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Note

A remark on a result on odd colorings of planar graphs

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ABSTRACT

A proper k-coloring of a graph is said to be odd if every non-isolated vertex has a color that appears an odd number of times on its neighborhood. Miao et al. (2024) [2] claimed that every planar graph without adjacent 3-cycles is odd 7-colorable and every triangle-free planar graph without intersecting 4-cycles is odd 5-colorable. Here, we point out that their published proof contains a fundamental flaw which affects the validity of the main results.

1. Introduction

All graphs considered in this note are finite, simple, and undirected. We follow the terminology and notation of [1] without redefining them here. A *proper* k*-coloring* of a graph G is an assignment $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The *chromatic number*, $\chi(G)$, of a graph G is the minimum k such that G admits a proper k-coloring. An *odd* k*-coloring* of a graph G is a proper k-coloring such that every non-isolated vertex of the graph G has a color that appears an odd number of times on its neighborhood. This notion was introduced by Petruševski and Škrekovski [3] in 2022. The *odd chromatic number*, $\chi_o(G)$, of a graph G is the minimum k such that G admits an odd k-coloring.

We refer to any vertex of degree d as a d*-vertex*. Similarly, a vertex of degree at least d (resp. at most d) is a d^+*-vertex* (resp. a d^-*-vertex*). Analogous terminology applies to faces with respect to a planar embedding of G. We take $V_{3^+}(G)$ as the set of all 3^+-vertices of the graph G. We call a vertex of odd degree an *odd vertex*. An *odd neighbor* of a vertex v is a neighbor of v with odd degree. Two cycles are *adjacent* if they share a common edge and two cycles are *intersecting* if they share a common vertex.

In 2024, Miao et al. [2] proved that every planar graph without adjacent 3-cycles is odd 7-colorable and every triangle-free planar graph without intersecting 4-cycles is odd 5-colorable; however, their published proof contains a fundamental flaw which affects the validity of both the results mentioned above. In [2], the authors define a graph G_1 to be *sparser* than a graph G_2 if $\mu(G_1) < \mu(G_2)$, where $\mu(G) = \frac{|V_{3^+}(G)|}{|V(G)|}$. We state the results from [2].

Theorem 1. *Every planar graph without adjacent* 3*-cycles is odd* 7*-colorable.*

Theorem 2. *Every triangle-free planar graph without intersecting* 4*-cycles is odd* 5*-colorable.*

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1.1. A remark on the non-existence of the counterexample with minimum μ

The authors of the paper [2] assumed that there exists a counterexample G to Theorem 1 with minimum μ. However, they have not justified the existence of such a counterexample. Let S be the set of all counterexamples to Theorem 1 and let $M = \{\mu(G) \mid G \in S\}$. Since $0 \leq |V_{3^+}(G)| \leq |V(G)|$, $\mu(G) \in \mathbb{Q} \cap [0, 1]$.

Observation 1. *The set M does not contain* 0.

Proof. For the sake of contradiction, assume that the set M contains 0. Then, there exists a counterexample G to Theorem 1 with $|V_{3^+}(G)| = 0$. So $\Delta(G) \leq 2$. This implies that G is either a cycle or a path; thus G is odd 5-colorable. This contradicts that G is a counterexample to Theorem 1. □

By Observation 1, $M \subseteq \mathbb{Q} \cap (0, 1]$. Since the set of finite planar graphs is infinite, the number of possible counterexamples is infinite. Therefore, the set M may not attain a minimum. Thus, we cannot guarantee the existence of a counterexample to Theorem 1 with minimum μ.

Using the same argument, we can conclude that a counterexample to Theorem 2 with minimum μ may not exist. Thus, there is a gap in the argument presented in [2].

In the sequel, we assure that the set M does not attain a minimum by constructing an infinite sequence of counterexamples with decreasing μ with the help of the following claims.

Claim 1. *For every counterexample G to Theorem 1, there exists another counterexample G' to Theorem 1 with smaller μ.*

Claim 2. *For every counterexample G to Theorem 2, there exists another counterexample G' to Theorem 2 with smaller μ.*

If Theorem 1 does not hold, then there exists a counterexample, say G. Using Claim 1 repeatedly, we get an infinite sequence of counterexamples to Theorem 1 with decreasing μ. Thus, there is no counterexample to Theorem 1 with minimum μ. Similarly, we argue for Theorem 2.

2. Properties of the possible counterexamples

Now we show some properties of the possible counterexamples.

Proposition 1. *Every planar graph G without adjacent 3-cycles contains:*

(a) *adjacent odd vertices, or*
(b) *a 3^--vertex, or*
(c) *a 4-vertex with either two or more odd neighbors or an odd neighbor and a 4-neighbor, or*
(d) *a 3-face f incident to at least two 4-vertices, say v and x, such that v has a 4-neighbor $w \notin V(f)$.*

Proof. Let G be a planar graph without adjacent 3-cycles. For the sake of contradiction, assume that G does not contain any of the mentioned structures. We assign the initial charge ch_0 to all the vertices and faces such that $\mathrm{ch}_0(v) = d(v) - 6$ for every vertex $v \in V(G)$ and $\mathrm{ch}_0(f) = 2d(f) - 6$ for every face $f \in F(G)$. Using Euler's formula, we have $\sum_{v \in V(G)} \mathrm{ch}_0(v) + \sum_{f \in F(G)} \mathrm{ch}_0(f) = -12$. Therefore, the total sum of the initial charge assigned to all the vertices and faces is negative.

We apply the following discharging rules, which are same as the discharging rules used in the proof of Theorem 1.3 in [2]:

(R1) Every face $f \in F(G)$ sends $\frac{2d(f)-6}{d(f)}$ to each vertex that is incident with f.
(R2) For $uv \in E(G)$ with $d(u) = 5$ and $d(v) = 4$, if uv is incident with a 3-face in G, then u sends $\frac{1}{4}$ to v.
(R3) Let uv be an edge in G with $d(u) \geq 6$ and $d(v) = 4$.
- **(R3.1)** If the edge uv is incident to two 4^+-faces, then u sends $\frac{1}{4}$ to v.
- **(R3.2)** If uvw is a 3-face, then u sends $\frac{1}{4}$ to v if w is a 4-vertex and $\frac{1}{2}$ otherwise.

We obtain the final charge of every vertex and every face, denoted by ch.

Claim 3 *(Claims 5–7, [2]). The following hold.*

(a) *The final charge of every 6^+-vertex in G is non-negative.*
(b) *The final charge of every 5-vertex in G is non-negative.*
(c) *The final charge of every 4-vertex in G is non-negative.*

By Claim 3, the final charge of every vertex is non-negative. Since every face donates charge $\frac{2d(f)-6}{d(f)}$ to each vertex incident with f, the final charge of the face f is $\mathrm{ch}(f) = \mathrm{ch}_0(f) - \frac{2d(f)-6}{d(f)} \cdot d(f) = 2d(f) - 6 - (2d(f) - 6) = 0$. Therefore, the final charge of every face is non-negative. Note that the total charge is preserved in the discharging procedure. This is a contradiction to the fact that the total initial charge is negative. This completes the proof of Proposition 1. □

Proposition 2. *Every triangle-free planar graph G without intersecting 4-cycles contains:*

(a) *a 2^--vertex, or*
(b) *adjacent odd vertices.*

Proof. Let G be a triangle-free planar graph without intersecting 4-cycles. For the sake of contradiction, assume that G contains neither two adjacent odd vertices nor a 2^--vertex.

We assign initial charge ch_0 to all the vertices and faces such that $\mathrm{ch}_0(v) = 2d(v) - 6$ for every vertex $v \in V(G)$ and $\mathrm{ch}_0(f) = d(f) - 6$ for every face $f \in F(G)$. Using Euler's formula, we have $\sum_{v \in V(G)} \mathrm{ch}_0(v) + \sum_{f \in F(G)} \mathrm{ch}_0(f) = -12$. Therefore, the total sum of the initial charge assigned to all the vertices and faces is negative.

We apply the following discharging rules, which are same as the discharging rules used in the proof of Theorem 1.4 in [2]:

(R1) Every 4^+-vertex sends $\frac{1}{2}$ to each incident face.
(R2) Every 5^+-face sends $\frac{1}{4}$ to each its adjacent face.

We obtain the final charge of every vertex and every face, denoted by ch.

Claim 4 *(Claims 10–12, [2]). The following hold.*

(a) *The final charge of every vertex in G is non-negative.*
(b) *The final charge of every 4-face in G is non-negative.*
(c) *The final charge of every 5^+-face in G is non-negative.*

By Claim 4, the final charge of every vertex and every face is non-negative. Note that the total charge is preserved in the discharging procedure. This is a contradiction to the fact that the total initial charge is negative. This completes the proof of Proposition 2. □

3. Proof of Claim 1

Let G be a counterexample to Theorem 1. By Proposition 1, G contains:

(a) adjacent odd vertices, or
(b) a 3^--vertex, or
(c) a 4-vertex with either two or more odd neighbors or an odd neighbor and a 4-neighbor, or
(d) a 3-face f incident to at least two 4-vertices, say v and x, such that v has a 4-neighbor $w \notin V(f)$.

Construct a sparser graph G_1 from G as obtained in the proof of Claims 1–4 in [2]. For an illustration, see Figs. 1, 2, and 3. Fig. 1(a) refers to the case when there exist adjacent odd vertices u and v in G. Fig. 1(b) refers to the case when there exists a 1-vertex v in G. Fig. 1(c)–(e) refers to the different cases when there exists a 2-vertex v; in particular (c) corresponds to the case when v has a 2-neighbor v_1; (d) corresponds to the case when the neighbors of v have a common 2-neighbor w'; and (e) corresponds to the case when v has no 2-neighbor and its neighbors does not have a common 2-neighbor. Fig. 1(f) refers to the case when there exists a 3-vertex v in G. Fig. 2 refers to the case when there exists a 4-vertex v with either two or more odd neighbors or an odd neighbor and a 4-neighbor and Fig. 3 refers to the case when there exists a 3-face f incident to at least two 4-vertices v and x such that v has a 4-neighbor $w \notin V(f)$.

Next, we show that G_1 is a counterexample to Theorem 1. For the sake of contradiction, assume that G_1 is not a counterexample. Thus, G_1 admits an odd 7-coloring, say c. We obtain an odd 7-coloring of G using c as obtained in the proof of Claims 1–4 in [2]. This is a contradiction to the fact that G is a counterexample. Thus, G_1 is a counterexample which is sparser than G.

Thus, for every counterexample G to Theorem 1, there exists another counterexample G' to Theorem 1 with $\mu(G') < \mu(G)$. This completes the proof of Claim 1.

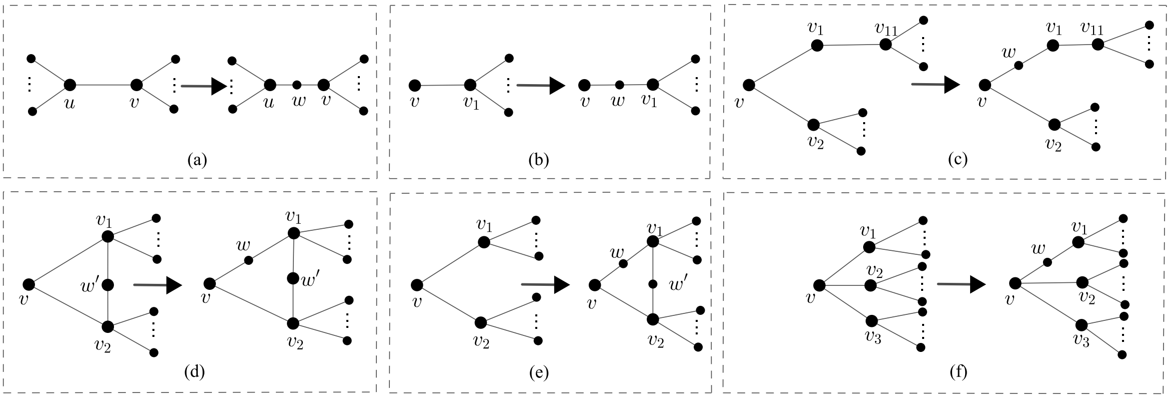

Fig. 1. Construction of the graph G_1 from G.

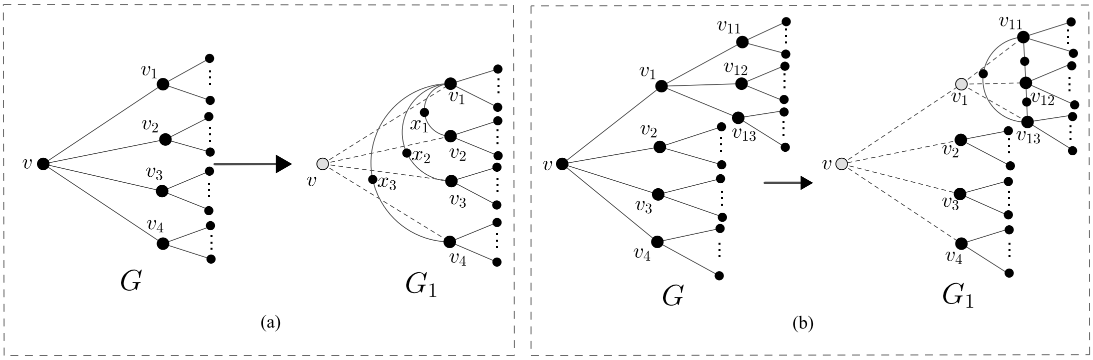

Fig. 2. Construction of the graph G_1 from G. Dashed edges and gray vertices represent deleted edges and vertices, respectively.

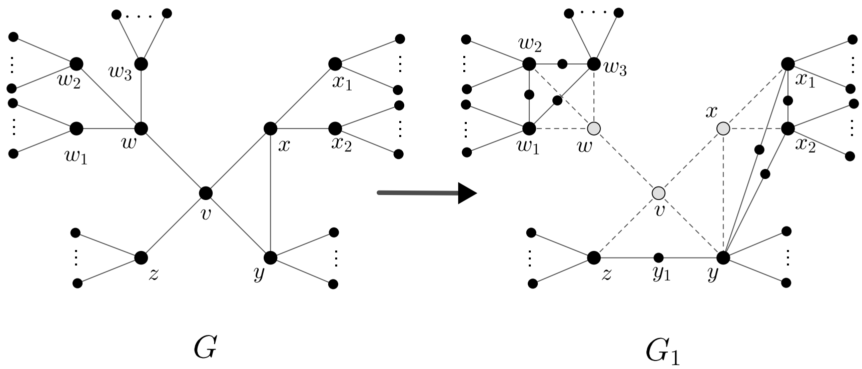

Fig. 3. Construction of the graph G_1 from G. Dashed edges and gray vertices represent deleted edges and vertices, respectively.

4. Proof of Claim 2

Let G be a counterexample to Theorem 2. By Proposition 2, G contains a 2^--vertex or adjacent odd vertices. Construct a sparser graph G_1 from G as obtained in the proof of Claims 8–9 in [2]. For an illustration, see Fig. 1(a)–(e).

Next, we show that G_1 is a counterexample to Theorem 2. For the sake of contradiction, assume that G_1 is not a counterexample. Thus, G_1 admits an odd 7-coloring, say c. We obtain an odd 7-coloring of G using c as obtained in the proof of Claims 8–9 in [2]. This is a contradiction to the fact that G is a counterexample. Thus, G_1 is a counterexample which is sparser than G.

Thus, for every counterexample G to Theorem 2, there exists another counterexample G' to Theorem 2 with $\mu(G') < \mu(G)$. This completes the proof of Claim 2.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

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