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Remarks on proper conflict-free degree-choosability of graphs with prescribed degeneracy

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ABSTRACT

A proper coloring ϕ of G is called a proper conflict-free coloring of G if for every non-isolated vertex v of G, there is a color c such that $|\phi^{-1}(c) \cap N_G(v)| = 1$. As an analogy of degree-choosability of graphs, we introduced the notion of proper conflict-free (degree $+\,k$)-choosability of graphs. For a non-negative integer k, a graph G is proper conflict-free (degree$\,+\,k$)-choosable if for any list assignment L of G with $|L(v)| \geq d_G(v)+k$ for every vertex $v \in V(G)$, G admits a proper conflict-free coloring ϕ such that $\phi(v) \in L(v)$ for every vertex $v \in V(G)$. In this note, we first remark if a graph G is d-degenerate, then G is proper conflict-free (degree $+ d + 1$)-choosable. Furthermore, when $d = 1$, we can reduce the number of colors by showing that every tree is proper conflict-free (degree$\,+\,1$)-choosable. This motivates us to state a question.

1. Introduction

Throughout the paper, we only consider simple, finite, and undirected graphs. Let $\mathbb{N}$ be the set of positive integers. For a positive integer k, let $[k]$ denote the set of integers $\{1, 2, \ldots, k\}$.

For a graph G, a mapping ϕ from $V(G)$ to $\mathbb{N}$ is called a *proper coloring of* G if $\phi(u) \neq \phi(v)$ for every edge $uv \in E(G)$. A proper coloring of a graph G in which every vertex of G maps to an integer in $[k]$ is called a proper k-coloring of G.

Recently, Fabrici, Lužar, Rindošová, and Soták [6] introduced a new variation of coloring named proper conflict-free coloring of graphs. For a graph G, a mapping ϕ from $V(G)$ to $\mathbb{N}$ is called a *proper conflict-free coloring of* G if ϕ is a proper coloring of G and every non-isolated vertex $v \in V(G)$ has a color c such that $|\phi^{-1}(c) \cap N_G(v)| = 1$, where $N_G(v)$ is the (open) neighborhood of v. A proper conflict-free coloring of a graph G such that every vertex of G maps to an integer in $[k]$ is called a *proper conflict-free* k*-coloring of* G. For a (partial) coloring ϕ of G and a vertex $v \in V(G)$, let $\mathcal{U}_\phi(v, G)$ denote the set of colors that appear exactly once in the neighborhood of v. Using this notation, a proper conflict-free coloring ϕ of G is a proper coloring ϕ of G such that $\mathcal{U}_\phi(v, G) \neq \emptyset$ for every non-isolated vertex $v \in V(G)$. The *proper conflict-free chromatic number* of a graph G, denoted by $\chi_{\mathrm{pcf}}(G)$, is the least integer k such that G admits a proper conflict-free k-coloring.

One major problem in proper conflict-free coloring is the following Brooks-type conjecture, which was posed by Caro, Petruševski, and Škrekovski [2].

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Conjecture 1. *For every graph G with the maximum degree $\Delta \geq 3$, $\chi_{\mathrm{pcf}}(G) \leq \Delta + 1$.*

This conjecture is widely open except for the case $\Delta = 3$ by Liu and Yu [13] and some asymptotic results in the literature [3,4,7,11,12].

It is well known that the original Brooks' theorem was generalized to degree-choosability of graphs in Borodin [1] and Erdős, Rubin, and Taylor [5]. Following the same direction, we introduced the concept of proper conflict-free (degree $+\, k$)-choosability of graphs in [9], as follows.

A list assignment L of a graph G maps each vertex of G to a set of integers. For a mapping f from $V(G)$ to positive integers, an f-list assignment of G is a list assignment L of G with $|L(v)| \geq f(v)$ for every vertex $v \in V(G)$. In particular, if f is the constant map from $V(G)$ to a positive integer k, an f-list assignment of G is called a k-list assignment of G.

For a given graph G and a list assignment L of G, a *proper conflict-free L-coloring* of G is a proper conflict-free coloring ϕ of G such that $\phi(v) \in L(v)$ for every vertex $v \in V(G)$. For a non-negative integer k, a graph G is *proper conflict-free* (degree $+\, k$)*-choosable* if G admits a proper conflict-free L-coloring for any f-list assignment of G, where $f(v) = d_G(v) + k$ for every vertex $v \in V(G)$. It is natural to ask whether there is an absolute constant k such that every graph is proper conflict-free (degree $+\, k$)-choosable, but in fact, even the existence of a constant k such that $\chi_{\mathrm{pcf}}(G) \leq \Delta(G) + k$ for every graph G is unknown.

In this note, we focus on the proper conflict-free (degree $+\, k$)-choosability of graphs with a given degeneracy. For a positive integer d, a graph G is d-degenerate if every subgraph H of G satisfies $\delta(H) \leq d$. We first remark the following simple bound, which states the relationship between the degeneracy and proper conflict-free (degree $+\, k$)-choosability of graphs.

Proposition 2. *If G is a d-degenerate graph for some positive integer d, then G is proper conflict-free* (degree $+\, d + 1$)*-choosable.*

Proof. Let G be a d-degenerate graph of order n. Let L be a list assignment of G satisfying $|L(v)| \geq d_G(v) + d + 1$ for each vertex $v \in V(G)$. Since G is d-degenerate, we label the vertices of G as $v_1, v_2, \ldots, v_n$ so that each vertex has at most d neighbors with smaller indices.

We color the vertices greedily in the order $v_1, v_2, \ldots, v_n$ as follows: For each $i \in \{1, 2, \ldots, n\}$, we assign v_i a color from $L(v_i)$ that differs from the colors of all smaller-indexed vertices that are either adjacent to v_i or are the smallest-indexed neighbor of a vertex adjacent to v_i. Note that at most d colors are forbidden by the previously colored neighbors and at most $d_G(v)$ colors are forbidden by the smallest-indexed neighbors of $N_G(v_i)$, and hence at least one color is available for v_i. It is obvious that the obtained coloring is a proper coloring of G. Furthermore, for each vertex of G, the color of the smallest-indexed neighbor appears exactly once in its neighbors, and hence we obtain a proper conflict-free L coloring of G. □

This improves a bound from Cranston and Liu [4] of a Brooks-type statement. One may ask whether the bound is best possible. As we saw previously, the 5-cycle is 2-degenerate and not proper conflict-free (degree $+\, 2$)-choosable, and hence we cannot improve the bound in Proposition 2 in general. On the other hand, when $d = 1$, we show the following result, that states that the bound can be reduced to (degree $+\, d$).

Theorem 3. *Every tree is proper conflict-free* (degree $+\, 1$)*-choosable.*

The proof of the above theorem is given in the next section. The bound (degree $+\, 1$) cannot be reduced to (degree $+\, 0$). Indeed, let us consider a star $K_{1,n-1}$ ($n \geq 3$) with the center v_0 and leaves $v_1, v_2, \ldots, v_{n-1}$, and let L be a list assignment of $K_{1,n-1}$ such that $|L(v_0)| = n - 1$ and $L(v_1) = L(v_2) = \cdots = L(v_{n-1}) = \{1\}$. Then the center v_0 must have $n - 1$ neighbors with color 1 no matter what list v_0 has, and hence $K_{1,n-1}$ is not proper conflict-free L-colorable.

Similarly, we can construct a 2-degenerate graph that is not proper conflict-free (degree $+\, 1$)-choosable. Let G be a graph obtained by n copies $C_1, C_2, \ldots, C_n$ of the 4-cycle ($n \geq 1$) by identifying the vertices $v_1, v_2, \ldots, v_n$ into a vertex v, where v_i is a vertex of C_i. Obviously G is 2-degenerate. Let L be a list assignment of G such that $L(v) = \{1, 2, \ldots, 2n + 1\}$ and each vertex of $V(C_i) \setminus \{v_i\}$ has a list $\{1, 2i, 2i + 1\}$. Then, by the conflict-free condition of vertices of the ith copy, colors $\{1, 2i, 2i + 1\}$ are forbidden for v, and hence there is no color left for v. Hence, G is not proper conflict-free L-colorable.

Considering Theorem 3 and these constructions, we ask whether the following may hold.

Conjecture 4. *If G is a d-degenerate graph for some positive integer d and G is not isomorphic to the* 5*-cycle, then G is proper conflict-free* (degree $+\, d$)*-choosable.*

The above constructions imply that the bound in Conjecture 4 is best possible for $d = 1, 2$. Also, by Theorem 3, Conjecture 4 holds for the case $d = 1$. For other values of d, we know that every connected outerplanar graph other than the 5-cycle is 2-degenerate and proper conflict-free (degree $+\, 2$)-choosable [8], and that every planar graph is 5-degenerate and proper conflict-free (degree $+\, 5$)-choosable [10], which will appear in separate papers.

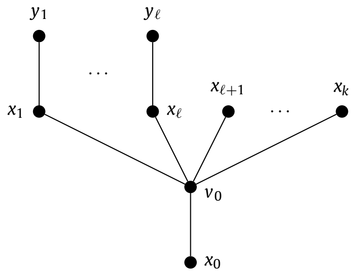

Fig. 1. A reducible structure of T.

2. Proof of Theorem 3

Suppose that the statement is false, and let T be a minimum counterexample. Obviously, $|V(T)| \geq 3$. Let L be a list assignment of T such that $|L(v)| = d_T(v) + 1$ for every vertex $v \in V(T)$ and T is not proper conflict-free L-colorable.

Claim 1. *Let $v_1v_2v_3$ be a path of length 2 of T with $d_T(v_1) = 1$ and $d_T(v_2) = 2$. Then $L(v_1) \subseteq L(v_2)$.*

Proof. Assume to the contrary that $L(v_1) \setminus L(v_2) \neq \emptyset$. Take a color $\alpha \in L(v_1) \setminus L(v_2)$. Let $T' = T - \{v_1, v_2\}$ and let L' be a list assignment of T' defined by $L'(v_3) = L(v_3) \setminus \{\alpha\}$ and $L'(v) = L(v)$ for every $v \in V(T') \setminus \{v_3\}$. Note that $|L'(v)| \geq d_{T'}(v) + 1$ for every $v \in V(T')$. By the minimality of T, T' admits a proper conflict-free L'-coloring ϕ. Let $\phi(v_3) = \beta \neq \alpha$ and let γ be a color in $\mathcal{U}_\phi(v_3, T')$. By setting $\phi(v_1) = \alpha$ and choosing $\phi(v_2) \in L(v_2) \setminus \{\beta, \gamma\}$, ϕ can be extended to a proper conflict-free L-coloring of T, a contradiction. □

Claim 2. *T does not have a path $v_1v_2v_3v_4$ of length 3 with $d_T(v_1) = 1$ and $d_T(v_2) = d_T(v_3) = 2$.*

Proof. Assume to the contrary that T has such a path $v_1v_2v_3v_4$. By Claim 1, we know that $L(v_1) \subseteq L(v_2)$. As $|L(v_1)| = 2$ and $|L(v_2)| = 3$, let α be a color in $L(v_2) \setminus L(v_1)$. Let $T' = T - \{v_1, v_2, v_3\}$ and let L' be a list assignment of T' defined by $L'(v_4) = L(v_4) \setminus \{\alpha\}$ and $L'(v) = L(v)$ for every $v \in V(T') \setminus \{v_4\}$. By the minimality of T, T' admits a proper conflict-free L'-coloring ϕ. Let $\phi(v_4) = \beta \neq \alpha$ and let γ be a color in $\mathcal{U}_\phi(v_4, T')$. Note that it is possible that $\gamma = \alpha$. We choose $\phi(v_3) \in L(v_3) \setminus \{\beta, \gamma\}$. We let $\phi(v_2) = \alpha$ if $\phi(v_3) \neq \alpha$, and let $\phi(v_2)$ be a color in $L(v_2) \setminus \{\phi(v_3), \beta\}$ otherwise. In either case, one of v_2 and v_3 is colored with α, which is not in $L(v_1)$. Since $|L(v_1) \setminus \{\phi(v_2), \phi(v_3)\}| \geq 1$, we can choose $\phi(v_1) \in L(v_1) \setminus \{\phi(v_2), \phi(v_3)\}$, and hence ϕ can be extended to a proper conflict-free L-coloring of T, a contradiction. □

By Claim 2, T has a vertex v_0 of degree at least 3 such that each component of $T - v_0$ except one component is isomorphic to K_1 or K_2. Let $N_T(v_0) = \{x_0, x_1, \dots, x_k\}$. Note that $k = d_T(v_0) - 1 \geq 2$. For each $i \in \{0, 1, \dots, k\}$, let T_i denote the component of $T - v_0$ that contains x_i. Without loss of generality, we may assume that T_i is isomorphic to K_2 for every $i \in \{1, 2, \dots, \ell\}$ and T_i is isomorphic to K_1 for every $i \in \{\ell+1, \ell+2, \dots, k\}$, where ℓ is a positive integer at most k. For each $i \in \{1, 2, \dots, \ell\}$, let $V(T_i) = \{x_i, y_i\}$ (Fig. 1).

By Claim 1, we have $L(y_i) \subseteq L(x_i)$ for each $i \in \{1, 2, \dots, \ell\}$. Thus, we let $L(x_i) = \{\alpha_i, \beta_i, \gamma_i\}$ and $L(y_i) = \{\beta_i, \gamma_i\}$ for each $i \in \{1, 2, \dots, \ell\}$, and let $L(x_i) = \{\alpha_i, \beta_i\}$ for each $i \in \{\ell+1, \ell+2, \dots, k\}$.

In the rest of the proof, we take a proper conflict-free coloring of $T' := T - \left(\bigcup_{i=1}^{k} V(T_i) \cup \{v_0\}\right)$ and extend it to T.

We first consider relatively simple three cases. In the following three cases, let ϕ be a proper conflict-free L-coloring of T'. Let $\alpha = \phi(x_0)$ and let β be a color in $\mathcal{U}_\phi(x_0, T')$.

Case 1. $k = 2$.

Note that $d_T(v_0) = 3$ in this case. If $\ell = 0$, then we let $\phi(x_2) \in L(x_2) \setminus \{\alpha\}$, $\phi(v_0) \in L(v_0) \setminus \{\alpha, \beta, \phi(x_2)\}$, and $\phi(x_1) \in L(x_1) \setminus \{\phi(v_0)\}$. If $\ell = 1$, then we let $\phi(x_2) \in L(x_2) \setminus \{\alpha\}$, $\phi(v_0) \in L(v_0) \setminus \{\alpha, \beta, \phi(x_2)\}$, $\phi(y_1) \in L(y_1) \setminus \{\phi(v_0)\}$, and $\phi(x_1) \in L(x_1) \setminus \{\phi(v_0), \phi(y_1)\}$. In either case, since $d_T(v_0) = 3$ and at least two colors appear in the neighbors of v_0, we have $\mathcal{U}_\phi(v_0, T) \neq \emptyset$. Thus, we obtain a proper conflict-free L-coloring of T, a contradiction.

Now assume that $\ell = 2$. We consider another list assignment L' of T'. If $\alpha_1 = \alpha_2$, then let L' be a list assignment of T' defined by $L'(x_0) = L(x_0) \setminus \{\alpha_1\}$ and $L'(v) = L(v)$ for every $v \in V(T') \setminus \{x_0\}$. Otherwise, let $L' = L$. Note that $|L'(v)| \geq d_{T'}(v) + 1$ for every $v \in V(T')$. By the minimality of T, T' admits a proper conflict-free L'-coloring ϕ'. Let $\phi'(x_0) = \alpha'$ and let β' be a color in $\mathcal{U}_{\phi'}(x_0, T')$. By the definition of L', either $\alpha_1 \neq \alpha'$ or $\alpha_2 \neq \alpha'$ holds. Without loss of generality, we may assume that $\alpha_1 \neq \alpha'$. Then let $\phi'(x_1) = \alpha_1$, $\phi'(v_0) \in L(v_0) \setminus \{\alpha', \beta', \alpha_1\}$, $\phi'(y_i) \in L(y_i) \setminus \{\phi'(v_0)\}$ for $i = 1, 2$, and $\phi'(x_2) \in L(x_2) \setminus \{\phi'(v_0), \phi'(y_2)\}$. It is easy to verify that ϕ' is a proper conflict-free L-coloring of T, a contradiction.

Case 2. $k \geq 3$ and $\ell = k$.

For each color $c \in L(v_0) \setminus \{\alpha, \beta\}$, let $I_c = \{i \mid 1 \leq i \leq \ell, c \in L(y_i)\}$. Since $|L(v_0) \setminus \{\alpha, \beta\}| \geq k$ and $\sum_{c \in L(v_0)\setminus\{\alpha,\beta\}} |I_c| \leq \sum_{i=1}^{\ell} |L(y_i)| = 2\ell$, there is a color $\gamma \in L(v_0) \setminus \{\alpha, \beta\}$ such that $|I_\gamma| \leq \frac{2\ell}{k} = 2$. Set $\phi(v_0) = \gamma$. For each $i \in I_\gamma$, let $\phi(y_i) \in L(y_i) \setminus \{\gamma\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\gamma, \phi(y_i)\}$. Since $|I_\gamma| \leq 2 < \ell$, we may assume that $1 \notin I_\gamma$.

Now we color remaining vertices. Since there are at most three colored neighbors of v_0 including x_0, either (a) there is a color α' that appears exactly once in the colored neighbors of v_0, or (b) all neighbors of v_0 are colored by α. For each case, we color the neighbors of v_i in the following manner:

- If (a), then let $\phi(x_i) \in L(x_i) \setminus \{\gamma, \alpha'\}$ for each $i \in \{1, 2, \ldots, \ell\} \setminus I_\gamma$.
- If (b), then let $\phi(x_1) \in L(x_1) \setminus \{\gamma, \alpha\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\gamma, \phi(x_1)\}$ for each $i \in \{2, 3, \ldots, \ell\} \setminus I_\gamma$.

Then, we have $\alpha' \in \mathcal{U}_\phi(v_0, G)$ if (a), and $\phi(x_1) \in \mathcal{U}_\phi(v_0, G)$ if (b). Finally, for each $i \in \{1, 2, \ldots, \ell\} \setminus I_\gamma$, we choose $\phi(y_i) \in L(y_i) \setminus \{\phi(x_i)\}$. Since $\gamma \notin L(y_i)$ for each $i \in \{1, 2, \ldots, \ell\} \setminus I_\gamma$, we know that $\phi(y_i) \neq \gamma = \phi(v_0)$ and hence $\gamma \in \mathcal{U}_\phi(x_i, G)$. Thus, ϕ is a proper conflict-free L-coloring of T, a contradiction.

Case 3. $k \geq 3$ and $\ell = k - 1$.

We define the color γ and the set I_γ similarly to Case 2. Note that the colors in $L(x_k)$ are not considered when we define γ and I_γ in this case. By the assumption of this case and the choice of the color γ, we know that $|I_\gamma| \leq \left\lfloor \frac{2\ell}{k} \right\rfloor = 1$. Note that $d_G(x_k) = 1$ and $d_G(x_i) = 2$ for each $i \leq k - 1$.

Set $\phi(v_0) = \gamma$. We let $\phi(y_i) \in L(y_i) \setminus \{\gamma\}$ and $\phi(x_i) \in L(x_i) \setminus \{\gamma, \phi(y_i)\}$ for $i \in I_\gamma$, and let $\phi(x_k) \in L(x_k) \setminus \{\gamma\}$. Then, the number of colored neighbors of v_0 is equal to $|I_\gamma \cup \{x_0, x_k\}| = |I_\gamma| + 2 \leq 3$. Therefore, by a similar argument as in Case 2, we can extend ϕ to a proper conflict-free L-coloring of T, a contradiction.

By the above three cases, we may assume that $k \geq 3$ and $\ell \leq k - 2$. We set $X = \{x_1, x_2, \ldots, x_k\}$. Let $L(X) = \bigcup_{x \in X} L(x)$ and let $\tilde{L}(v_0) = L(v_0) \setminus L(X)$. We consider two cases depending on whether $\left|\tilde{L}(v_0)\right| \geq 2$ or not.

Case 4. $\left|\tilde{L}(v_0)\right| \geq 2$.

By Claim 1, $\tilde{L}(v_0) \cap L(y_i) = \tilde{L}(v_0) \cap L(x_i) = \emptyset$ for every $i \in \{1, 2, \ldots, \ell\}$. We fix a color $\gamma \in \tilde{L}(v_0)$, and let L' be a list assignment of T' defined by $L'(x_0) = L(x_0) \setminus \{\gamma\}$ and $L'(v) = L(v)$ for every $v \in V(T') \setminus \{x_0\}$. By the minimality of T, T' admits a proper conflict-free L'-coloring ϕ. Let $\phi(x_0) = \alpha$ and let β be a color in $\mathcal{U}_\phi(x_0, T')$. Note that it is possible that $\beta = \gamma$.

If $\tilde{L}(v_0) \setminus \{\alpha, \beta\} \neq \emptyset$, then let $\phi(v_0) \in \tilde{L}(v_0) \setminus \{\alpha, \beta\}$, let $\phi(x_i) \in L(x_i) \setminus \{\alpha\}$ for each $i \in \{1, 2, \ldots, k\}$ and let $\phi(y_i) \in L(y_i) \setminus \{\phi(x_i)\}$ for each $i \in \{1, 2, \ldots, \ell\}$. This extends ϕ to a proper conflict-free L-coloring of T, a contradiction.

Thus, we infer that $\tilde{L}(v_0) \setminus \{\alpha, \beta\} = \emptyset$, which implies that $\alpha \in \tilde{L}(v_0)$ and $\beta = \gamma$. Then we choose $\phi(v_0) \in L(v_0) \setminus \{\alpha, \beta\}$ arbitrarily, let $\phi(y_i) \in L(y_i) \setminus \{\phi(v_0)\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0), \phi(y_i)\}$ for each $i \in \{1, 2, \ldots, \ell\}$, and let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0)\}$ for each $i \in \{\ell + 1, \ell + 2, \ldots, k\}$. Since $\alpha \notin L(X)$, we have $\alpha \in \mathcal{U}_\phi(v_0, T)$ and hence ϕ is a proper conflict-free L-coloring of T, a contradiction.

Case 5. $\left|\tilde{L}(v_0)\right| \leq 1$.

The assumption of this case implies that $|L(X)| \geq |L(v_0)| - 1 \geq k + 1$. For each color $c \in L(X)$, let $J_c = \{i \mid 1 \leq i \leq k, c \in L(x_i)\}$. Note that $J_c \neq \emptyset$ for any color $c \in L(X)$. Let γ be a color in $L(X)$ such that $|J_\gamma|$ is the smallest among all colors in $L(X)$. Since $|L(X)| \geq k + 1$ and $\sum_{c \in L(X)} |J_c| = \sum_{i=1}^{k} |L(x_i)| = 2k + \ell$, we have $|J_\gamma| \leq \left\lfloor \frac{2k+\ell}{k+1} \right\rfloor \leq 2$. In particular, if $\ell \leq 1$, then $|J_\gamma| = 1$. Let L' be a list assignment of T' defined by $L'(x_0) = L(x_0) \setminus \{\gamma\}$ and $L'(v) = L(v)$ for every $v \in V(T') \setminus \{x_0\}$. By the minimality of T, T' admits a proper conflict-free L'-coloring ϕ. Let $\alpha = \phi(x_0)$ and let β be a color in $\mathcal{U}_\phi(x_0, T')$.

Subcase 5.1. $\ell \leq 1$.

Let $J_\gamma = \{p\}$. We set $\phi(x_p) = \gamma$ and let $\phi(y_p) \in L(y_p) \setminus \{\phi(x_p)\}$ if y_p exists. Since $|L(v_0)| = k + 2 \geq 5$, we choose $\phi(v_0) \in L(v_0) \setminus \{\alpha, \beta, \gamma, \phi(y_p)\}$. For $i \in \{1, 2, \ldots, \ell\} \setminus \{p\}$, we let $\phi(y_i) \in L(y_i) \setminus \{\phi(v_0)\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0), \phi(y_i)\}$. For $i \in \{\ell + 1, \ell + 2, \ldots, k\} \setminus \{p\}$, let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0)\}$. Since $\gamma \in \mathcal{U}_\phi(v_0, T)$, ϕ is a proper conflict-free L-coloring of T, a contradiction.

Subcase 5.2. $\ell \geq 2$.

The assumption of the subcase, together with the assumption $\ell \leq k - 2$, implies that $k \geq \ell + 2 \geq 4$. If there is a color $c \in L(X)$ with $|J_c| \leq 1$, then we argue in a similar way as in the previous Subcase 5.1. We may now assume that $|J_c| \geq 2$ for every color $c \in L(X)$, and in particular we know that $|J_\gamma| = 2$. Let $J_\gamma = \{p, q\}$ for some $1 \leq p < q \leq k$.

Suppose first that $k \geq 5$. We let $\phi(x_p) = \gamma$, $\phi(x_q) \in L(x_q) \setminus \{\gamma\}$, and for each $i \in \{p, q\}$, let $\phi(y_i) \in L(y_i) \setminus \{\phi(x_i)\}$ if y_i exists. Since $|L(v_0)| = k + 2 \geq 7$, we choose $\phi(v_0) \in L(v_0)$ distinct from $\alpha, \beta, \gamma, \phi(x_q)$, and also distinct from $\phi(y_p)$ and $\phi(y_q)$ in case they are defined. For $i \in \{1, 2, \ldots, \ell\} \setminus \{p, q\}$, we let $\phi(y_i) \in L(y_i) \setminus \{\phi(v_0)\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0), \phi(y_i)\}$. For $i \in \{\ell + 1, \ell + 2, \ldots, k\} \setminus \{p, q\}$, let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0)\}$. Since $\gamma \in \mathcal{U}_\phi(v_0, T)$, ϕ is a proper conflict-free L-coloring of T, a contradiction.

Now we may assume that $k = 4$. Since $\ell \leq k-2 = 2$ and $2(k+1) \leq \sum_{c \in L(X)} |J_c| = \sum_{i=1}^{k} |L(x_i)| = 2k + \ell$, we infer that $\ell = 2$ and $|J_c| = 2$ for every $c \in L(X)$. Thus, without loss of generality, we may assume that $q = 4$, which implies that $d_T(x_q) = 1$. Then we let $\phi(x_q) = \gamma$, $\phi(x_p) \in L(x_p) \setminus \{\gamma\}$, and let $\phi(y_p) \in L(y_p) \setminus \{\phi(x_p)\}$ if y_p exists. Since $|L(v_0)| = k+2 = 6$, we choose $\phi(v_0) \in L(v_0)$ distinct from $\alpha, \beta, \gamma, \phi(x_p)$, and also distinct from $\phi(y_p)$ in case it is defined. For $i \in \{1, 2\} \setminus \{p\}$, we let $\phi(y_i) \in L(y_i) \setminus \{\phi(v_0)\}$ and let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0), \phi(y_i)\}$. For $i \in \{3, 4\} \setminus \{p, q\}$, let $\phi(x_i) \in L(x_i) \setminus \{\phi(v_0)\}$. Since $\gamma \in \mathcal{U}_\phi(v_0, T)$, ϕ is a proper conflict-free L-coloring of T, a contradiction.

This completes the proof of Theorem 3.

Declaration of competing interest

There is no competing interest.

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Data availability

No data was used for the research described in the article.

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