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List strong and list normal edge-coloring of (sub)cubic graphs

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ABSTRACT

A *strong edge-coloring* of a graph is a proper edge-coloring, in which the edges of every path of length 3 receive distinct colors; in other words, every pair of edges at distance at most 2 must be colored differently. The least number of colors needed for a strong edge-coloring of a graph is the *strong chromatic index*. We consider the list version of the coloring and prove that the list strong chromatic index of graphs with maximum degree 3 is at most 10. This bound is tight and improves the previous bound of 11 colors.

We also consider the question whether the strong chromatic index and the list strong chromatic index always coincide. We answer it in negative by presenting an infinite family of graphs for which the two invariants differ. For the special case of the Petersen graph, we show that its list strong chromatic index equals 7, while its strong chromatic index is 5. Up to our best knowledge, this is the first known edge-coloring for which there are graphs with distinct values of the chromatic index and its list version.

In relation to the above, we also initiate the study of the list version of the normal edge-coloring. A *normal edge-coloring* of a cubic graph is a proper edge-coloring, in which every edge is adjacent to edges colored with 4 distinct colors or to edges colored with 2 distinct colors. It is conjectured that 5 colors suffice for a normal edge-coloring of any bridgeless cubic graph and this statement is equivalent to the Petersen Coloring Conjecture.

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It turns out that similarly to strong edge-coloring, list normal edge-coloring is much more restrictive and consequently for many graphs the list normal chromatic index is greater than the normal chromatic index. In particular, we show that there are cubic graphs with list normal chromatic index at least 9, there are bridgeless cubic graphs with its value at least 8, and there are cyclically 4-edge-connected cubic graphs with value at least 7.

1. Introduction

A *strong edge-coloring* of a graph G is a proper edge-coloring in which the edges at distance at most 2 receive distinct colors. Here, we define the *distance between two edges* in a graph G, as the distance between their corresponding vertices in the line graph of G; thus, two adjacent edges are at distance 1, and two non-adjacent edges, which are adjacent to a common edge, are at distance 2. The least number of colors for which G admits a strong edge-coloring is called the *strong chromatic index*, and is denoted by $\chi_s'(G)$.

In 1985, Erdős and Nešetřil [6] proposed the following conjecture; in 1990, it was updated to its current form by Faudree et al. [8], in order to fit the graphs with an even or odd maximum degree.

Conjecture 1.1 (*Erdős, Nešetřil [6]*). *The strong chromatic index of an arbitrary graph G satisfies*

$$\chi_s'(G) \leq \begin{cases} \frac{5}{4}\Delta(G)^2\,, & \text{if } \Delta(G) \text{ is even} \\ \frac{1}{4}(5\Delta(G)^2 - 2\Delta(G) + 1)\,, & \text{if } \Delta(G) \text{ is odd.} \end{cases}$$

We are still far from resolving the conjecture in general as the best known upper bound is $1.772\Delta(G)^2$ (provided that $\Delta(G)$ is large enough) due to Hurley et al. [16]. However, when limited to graphs of small maximum degree, we know a bit more; e.g., for graphs with maximum degree 3 (we refer to them as to *subcubic graphs*), the tight upper bound is established.

Theorem 1.2 (*Andersen [2]; Horák, Qing, and Trotter [14]*)**.** *For any subcubic graph G, it holds that*

$$\chi_s'(G) \leq 10.$$

There are only two known connected bridgeless subcubic graphs that need 10 colors for a strong edge-coloring: the Wagner graph (the left graph in Fig. 10) and the complete bipartite graph $K_{3,3}$ with one subdivided edge. Moreover, there is also no known connected bridgeless subcubic graph on more than 12 vertices with the strong chromatic index more than 8, and based on that also the following, stronger conjecture was proposed.

Conjecture 1.3 (*Lužar, Máčajová, Škoviera, and Soták [25]*)**.** *For any connected bridgeless subcubic graph G on at least 13 vertices, it holds that*

$$\chi_s'(G) \leq 8.$$

On the other hand, the lower bound of 5 colors for the strong chromatic index of cubic graphs (i.e., 3-regular graphs) is attained precisely by the covers of the Petersen graph [25].

1.1. List strong edge-coloring

Our research reported in this paper revolves about the list version of the strong edge-coloring of subcubic graphs. We say that L is a *list assignment* for a graph G if it assigns a list $L(e)$ of possible colors to each edge e of G. If G admits a strong edge-coloring φ such that $\varphi(e) \in L(e)$ for all edges in $E(G)$, then we say that G is *strong L-edge-colorable* or that φ is a *strong L-edge-coloring* of G. The graph G is *strong k-edge-choosable* if it is strong L-edge-colorable for every list assignment L, where $|L(e)| \geq k$ for every $e \in E(G)$. The *list strong chromatic index* $\chi'_{s,l}(G)$ of G is the least k such that G is strong k-edge-choosable.

For graphs with small maximum degrees, a number of results are already known. Horňák and Woźniak [15] showed that for any cycle, its list strong chromatic index and its strong chromatic index coincide. Dai et al. [5] proved that the list strong chromatic index of subcubic graphs is at most 11, and at most 10 in the case of subcubic planar graphs. The latter result was later extended to toroidal graphs by Pang et al. [28]. For graphs of maximum degree 4 it was shown that the list strong chromatic index is at most 22 [32], and at most 19 in the case of planar graphs [4].

In this paper, we generalize the results from [5,28] by establishing a tight upper bound for subcubic graphs.

Theorem 1.4. *For any subcubic graph G, it holds that*

$$\chi'_{s,l}(G) \leq 10.$$

As in the non-list version, only the Wagner graph and the complete bipartite graph $K_{3,3}$ with one subdivided edge are known to attain the upper bound.

The second question regarding the list strong edge-coloring is whether the values of the list strong chromatic index and the strong chromatic index of subcubic graphs coincide. In particular, we are interested in a special case of the question proposed by Dai et al. [5].

Question 1.5 (*Dai, Wang, Yang, and Yu [5], Question 4.1*)**.** *Is it true that for any graph G, it holds that*

$$\chi'_{s,l}(G) = \chi'_s(G)\,?$$

The motivation for the question comes from the List (Edge) Coloring Conjecture stating that the values of the chromatic index and the list chromatic index of any graph coincide. The conjecture was stated independently by several researchers (see [20, Problem 12.20] for more details) and in general it is still widely open; cf., e.g., [3] for a short survey.

In other words, the List Coloring Conjecture states that the chromatic number of any line-graph is equal to its list chromatic number, which is not true for graphs in general. Therefore, it seems that the structural properties of line-graphs are the ones that guarantee the equality of the two invariants. One can thus ask what are other structural properties of graphs that would also guarantee equality. In this sense, Kostochka and Woodall [24] conjectured that the chromatic number and the list chromatic number are equal for every square graph, where the *square graph G^2* is obtained from a graph G by connecting all pairs of vertices at distance 2. The conjecture was refuted in general by Kim and Park [23], but it is open for specific graph classes; for example, whether the two chromatic numbers are equal for the squares of line graphs. Since a (list) strong edge-coloring of a graph G is exactly a (list) coloring of vertices of the square of the line graph of G, Question 1.5 asks exactly that. We answer it in negative by presenting an infinite family of graphs G for which $\chi'_{s,l}(G) > \chi'_s(G)$.

Theorem 1.6. *There is an infinite family of connected cubic graphs G with*

$$\chi'_s(G) = 5 \quad \text{and} \quad \chi'_{s,l}(G) > 5.$$

Interestingly, there are also some planar graphs (e.g., the dodecahedron) and bipartite graphs (e.g., the generalized Petersen graph $GP(10,3)$) among the graphs with different values for the two invariants. Note that the above results are independently obtained also by Hasanvand [13].

Finally, for the case of the Petersen graph, we prove the exact value of the list strong chromatic index.

Theorem 1.7. *For the Petersen graph P, it holds that*

$$\chi'_{s,l}(P) = 7.$$

After publishing the preprint of this paper, we were notified by M. Hasanvand that the result of Theorem 1.7 follows also from the result of Kierstead on complete multipartite graphs [22].

1.2. List normal edge-coloring

The second part of this paper is dedicated to initiating the study of the list version of the normal edge-coloring of cubic graphs.

A *normal edge-coloring* of a cubic graph is a proper edge-coloring, in which every edge is adjacent to edges colored with four distinct colors (such edges are called *rich*) or to edges colored with two distinct colors (such edges are called *poor*). If at most k colors are used, we call the coloring a *normal k-edge-coloring*. The smallest k, for which a graph G admits a normal k-edge-coloring is the *normal chromatic index*, denoted by $\chi'_n(G)$. Clearly, every strong edge-coloring is also a normal edge-coloring, since every edge is rich. On the other hand, if a cubic graph admits a proper edge-coloring with 3 colors, then every edge is poor, and hence the coloring is also normal.

The normal edge-coloring was defined by Jaeger [17] as an equivalent way of formulating the Petersen Coloring Conjecture [19], which asserts that the edges of every bridgeless cubic graph G can be colored by using the edges of the Petersen graph P as colors in such a way that adjacent edges of G are colored by adjacent edges of P; in particular, a bridgeless cubic graph admits a normal 5-edge-coloring if and only if it admits a Petersen coloring.

Conjecture 1.8 (*Jaeger [17]*)**.** *For any bridgeless cubic graph G, it holds that*

$$\chi'_n(G) \leq 5.$$

Resolving Conjecture 1.8 would have a huge impact to the theory as it implies two famous conjectures; namely, the Cycle Double Cover Conjecture [18] and the Berge–Fulkerson Conjecture [10]; cf. [21] for more details.

In general, it is known that every cubic graph (with the bridgeless condition omitted) admits a normal 7-edge-coloring [27], and the bound is tight, e.g., by any cubic graph that contains as a subgraph the complete graph K_4 with one edge subdivided. When considering only bridgeless cubic graphs, Mazzuoccolo and Mkrtchyan [26] proved that all claw-free cubic graphs, tree-like snarks, and permutation snarks [26] admit a normal 6-edge-coloring; the latter result was generalized to bridgeless cubic graphs of oddness 2 by Fabrici et al. [7]. With at most 5 colors available, only very particular graphs are known to admit a normal edge-coloring, see, e.g., [9,11,29,30]. Hence, Conjecture 1.8 remains widely open in general.

In this paper, in relation to the list strong edge-colorings, we also study the properties of the list version of the normal edge-coloring. For a cubic graph G, list normal edge-coloring and the *list normal chromatic index*, $\chi'_{n,l}(G)$, are defined analogously to the list strong variants.

Clearly, the upper bound for the list normal chromatic index of cubic graphs is implied by Theorem 1.4.

Corollary 1.9. *For any subcubic graph G, it holds that*

$$\chi'_{n,l}(G) \leq 10.$$

We show that, similarly to the list strong edge-coloring, also in the list normal edge-coloring there are graphs G with $\chi'_{n,l}(G) > \chi'_n(G)$. In particular, there is an infinite family of cubic graphs with list normal chromatic index at least 9, there are bridgeless cubic graphs with list normal chromatic index at least 8, and there is an infinite family of cyclically 4-edge-connected cubic graphs with list normal chromatic index at least 7. Interestingly, our examples of bridgeless graphs for the above results are all from class I, and therefore they all have the normal chromatic index equal to 3.

The paper is structured as follows. In Section 2, we introduce notation, terminology, and auxiliary results. In Sections 3 and 4, we prove results regarding the list strong chromatic index, and in Section 5, we present constructions of graphs with distinct normal and list normal chromatic indices. We conclude the paper with some open problems in Section 6.

2. Preliminaries

In this section, we introduce the terminology and auxiliary results used in the paper.

As usual, for a sequence of consecutive integers, we use the abbreviation $[i, j] = \{i, i+1, \dots, j\}$. We call a cycle of length k a *k-cycle*. The *edge-neighborhood* $N(e)$ of an edge e is the set of edges adjacent to e, and the *2-edge-neighborhood* $N_2(e)$ is the set of edges at distance 1 or 2 from e. An *induced matching* is a set of edges M such that any pair of edges in M is at distance at least 3; i.e., the graph induced on the endvertices of the edges of M is a matching.

For a given list assignment L, a *partial strong L-edge-coloring* φ of a graph G is a strong edge-coloring of a subset of edges of G such that any pair of colored edges e and f; i.e., we have $\varphi(e) \in L(e)$, $\varphi(f) \in L(f)$ and $\varphi(e) \neq \varphi(f)$ if e and f are at distance at most 2 in G.

Given a list assignment L and a partial strong L-edge-coloring, we say that a color $c \in L(e)$ is *available* for the edge e if no edge in $N_2(e)$ is colored with c. We denote the set of all available colors for an edge e with $A(e)$. Clearly, $A(e) \subseteq L(e)$.

In our proofs, we use the following application of Hall's Marriage Theorem [12].

Theorem 2.1. *Let G be a graph and φ a partial (strong) edge-coloring of G. Let $X = \{e_1, \dots, e_k\}$ be the set of non-colored edges of G. Let $\mathcal{F} = \{A(e_1), \dots, A(e_k)\}$. If for every subset $\mathcal{X} \subseteq \mathcal{F}$ it holds that*

$$|\mathcal{X}| \leq \Big| \bigcup_{X \in \mathcal{X}} X \Big|,$$

then one can choose an available color for every edge in X such that all the edges receive distinct colors.

One of the strongest tools for determining whether colors from the sets of available colors can always be found such that the given conditions are satisfied is the following result due to Alon [1].

Theorem 2.2 (*Combinatorial Nullstellensatz [1]*)**.** *Let $\mathbb{F}$ be an arbitrary field, and let $P = P(X_1, \dots, X_n)$ be a polynomial in $\mathbb{F}[X_1, \dots, X_n]$. Suppose that the coefficient of the monomial $X_1^{k_1} \dots X_n^{k_n}$, where each k_i is a non-negative integer, is non-zero in P and the degree $\deg(P)$ of P equals $\sum_{i=1}^{n} k_i$. If moreover $S_1, \dots, S_n$ are any subsets of $\mathbb{F}$ with $|S_i| > k_i$ for $i = 1, \dots, n$, then there are $s_1 \in S_1, \dots, s_n \in S_n$ such that $P(s_1, \dots, s_n) \neq 0$.*

In short, for P_G being the graph polynomial of a graph G, if there is a monomial m of P_G with degree $\deg(P_G)$ and a non-zero coefficient, and moreover in m the degree of every variable is less than the number of available colors for the vertex represented by the variable, then there exists a coloring of G. For a monomial m, we denote the coefficient of m in the polynomial P_G by $\text{coef}(P_G;\ m)$.

Usually, we only consider edge-coloring of a subgraph H of a graph G, with some of the other edges in G already being precolored and hence the lists of available colors for edges in H are reduced accordingly. In order to apply Theorem 2.2, we construct an auxiliary conflict graph $C(H)$, in which every vertex represents an edge to be colored, and two vertices are adjacent whenever the corresponding edges need to be colored with distinct colors. Clearly, the input to Theorem 2.2 is the graph polynomial of $C(G)$, but to avoid this step, we simply say that we consider a *conflict graph polynomial* for H.

Note that in this paper, every conflict graph polynomial is homogeneous, i.e., it is a sum of monomials of the same degree, and therefore the degree condition of Theorem 2.2 for monomials is always fulfilled.

3. Upper bound on the list strong chromatic index

In this section, we prove the tight upper bound for the list strong chromatic index.

In the first part of our proof, we follow the proof of the result of Dai et al. [5] that the list strong chromatic index of subcubic graphs is at most 11. In particular, they showed that for eliminating cycles of length at most 5 from the minimal counterexample, one can even assume lists of length 10.

Proof of Theorem 1.4. Suppose the contrary and let G be a minimal counterexample to the theorem; i.e., a graph with maximum degree 3, which has the list strong chromatic index greater than 10.

Clearly, G is connected. Moreover, from [5], we have the following structural properties of G (since lists of size 10 are assumed in these lemmas).

Claim 1 (*[5, Lemma 2.1]*). *G is 3-regular.* ♦

Claim 2 (*[5, Lemma 2.2]*). *G does not contain any 3-cycle.* ♦

Claim 3 (*[5, Lemma 2.3]*). *G does not contain any 4-cycle.* ♦

Claim 4 (*[5, Lemma 2.4]*). *G does not contain any 5-cycle.* ♦

Next, we reduce cycles of length at least 6.

Claim 5. *G does not contain any 6-cycle.*

Proof. Suppose the contrary and let $C = v_0 \dots v_5$ be a 6-cycle in G. For every $i \in \{0, \dots, 5\}$, call the edge $x_i = v_i v_{i+1}$ (indices modulo 6) a *cycle edge*, and every non-cycle edge y_i incident to v_i a *pendant edge* (see Fig. 1).

By the minimality of G, there exists a list strong edge-coloring φ' of $G' = G \setminus \{v_1, v_2, v_3, v_4, v_5\}$ for any list assignment L with lists of size at least 10. Let φ be the coloring of G induced by φ'. Then, only the edges of C and the pendant edges except y_0 are non-colored in φ. The edges x_0, x_1, x_4 and x_5 have at least 5 available colors, the edges x_2 and x_3 have at least 6, y_1 and y_5 have at least 3, and y_2, y_3, y_4 have at least 4 available colors.

Claims 2–4 imply that no two pendant edges are the same or adjacent; it may however happen that the edges y_1 and y_4 (and similarly, y_2 and y_5) are connected by an edge; we thus assume also these two edges. So, the conflict graph polynomial P_{C_6} created on the non-colored edges with conflicts between edges at distance at most 2 is the following (taking indices modulo 6):

$$
\begin{aligned}
P_{C_6}(x_0, \dots, x_5, y_1, \dots, y_5) = & \left[\prod_{i=0}^{5} (x_i - x_{i+1}) \cdot (x_i - x_{i+2})\right] \\
& \cdot (x_0 - y_1) \cdot (x_0 - y_2) \cdot (x_0 - y_5) \\
& \cdot (x_1 - y_2) \cdot (x_1 - y_3) \cdot (x_1 - y_1) \\
& \cdot (x_2 - y_3) \cdot (x_2 - y_4) \cdot (x_2 - y_2) \cdot (x_2 - y_1) \\
& \cdot (x_3 - y_4) \cdot (x_3 - y_5) \cdot (x_3 - y_3) \cdot (x_3 - y_2) \\
& \cdot (x_4 - y_5) \cdot (x_4 - y_4) \cdot (x_4 - y_3) \\
& \cdot (x_5 - y_1) \cdot (x_5 - y_5) \cdot (x_5 - y_4) \\
& \cdot (y_1 - y_2) \cdot (y_2 - y_3) \cdot (y_3 - y_4) \cdot (y_4 - y_5) \\
& \cdot (y_1 - y_4) \cdot (y_2 - y_5)
\end{aligned}
$$

Using the function `Coefficient` in Wolfram Mathematica [31], we infer that in P_{C_6}, we have the coefficient

$$\operatorname{coef}(P_{C_6};\ x_0^4\, x_1^4\, x_2^5\, x_3^5\, x_4^4\, x_5^4\, y_1^2\, y_2^3\, y_3^2\, y_4^3\, y_5^2) = -2,$$

which, by Theorem 2.2, means that we can extend the coloring φ to all the edges of G, a contradiction.♦

We continue by showing that in G any cycle is reducible.

Claim 6. *G does not contain any cycle.*

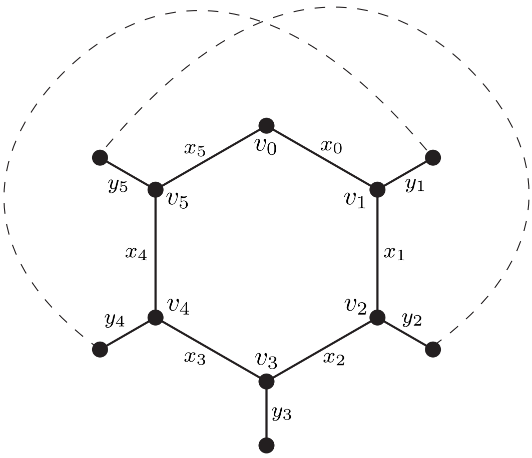

Fig. 1. The hypothetical 6-cycle C in G. The edges y_1 and y_4 (and also y_2 and y_5) might be connected by an edge (depicted dashed).

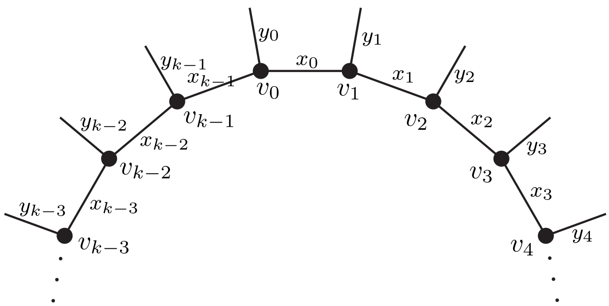

Fig. 2. The hypothetical k-cycle C in G.

Proof. Let $C = v_0 \dots v_{k-1}$ be a shortest cycle in G. For every $i \in \{0, \dots, k-1\}$, call the edge $x_i = v_i v_{i+1}$ (indices modulo k) a *cycle edge*, and every non-cycle edge y_{i+1} incident to v_i a *pendant edge* (see Fig. 2). By Claims 2–5, we have that $k \geq 7$. Moreover, since there is no $(k-1)$-cycle in G, we have that no pair of pendant edges is connected by any edge except by a cycle edge.

Let L be a list assignment for the edges of G with lists of size at least 10 for which G is not strongly L-edge-choosable. Let G' be the graph obtained from G by removing the vertices of C. By the minimality, G' admits a list strong edge-coloring φ' with color of every edge $e \in E(G')$ from $L(e)$. Let φ be the coloring of G induced by φ', where only the edges incident to the vertices in $V(C)$ are non-colored. In particular, every cycle edge x_i has at least 6 available colors, and every pendant edge y_i has at least 4. We will show that we can extend φ to all the edges of G.

First, let P_{C_k} be the conflict graph polynomial created on the non-colored edges with conflicts between edges at distance at most 2; taking indices modulo k, we have the following:

$$P_{C_k}(x_0, \dots, x_{k-1}, y_0, \dots, y_{k-1}) =$$
$$\prod_{i=0}^{k-1} (x_i - x_{i+1}) \cdot (x_i - x_{i+2}) \cdot (y_i - y_{i+1}) \cdot (x_i - y_{i-1}) \cdot (x_i - y_i) \cdot (x_i - y_{i+1}) \cdot (x_i - y_{i+2}).$$

Next, we prove that

$$\operatorname{coef}(P_{C_k};\ x_0^4y_0^3\ x_1^2y_1^2\ x_2^5y_2^3\ x_3^5y_3^3\ x_4^4y_4^3\ x_5^5y_5^3 \cdot \prod_{i=6}^{k-1} x_i^4y_i^3) = (-1)^k.$$

In order to compute the coefficient in general, we use Wolfram Mathematica [31]. Due to the limitations of the software, we need to split our computations into several steps; in particular, we compute coefficients of selected subpolynomials.

We begin by considering the subpolynomial $P_{C_k}^{2,3,4}$ of P_{C_k}, comprised of all factors containing x_i or y_i for $i \in \{2, 3, 4\}$. The polynomial $P_{C_k}^{2,3,4}$ has degree 29 and we infer that

$$\operatorname{coef}(P_{C_k}^{2,3,4};\ x_2^5y_2^3x_3^5y_3^3x_4^4y_4^3) = x_0^2x_5^2y_1^2 + 2x_0^2x_5y_1^2y_5 - x_0^2x_5^2y_5^2 - 2x_0x_5^2y_1y_5^2 + x_0^2y_1^2y_5^2 - x_5^2y_1^2y_5^2.$$

Here and in several subsequent cases, we slightly abuse the notation as the value of the coefficient is a polynomial, which appears in the conflict graph polynomial multiplied with the monomial given as an argument. Note that in the resulting polynomial, no variable from the monomial appears.

In the second step, we create polynomial $P_{C_k}^{5,6}$, comprised of $\operatorname{coef}(P_{C_k}^{2,3,4};\ x_2^5y_2^3x_3^5y_3^3x_4^4y_4^3)$ and multiplied with all factors containing x_i or y_i for $i \in \{5, 6\}$, which were not yet used in $P_{C_k}^{2,3,4}$. We infer that

$$\operatorname{coef}(P_{C_k}^{5,6};\ x_5^5y_5^3x_6^4y_6^3) = x_0^2y_1^2(x_7 + y_7).$$

Therefore, $x_0^2y_1^2(x_7 + x_7)$ is also the coefficient of the monomial $x_2^5y_2^3x_3^5y_3^3x_4^4y_4^3x_5^5y_5^3x_6^4y_6^3$ in the subpolynomial of P_{C_k} containing x_i or y_i for all $i \in \{2, \dots, 6\}$.

Now, we define (again, indices modulo k) an auxiliary polynomial

$$\begin{aligned} A_i(x_i, x_{i+1}, x_{i+2}, y_i, y_{i+1}, y_{i+2}) =& (x_i - x_{i+1})(x_i - x_{i+2})(x_i - y_i) \cdot \\ & \cdot (x_i - y_{i+1})(x_i - y_{i+2})(x_{i+1} - y_i)(y_i - y_{i+1}), \end{aligned}$$

used for defining partial polynomials for each of the remaining pairs x_i, y_i. Let

$$P_{C_k}^{7}(x_0, x_7, x_8, x_9, y_1, y_7, y_8, y_9) = \operatorname{coef}(P_{C_k}^{5,6};\ x_5^5y_5^3x_6^4y_6^3) \cdot A_7(x_7, x_8, x_9, y_7, y_8, y_9).$$

Then,

$$\operatorname{coef}(P_{C_k}^{7};\ x_7^4y_7^3) = -x_0^2y_1^2(x_8 + y_8).$$

Finally, for every i, $8 \le i \le k-1$, let

$$P_{C_k}^{i}(x_0, x_i, x_{i+1}, x_{i+2}, y_1, y_i, y_{i+1}, y_{i+1}) = (-1)^i \cdot x_0^2y_1^2(x_i + y_i) \cdot A_i(x_i, x_{i+1}, x_{i+2}, y_i, y_{i+1}, y_{i+1}),$$

obtaining

$$\operatorname{coef}(P_{C_k}^{i};\ x_i^4y_i^3) = (-1)^i \cdot x_0^2y_1^2(x_{i+1} + y_{i+1}).$$

In the last step, we consider the non-used factors with x_0, x_1, y_0, and y_1; we have

$$\begin{aligned} P_{C_k}^{0}(x_0, x_1, y_0, y_1) =& \operatorname{coef}(P_{C_k}^{k-1};\ x_{k-1}^4y_{k-1}^3) \cdot \\ & \cdot (x_0 - x_1)(x_0 - y_0)(x_0 - y_1)(x_1 - y_0)(x_1 - y_1)(y_0 - y_1), \end{aligned}$$

giving us

$$\operatorname{coef}(P_{C_k}^{0};\ x_0^4y_0^3) = (-1)^k \cdot (x_1^2y_1^2 - y_1^4).$$

This means that

$$\operatorname{coef}(P_{C_k};\ x_0^4y_0^3\ x_1^2y_1^2\ x_2^5y_2^3\ x_3^5y_3^3\ x_4^4y_4^3\ x_5^5y_5^3 \cdot \prod_{i=6}^{k-1} x_i^4y_i^3) = (-1)^k,$$

which implies, by Theorem 2.1, that we can always extend the coloring φ to all the edges of G, a contradiction.♦

Since G must be 3-regular by Claim 1, but it does not contain any cycle by Claims 2 to 6, we obtain a contradiction establishing the theorem. □

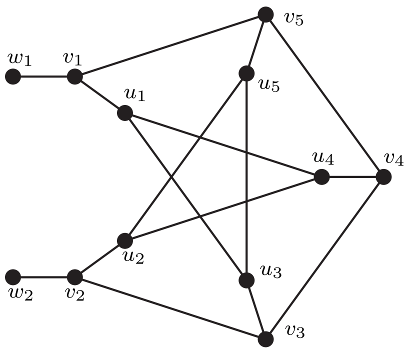

Fig. 3. The graph P' obtained from the Petersen graph by replacing one edge with two pendant edges.

4. Graphs G with $\chi'_s(G) < \chi'_{s,l}(G)$

In this section, we give a negative answer to Question 1.5 by proving Theorem 1.6. First, we recall the result about cubic graphs with strong chromatic index equal to 5. It uses the notion of covering graphs defined as follows. A surjective graph homomorphism $f : \tilde{G} \to G$ is called a *covering projection* if for every vertex $\tilde{v}$ of $\tilde{G}$ the set of edges incident with $\tilde{v}$ is bijectively mapped onto the set of edges incident with $f(\tilde{v})$. The graph G is usually referred to as the *base graph* and $\tilde{G}$ as a *covering graph* or a *lift* of G. A graph $\tilde{G}$ *covers* G if there exists such a covering projection.

Theorem 4.1 (*Lužar, Máčajová, Škoviera, and Soták [25]*)**.** *The strong chromatic index of a cubic graph G equals* 5 *if and only if G covers the Petersen graph.*

Let P' be the graph obtained from the Petersen graph by replacing one edge with two pendant edges (see Fig. 3). Consider the labeling of its vertices as given in the figure. For $1 \le i \le 5$, we call the edges $u_i v_i$ the *spokes* of P', the edges $u_i u_{i+2}$ (indices modulo 5) the *inner edges*, and the edges $v_i v_{i+1}$ (indices modulo 5 and $i = 1$ skipped) the *outer edges*.

We are now ready to prove Theorem 1.6.

Proof of Theorem 1.6. Let R be a covering graph of the Petersen graph P. By Theorem 4.1, we have that $\chi'_s(R) = 5$; let φ_R be a strong 5-edge-coloring of R.

Consider the graph G obtained from $R - uv$ (for some edge uv of R) and P' by identifying the vertices u and w_1, and v and w_2.

We first show that $\chi'_s(G) = 5$. Let π be a strong 5−edge-coloring of P'. with the two pendant edges having the same color. We obtain a strong 5-edge-coloring φ of G by keeping the colors from φ_R on the edges of $R - uv$, setting $\varphi(uv_1) = \varphi(vv_2) = \varphi_R(uv)$, permuting the colors of π such that $\varphi_R(uv) = \pi(v_1 w_1)$ and such that the colors on the edges incident to u (v) in φ_R are distinct from the colors incident to v_1 (v_2) in π (this can be done, since the same color c of the two pendant edges guarantees that c is the only color incident to both vertices v_1 and v_2), and finally setting $\varphi'(e') = \pi(e)$ for every edge $e' \in E(G)$ that corresponds to an edge $e \in E(P')$. Note that, by Theorem 4.1, this means that G is also a covering graph of P.

Next, we show that $\chi'_{s,l}(G) > 5$. Let L be a list assignment for G such that $L(e) = \{1, 2, 3, 4, 5\}$ for every edge e of G corresponding to an inner edge of P', $L(e) = \{1, 2, 3, 4, 6\}$ for every edge e of G corresponding to a spoke of P', and $L(e) = \{1, 2, 3, 5, 6\}$ for the remaining edges of G.

Let G' be the graph obtained from G by removing all the edges of P' except uv_1 and vv_2. Clearly, G' is the graph R with one edge removed and replaced with two pendant edges, and thus it admits a strong 5-edge-coloring φ^* induced by the coloring φ of R. Note that in φ, the edges uv_1 and vv_2 receive the same color (the color $\varphi(uv)$). Now, we show that in any strong 5-edge-coloring of G'

these two edges must be colored with the same color. Since the edges of G' in L have the same lists of size 5, this will imply that the two edges must receive the same color in any strong L-edge-coloring.

First, observe that in φ^* the only common color the vertices u and v are incident with is color $a = \varphi(uv)$. Let $b \neq a$ be a color incident with u, and $c \neq a$ a color incident with v. Let k be the number of edges of color b in φ^*. Since every edge of color b is adjacent to edges of all other four colors, the edges of G' colored with b in φ^* together with their adjacent edges cover all the edges of G' (every edge exactly once) except vv_2; we denote this (almost) covering C_b. Similarly, the edges of G' colored with c in φ^* together with their adjacent edges cover all the edges of G' (every edge exactly once) except uv_1; we denote this (almost) covering C_c.

Now, let σ be a strong 5-edge-coloring of G'. On the edges of C_b, every color appears k times, so together with the edge vv_2, the color $\sigma(vv_2)$ appears $k+1$ times. Similarly we deduce that using the covering C_c, the color $\sigma(uv_1)$ appears on $k+1$ edges, and so $\sigma(uv_1) = \sigma(vv_2)$. Hence, in every strong 5-edge-coloring of G' the edges uv_1 and vv_2 must be colored with the same color.

Similarly, we can show that around the vertices u and v in G, all five colors appear (i.e., the only common incident color is the color of the edges uv_1 and vv_2). Observe that the edges of G' colored with a in φ^* (except the edges uv_1 and vv_2) together with their adjacent edges cover all the edges of G' (every edge exactly once) except the edges incident with u and v; we denote this covering C_a. Again, in every strong 5-edge-coloring of G', on the edges of C_a every color appears $k-1$ times, while in the whole graph every color appears on k edges, except the color of uv_1 and vv_2, which appears $k+1$ times. This means, that u and v together are incident with edges of all five colors.

Now consider the coloring of the edges of P'. Clearly, in any strong L-edge-coloring, all the five colors from $\{1, 2, 3, 4, 5\}$ appear on the inner edges of P'. Similarly, since every spoke edge of P' sees 4 distinct colors on the inner edges of P', every spoke edge can be colored with precisely one of the colors from $\{1, 2, 3, 4\}$ or color 6, except for the spoke edge that does not have color 5 in its 2-edge-neighborhood—that edge must be colored with 6. Moreover, the edges v_1w_1 and v_2w_2 must be colored with the same color, so that we can combine the colorings of G' and P'.

There are three non-isomorphic possibilities on which inner edge color 5 appears. First, suppose that u_1u_4 is colored with 5. Then, u_5v_5 must be colored with 6 and therefore v_2v_3 is the only outer edge of P' which can be colored with 5 or 6. Therefore the colors 1, 2, and 3 must be used on the remaining outer edges, and consequently v_1w_1 and v_2w_2 must also both be colored with either 5 or 6. This is not possible, since v_1w_1 has both colors in it<s 2-edge-neighborhood.

Second, suppose that u_3u_5 is colored with 5. Then, u_4v_4 must be colored with 6 and thus no outer edge of P' can have color 6. Since every outer edge of P' also has color 5 in the 2-edge-neighborhood, it follows that the remaining four outer edges must be colored with colors 1, 2, and 3. This means that v_1v_5 and v_2v_3 receive the same color, say 1. But then, color 1 cannot be incident with u and v, and consequently, u and v together will not be incident with all five colors, which is not possible by the argument above.

So, we may assume that u_1u_3 is colored with 5. Then, u_2v_2 must be colored with 6, and v_1w_1 and v_2w_2 must both be colored with the same color as u_3u_5, which cannot be 4—say it is 1. Then, the outer edges of P' must be colored with colors from $\{2, 3, 5, 6\}$. Since only v_4v_5 can be colored with 5, it follows that v_1v_5 must have color 6. Therefore, u_2v_2 and v_1v_5 both have color 6, which means that, since uv_1 and vv_2 both have color 1, some color, different 1, must be incident with u and v. As we showed above, this is not possible, and therefore a strong L-edge-coloring of G does not exist. □

As already mentioned, there are planar graphs and bipartite graphs with different values of the strong chromatic index and the list strong chromatic index. Two representative examples are the dodecahedron and the generalized Petersen graph $GP(10, 3)$ (see Fig. 4). Both graphs cover the Petersen graph and thus their strong chromatic indices are 5, while neither of them is colorable from the list assignment assigning the list $\{1, 2, 3, 4, 5\}$ to the solid edges, the list $\{1, 2, 3, 4, 6\}$ to the dotted edges, and the list $\{1, 2, 3, 5, 6\}$ to the dashed edges, as they are depicted in the figure. We omit the proof.

Theorem 1.6 guarantees the difference between the strong chromatic index and its list version, but it is not clear what is the exact value of the latter. For the special case of the Petersen graph, we are able to prove the exact bound.

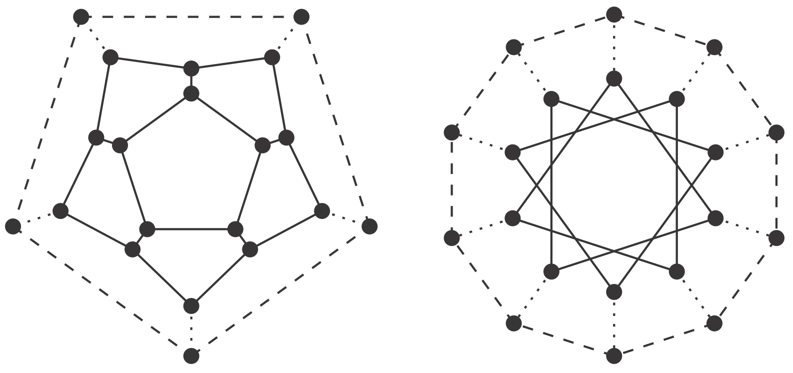

Fig. 4. The dodecahedron (left) and the generalized Petersen graph $GP(10, 3)$ (right).

Proof of Theorem 1.7. We first prove that the list strong chromatic index of the Petersen graph P is at least 7. Consider the drawing of P in Fig. 5. Let L be the list assignment assigning the list $\{1, 2, 4, 5, 7, 8\}$ to the outer cycle (the dashed edges), the list $\{1, 3, 4, 6, 7, 9\}$ to the spokes (the dotted edges), and the list $\{2, 3, 5, 6, 8, 9\}$ to the inner cycle (the solid edges). Recall that every maximum induced matching in P is of size 3 and it contains precisely one edge of the outer cycle, one spoke, and one edge of the inner cycle (in Fig. 5, we depict one with bolder edges). Moreover, any pair of edges at distance 3 belongs to exactly one maximum induced matching. Since there are five disjoint maximum induced matchings in P, one color can appear only on the edges of the same matching, but on at most two of its edges. Hence, we need at least 5 colors to color at most 10 edges, and at least 5 other colors to color the remaining 5 edges. However, we only have 9 distinct colors in the union of lists of L, thus we cannot color the edges of P from L.

Now, we show that the list strong chromatic index of the Petersen graph P is at most 7. Let M_k denote the five disjoint maximum induced matchings in P induced by the edges k_1, k_2, k_3, for $k \in \{a, b, c, d, e\}$, with the labeling of the edges as shown in Fig. 6.

In what follows, we will analyze the conflict graph polynomial P_P of the Petersen graph. We first define an auxiliary polynomial (representing a conflict graph polynomial of two maximum induced matchings)

$$C(x_1, x_2, x_3, y_1, y_2, y_3) = \prod_{i=1}^{3}\prod_{j=1}^{3}(x_i - y_j).$$

Next, observe that only the edges of a particular matching can be colored by the same color, and therefore each edge needs to receive a color distinct from colors of all other edges (from the other matchings). Hence, we have that

$$\begin{aligned}
P_P(a_1, a_2, a_3, b_1, \ldots, e_2, e_3) = {} & C(a_1, a_2, a_3, b_1, b_2, b_3) \cdot C(a_1, a_2, a_3, c_1, c_2, c_3) \\
& \cdot C(a_1, a_2, a_3, d_1, d_2, d_3) \cdot C(a_1, a_2, a_3, e_1, e_2, e_3) \\
& \cdot C(b_1, b_2, b_3, c_1, c_2, c_3) \cdot C(b_1, b_2, b_3, d_1, d_2, d_3) \\
& \cdot C(b_1, b_2, b_3, e_1, e_2, e_3) \cdot C(c_1, c_2, c_3, d_1, d_2, d_3) \\
& \cdot C(c_1, c_2, c_3, e_1, e_2, e_3) \cdot C(d_1, d_2, d_3, e_1, e_2, e_3)\,.
\end{aligned}$$

Now, we consider several cases regarding the possible colorings of the maximum induced matchings. Note that throughout the process of coloring the edges, as soon as some color is picked for an edge $e \in M_i$, this color is removed from the lists of the edges which are in conflict with e, i.e., the edges of the maximum induced matchings different from M_i.

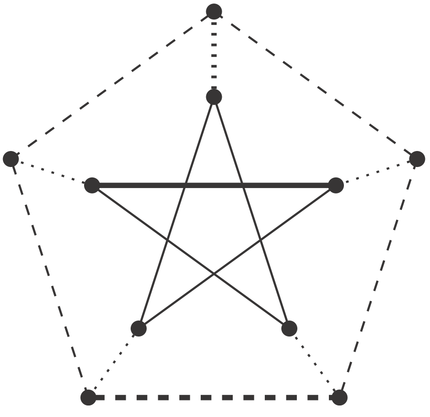

Fig. 5. The Petersen graph P.

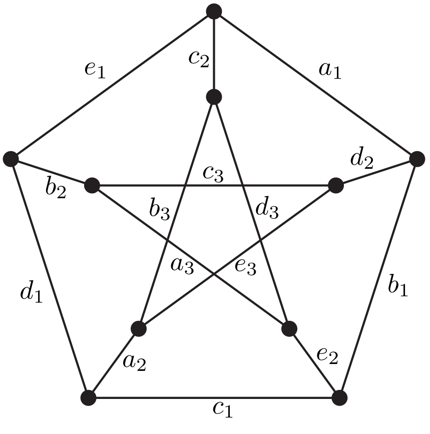

Fig. 6. Five maximum induced matchings of the Petersen graph P.

Case 1. *Suppose that one maximum induced matching can be colored monochromatically.*

First, we color the edges of M_a, say by color 1. We distinguish three possible subcases regarding the coloring of the remaining maximum induced matchings.

Case 1.1. *Suppose that one another maximum induced matching, say M_b, can be colored monochromatically (by color different from 1).*

Without loss of generality, we color M_b by 2. The remaining nine edges of P have each at least 5 colors available. Now, for the non-colored edges, we have the following conflict graph polynomial:

$$P_{P-2}(c_1, c_2, \dots, e_2, e_3) = C(c_1, c_2, c_3, d_1, d_2, d_3) \cdot C(c_1, c_2, c_3, e_1, e_2, e_3) \cdot C(d_1, d_2, d_3, e_1, e_2, e_3)\,.$$

In P_{P-2}, we have coefficient

$$\mathrm{coef}(P_{P-2};\ c_1^4 c_2^3 c_3^3 d_1^3 d_2^3 d_3^3 e_1^3 e_2^3 e_3^2) = 94,$$

which means, by Theorem 2.2, that it is possible to color the edges $c_1, c_2, c_3, d_1, d_2, d_3, e_1, e_2, e_3$ using the remaining colors of their lists.

Case 1.2. *Suppose that no maximum induced matching except M_a can be colored monochromatically, and at least one maximum induced matching, say M_b, can be colored using exactly two colors.*

Without loss of generality, we may assume that M_b is colored by colors 2 and 3. Since it is not possible to color any other maximum induced matching except M_a with just one color (different from 1), among the remaining edges, there is at least one such edge whose list does not contain both colors 2 and 3. It follows that all edges have lists of size at least 4, and at least one edge, say c_1, has list of size at least 5. For the non-colored edges, we again use the conflict graph polynomial as in Case 1, containing the coefficient

$$\mathrm{coef}(P_{P-2};\ c_1^4c_2^3c_3^3d_1^3d_2^3d_3^3e_1^3e_2^3e_3^2) = 94,$$

which means that it is possible to color the non-colored edges using the colors from their lists.

Case 1.3. *Suppose that no maximum induced matching except M_a can be colored monochromatically, and no other maximum induced matching is colorable by* 2 *colors.*

From now on we consider only the edges not included in M_a. Each of the remaining maximum induced matchings consists of three edges with pairwise disjoint lists of colors of size at least 6. This means that the union of lists of any two edges of a maximum induced matching is of size at least 12, and the union of lists of three edges of a maximum induced matching is of size at least 18. Consequently, for every set of at most 4 edges it holds that the union of their color lists is of size at least 6; for every set of 5 to 8 edges it holds that the union of their color lists is of size at least 12; and for every set of 9 to 12 edges it holds that the union of their color lists is of size at least 18. Hence, we can apply Theorem 2.1, according to which it is possible to color all the remaining edges.

Case 2. *Suppose that none of the maximum induced matchings can be colored by either* 1 *or* 2 *colors.*

Thus, each of the maximum induced matchings consists of three edges with pairwise disjoint lists of colors of size 7. This means that the union of lists of any two edges of a maximum induced matching is of size 14, and the union of lists of three edges of a maximum induced matching is of size 21. Therefore, for every set of at most five edges it holds that the union of their color lists is of size at least 7; for every set of 6 to 10 edges it holds that the union of their color lists is of size at least 14; and for every set of 11 to 15 edges it holds that the union of their color lists is of size at least 21. It follows that we can apply Theorem 2.1, and hence color all the edges by different colors.

Case 3. *Suppose that one maximum induced matching can be colored using* 2 *colors (and none of them can be colored monochromatically).*

We color M_a, say by colors 1 and 2. We consider two possible subcases regarding the coloring of the maximum induced matchings different from M_a.

Case 3.1. *Suppose that none of the remaining maximum induced matchings can be colored using* 2 *colors.*

From now on, we only consider the edges not included in M_a. Regarding any two (three) edges of any other maximum induced matching, we infer that the union of their color lists contains at least 10 (15) colors.

Therefore, for every set of at most 4 edges it holds that the union of their color lists is of size at least 5; for every set of 5 to 8 edges it holds that the union of their color lists is of size at least 10; and for every set of 9 to 12 edges it holds that the union of their color lists is of size at least 15. It follows that we can apply Theorem 2.1, and hence color all the remaining edges by different colors.

Case 3.2. *Suppose that we can color at least one other maximum induced matching using* 2 *colors.*

Without loss of generality, we may assume that M_b is colored by colors 3 and 4. Note that all these four colors can occur in the lists of other edges, but at most twice per maximum induced matching. Regarding the setup of these colors, it follows that lists of edges of any particular maximum induced matching are of size: at least 3, at least 3, and at least 7; or at least 3, at least 4, and at least 6; or at least 3, at least 5, and at least 5; or at least 4, at least 4, and at least 5. Note that these color lists may not be disjoint.

As above, for the non-colored edges we have the following conflict graph polynomial:

$$P_{P-2}(c_1, c_2, \ldots, e_2, e_3) = C(c_1, c_2, c_3, d_1, d_2, d_3) \cdot C(c_1, c_2, c_3, e_1, e_2, e_3) \cdot C(d_1, d_2, d_3, e_1, e_2, e_3)\,.$$

We consider the four cases regarding the sizes of the lists of the edges of maximum induced matchings as listed in the previous paragraph.

First, suppose there exists a maximum induced matching with lists of colors of the edges of sizes at least 3, at least 3, and at least 7. Then, in P_{P-2}, we have the coefficients

$$\mathrm{coef}(P_{P-2};\ c_1^2c_2^2c_3^6d_1^2d_2^2d_3^4e_1^2e_2^3e_3^4) = -14,$$

and

$$\mathrm{coef}(P_{P-2};\ c_1^2c_2^2c_3^6d_1^2d_2^2d_3^4e_1^2e_2^2e_3^5) = -6,$$

regarding the monomials fitting the possible sizes of the lists of remaining maximum induced matchings. Therefore, by Theorem 2.2, it is possible to color the remaining edges.

If there is no maximum induced matching with the properties as in the previous case, then suppose that there is one with lists of colors of sizes at least 3, at least 5, and at least 5. Then, in P_{P-2}, we have the coefficient

$$\mathrm{coef}(P_{P-2};\ c_1^2c_2^4c_3^4d_1^2d_2^3d_3^4e_1^2e_2^2e_3^4) = 60,$$

which means that it is possible to color the remaining edges.

Now, we may assume that there exists no maximum induced matching which satisfies properties of previous cases. Suppose that there is one maximum induced matching with lists of colors of sizes at least 3, at least 4, and at least 6. Then, in P_{P-2}, there is the coefficient

$$\mathrm{coef}(P_{P-2};\ c_1^2c_2^3c_3^5d_1^2d_2^3d_3^4e_1^2e_2^2e_3^4) = 33,$$

which means that it is possible to color the remaining edges.

Lastly, suppose that all of the remaining maximum induced matchings have lists of colors of the edges of sizes at least 4, at least 4, and at least 5. Then, in P_{P-2}, we have the coefficient

$$\mathrm{coef}(P_{P-2};\ c_1^3c_2^3c_3^4d_1^2d_2^3d_3^4e_1^1e_2^3e_3^4) = 36,$$

which again means that it is possible to color the remaining edges.

Thus, the list strong chromatic index of P is 7. □

5. Graphs G with $\chi'_n(G) < \chi'_{n,l}(G)$

In this section, we consider the results on list normal edge-coloring. As already mentioned, lists of size at least 10 are always enough to find a normal list edge-coloring of a cubic graph. We do not know whether this bound is tight; currently, there are only examples of graphs with list normal chromatic index equal to 9.

Theorem 5.1. *There is an infinite family of cubic graphs with list normal chromatic index at least* 9.

Proof. In order to prove the theorem, we will show that if a cubic graph G contains the configuration H_l depicted in Fig. 7, then there is a list assignment for the edges of H_l, for which G does not admit a list normal edge-coloring. We use the labeling of the vertices as given in the figure.

Let L be a list assignment for G such that for every edge e of H_l, except v_1v_6 and v_7v_8, we have $L(e) = [1, 8]$. For the two special edges, we use $L(v_1v_6) = [9, 16]$ and $L(v_7v_8) = [17, 24]$. Without loss of generality, we may assume that v_1v_6 is colored by 9, and v_7v_8 with 17. There are nine remaining (thin) edges, which can altogether receive the eight distinct colors from $[1, 8]$, so at least one pair must receive the same color. We will show that this is not possible. Note that the edges adjacent to the edges v_1v_6 and v_7v_8 must all be rich, since they are adjacent to an edge with a unique color.

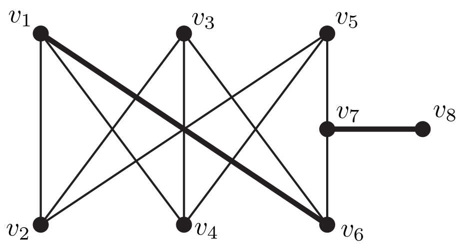

Fig. 7. The configuration H_l which is not list normal 8-colorable if the two bold edges each receive a list disjoint from all other lists in a given list assignment.

First, without loss of generality, we color the edge v_1v_2 by 1. Since v_1v_4 must be rich, the edges v_3v_4 and v_4v_5 must be colored differently, with colors distinct from 1, say with 2 and 3, respectively.

Suppose now that the edge v_3v_6 also receives color 1. Then, the edge v_1v_6 must be poor, meaning that the edges v_1v_4 and v_6v_7 receive the same color, say 4. Similarly, the edge v_2v_3 must be poor, and so the edges v_2v_5 and v_3v_4 must be colored the same, which means that v_2v_5 also receives color 2. But now v_4v_5 must be poor, and thus v_5v_7 must be colored with 4, which is not possible.

Next, suppose that v_3v_6 is colored with 3. Then v_3v_4 must be poor and so the edges v_1v_4 and v_2v_3 must receive the same color, which is not possible, since v_1v_2 is rich.

So, we may assume that v_3v_6 is colored with, say, 4. Suppose first that v_1v_4 is colored with 4. Then, v_3v_4 is poor and v_2v_3 is colored with 3. But now both v_2v_5 and v_1v_6 must be poor, meaning that both v_5v_7 and v_6v_7 must be colored with 1, a contradiction. Therefore, we may assume that v_1v_4 is colored with a new color, say 5, and consequently that v_3v_4 is rich, giving that v_2v_3 receives a new color, say 6. Since v_1v_2 and v_2v_3 are rich, a new color is given also to v_2v_5, say 7. Finally, note that the edges adjacent to v_5v_7 are all rich and consequently it must be colored with 8, which means that there is no available color for v_6v_7. Thus, for the given L, the graph G does not admit a list normal edge-coloring, and therefore $\chi'_{n,l}(G) > 8$. □

In the above described family, every graph contains a bridge. As Conjecture 1.8 considers bridgeless cubic graphs only, it is natural to ask whether there are graphs with the list normal chromatic index greater than 5. The next example shows that even lists of size 7 in the list assignment are sometimes not sufficient.

Theorem 5.2. *There are bridgeless cubic graphs with list normal chromatic index at least* 8.

Proof. As an example of a bridgeless cubic graph with the list normal chromatic index at least 8, we use the graph G depicted in Fig. 8. We will present a list assignment for the edges of G, for which G does not admit a list normal 7-edge-coloring. We use the labeling of the vertices as given in the figure.

Let L be a list assignment for G such that every edge e of G, except v_3v_6, v_5u_5, and u_4u_5, has $L(e) = [1, 7]$. The three special edges have $L(v_3v_6) = [8, 14]$, $L(v_5u_5) = [15, 21]$, and $L(u_4u_5) = [22, 28]$. Note that this setting implies that all the edges adjacent to these three edges must be rich.

First, we color the three special edges; without loss of generality, we color v_3v_6 with 8, v_5u_5 with 15, and u_4u_5 with 22. Next, we color the edges v_1v_2, v_1v_4, and v_1v_6 with, say, 1, 2, and 3, respectively.

Now, the edge v_4v_5 cannot receive color 1, since the edge v_2v_5 must be rich. If we color v_4v_5 with 3, then v_1v_4 must be poor and v_3v_4 colored with 1, which is not possible, since v_2v_3 must be rich. So, we assign to v_4v_5 color 4. Next, the edge v_3v_4 cannot receive any color from $\{1, 2, 3, 4\}$ (since v_1v_4 must be rich), and thus we color it with 5. Since v_3v_4 must be rich, v_2v_3 cannot be colored with 2, and since v_4v_5 must be rich, v_2v_5 cannot be colored with 2. This means that v_1v_2 must be rich and consequently, v_2v_3 and v_2v_5 cannot receive any color from the set $\{1, 2, 3, 4, 5\}$. Therefore, we assign, say, color 6 to v_2v_3 and color 7 to v_2v_5.

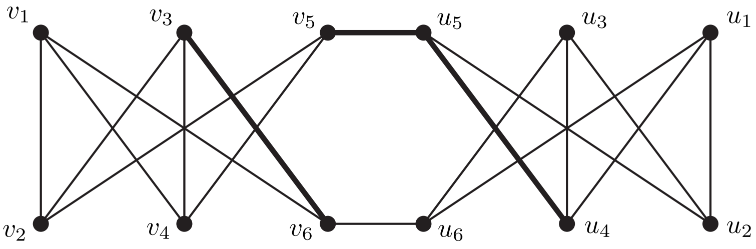

Fig. 8. A bridgeless cubic graph which is not list normal 7-colorable if the three bold edges each receive lists disjoint from all other lists.

At this point, the only possible colors the edge v_6u_6 can receive are 4 and 7. Note that these are exactly the colors already assigned to the edges adjacent to v_5u_5.

Now consider the other non-colored edges of the graph. Let $\{a, b, c, d, e, f, g\} = [1, 7]$. We can color the edges u_1u_2, u_1u_4, and u_1u_6 by three distinct colors, say a, b, and c, respectively. Since u_1u_4 must be rich, u_3u_4 must be colored with a color distinct from the previous three, say with d. Since the edges u_3u_4 and u_4u_5 are rich, none of the edges u_2u_3 and u_2u_5 can be colored with b, and therefore the edge u_1u_2 must be rich. Moreover, u_2u_5 cannot receive color d, since u_4u_5 must be rich. Therefore, we color u_2u_3 by e and u_2u_5 by f. The edge u_3u_6 cannot receive any color from the set $\{a, b, c, d, e, f\}$ (since u_2u_3 and u_3u_4 must be rich), so we must color it with the only remaining color g. Finally observe that the only possible color the edge v_6u_6 can receive is f, which is the same color as the color of u_2u_5. This means that two of the edges adjacent to v_5u_5 (which must be rich) must receive the same color, this contradicts the fact that v_5u_5 must be rich. □

The graph in the proof of Theorem 5.2 has a 2-edge-cut. So, again a question arises whether there are cubic graphs with high list normal chromatic index and high connectivity. We focused on cyclically 4-edge-connected cubic graphs and surprisingly there are such graphs with list normal chromatic index at least 7.

Theorem 5.3. *There is an infinite family of cyclically* 4*-edge-connected cubic graphs with list normal chromatic index at least* 7*.*

Proof. We will show that for a cubic graph L_{2k} depicted in Fig. 9 and any $k \geq 5$, there is a list assignment for the edges of L_{2k}, for which L_{2k} does not admit a list normal edge-coloring. Clearly, L_{2k} is cyclically 4-edge-connected. We use the labeling of the vertices as given in the figure.

Let L be a list assignment for L_{2k} such that for its every edge e, except v_1v_3 and v_2v_4, we have $L(e) = [1, 6]$. For the two special edges, we use $L(v_1v_3) = [7, 12]$ and $L(v_2v_4) = [13, 18]$. Without loss of generality, we may assume that v_1v_3 is colored by 7, and v_2v_4 with 13.

Without loss of generality, we can assign colors 1, 2, and 3 to the edges v_3v_5, v_4v_5, and v_5v_7, respectively. Since the edges adjacent to the edges v_1v_3 and v_2v_4 must all be rich, the edges v_4v_6 and v_3v_6 must obtain colors that were not used yet, say 4 and 5, respectively. Now, we consider two cases regarding the color of v_6v_8.

Suppose first that v_6v_8 is colored with 3. Then the edge v_7v_8 must be poor, and consequently v_7v_9 and v_8v_{10} must receive the same color. Now, following an analogous argument for coloring the remaining edges v_iv_{i+1}, v_iv_{i+2}, and $v_{i+1}v_{i+3}$, for $i \in \{9, 11, \ldots, 2k-1\}$, we infer that also u_3u_4 and u_1u_2 must be poor and thus u_1v_1 and u_2v_2 must receive the same color. This is not possible, since v_1v_2 must be rich.

So, v_6v_8 must be colored with color 6 (the only color not used yet). Then, v_7v_8 must be rich and by the symmetry, we may assume that v_7v_8 is colored with 1. It follows that v_7v_9 is colored with 2, and v_8v_{10} must be colored with 4 or 5. But this is not possible, since v_6v_8 must be rich. This completes the proof. □

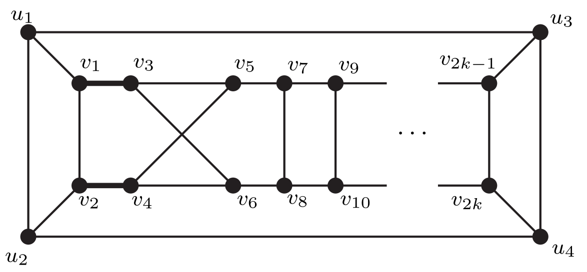

Fig. 9. A cyclically 4-edge-connected cubic graph L_{2k} which is not list normal 6-colorable if the two bold edges receive lists disjoint from all other lists.

6. Conclusion

One of the main results of this paper is the tight upper bound of 10 colors for the strong chromatic index of subcubic graphs. However, since this bound is only known to be attained by the Wagner graph and graph containing the $K_{3,3}$ with a subdivided edge as a subgraph, one may ask whether there are other examples of such graphs, perhaps with smaller chromatic index.

Question 6.1. *Is it true that for any subcubic graph G with* $\chi'_{s,l}(G) = 10$*, it holds that* $\chi'_s(G) = 10$*?*

As the second main result, we proved that the strong chromatic index and the list strong chromatic index differ for some graphs; we provided an infinite family of such graphs, but the family only contains graphs with the minimum possible value of the strong chromatic index, and it does not seem likely that for graphs with strong chromatic index closer to the general upper bound of 10 colors, their list strong chromatic index will be different. Therefore, we propose a rather bold statement, which is in line with Conjecture 1.3.

Conjecture 6.2. *For any connected bridgeless subcubic graph G on at least* 13 *vertices, it holds that*

$$\chi'_{s,l}(G) \leq 8.$$

The first step towards proving this conjecture would be proving that the list strong chromatic index of any connected bridgeless subcubic graph, not isomorphic to the Wagner graph, is at most 9. Or even more specifically, finding the exact upper bounds for the list strong chromatic indices of special graph families such as planar graphs and bipartite graphs would also give a relevant insight into the topic.

On the other hand, we do believe that there are cubic graphs with strong chromatic index 6 and greater list strong chromatic index.

Problem 6.3. Find an infinite family of cubic graphs G with $\chi'_s(G) = 6$ and

$$\chi'_{s,l}(G) > \chi'_s(G).$$

Also, we are confident that Theorem 1.6 can be extended to all cubic graphs with strong chromatic index equal to 5.

Conjecture 6.4. *For every cubic graph G with* $\chi'_s(G) = 5$ *we have that* $\chi'_{s,l}(G) > 5$.

The strong edge-coloring is an important concept; the study of (sub)cubic graphs is popular as these graphs are somewhat easier to handle than general graphs. The properties of list strong edge-coloring for general graphs are thus also of interest. In [25], it was shown that k-regular graphs attaining the lowest possible value $2k-1$ of the strong chromatic index are precisely the covers of the Kneser graphs $K(2k-1, k)$. It seems that Theorem 1.6, with some additional effort, could be extended to regular graphs of greater degree. Along these lines, we suggest the following question.

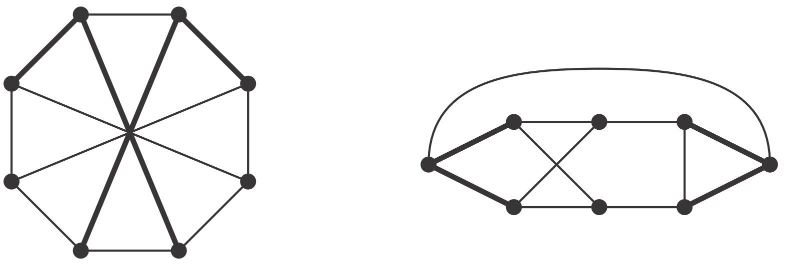

Fig. 10. Two bridgeless cubic graphs with list strong chromatic index at least 8. If the bold edges receive unique lists of seven colors and all the other edges the same lists of seven colors, then one cannot realize a list normal edge-coloring.

Question 6.5. *Is it true that for a given integer $k \geq 4$, there is an infinite family of graphs G of maximum degree k such that*

$$\chi'_{s,l}(G) > \chi'_s(G)\,?$$

In the case of list normal edge-coloring, we have the upper bound given by Theorem 1.4, but we do not have an example of a graph attaining the bound; in fact, we do not believe one exists.

Conjecture 6.6. *For any cubic graph G, it holds that*

$$\chi'_{n,l}(G) \leq 9.$$

Conjecture 1.8 assumes only bridgeless cubic graphs. We showed in Theorem 5.2 that in the list version, there are bridgeless cubic graphs with list normal chromatic index at least 8. In the proof of the theorem, we only provided one graph of order 12. However, we are only aware of two other graphs with list normal chromatic index at least 8; namely the Wagner graph and the graph obtained from $K_{3,3}$ in which one vertex is truncated (see Fig. 10). We also remark here without a proof that with some additional effort, one can show that the list normal chromatic index of the Wagner graph is equal to 8.

Based on our results and additional computer tests on small graphs, we confidently propose also the following.

Conjecture 6.7. *For any connected bridgeless cubic graph G on at least 14 vertices, it holds that*

$$\chi'_{n,l}(G) \leq 7.$$

As opposed to the normal edge-coloring, in its list version, the property of being a class I graph does not resolve the problem trivially. In fact, it seems that the following are highly non-trivial questions.

Question 6.8. *What is the tight upper bound for the list normal chromatic index of a cubic graph, which is:*

(a) (bridgeless) planar;
(b) class I;
(c) bipartite;
(d) with girth at least C, for some large enough constant C;
(e) cyclically k-edge-connected, for some integer k?

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