)x}\ P\ u^i_{a+i\,x}$ in H_i to the path $v^{i+1}_{a+i\,x}\ P\ u^{i+1}_{a+(i+1)x}$ in H_{i+1} by the spoke $u^i_{a+i\,x}\ v^{i+1}_{a+i\,x}$. We obtain two vertex-disjoint paths – the former from v^1_0 to $u^{\lambda-1}_{(\lambda-1)x}$ and the latter from v^1_a to $u^{\lambda-1}_{a+(\lambda-1)x}$ – whose union covers all the vertices in $G - (H_0 \cup H_\lambda)$. We connect the two paths to the Hamilton paths $u^0_0\ P\ u^0_a$ in H_0 and $v^{\lambda}_{(\lambda-1)x}\ P\ v^{\lambda}_{a+(\lambda-1)x}$ in H_λ by adding the spokes $u^0_0\ v^1_0$, $u^0_a\ v^1_a$ and $u^{\lambda-1}_{(\lambda-1)x}\ v^{\lambda}_{(\lambda-1)x}$, $u^{\lambda-1}_{a+(\lambda-1)x}\ v^{\lambda}_{a+(\lambda-1)x}$. We thus obtain a Hamilton cycle in G. We summarize the construction with the diagram in Figure 6.

For the case where the 2-hooked cycle in H_0 provides a Hamilton path connecting u_0 and u_b, we can repeat the same argument as above (it suffices to replace the parameter a with the parameter b). The assertion follows. □

Remark 4.3 We can apply the 2-hooked construction described in Proposition 4.2 even when we have a Hamilton path from u_0 to u_p in H (denoted by $u_0\ P\ u_p$), where u_p is an arbitrary vertex of H, a Hamilton path from v_0 to v_p, (denoted by $v_0\ P\ v_p$), and two paths whose union partitions the vertices of H, namely the paths $u_0\ P\ v_p$ and $u_p\ P\ v_0$ from u_0 to v_p and from u_p to v_0, or the paths $u_0\ P\ v_0$ and $u_p\ P\ v_p$ from u_0 to v_0 and from u_p to v_p, respectively. In fact, in the construction described in the proof of Proposition 4.2, we can replace the subpaths $v_0\ P\ u_x$ and $v_a\ P\ u_{x+a}$ in H_0 with the

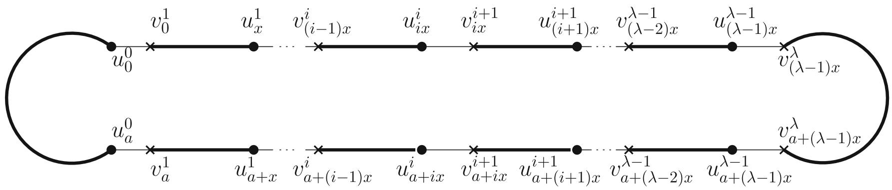

Fig. 6 The 2-hooked construction for the generalized rose window graphs described in Proposition 4.2. The bold lines represent the paths $v^i_{(i-1)x}\ P\ u^i_{i\,x}$, $v^i_{a+(i-1)x}\ P\ u^i_{a+i\,x}$ in H_i with $1 \le i \le \lambda - 1$, and the Hamilton paths $u^0_0\ P\ u^0_a$ in H_0, $v^\lambda_{(\lambda-1)x}\ P\ v^\lambda_{a+(\lambda-1)x}$ in H_λ. The paths are joined by spokes $u^i_{i\,x}\,v^{i+1}_{i\,x}$, $u^i_{a+i\,x}\,v^{i+1}_{a+i\,x}$ for $0 \le i \le \lambda - 1$

subpaths $u_0\ P\ v_p$ and $u_p\ P\ v_0$, or $u_0\ P\ v_0$ and $u_p\ P\ v_p$; consequently, in each H_i, with $1 \le i \le \lambda - 1$, we will consider the corresponding copies; we will take the Hamilton path from u_0 to u_p in H_0, and the copy of the Hamilton path from v_0 to v_p in H_λ. Roughly speaking, in order to obtain a Hamilton cycle in G, we will join the outer vertices of H_i to the inner vertices of H_{i+1} having the same subscripts. For instance, if we have the paths $u_0\ P\ v_p$ and $u_p\ P\ v_0$ partitioning the vertices in H_0, then in H_i, with $1 \le i \le \lambda - 1$, we can consider the corresponding copies $u^i_0\ P\ v^i_p$ and $u^i_p\ P\ v^i_0$ - we recall that $u^i_0 = u_{ic}$, $u^i_p = u_{p+ic}$ and $v^i_0 = v_{ic}$, $v^i_p = v_{p+ic}$; we find a Hamilton cycle in G by connecting the vertices u^i_p and u^i_0, with $1 \le i \le \lambda - 2$, to the vertices v^{i+1}_p and v^{i+1}_0, respectively; we also add the edges $u^0_0 v^1_0$, $u^0_p v^1_p$, and $u^{\lambda-1}_0 v^\lambda_0$, $u^{\lambda-1}_p v^\lambda_p$.

In Example 4.4, we show how to use the 2-hooked construction, described in Remark 4.3. We will need this example in the proof of Proposition 4.5.

Example 4.4 Let $b \equiv -2a \pmod m$, and assume that a Hamilton cycle C of an I-graph $H = I(m; a, b)$ contains the subpaths $(v_b, v_0, u_0, u_a) = (v_{-2a}, v_0, u_0, u_a)$ and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b}) = (u_{-2a}, u_{-a}, v_{-a}, v_a, v_{3a}, u_{3a})$ occurring in this order in C. Then we find a Hamilton path from u_0 to u_{3a}, a Hamilton path from v_0 to v_{3a}, and the paths from u_0 to v_{3a} and from u_{3a} to v_0, whose union partitions the vertices of H. In detail, the existence of the two paths from u_0 to v_{3a} and from u_{3a} to v_0 is straightforward (remove the edges u_0v_0 and $u_{3a}v_{3a}$); the Hamilton path from v_0 to v_{3a} can be obtained as follows: remove the edges v_0v_{-2a}, u_0u_a, $u_{-a}u_{-2a}$, v_av_{3a} from C, and add the edges u_0u_{-a}, u_av_a, $u_{-2a}v_{-2a}$. For the Hamilton path from u_0 to u_{3a}, we first note that C also contains the subpath $(u_a, u_{2a}, v_{2a}, v_{4a})$ and the edge $u_{3a}u_{4a}$. Then we remove the edges u_0v_0, $v_{2a}v_{4a}$, $u_{3a}u_{4a}$, and add the edges v_0v_{2a}, $u_{4a}v_{4a}$.

Let $a \equiv -2b \pmod m$ and again assume that a Hamilton cycle C of an I-graph $H = I(m; a, b)$ contains the subpaths(v_b, v_0, u_0, u_a) and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b})$ occuring in this order in C. Notice that in this case the cycle C also contains the subpaths $(u_a, u_0, v_0, v_b, v_{2b}, u_{2b}, u_{4b})$ and $(u_{a-b}, v_{a-b}, v_a, v_{a+b}, u_{a+b}, u_b, u_{3b}, v_{3b}, v_{4b})$ in this order and we can repeat arguments similar to the previous ones due to the symmetry between the parameters a and b and find a Hamilton path from u_0 to u_{3b}, a Hamilton path from v_0 to v_{3b}, and the paths from u_0 to v_{3b} and from u_{3b} to v_0, whose union partitions the vertices of H.

Proposition 4.5 The 4-hooked construction. *Let $G = R(m; a, b, c)$ be a connected generalized rose window graph with $a \neq \pm b$. Let H be the graph obtained from G by removing the spokes of type c, and let H_0 be the connected component of H containing the vertex u_0. Assume $\lambda = \gcd(m, a, b) - 1 > 0$. If H_0 has a 4-hooked Hamilton cycle, then G is hamiltonian.*

Proof Assume that the graph H_0 has a 4-hooked Hamilton cycle; denote it by C. Since C is a 4-hooked cycle, it contains the edges $u_0\,u_a$, $u_b\,u_{a+b}$, $v_0\,v_b$, $v_a\,v_{a+b}$. The outer vertices u_0, u_a, u_b, u_{a+b} appear in C in the sequence u_0, u_a, u_{a+b}, u_b, or u_0, u_a, u_b, u_{a+b}. The edge $v_0\,v_b$ is placed in one of the subpaths of C we obtain by removing the edges $u_0\,u_a$, $u_b\,u_{a+b}$; the same holds for the edge $v_a\,v_{a+b}$, and it may or may not belong to the same subpath as $v_0\,v_b$. Together there are, up to symmetry, 48 different orderings of the vertices $u_0, u_a, u_b, u_{a+b}, v_0, v_b, v_a, v_{a+b}$ on C.

We show how the 4-hooked construction works in the hypothesis that the vertices u_0, u_a, u_b, u_{a+b} are ordered in C in the sequence u_0, u_a, u_{a+b}, u_b, and that $v_0\,v_b$ belongs to the subpath $u_0\,P\,u_b$, whereas $v_a\,v_{a+b}$ is in $u_{a+b}\,P\,u_a$ ($u_0\,P\,u_b$, $u_{a+b}\,P\,u_a$ are the subpaths of C we obtain by removing the edges $u_0\,u_a$, $u_b\,u_{a+b}$). We also assume that v_0 precedes v_b in the path $u_0\,P\,u_b$, and v_{a+b} precedes v_a in the path $u_{a+b}\,P\,u_a$. Then, by removing the edges $v_0\,v_b$, $v_a\,v_{a+b}$ in $C - \{u_0\,u_a, u_b\,u_{a+b}\}$, we obtain the following four subpaths: $v_0\,P\,u_0$, $v_b\,P\,u_b$, $v_{a+b}\,P\,u_{a+b}$, $v_a\,P\,u_a$. We will also consider the subpaths $v_0\,P\,v_a$, $v_b\,P\,v_{a+b}$ we obtain from C by removing the edges $v_0\,v_b$, $v_a\,v_{a+b}$.

In H_i, with $1 \leq i \leq \lambda - 1$, we consider the subpaths $v^i_j\,P\,u^i_j$, $j \in \{0, b, a, a+b\}$, which correspond to the above four subpaths of C. In H_λ, we consider the subpaths $v^\lambda_0\,P\,v^\lambda_a$, $v^\lambda_b\,P\,v^\lambda_{a+b}$, which corresponds to the subpaths $v_0\,P\,v_a$, $v_b\,P\,v_{a+b}$ of C.

For $1 \leq i \leq \lambda - 2$ and $j \in \{0, b, a, a+b\}$, we join the path $v^i_j\,P\,u^i_j$ in H_i to the path $v^{i+1}_j\,P\,u^{i+1}_j$ in H_{i+1} by the edge $u^i_j\,v^{i+1}_j$, and get a path $v^1_j\,P\,u^{\lambda-1}_j$ from v^1_j in H_1 to $u^{\lambda-1}_j$ in $H_{\lambda-1}$. The union of the four paths is a disconnected graph covering all the vertices of G, with the exception for the vertices in $H_0 \cup H_\lambda$. We connect the four paths $v^1_j\,P\,u^{\lambda-1}_j$, $j \in \{0, b, a, a+b\}$ to the paths $u^0_0\,P\,u^0_b$, $u^0_{a+b}\,P\,u^0_a$ in H_0 and to the paths $v^\lambda_0\,P\,v^\lambda_a$, $v^\lambda_b\,P\,v^\lambda_{a+b}$ in H_λ by adding the spokes $u^0_j\,v^1_j$, $u^{\lambda-1}_j\,v^\lambda_j$, $j \in \{0, b, a, a+b\}$. We thus obtain a Hamilton cycle in G, and the assertion follows. We summarize the construction in the diagram in Figure 7.

We can repeat the same construction even if the four edges $u_0\,u_a$, $u_b\,u_{a+b}$, $v_0\,v_b$, $v_a\,v_{a+b}$ are arranged on C in a different way from that considered above when the outer and inner edges among these edges alternate on C, with the exceptions of the following four orderings: $u_0, u_a, v_0, v_b, u_b, u_{a+b}, v_a, v_{a+b}$ and $u_0, u_a, v_b, v_0, u_b, u_{a+b}, v_{a+b}, v_a$ and $u_0, u_a, v_{a+b}, v_a, u_b, u_{a+b}, v_b, v_0$ and $u_0, u_a, v_a, v_{a+b}, u_b, u_{a+b}, v_0, v_b$. In such exceptions we find either a Hamilton path $v_0\,P\,v_a$ or a Hamilton path $u_0\,P\,u_b$ in H_0. For instance, in the case of the sequence $u_0, u_a, v_0, v_b, u_b, u_{a+b}, v_a, v_{a+b}$, we find a Hamilton path from u_0 to u_b by removing the edges $u_0\,u_a$, $u_b\,u_{a+b}$, $v_a\,v_{a+b}$ and adding the edges $u_a\,v_a$, $u_{a+b}\,v_{a+b}$. The assertion then follows from Proposition 4.2.

When the outer and the inner edges from $\{u_0\,u_a, u_b\,u_{a+b}, v_0\,v_b, v_a\,v_{a+b}\}$ do not alternate on C, we can almost always find either a Hamilton path $v_0\,P\,v_a$ or a Hamilton

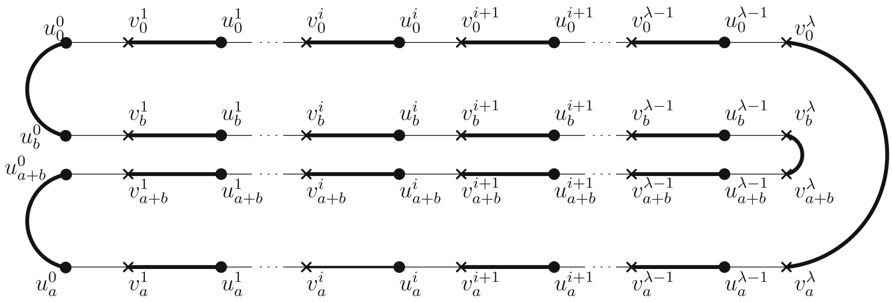

Fig. 7 The 4-hooked construction for the generalized rose window graphs described in Proposition 4.5. The bold lines represent the paths $u_0^0\, P\, u_b^0$, $u_{a+b}^0\, P\, u_a^0$ in H_0, $v_0^\lambda\, P\, v_a^\lambda$, $v_b^\lambda\, P\, v_{a+b}^\lambda$ in H_λ, and the paths $u_j^i\, P\, v_j^i$ in H_i, with $j \in \{0, b, a, a+b\}$ for $1 \le i \le \lambda - 1$; the paths are connected by the spokes $u_j^i\, v_j^{i+1}$, for $0 \le i \le \lambda - 1$

path $u_0\, P\, u_b$ in H_0 (sometimes by relabeling the vertices appropriately) and then use Proposition 4.2 to find a Hamilton cycle in the graph G.

We cannot find such paths directly only if the cycle C is elusive. By Remark 3.5 we may assume that the cycle C is elusive of type 1. Then by Lemma 3.7 we can use the 2-hooked construction or the 4-hooked construction described above to obtain a Hamilton cycle in the graph G. In the special case where $b \equiv -2a \pmod m$ or $a \equiv -2b \pmod m$, and the Hamilton cycle C contains the subpaths (v_b, v_0, u_0, u_a) and $(u_b, u_{a+b}, v_{a+b}, v_a, v_{a-b}, u_{a-b})$ occurring in this order in C, we may use the 2-hooked construction as described in Remark 4.3 and Example 4.4. □

By combining Lemmas 3.3, 3.4 and Propositions 4.1, 4.2, 4.5, we can prove our main result that every connected generalized rose window graph is hamiltonian.

Proof of Theorem 1.2 Let $G = R(m; a, b, c)$ be a connected generalized rose window graph and let H be the graph obtained from G by removing the spokes of type c. Let $\lambda = \gcd(m, a, b) - 1$.

First, we consider the case where $\lambda = 0$, that is, H is a connected spanning subgraph of G. By Theorem 3.2, we know that a connected I-graph is hamiltonian, with the exception of the generalized Petersen graphs $G(n, 2)$, with $n \equiv 5 \pmod 6$. Therefore, if H is not isomorphic to a graph $G(n, 2)$, then a Hamilton cycle of H is also a Hamilton cycle of G. We find a Hamilton cycle in G even if H is a generalized Petersen graph $G(n, 2)$: Theorem 3.1 assures the existence of a Hamilton path in H connecting the vertices u_0 and v_c, which are adjacent in G but not in H, since H contains no spokes of type c; adding the spoke $u_0 v_c$ yields a Hamilton cycle in G. Thus, the assertion follows if $\lambda = 0$. In the rest of the proof we consider $\lambda > 0$.

Let H_0 be the connected component of H containing the vertex u_0. If H_0 is not isomorphic to a generalized Petersen graph $G(n, 2)$, with $n \equiv 5 \pmod 6$, then the assertion follows from Lemmas 3.3, 3.4 and Propositions 4.1, 4.2, 4.5.

Let us now consider the case where H_0 is the generalized Petersen graph $G(n, 2)$, with $n \equiv 5 \pmod 6$. Notice that $n = m/\gcd(m, a, b)$ and that the indices of the vertices of H_0 are all multiples of $\gcd(m, a, b) = \lambda + 1$. By Theorem 3.1, we can

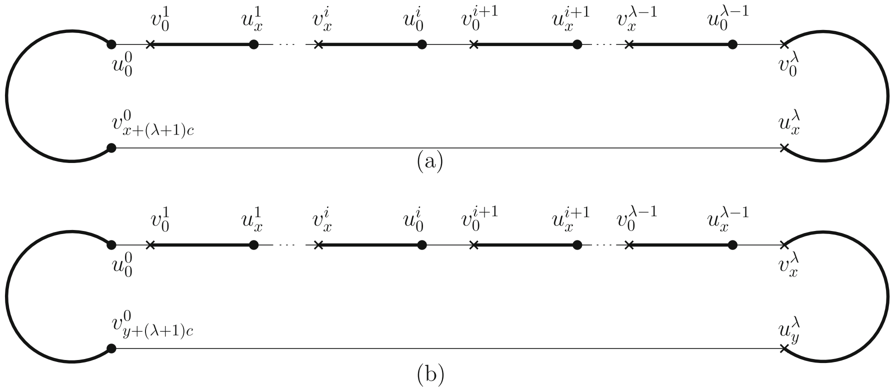

Fig. 8 The construction of a Hamilton cycle in a generalized rose window graph G when the connected components of H are isomorphic to a generalized Petersen graph $G(n, 2)$, $n \equiv 5 \pmod 6$; see the proof of Theorem 1.2. We apply case (a) for odd values of $\lambda = \gcd(m, a, b) - 1$, and case (b) for even values of λ. The bold lines represent the paths $v_0^i\ P\ u_x^i$, $v_x^i\ P\ u_0^i$ with $1 \leq i \leq \lambda - 1$, the Hamilton paths $u_0^0\ P\ v_{x+(\lambda+1)c}^0$ and $v_0^\lambda\ P\ u_x^\lambda$ in case (a) and the Hamilton paths $u_0^0\ P\ v_{y+(\lambda+1)c}^0$ and $v_x^\lambda\ P\ u_y^\lambda$ in case (b)

find a Hamilton path $v_0\ P\ u_x$ in H_0 connecting v_0 to u_x, and also a Hamilton path $v_x\ P\ u_0$, for every integer $x \in (\lambda+1)\mathbb{Z}_m$, $x \not\equiv 0 \pmod m$. For odd values of λ, we select $x \in (\lambda+1)\mathbb{Z}_m$ such that $x + (\lambda+1)c \not\equiv 0 \pmod m$. For even values of λ, we do not add additional conditions on x, but we select another integer $y \in (\lambda+1)\mathbb{Z}_m$ such that $x \not\equiv y \pmod m$, $y + (\lambda+1)c \not\equiv 0 \pmod m$, and consider the vertices $u_y^\lambda = u_{y+\lambda c} \in H_\lambda$, $v_{y+(\lambda+1)c} \in H_0$. Notice that the choice of x and y is always possible, since $n \geq 5$.

We now construct a Hamilton cycle in G as follows. We take the path $v_0^i\ P\ u_x^i$ in H_i with i odd, $1 \leq i \leq \lambda - 1$, and the path $v_x^i\ P\ u_0^i$ in each H_i with i even, $1 \leq i \leq \lambda - 1$. We join the paths by the spokes $u_x^i\ v_x^{i+1}$ for $1 \leq i \leq \lambda - 2$ with i odd, and $u_0^i\ v_0^{i+1}$ for $1 \leq i \leq \lambda - 2$ with i even. For odd values of λ, we obtain a path $v_0^1\ P\ u_0^{\lambda-1}$ connecting the vertices v_0^1 and $u_0^{\lambda-1}$; for even values of λ, we have a path $v_0^1\ P\ u_x^{\lambda-1}$ connecting the vertices v_0^1 and $u_x^{\lambda-1}$; both paths $v_0^1\ P\ u_0^{\lambda-1}$ and $v_0^1\ P\ u_x^{\lambda-1}$ cover all the vertices in $G - (H_0 \cup H_\lambda)$.

For odd values of λ, we take the Hamilton path $v_0^\lambda\ P\ u_x^\lambda$ in H_λ, and the Hamilton path $v_{x+(\lambda+1)c}^0\ P\ u_0^0$ in H_0 (whose existence follows from Theorem 3.1 by the assumptions on x). We join the paths $v_{x+(\lambda+1)c}^0\ P\ u_0^0$, $v_0^1\ P\ u_0^{\lambda-1}$, $v_0^\lambda\ P\ u_x^\lambda$ by the spokes $u_0^0\ v_0^1$, $u_0^{\lambda-1}\ v_0^\lambda$, $u_x^\lambda\ v_{x+(\lambda+1)c}^0$, and obtain a Hamilton cycle in G.

For even values of λ, we take the Hamilton path $v_x^\lambda\ P\ u_y^\lambda$ in H_λ and the Hamilton path $v_{y+(\lambda+1)c}^0\ P\ u_0^0$ in H_0 (whose existence follows from Theorem 3.1 by the assumptions on y). We join the paths $v_{y+(\lambda+1)c}^0\ P\ u_0^0$, $v_0^1\ P\ u_x^{\lambda-1}$, $v_x^\lambda\ P\ u_y^\lambda$ by the spokes $u_0^0\ v_0^1$, $u_x^{\lambda-1}\ v_x^\lambda$, $u_y^\lambda\ v_{y+(\lambda+1)c}^0$, and obtain a Hamilton cycle in G, which completes the proof. We summarize the construction in the diagram in Figure 8. □

5 Concluding remarks

Proving that all generalized rose window graphs are hamiltonian could be the first step to proving Conjecture 1.1. The next step would be to consider the pentavalent generalized Tabačjn graphs, which are obtained from the generalized rose window graphs by adding an additional set of spokes, similarly as the generalized rose window graphs are obtained from the I-graphs, see [5, 23]. Given $m \geq 3$ and $a, b, c, d \in \mathbb{Z}_m \setminus \{0\}$ with $a, b \neq m/2$, the *generalized Tabačjn graph* $T(m; a, b, c, d)$ is defined to be the bicirculant graph $B(m; \{a, -a\}, \{0, c, d\}, \{b, -b\})$. Every connected generalized Tabačjn graph $T(m; a, b, c, d)$ contains three generalized rose window graphs as subgraphs, namely $R(m; a, b, c)$, $R(m; a, b, d)$ and $B(m; R, S \setminus \{0\}, T)$, which is isomorphic to $R(m; a, b, d - c)$ by Proposition 2.3. It may happen that at least one of these is connected. In this case, also the graph $T(m; a, b, c, d)$ is hamiltonian by Theorem 1.2. Moreover, the graph $T(m; a, b, c, d)$ contains the cubic Haar graph $B(m; \emptyset, \{0, c, d\}, \emptyset)$ as a subgraph. If that graph is connected, it is hamiltonian by [4, Theorem 3.1]; this happens when $\gcd(m, c, d) = 1$.

We can apply the same reasoning to more general bicirculant graphs: if a bicirculant contains a connected generalized rose window graph as a subgraph, then it is hamiltonian by Theorem 1.2; if it contains a connected cubic Haar graph as a subgraph, then it is hamiltonian by [4, Theorem 3.1].

Proposition 5.1 *Let $G = H(m; S)$ be a connected cyclic Haar graph with $|S| \geq 4$. If m is a product of at most three prime powers, then G is hamiltonian.*

Proof Let $S = \{0, c_1, \ldots, c_{s-1}\}$, where $s = |S| \geq 4$. Since the graph G is connected, we have $\gcd(m, S) = 1$. If G contains a connected cubic Haar graph as a subgraph, then it is hamiltonian. Therefore we assume that G does not contain a connected cubic Haar graph as a subgraph and we will show that in this case m needs to be a product of at least four prime powers.

Since $\gcd(m, c_1, c_2) > 1$, it is divisible by some prime, say p. Since $\gcd(m, S) = 1$, there exists an element of S, say c_i, that is not divisible by p. Therefore there exists another prime, say q, such that $\gcd(m, c_1, c_i)$ is divisible by q. Now there exists some element of S that is not divisible by q, say c_j (it may happen that $c_j = c_2$). Therefore there exists another prime, say r, such that $\gcd(m, c_i, c_j)$ is divisible by r. Thus m is a product of at least three prime powers.

Suppose that m is a product of exactly three prime powers, namely, the powers of p, q and r. Since $\gcd(m, S) = 1$, again there exists an element of S, say c_k, that is not divisible by r (it may happen that $c_j = c_1$ or $c_j = c_2$). Now we have elements c_i, c_j, c_k from S such that c_i is not divisible by p, c_j is not divisible by q and c_k is not divisible by r. On the other hand all of $\gcd(m, c_i, c_j)$, $\gcd(m, c_i, c_k)$, $\gcd(m, c_j, c_k)$ are greater than one. That means that c_i, c_j are both divisible by r, c_i, c_k are both divisible by q and c_j, c_k are both divisible by p. But then $c_i - c_k$ is not divisible by any of p, r and $c_j - c_k$ is not divisible by q. It follows that $\gcd(m, c_i - c_k, c_j - c_k) = 1$ and G contains a connected cubic Haar graph $H(m; \{0, c_i - c_k, c_j - c_k\})$ as a subgraph, a contradiction. Therefore m is a product of at least four prime powers. □

Proposition 5.2 *Let $G = B(m; a, S, b)$ be a connected bicirculant with $|S| \geq 3$ and $a, b \neq m/2$. If m is a product of at most three prime powers, then G is hamiltonian.*

Proof Let $S = \{0, c_1, \ldots, c_{s-1}\}$, where $s = |S| \geq 3$. Since the graph G is connected, we have $\gcd(m, a, S, b) = 1$. If the graph G contains a connected generalized rose window graph or a connected cubic Haar graph as a subgraph, then it is hamiltonian. Therefore we assume that this is not the case and we will show that then m needs to be a product of at least four prime powers.

We may assume that $\gcd(m, S) = d > 1$, otherwise already $B(m; \emptyset, S, \emptyset)$ is connected and the claim follows from Proposition 5.1. Therefore there exists a prime r that divides d. Since the graph G is connected and it does not contain a connected generalized rose window graph as a subgraph, at least one of a, b, say a, must be coprime to r; therefore there exists a prime p that is coprime to d such that p is coprime to r and p divides $\gcd(m, a, b, c_1)$. In particular $p \neq r$. Since p is coprime to d, there exists $c_i \in S \setminus \{c_1\}$ that is not divisible by p. Therefore there exists a third prime, say q, that divides $\gcd(m, a, b, c_i)$. Thus m is a product of at least three prime powers.

Suppose that m is a product of exactly three prime powers, namely, the powers of p, q and r. Since the graph G is connected, there exists $c_j \in S \setminus \{c_i\}$ that is not divisible by q (it may happen that $c_j = c_1$). Since $\gcd(m; a, b, c_j) > 1$, it follows that c_j must be divisible by p. Now we have elements c_i, c_j from S such that c_i is divisible by q and is coprime to p, c_j is divisible by p and is coprime to q. But then $c_i - c_j$ is not divisible by any of p, q and a is not divisible by r. It follows that $\gcd(m, a, c_i - c_j, b) = 1$ and G contains a connected rose window graph $S(m; a, \{0, c_i - c_j\}, b)$ as a subgraph, a contradiction. Therefore m is a product of at least four prime powers. □

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Data Availability There is no associated data to this work.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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