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Families of Proper Holomorphic Maps

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Abstract
Given a smooth, open, oriented surface X endowed with a family of complex structures $\{J_b\}_{b\in B}$ depending continuously on the parameter b in a metrisable space B, we construct a continuous family of proper holomorphic maps $F_b : (X, J_b) \to \mathbb{C}^2$, $b \in B$.

Keywords Riemann surface · Proper holomorphic map

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1 Introduction

Every smooth, open, oriented surface X endowed with an almost complex structure J is a Riemann surface. Therefore, by choosing a continuously varying family of almost complex structures $(J_b)_{b\in B}$ for some parameter space B, we determine a family of open Riemann surfaces $(X, J_b)_{b\in B}$. In 2025, Forstnerič [7] initiated the study of continuous maps F from $B \times X$ to Euclidean space, or more generally, to an Oka manifold, such that for each $b \in B$ the map $F(b, \cdot)$ is holomorphic on the Riemann surface (X, J_b). In this framework, he obtained the Runge and Mergelyan approximation theorems, as well as the Weierstrass interpolation theorem.

Our main result answers in part the question raised by Forstnerič [7, Problem 8.7 (a)] concerning the existence of proper holomorphic maps in this setting:

Theorem 1.1 *Let X be a smooth, connected, open, oriented surface, B a metrisable space, and $\{J_b\}_{b\in B}$ a continuous family of complex structures on X of class $C^{(k,\alpha)}$*

Dedicated to Josip Globevnik

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with $k \in \mathbb{Z}_+, 0 < \alpha < 1$. *Then there exists a continuous map* $F : B \times X \to \mathbb{C}^2$ *such that for every* $b \in B$ *the map* $F(b, \cdot) : (X, J_b) \to \mathbb{C}^2$ *is proper holomorphic.*

Precise definitions will be given in the next section. It is classical that for every open Riemann surface there is a proper holomorphic immersion into $\mathbb{C}^2$ and a proper holomorphic embedding into $\mathbb{C}^3$, see [6, Theorem 2.4.1] and the references therein.

By increasing the dimension of the target Euclidean space by one, we obtain a family of proper holomorphic immersions:

Corollary 1.2 *Let* X *be a smooth, connected, open, oriented surface,* B *a finite CW complex or a smooth manifold, and* $\{J_b\}_{b\in B}$ *a continuous family of complex structures on* X *of class* $C^{(k,\alpha)}$ *with* $k \geq 1, 0 < \alpha < 1$. *Then there exists a continuous map* $G : B \times X \to \mathbb{C}^3$ *such that for every* $b \in B$ *the map* $G(b, \cdot) : (X, J_b) \to \mathbb{C}^3$ *is a proper holomorphic immersion.*

Proof By [7, Corollary 8.3] there exists a continuous function $h : B \times X \to \mathbb{C}$ such that $h(b, \cdot) : (X, J_b) \to \mathbb{C}$ is a holomorphic immersion for every $b \in B$. Let $F : B \times X \to \mathbb{C}^2$ be a continuous map such that for every $b \in B$ the map $F(b, \cdot) : (X, J_b) \to \mathbb{C}^2$ is proper holomorphic, provided by Theorem 1.1. Then the map $(F, h) : B \times X \to \mathbb{C}^3$ is continuous and for every $b \in B$ the map $(F, h)(b, \cdot) : (X, J_b) \to \mathbb{C}^3$ is a proper holomorphic immersion. □

We extend to families the result of Forstnerič and Globevnik [8, Theorem 1.4], Alarcón and López [3, Corollary 1.1], and Andrist and Wold [4, Theorem 5.6] on proper harmonic maps from open Riemann surfaces to $\mathbb{R}^2$:

Theorem 1.3 *Let* X *be a smooth, connected, open, oriented surface,* B *a metrisable space, and* $\{J_b\}_{b\in B}$ *a continuous family of complex structures on* X *of class* $C^{(k,\alpha)}$ *with* $k \in \mathbb{Z}_+, 0 < \alpha < 1$. *There exists a continuous map* $H : B \times X \to \mathbb{R}^2$ *such that for every* $b \in B$ *the map* $H(b, \cdot) : (X, J_b) \to \mathbb{R}^2$ *is proper harmonic.*

The proof relies on the proof of Theorem 1.1 and we postpone it to Section 3.

By Remmert's proper mapping theorem, the image of an analytic subvariety under a proper holomorphic map is an analytic subvariety. Therefore, the following corollary provides, in particular, a path of complex analytic subvarieties in $\mathbb{C}^2$ from the one parametrised by the complex line to the one parametrised by the unit disc.

Corollary 1.4 *Let* X *be a smooth, connected, open, oriented surface. Let* J_0, J_1 *be complex structures on* X *of class* $C^{(k,\alpha)}$ *with* $k \in \mathbb{Z}_+, 0 < \alpha < 1$. *There exist a continuous family* $\{J_b\}_{b\in[0,1]}$ *of complex structures on* X *of class* $C^{(k,\alpha)}$ *and a continuous map* $F : [0, 1] \times X \to \mathbb{C}^2$ *such that for every* $b \in [0, 1]$ *the map* $F(b, \cdot) : (X, J_b) \to \mathbb{C}^2$ *is proper* J_b*-holomorphic.*

Proof Each complex structure determines a compatible Riemannian metric on X of the same smoothness class. Convex combinations of these metrics yield a path connecting the two, which in turn induces a corresponding path of almost complex structures on X of the same smoothness class; see, for example [2, Lemma 1.9.1]. Then the conclusion follows from Theorem 1.1. □

The main idea in the proof is constructing a convergent sequence of maps on an exhausting sequence of Runge compact sets of X in a way similar to constructions in [1, 3, 5]. In [3], Alarcón and López constructed a proper conformal minimal immersion from any open Riemann surface M into $\mathbb{R}^3$ with its image in a wedge, and in [1], Alarcón and Forstnerič obtained a proper holomorphic immersion from any open Riemann surface M into $\mathbb{C}^2$ directed by an Oka cone. The main tool in our construction is the Mergelyan approximation theorem for proper families of compact Runge sets recently proven by Forstnerič [7]. When the parameter space B is not compact, one has to deal with nonconstant proper families of compact Runge subsets of X, which are present already in the noncritical case, i.e., when the topology of X is trivial.

2 Preliminaries

We use the notations $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, 3, \ldots\}$ for the set of natural numbers, respectively the set of nonnegative integers. If K is a compact topological space and $f : K \to \mathbb{C}$ is a continuous function, we denote by $\|f\|_K$ the supremum norm of f on K.

Throughout the paper, we denote by X a smooth, connected, open, oriented, Hausdorff, second countable surface. We are interested in families of complex structures on X, parametrised by some topological space B as defined in [7]. A complex structure on X is given by a section $J \in \Gamma(\mathrm{End}(TX))$ of the bundle of endomorphisms $\mathrm{End}(TX)$ of the tangent bundle TX of X that satisfies the condition $J^2 = -\mathrm{Id}$. We always assume that J induces on X the given orientation of X. Since the tangent bundle TX is trivial, the bundle $\mathrm{End}(TX)$ is isomorphic to the trivial bundle $X \times \mathrm{End}(\mathbb{R}^2)$. If we choose a trivialisation of $\mathrm{End}(TX)$, we can identify sections of $\mathrm{End}(TX)$ with functions from X to $\mathrm{End}(\mathbb{R}^2)$. If we furthermore choose a Riemannian metric on X, we can define Banach spaces $\Gamma^{(k,\alpha)}(\mathrm{End}(TX)|_\Omega)$ of sections of $\mathrm{End}(TX)$ of Hölder class $C^{(k,\alpha)}(\Omega)$ for any $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and any relatively compact domain $\Omega \subset X$. A complex structure $J \in \Gamma(\mathrm{End}(TX))$ is locally of class $C^{(k,\alpha)}$ if $J|_\Omega \in \Gamma^{(k,\alpha)}(\mathrm{End}(TX)|_\Omega)$ for every relatively compact domain $\Omega \subset X$.

Definition 2.1 Let B be a topological space, $k \in \mathbb{Z}_+$ and $0 < \alpha < 1$. A **continuous family of complex structures on** X of class $C^{(k,\alpha)}$, parametrised by B, is a family of complex structures $J = \{J_b\}_{b\in B}$ on X, which are locally of class $C^{(k,\alpha)}$, such that for every relatively compact domain $\Omega \subset X$ the map $b \mapsto J_b|_\Omega \in \Gamma^{(k,\alpha)}(\mathrm{End}(TX)|_\Omega)$ is continuous.

A continuous family $J = (J_b)_{b\in B}$ of complex structures on X furnishes us with a Riemann surface (X, J_b) for every $b \in B$. A function $f : X \to \mathbb{C}$ is J_b**-holomorphic** if it is holomorphic with respect to the complex structure J_b on X and the standard complex structure on $\mathbb{C}$.

Let B be a topological space and let A be a subset of $B \times X$. For every $b \in B$ we denote

$$A_b = \{x \in X : (b, x) \in A\}.$$

If $f : A \to \mathbb{C}$ is a function and $b \in B$, we denote by $f_b : A_b \to \mathbb{C}$ the function, given by

$$f_b(x) = f(b, x)$$

for $x \in A_b$. We are interested in continuous families of holomorphic functions.

Definition 2.2 Let B be a topological space, $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and let $J = (J_b)_{b \in B}$ be a continuous family of complex structures on X of class $C^{(k,\alpha)}$, parametrised by B.

(1) Let $U \subset B \times X$ be an open subset. A continuous function $f : U \to \mathbb{C}$ is **J-holomorphic**, if the function $f_b : U_b \to \mathbb{C}$ is J_b-holomorphic for every $b \in B$. The vector space of all J-holomorphic functions on U is denoted by $\mathcal{O}_J(U)$. We similarly define a J-holomorphic map $f : U \to M$, where M is a complex manifold.

(2) Now let $Z \subset B \times X$ be a closed subset. A continuous function $f : Z \to \mathbb{C}$ is **J-holomorphic** if there exist an open set $U \subset B \times X$, containing Z, and $\tilde{f} \in \mathcal{O}_J(U)$ such that $\tilde{f}|_Z = f$. The vector space of all J-holomorphic functions on Z is denoted by $\mathcal{O}_J(Z)$. The vector space of all continuous functions $f : Z \to \mathbb{C}$, for which the function $f_b : \mathrm{Int}\,(Z_b) \to \mathbb{C}$ is J_b-holomorphic for every $b \in B$, is denoted by $\mathcal{A}_J(Z)$.

For our construction, we need to consider continuous functions on proper families of compact subsets of X, which we recall below.

Definition 2.3 Let B be a topological space and let $\pi : B \times X \to B$ be the projection onto the first factor. A **family of compact subsets of** X, parametrised by B, is given by a closed subset $K \subset B \times X$, for which K_b is a compact subset of X for every $b \in B$ (note that K_b may be empty). A family of compact subsets K is **proper** if the map $\pi|_K : K \to B$ is proper, it is **wide** if K_b is non-empty for every $b \in B$, and it is called **Runge** if K_b is Runge for every $b \in B$.

Recall that a continuous map between topological spaces is proper if the preimage of every compact subset is compact, and that a compact subset $K \subset X$ is Runge if the complement $X \setminus K$ has no relatively compact connected components. A Runge compact set K is holomorphically convex in every complex structure on X.

As noted in [7], we have the following characterization of proper families of compact subsets:

Proposition 2.4 *Let B be a Hausdorff topological space and let $K \subset B \times X$ be a closed subset. Then K is a proper family of compact subsets of X if and only if the following two conditions hold:*

(1) For every $b \in B$ the fiber K_b is compact,

(2) For every $b_0 \in B$ and every open subset $U \subset X$ containing K_{b_0} there is a neighbourhood B_0 of b_0 in B such that $K_b \subset U$ for every $b \in B_0$.

Let us now take a look at some examples.

Example 2.5 (1) For every compact subset $K_0 \subset X$ we have the constant family $K = B \times K_0$ of compact subsets of X for which $K_b = K_0$ for every $b \in B$. More generally, let $K \subset B \times X$ be a closed subset for which $\pi|_K : K \to B$ is a fiber bundle with a compact fiber. Then K is a proper family of compact subsets of X.

(2) A proper family of compact subsets of X need not have all fibers homeomorphic. As an example, consider the case when $B = \mathbb{R}$, $X = \mathbb{C} = \mathbb{R}^2$ and denote by $\overline{\mathbb{D}} \subset \mathbb{C}$ the closed unit disk. The set

$$K = \left((-\infty, 0] \times \overline{\mathbb{D}}\right) \cup ([0, \infty) \times \{0\})$$

is then a proper family of compact subsets of X. On the other hand, let us define the set

$$\tilde{K} = ((-\infty, 0] \times \{0\}) \cup \left\{(x, \tfrac{1}{x}) \,|\, x \in (0, \infty)\right\}.$$

The set $\tilde{K}$ defines a family of compact subsets of X which is not a proper family.

To show that a given set is a proper family of compact subsets of X we easily obtain the following useful criteria.

Proposition 2.6 *Let B be a topological space.*

(1) Let $K \subset B \times X$ be a proper family of compact subsets of X and let K' be a closed subset of K. Then K' is a proper family of compact subsets of X as well.
(2) Let $K_1, K_2 \subset B \times X$ be proper families of compact subsets of X. Then $K_1 \cup K_2$ is a proper family of compact subsets as well.

Let $K \subset B \times X$ be a wide, proper family of compact subsets of X and let $\eta : K \to (0, \infty)$ be a positive continuous function. Since K_b is non-empty for every $b \in B$, there exists the minimum

$$\min(\eta)(b) = \min\{\eta(b, x) \,:\, x \in K_b\} > 0.$$

We thus obtain a function $\min(\eta) : B \to (0, \infty)$, which is continuous if K is a constant or a locally trivial family. In general, however, the function $\min(\eta)$ is only lower semicontinuous, but we can always find a continuous minorant $m(\eta) : B \to (0, \infty)$ of $\min(\eta)$:

Proposition 2.7 *Let B be a metrisable space, $K \subset B \times X$ a wide, proper family of compact subsets of X and let $\eta : K \to (0, \infty)$ be a continuous function.*

(1) The function $\min(\eta) : B \to (0, \infty)$ is lower semicontinuous.
(2) There exists a continuous function $m(\eta) : B \to (0, \infty)$, such that $m(\eta)(b) < \min(\eta)(b)$ holds for every $b \in B$. If B is a smooth manifold, we can in addition ensure that the function m is smooth.

Proof (a) Let $\epsilon > 0$ and $b_0 \in B$. We have to prove that there exists an open neighbourhood U_{b_0} of b_0 in B such that $\min(\eta)(b) > \min(\eta)(b_0) - \epsilon$ for every $b \in U_{b_0}$. Suppose, on the contrary, that such a neighbourhood does not exist for some b_0. Then there exists a sequence (b_n, x_n) of points in K such that $\eta(b_n, x_n) \leq \min(\eta)(b_0) - \epsilon$ for every $n \in \mathbb{N}$ and $\lim\limits_{n\to\infty} b_n = b_0$. The set $L = \{b_n \,|\, n \in \mathbb{Z}_+\}$ is a compact subset of B, hence $(\pi|_K)^{-1}(L)$ is a compact subset of K. We may therefore assume, that the sequence (b_n, x_n) is convergent with limit $(b_0, x_0) \in K$. But then we have

$$\min(\eta)(b_0) \leq \eta(b_0, x_0) \leq \min(\eta)(b_0) - \epsilon,$$

which leads us to a contradiction.

(b) We have shown that for every $b_0 \in B$ we can find a neighbourhood U_{b_0} of b_0 in B and a number $\epsilon_{b_0} > 0$ such that $\epsilon_{b_0} < \min(\eta)(b)$ for every $b \in U_{b_0}$. Since B is paracompact, we can find a subset $B' \subset B$ and for every $b \in B'$ an open subset $V_b \subset U_b$ such that $\{V_b\}_{b \in B'}$ is a locally finite open cover of B. Choose a continuous partition of unity $\{\rho_b\}_{b \in B'}$, subordinated to the cover $\{V_b\}_{b \in B'}$. The function $m(\eta) : B \to (0, \infty)$, defined by

$$m(\eta) = \sum_{b \in B'} \epsilon_b \rho_b$$

is then continuous and satisfies $0 < m(\eta)(b) < \min(\eta)(b)$ for every $b \in B$. □

Classical versions of Runge and Mergelyan approximation theorems show us that we can approximate a holomorphic function on a Runge compact set K arbitrarily closely on K with global holomorphic functions on X. In the construction of families of proper holomorphic maps, we use the following Mergelyan theorem for proper families of compact Runge sets (see Corollary 5.2 and Remark 5.7 in [7]).

Theorem 2.8 (Mergelyan theorem for proper families of Runge compacts) *Let B be a paracompact Hausdorff space, $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and let $J = \{J_b\}_{b \in B}$ be a continuous family of complex structures on X of class $C^{(k,\alpha)}$, parametrised by B. Let $K \subset B \times X$ be a proper family of Runge compacts in X and let $\epsilon : B \to (0, \infty)$ be a continuous function. Then for every $f \in \mathcal{A}_J(K)$ there exists a function $F \in \mathcal{O}_J(B \times X)$ such that $\|F_b - f_b\|_{K_b} < \epsilon(b)$ for every $b \in B$.*

In our construction, we need the following combination of the Mergelyan theorem and Proposition 2.7.

Proposition 2.9 *Let B be a metrisable space, $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and let $J = \{J_b\}_{b \in B}$ be a continuous family of complex structures on X of class $C^{(k,\alpha)}$, parametrised by B. Let $n \in \mathbb{N}$ and let $K_1, K_2, \ldots, K_n \subset B \times X$ be wide, proper families of compact subsets of X such that their union is contained in a proper family K of Runge compacts in X. Suppose $f \in \mathcal{A}_J(K)$ is a function that satisfies conditions $\Re f > C_i$ on K_i for some positive constants C_i for $1 \le i \le n$. Then for every continuous function $\epsilon : B \to (0, \infty)$ there exists a function $F \in \mathcal{O}_J(B \times X)$ such that $\Re F > C_i$ on K_i for $1 \le i \le n$ and $\|F_b - f_b\|_{K_b} < \epsilon(b)$ for every $b \in B$.*

Proof For $1 \le i \le n$ the function $\Re f - C_i$ is continuous and positive on K_i. By Proposition 2.7, there exist continuous functions $\delta_i : B \to (0, \infty)$ such that for every $(b, x) \in K_i$ we have $\Re f(b, x) - \delta_i(b) > C_i$. Now define a continuous function $\delta = \min\{\delta_1, \delta_2, \ldots, \delta_n, \epsilon\} : B \to (0, \infty)$. From Mergelyan's theorem, it follows that there exists a function $F \in \mathcal{O}_J(B \times X)$ such that $\|F_b - f_b\|_{K_b} < \delta(b)$ for every $b \in B$. This function F satisfies the conditions. □

3 Construction of families of proper holomorphic maps

We first recall how we can reduce the construction of a family of proper holomorphic maps to the construction of a converging sequence on an exhausting family of compact sets in X.

Proposition 3.1 *Let B be a topological space, $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and let $J = \{J_b\}_{b \in B}$ be a continuous family of complex structures on X of class $C^{(k,\alpha)}$, parametrised by B. Let $\emptyset = K_0 \subset K_1 \subset K_2 \subset K_3 \subset \ldots$ be an exhaustion of X by compact sets such that $K_n \subset \mathrm{Int}\, K_{n+1}$ for every $n \in \mathbb{N}$. Suppose that for every $n \in \mathbb{Z}_+$ we have functions $F_{n,1}, F_{n,2} \in \mathcal{A}_J(B \times K_n)$, such that for every $n \in \mathbb{N}$ it holds:*

$(a)_n$ $|F_{n,i}(b,x) - F_{n-1,i}(b,x)| < \frac{1}{2^{n-1}}$ *for every* $(b,x) \in B \times K_{n-1}$ *and* $i = 1, 2$,
$(b)_n$ $\max\{\Re F_{n,1}(b,x), \Re F_{n,2}(b,x)\} > n-1$ *for every* $(b,x) \in B \times (K_n \setminus \mathrm{Int}\, K_{n-1})$.

Then there exist functions $F_1, F_2 \in \mathcal{O}_J(B \times X)$ such that $F = (F_1, F_2) : B \times X \to \mathbb{C}^2$ is a continuous J-holomorphic map, for which $F_b : X \to \mathbb{C}^2$ is a proper map for every $b \in B$.

Proof It follows from the condition $(a)_n$ that the sequences $(F_{n,1})_{n \in \mathbb{N}}$ and $(F_{n,2})_{n \in \mathbb{N}}$ converge uniformly on the sets of the form $B \times K$, where $K \subset X$ is a compact subset. For the limit functions $F_1 = \lim_{n\to\infty} F_{n,1}$ and $F_2 = \lim_{n\to\infty} F_{n,2}$, we have that $F_1, F_2 \in \mathcal{O}_J(B \times X)$.

For $n > 1$ and $i = 1, 2$ it then follows from $(a)_n$ that

$$|F_i(b,x) - F_{n,i}(b,x)| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \ldots = \frac{1}{2^{n-1}} < 1 \text{ for } (b,x) \in B \times K_n$$

and further from $(b)_n$ that

$$\max\{\Re F_1(b,x), \Re F_2(b,x)\} > n-2 \text{ for } (b,x) \in B \times (K_n \setminus \mathrm{Int}\, K_{n-1}),$$

which implies that $F_b : X \to \mathbb{C}^2$ is a proper map for every $b \in B$. □

We now consider the case $X = \mathbb{R}^2$, where the topology is trivial:

Theorem 3.2 *Let B be a metrisable topological space, $k \in \mathbb{Z}_+$, $0 < \alpha < 1$ and let $J = \{J_b\}_{b \in B}$ be a continuous family of complex structures on $\mathbb{R}^2$ of class $C^{(k,\alpha)}$, parametrised by B. Then there exists a J-holomorphic map $F : B \times \mathbb{R}^2 \to \mathbb{C}^2$ for which $F_b : \mathbb{R}^2 \to \mathbb{C}^2$ is a proper map for every $b \in B$.*

To prove Theorem 3.2 we first introduce some notations, where we identify $\mathbb{R}^2$ and $\mathbb{C}$ for convenience. We choose the exhaustion of $\mathbb{C}$ by closed disks

$$K_n = n\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le n\}$$

for $n \in \mathbb{N}$, and denote by

$$A_n = K_n \setminus \mathrm{Int}\, K_{n-1}$$

the closed annulus in $\mathbb{C}$ between circles of radii $n-1$ and n. Also let $K_0 = \emptyset$.

Definition 3.3 (1) Let $n \in \mathbb{N}$ and suppose we have $k \in \mathbb{N}$ angles $0 \leq \phi_1 < \ldots < \phi_k < 2\pi$. We then define the following subsets of $\mathbb{C}$:

$$\gamma(n, \{\phi_1, \ldots, \phi_k\}) = \{re^{i\phi} : r \in [n, n+1], \phi \in \{\phi_1, \ldots, \phi_k\}\},$$
$$p(n, \{\phi_1, \ldots, \phi_k\}) = \{ne^{i\phi} : \phi \in \{\phi_1, \ldots, \phi_k\}\}.$$

The set $\gamma(n, \{\phi_1, \ldots, \phi_k\})$ is the union of radial line segments at angles in $\{\phi_1, \ldots, \phi_k\}$, while their inner endpoints form the set $p(n, \{\phi_1, \ldots, \phi_k\})$.

(2) Let $n \in \mathbb{N}$ and suppose $\phi_1, \phi_2 \in [0, 2\pi]$ are such that $0 < \phi_2 - \phi_1 < 2\pi$. We then define:

$$D(n, \phi_1, \phi_2) = \{re^{i\phi} : r \in [n, n+1], \phi \in [\phi_1, \phi_2]\},$$
$$\alpha(n, \phi_1, \phi_2) = \{ne^{i\phi} : \phi \in [\phi_1, \phi_2]\}.$$

The set $D(n, \phi_1, \phi_2) \subset \mathbb{C}$ is the part of the annulus A_n which lies between angles ϕ_1 and ϕ_2. Its inner boundary arc is denoted by $\alpha(n, \phi_1, \phi_2)$.

(3) Let $n \in \mathbb{N}$, let $\phi_1, \phi_2 \in [0, 2\pi]$ be such that $0 < \phi_2 - \phi_1 < 2\pi$ and suppose that $0 < \delta < \frac{1}{3}\min\{1, \phi_2 - \phi_1\}$. We then define:

$$L(n, \delta, \phi_1, \phi_2) = \{re^{i\phi} : r \in [n+\delta, n+1], \phi \in [\phi_1 + \delta, \phi_2 - \delta]\},$$
$$W(n, \delta, \phi_1, \phi_2) = \overline{D(n, \phi_1, \phi_2) \setminus L(n, \delta, \phi_1, \phi_2)}.$$

The set $W(n, \delta, \phi_1, \phi_2)$ is the closed δ-neighbourhood of $\gamma(n, \{\phi_1, \phi_2\}) \cup \alpha(n, \phi_1, \phi_2)$ in $D(n, \phi_1, \phi_2)$.

(4) Let $n, k \in \mathbb{N}$ and suppose $0 = \phi_1 < \phi_2 < \ldots < \phi_k < 2\pi$. Furthermore, let $0 < \delta < \frac{1}{3}\min\{1, \phi_2 - \phi_1, \phi_3 - \phi_2, \ldots, \phi_k - \phi_{k-1}, 2\pi - \phi_k\}$. We then define:

$$W(n, \delta, \{\phi_1, \phi_2, \ldots, \phi_k\}) = W(n, \delta, \phi_1, \phi_2) \cup \ldots \cup W(n, \delta, \phi_{k-1}, \phi_k) \cup W(n, \delta, \phi_k, 2\pi),$$
$$L(n, \delta, \{\phi_1, \phi_2, \ldots, \phi_k\}) = L(n, \delta, \phi_1, \phi_2) \cup \ldots \cup L(n, \delta, \phi_{k-1}, \phi_k) \cup L(n, \delta, \phi_k, 2\pi).$$

(5) Let $n \in \mathbb{N}$ and suppose $0 = \phi_1 < \phi_2 < \ldots < \phi_k < 2\pi$, where k is an even number. We then define:

$$D_{\text{odd}}(n, \{\phi_1, \phi_2, \ldots, \phi_k\}) = D(n, \phi_1, \phi_2) \cup D(n, \phi_3, \phi_4) \cup \ldots \cup D(n, \phi_{k-1}, \phi_k),$$
$$D_{\text{even}}(n, \{\phi_1, \phi_2, \ldots, \phi_k\}) = D(n, \phi_2, \phi_3) \cup D(n, \phi_4, \phi_5) \cup \ldots \cup D(n, \phi_k, 2\pi).$$

In a similar fashion we define the sets α_{odd}, α_{even}, L_{odd}, L_{even}, W_{odd} and W_{even}.

All of the above sets are compact subsets of $\mathbb{C}$ as shown in Figure 1.

Next we extend the above definitions to the setting of $B \times \mathbb{C}$, where B is a topological space. If $\phi_i : B \to [0, 2\pi]$ and $\delta : B \to (0, \frac{1}{3})$ are continuous functions that satisfy the conditions $(1) - (5)$ in Definition 3.3 pointwise, we can define families of compact subsets of $\mathbb{C}$ whose fibres are the corresponding sets. We denote such a family by

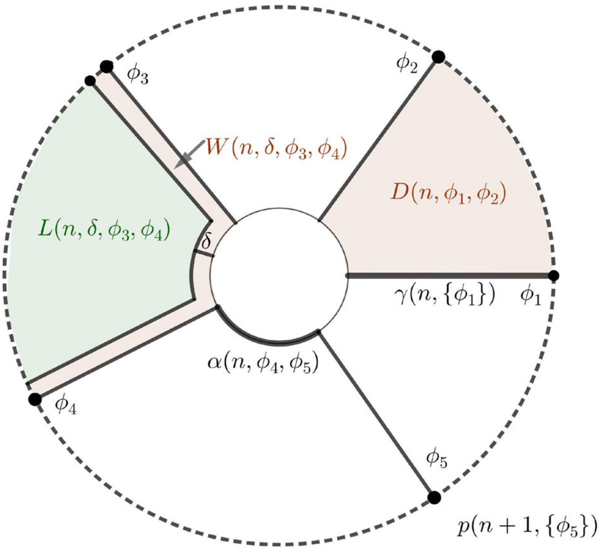

Fig. 1 Pictures of sets from Definition 3.3

adding a subscript B. For example, if $n \in \mathbb{N}$ and $\phi_1, \phi_2 : B \to [0, 2\pi]$ are continuous functions such that $0 < \phi_2(b) - \phi_1(b) < 2\pi$ for every $b \in B$, then $D_B(n, \phi_1, \phi_2)$ is a subset of $B \times \mathbb{C}$, which is implicitly defined by

$$(D_B(n, \phi_1, \phi_2))_b = D(n, \phi_1(b), \phi_2(b)) \subset \mathbb{C}$$

for every $b \in B$.

Proposition 3.4 *Let B be a topological space. The sets γ_B, p_B, D_B, α_B, L_B and W_B are all proper families of compact subsets of $\mathbb{C}$. The same is true for their odd and even versions.*

Proof All these sets are subsets of the constant family $B \times K_{n+1}$, so by Proposition 2.6 it suffices to prove they are closed subsets. This follows from the fact that their complements in $B \times \mathbb{C}$ are open since the functions ϕ_i and δ are continuous. □

To be able to use the Mergelyan theorem to construct the sequence of functions from Proposition 3.1, we need the following result.

Proposition 3.5 *Let B be a metrisable topological space, $n \in \mathbb{N}$ and let $\phi_1, \phi_2 : B \to [0, 2\pi]$ be continuous functions such that $0 < \phi_2(b) - \phi_1(b) < 2\pi$ for every $b \in B$. Suppose $f : D_B(n, \phi_1, \phi_2) \to \mathbb{C}$ is a continuous function such that:*

(1) $\Re f > n$ on $\gamma_B(n, \{\phi_1, \phi_2\}) \cup \alpha_B(n, \phi_1, \phi_2)$,
(2) $\Re f > n + 1$ on $p_B(n + 1, \{\phi_1, \phi_2\})$.

Then there exists a continuous function $\delta : B \to (0, \frac{1}{3})$, satisfying $\delta < \frac{1}{3} \min\{1, \phi_2 - \phi_1\}$ and such that:

(1) $\Re f > n$ *on* $W_B(n, \delta, \phi_1, \phi_2)$,
(2) $\Re f > n+1$ *on* $\alpha_B(n+1, \phi_1, \phi_1+\delta) \cup \alpha_B(n+1, \phi_2-\delta, \phi_2)$.

Proof Let us define

$$\begin{aligned} K' =&\{(b,x) \in D_B(n, \phi_1, \phi_2) : \Re f(b,x) \le n\} \cup \{(b,x) \in \alpha_B(n+1, \phi_1, \phi_2) \\ &: \Re f(b,x) \le n+1)\}. \end{aligned}$$

The set K' is a closed subset of the proper family of compact subsets $D_B(n, \phi_1, \phi_2)$, hence K' is a proper family as well. Since K' may have empty fibers, we enlarge it to a proper family of compact subsets

$$K = K' \cup p_B(n+1, \{\tfrac{1}{2}(\phi_1+\phi_2)\}),$$

for which $K_b \neq \emptyset$ for every $b \in B$. Then K is a wide, proper family of compact subsets of $\mathbb{C}$ which is disjoint from $\gamma_B(n, \{\phi_1, \phi_2\}) \cup \alpha_B(n, \phi_1, \phi_2)$.

For any $(b,x) \in K$ we can write x in the form $x = re^{i\phi}$ for unique $r \in (n, n+1]$ and $\phi \in (\phi_1(b), \phi_2(b))$. The functions $\phi_2 - \phi$, $\phi - \phi_1$ and $r - n$ are continuous and positive on K. By Proposition 2.7, we can find a function $\delta : B \to (0, \infty)$ such that for every $(b, re^{i\phi}) \in K$ we have:

$$\begin{aligned} &r \in (n+\delta(b), n+1], \\ &\phi \in (\phi_1(b)+\delta(b), \phi_2(b)-\delta(b)). \end{aligned}$$

If needed, we can make δ smaller, so that $\delta < \frac{1}{3}\min\{1, \phi_2-\phi_1\}$. We then have $\Re f > n$ on $W_B(n, \delta, \phi_1, \phi_2)$ and $\Re f > n+1$ on $\alpha_B(n+1, \phi_1, \phi_1+\delta) \cup \alpha_B(n+1, \phi_2-\delta, \phi_2)$. □

Proof of Theorem 3.2 Let $l_n = 3^{n-1}$ for $n \in \mathbb{N}$. According to Proposition 3.1, it suffices to construct a sequence of functions $F_{n,1}, F_{n,2} \in \mathcal{A}_J(B \times K_n)$ that for every $n \in \mathbb{N}$ satisfy the conditions:

$(a)_n$ $|F_{n,i}(b,x) - F_{n-1,i}(b,x)| < \frac{1}{2^{n-1}}$ for every $(b,x) \in B \times K_{n-1}$ and $i = 1, 2$,
$(b)_n$ $\max\{\Re F_{n,1}(b,x), \Re F_{n,2}(b,x)\} > n-1$ for every $(b,x) \in B \times A_n$.

We construct such a sequence inductively, together with the sequence of continuous families of angles: These angles are defined by continuous functions $\phi_{n,j} : B \to [0, 2\pi]$ for $j \in \{1, \ldots, 2l_n+1\}$, which satisfy for every $n \in \mathbb{N}$ the following conditions:

$(c)_n$ $0 = \phi_{n,1}(b) < \phi_{n,2}(b) < \cdots < \phi_{n,2l_n}(b) < 2\pi = \phi_{n,2l_n+1}(b)$ for every $b \in B$,
$(d)_n$ $\Re F_{n,1} > n$ on $(\alpha_{\text{odd}})_B(n, \{\phi_{n,1}, \phi_{n,2}, \ldots, \phi_{n,2l_n}\})$,
$\Re F_{n,2} > n$ on $(\alpha_{\text{even}})_B(n, \{\phi_{n,1}, \phi_{n,2}, \ldots, \phi_{n,2l_n}\})$.

To start with the induction, we define constant functions $\phi_{1,1}, \phi_{1,2}, \phi_{1,3} : B \to [0, 2\pi]$ by

$$\phi_{1,1} = 0,\ \phi_{1,2} = \pi,\ \phi_{1,3} = 2\pi$$

and choose any functions $F_{0,1}, F_{0,2} \in \mathcal{A}_J(B \times K_0)$ and $F_{1,1}, F_{1,2} \in \mathcal{A}_J(B \times K_1)$ that satisfy the conditions $(a)_1$, $(b)_1$ and $(d)_1$. (For example, we could just choose appropriate constant functions $F_{0,1}$, $F_{0,2}$, $F_{1,1}$ and $F_{1,2}$.)

Suppose that for some $n \in \mathbb{N}$ we have functions $F_{m,1}, F_{m,2} \in \mathcal{A}_J(B \times K_m)$ and $\phi_{m,j} : B \to [0, 2\pi]$ for $j \in \{1, 2, \ldots, 2l_m + 1\}$ which satisfy conditions $(a)_m$, $(b)_m$, $(c)_m$ and $(d)_m$ for $m \in \{1, 2, \ldots, n\}$. In the induction step, we construct continuous functions

$$\phi_{n+1,1}, \phi_{n+1,2}, \ldots, \phi_{n+1,2l_{n+1}+1} : B \to [0, 2\pi],$$

that satisfy

$$0 = \phi_{n+1,1} < \phi_{n+1,2} < \ldots < \phi_{n+1,2l_{n+1}} < 2\pi = \phi_{n+1,2l_{n+1}+1}$$

and functions $F_{n+1,1}, F_{n+1,2} \in \mathcal{A}_J(B \times K_{n+1})$ that satisfy $(a)_{n+1}$, $(b)_{n+1}$ and $(d)_{n+1}$. Before we turn to details let us quickly describe the main idea of the induction step. We need to construct functions $F_{n+1,1}, F_{n+1,2} \in \mathcal{A}_J(B \times K_{n+1})$ for which $\max\{\Re F_{n+1,1}, \Re F_{n+1,2}\} > n$ on $B \times A_{n+1}$. To do that, we split the inductive step into three parts. In the first part, we use Mergelyan's theorem to construct functions $\tilde{F}_{n,1}, \tilde{F}_{n,2} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ which satisfy $\max\{\Re \tilde{F}_{n,1}, \Re \tilde{F}_{n,2}\} > n$ on the subset $\gamma_B(n, \{\phi_{n,1}, \phi_{n,2}, \ldots, \phi_{n,2l_n}\})$ of $B \times A_{n+1}$. Next we use Proposition 3.5 to show that there exists a continuous function $\delta : B \to (0, \frac{1}{3})$ such that $\max\{\Re \tilde{F}_{n,1}, \Re \tilde{F}_{n,2}\} > n$ on the subset $W_B(n, \{\phi_{n,1}, \phi_{n,2}, \ldots, \phi_{n,2l_n}\})$ of $B \times A_{n+1}$. In the third part, we use the idea from the proofs in [1, 3] to obtain functions $F_{n+1,1}, F_{n+1,2} \in \mathcal{A}_J(B \times K_{n+1})$ for which $\max\{\Re F_{n+1,1}, \Re F_{n+1,2}\} > n$ on $B \times A_{n+1}$.

Let us now describe the details. First we construct functions $\tilde{F}_{n,1}, \tilde{F}_{n,2} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ that satisfy:

$(a^1)_{n+1}$ $|\tilde{F}_{n,i}(b, x) - F_{n,i}(b, x)| < \frac{1}{2^{n+1}}$ for $(b, x) \in B \times K_n$ and $i = 1, 2$,

$(b^1)_{n+1}$ $\Re \tilde{F}_{n,1} > n$ on $\gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup (\alpha_{\text{odd}})_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,
$\Re \tilde{F}_{n,2} > n$ on $\gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup (\alpha_{\text{even}})_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,

$(d^1)_{n+1}$ $\Re \tilde{F}_{n,i} > n + 1$ on $p_B(n + 1, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ for $i = 1, 2$.

To do that we first continuously extend the functions $F_{n,1}, F_{n,2}$ from the set $B \times K_n$ to the set $(B \times K_n) \cup \gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ so that $\Re F_{n,i} > n$ on $\gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ and $\Re F_{n,i} > n + 1$ on $p_B(n + 1, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ for $i = 1, 2$. Now note that $\Re F_{n,1} > n$ on the proper family $\gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup (\alpha_{\text{odd}})_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ and that $\Re F_{n,1} > n + 1$ on the proper family $p_B(n + 1, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ of compact subsets of $\mathbb{R}^2$. The union of these two proper families is contained in the proper family $(B \times K_n) \cup \gamma_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ of Runge compacts in $\mathbb{R}^2$, so by Proposition 2.9 we can find a function $\tilde{F}_{n,1} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ that satisfies $(a^1)_{n+1}$, $(b^1)_{n+1}$ and $(d^1)_{n+1}$. In a similar fashion we also obtain a function $\tilde{F}_{n,2} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ which approximates the function $F_{n,2}$.

We now proceed to the second part of the induction step. Consider the function $\tilde{F}_{n,1}$ on the proper family $(D_{\text{odd}})_B(n, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$ of compact subsets of $\mathbb{R}^2$. From Proposition 3.5 it follows that there exists a continuous function $\delta_1 : B \to (0, \frac{1}{3})$ such that:

- $\Re \tilde{F}_{n,1} > n$ on $(W_{\text{odd}})_B(n, \delta_1, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,

- $\Re\tilde{F}_{n,1} > n+1$ on $\bigcup_{k=1}^{l_n} \big(\alpha_B(n+1, \phi_{n,2k-1}, \phi_{n,2k-1}+\delta_1) \cup$
$\alpha_B(n+1, \phi_{n,2k}-\delta_1, \phi_{n,2k})\big)$.

In the sequel, we repeat this argument for the function $\tilde{F}_{n,2}$ on the $(D_{\text{even}})_B(n, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\})$ to obtain a function $\delta_2 : B \to (0, \frac{1}{3})$ such that $\tilde{F}_{n,2}$ satisfies conditions:

- $\Re\tilde{F}_{n,2} > n$ on $(W_{\text{even}})_B(n, \delta_2, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\})$,
- $\Re\tilde{F}_{n,2} > n+1$ on $\bigcup_{k=1}^{l_n} \big(\alpha_B(n+1, \phi_{n,2k}, \phi_{n,2k}+\delta_2) \cup$
$\alpha_B(n+1, \phi_{n,2k+1}-\delta_2, \phi_{n,2k+1})\big)$.

Let $\delta = \min\{\delta_1, \delta_2\}$ and define functions $\phi_{n+1,1}, \phi_{n+1,2}, \dots, \phi_{n+1,2l_{n+1}+1} : B \to [0, 2\pi]$ by:

$$\begin{aligned}
\phi_{n+1,3k+1} &= \phi_{n,k+1},\\
\phi_{n+1,3k+2} &= \phi_{n,k+1} + \delta,\\
\phi_{n+1,3k+3} &= \phi_{n,k+2} - \delta
\end{aligned}$$

for $k \in \{0, 1, \dots, 2l_n - 1\}$ and $\phi_{n+1,2l_{n+1}+1} = 2\pi$. Observe that these functions satisfy the condition $(c)_{n+1}$ while functions $\tilde{F}_{n,1}, \tilde{F}_{n,2}$ satisfy the conditions:

$(a^2)_{n+1}$ $|\tilde{F}_{n,i}(b,x) - F_{n,i}(b,x)| < \frac{1}{2^{n+1}}$ for all $(b,x) \in B \times K_n$ and $i = 1, 2$,

$(b^2)_{n+1}$ $\max\{\Re\tilde{F}_{n,1}(b,x), \Re\tilde{F}_{n,2}(b,x)\} > n$ for all $(b,x) \in W_B(n, \delta, \{\phi_{n,1}, \phi_{n,2}, \dots, \phi_{n,2l_n}\})$,

$(d^2)_{n+1}$ $\Re\tilde{F}_{n,1} > n+1$ on $\bigcup_{k=1}^{l_n} \big(\alpha_B(n+1, \phi_{n,2k-1}, \phi_{n,2k-1}+\delta) \cup$
$\alpha_B(n+1, \phi_{n,2k}-\delta, \phi_{n,2k})\big)$,
$\Re\tilde{F}_{n,2} > n+1$ on $\bigcup_{k=1}^{l_n} \big(\alpha_B(n+1, \phi_{n,2k}, \phi_{n,2k}+\delta) \cup$
$\alpha_B(n+1, \phi_{n,2k+1}-\delta, \phi_{n,2k+1})\big)$.

In the third part of the induction step, we correct the functions $\tilde{F}_{n,1}, \tilde{F}_{n,2}$ so that we obtain the condition $(b)_{n+1}$ on the set $L_B(n, \delta, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\})$ as well as the condition $(d)_{n+1}$ on the remaining arcs. Let us define a proper family of Runge compacts by

$$(A_{\text{odd}})_n = (B \times K_n) \cup (D_{\text{odd}})_B(n, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\}) \cup (L_{\text{even}})_B(n, \delta, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\}),$$

see the left part of Figure 2, and define the function $\overline{F}_{n,1} \in \mathcal{A}_J((A_{\text{odd}})_n)$ by

$$\overline{F}_{n,1}(b,x) = \begin{cases} \tilde{F}_{n,1} & ; (b,x) \in (B \times K_n) \cup (D_{\text{odd}})_B(n, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\}),\\ n+2 & ; (b,x) \in (L_{\text{even}})_B(n, \delta, \{\phi_{n,1}, \dots, \phi_{n,2l_n}\}). \end{cases}$$

Function $\overline{F}_{n,1}$ satisfies conditions:

- $|\overline{F}_{n,1}(b,x) - F_{n,1}(b,x)| < \frac{1}{2^{n+1}}$ for $(b,x) \in B \times K_n$,

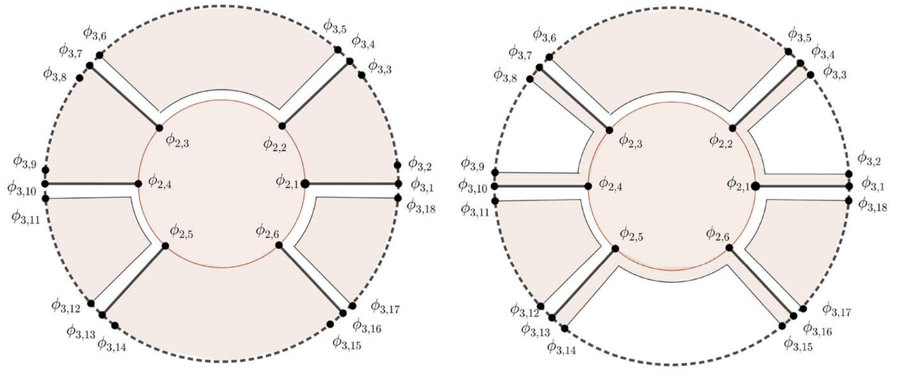

Fig. 2 Regions in the inductive step in the case $n = 2$

- $\Re\overline{F}_{n,1} > n$ on $(W_{\text{odd}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup (L_{\text{even}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,
- $\Re\overline{F}_{n,1} > n + 1$ on $(\alpha_{\text{odd}})_B(n + 1, \{\phi_{n+1,1}, \ldots, \phi_{n+1,2l_{n+1}}\})$.

By applying Proposition 2.9 to the function $\overline{F}_{n,1}$ with precision at least $\frac{1}{2^{n+1}}$ we obtain a function $F_{n+1,1} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ that satisfies:

$(a^{3,1})_{n+1}$ $|F_{n+1,1}(b, x) - F_{n,1}(b, x)| < \frac{1}{2^n}$ for $(b, x) \in B \times K_n$,
$(b^{3,1})_{n+1}$ $\Re F_{n+1,1} > n$ on $(W_{\text{odd}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup$
$(L_{\text{even}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,
$(d^{3,1})_{n+1}$ $\Re F_{n+1,1} > n + 1$ on $(\alpha_{\text{odd}})_B(n + 1, \{\phi_{n+1,1}, \ldots, \phi_{n+1,2l_{n+1}}\})$.

Similarly we obtain a function $F_{n+1,2} \in \mathcal{O}_J(B \times \mathbb{R}^2)$ which satisfies conditions:

$(a^{3,2})_{n+1}$ $|F_{n+1,2}(b, x) - F_{n,2}(b, x)| < \frac{1}{2^n}$ for $(b, x) \in B \times K_n$,
$(b^{3,2})_{n+1}$ $\Re F_{n+1,2} > n$ on $(W_{\text{even}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\}) \cup$
$(L_{\text{odd}})_B(n, \delta, \{\phi_{n,1}, \ldots, \phi_{n,2l_n}\})$,
$(d^{3,2})_{n+1}$ $\Re F_{n+1,2} > n + 1$ on $(\alpha_{\text{even}})_B(n + 1, \{\phi_{n+1,1}, \ldots, \phi_{n+1,2l_{n+1}}\})$.

The areas in $B \times A_{n+1}$ where $\Re F_{n+1,1} > n$ respectively $\Re F_{n+1,2} > n$ are shown in the right part of Figure 2. Condition $(a)_{n+1}$ now follows from conditions $(a^{3,1})_{n+1}$ and $(a^{3,2})_{n+1}$, condition $(b)_{n+1}$ follows from conditions $(b^{3,1})_{n+1}$ and $(b^{3,2})_{n+1}$ while condition $(d)_{n+1}$ follows from conditions $(d^{3,1})_{n+1}$ and $(d^{3,2})_{n+1}$. The proof is concluded by applying Proposition 3.1. □

Proof of Theorem 1.1 Let $K_0 = \emptyset$. Choose a point $b_0 \in B$, and a strongly J_{b_0}-subharmonic Morse exhaustion function $\tau : X \to (0, \infty)$. By a small perturbation, we may assume that there is exactly one critical point at every critical level set. Choose an increasing sequence $(c_n)_{n \in \mathbb{N}}$ of regular values of τ converging to ∞ such that the interval (c_n, c_{n+1}) contains at most one critical value of τ. Then $K_n = \{x \in X : \tau(x) \leq c_n\}$ is a smoothly bounded compact Runge set, and we may assume that c_1 is chosen so large that K_1 is nonempty and so small that it is simply connected. Then bK_n is a union of finitely many, say k_n, smooth closed Jordan curves. If τ has no critical values in (c_n, c_{n+1}), then $K_{n+1} \setminus \text{Int}\, K_n$ is a union of k_n annular regions, and we call this the *noncritical case*. In this case, there is no change in the topology, and the construction is similar to the construction in the proof of Theorem 3.2. We will explain the details below. In the *critical case*, τ has exactly one critical point in $K_{n+1} \setminus \operatorname{Int} K_n$ of index 0 or 1.

If its index is 0, then it is a minimum of τ and a new simply connected component appears. If its index is 1, then there is a compact Jordan arc $\gamma_n \subset \operatorname{Int} K_{n+1} \setminus \operatorname{Int} K_n$ transversally attached with both endpoints to K_n, and otherwise disjoint from K_n, such that $K_n \cup \gamma_n$ is a Runge set and a strong deformation retract of K_{n+1}. We need to distinguish two cases: either the endpoints of the arc γ_n lie on the same component of bK_n or the arc connects two different components of bK_n. We choose two distinct points, denoted by p_n^j and q_n^j on each boundary component of bK_n ($j = 1, \dots, k_n$) such that the endpoints of the arc γ_n are p_n^j and q_n^l for some $j, l \in \{1, \dots, k_n\}$. The map F is constructed inductively and at the critical case, we need to continuously extend the maps $F_{n,1}, F_{n,2} : B \times b\gamma_n \to \{z \in \mathbb{C} : \Re z > n\}$ to maps $F_{n,1}, F_{n,2} : B \times \gamma_n \to \{z \in \mathbb{C} : \Re z > n\}$. This is possible since the set $\{z \in \mathbb{C} : \Re z > n\}$ is contractible. Moreover, we will also obtain a continuously varying family of points on each boundary component of K_n, which corresponds to the continuous family of angles in the proof of Theorem 3.2, and the points p_n^j and q_n^j will correspond to the constant angles with the different parity: for this reason, we choose for each n and for each $j \in \{1, \dots k_n\}$ a continuous map φ_n^j from $[0, 2\pi]$ to the j-th component of bK_n which induces a homeomorphism from the quotient $[0, 2\pi]/(0 \sim 2\pi)$ to the j-th component of bK_n, inducing the given orientation. Furthermore, we may achieve that $\varphi_n^j(0) = \varphi_n^j(2\pi) = p_n^j$ and $\varphi_n^j(\pi) = q_n^j$.

We inductively construct functions $F_{n,1}, F_{n,2} \in \mathcal{A}_J(B \times K_n), n \in \mathbb{N} \cup \{0\}$, positive integers l_n^j, $j \in \{1, \dots, k_n\}, n \in \mathbb{N}$, continuous functions $\phi_{n,m}^j : B \to [0, 2\pi]$, $m \in \{1, \dots, 2l_n^j + 1\}$, $j \in \{1, \dots, k_n\}, n \in \mathbb{N}$, that satisfy the following conditions for every $n \in \mathbb{N}$:

$(a)_n$ $|F_{n,i}(b, x) - F_{n-1,i}(b, x)| < \frac{1}{2^{n-1}}$ for $(b, x) \in B \times K_{n-1}$ and $i = 1, 2$,

$(b)_n$ $\max\{\Re F_{n,1}(b, x), \Re F_{n,2}(b, x)\} > n-1$ for every $(b, x) \in B \times (K_n \setminus \operatorname{Int} K_{n-1})$,

$(c)_n$ $0 = \phi_{n,1}^j(b) < \phi_{n,2}^j(b) < \dots < \phi^j_{n,2l_n^j}(b) < 2\pi = \phi^j_{n,2l_n^j+1}(b)$ for each $b \in B$ and $j \in \{1, \dots, k_n\}$; for each $j \in \{1, \dots, k_n\}$ there is $m_n^j \in \{1, \dots, l_n^j\}$ such that $\phi^j_{n,2m_n^j} \equiv \pi$,

$(d)_n$ $\Re F_{n,1}(b, x) > n$ for $x \in \varphi_n^j([\phi_{n,2m-1}^j(b), \phi_{n,2m}^j(b)])$ and $\Re F_{n,2}(b, x) > n$ for $x \in \varphi_n^j([\phi_{n,2m}^j(b), \phi_{n,2m+1}^j(b)])$ for each $m \in \{1, \dots, l_n^j\}$, and $j \in \{1, \dots, k_n\}$.

Once we complete the construction, the proof is complete due to Proposition 3.1.

To start the induction take $F_{0,1} = F_{0,2} = F_{1,1} = F_{1,2} = 2$, $l_1^j = 1$, $m_1^j = 1$ for $j \in \{1, \dots, k_1\}$, which satisfy $(a)_1 - (d)_1$.

Assume we have already constructed $F_{i,1}, F_{i,2}, l_i^j, m_i^j, \phi_{i,m}^j, m \in \{1, \dots, 2l_i^j + 1\}$, $j \in \{1, \dots, k_i\}, i \in \{1, \dots n\}$, that satisfy $(a)_i - (d)_i$ for all $i \in \{1, \dots n\}$. We construct the functions $F_{n+1,1}, F_{n+1,2}$ by dividing each component of the set $K_{n+1} \setminus \operatorname{Int} K_n$ into two unions of simply connected regions that play the roles of $(D_{\text{odd}})_B$ and $(D_{\text{even}})_B$ in the proof of Theorem 3.2.

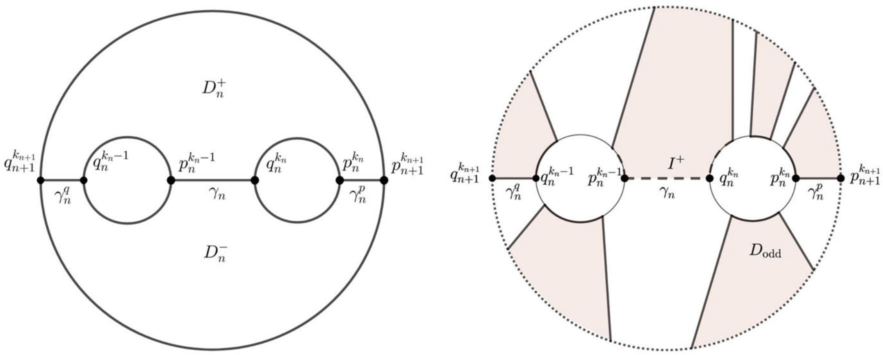

Fig. 3 Critical case 1

In the noncritical case, the set $K_{n+1} \setminus \operatorname{Int} K_n$ is homeomorphic to a disjoint union of $k_n = k_{n+1}$ annular components which we denote by A_n^j for $j = 1, 2, \ldots, k_n$. Suppose that the boundary components of A_n^j are parametrised by $\varphi_n^{j_{\mathrm{inner}}}$ and $\varphi_{n+1}^{j_{\mathrm{outer}}}$. We may choose a diffeomorphism $\psi_n^j : \{z \in \mathbb{C} : 1 \le |z| \le 2\} \to A_n^j$ such that $\psi_n^j(\mathrm{e}^{it}) = \varphi_n^{j_{\mathrm{inner}}}(t)$ and $\psi_n^j(2\mathrm{e}^{it}) = \varphi_{n+1}^{j_{\mathrm{outer}}}(t)$ for $t \in [0, 2\pi)$. Denote by $\gamma_n'^j$ the arc $\psi_n^j([1, 2])$ and by $\gamma_n''^j$ the arc $\psi_n^j([-2, -1])$. By the property $(d)_n$ we can continuously extend the maps $F_{n,1}$, $F_{n,2}$ from $B \times K_n$ to maps from $B \times (K_n \cup \gamma_n'^j \cup \gamma_n''^j)$ so that the image of $B \times (\gamma_n'^j \cup \gamma_n''^j)$ lies in $\{z \in \mathbb{C} : \Re z > n\}$ and the image of $B \times \{p_{n+1}^j\}$ and of $B \times \{q_{n+1}^j\}$ lies in $\{z \in \mathbb{C} : \Re z > n + 1\}$. Then we proceed as in the proof of Theorem 3.2 to obtain functions $F_{n+1,1}, F_{n+1,2} \in \mathcal{A}_J(B \times K_{n+1})$, integers l_{n+1}^j, m_{n+1}^j, and functions $\phi_{n+1,i}^j$ ($i \in \{1, \ldots, 2l_{n+1}^j + 1\}$, $j \in \{1, \ldots, k_{n+1}\}$) satisfying properties $(a)_{n+1} - (d)_{n+1}$.

In the critical case, we only need to consider critical points with the index 1, since we can treat the new appearing component in the case of critical points with index 0 in the same way as at the start of the inductive construction. Thus, we first consider the situation in which the arc γ_n connects two different components of bK_n. Then the number of components of bK_{n+1} is one less than the number of components of bK_n. By rearranging the notation, we may assume that γ_n connects $p_n^{k_n-1}$ and $q_n^{k_n}$. The set $K_{n+1} \setminus \operatorname{Int} K_n$ is a union of a two-connected domain D_n in X, and perhaps a finite number of annuli, where the arc $\gamma_n \subset D_n$ connects two components of the complement of D_n in X. In the annular regions of $K_{n+1} \setminus \operatorname{Int} K_n$ we proceed as in the noncritical case, thus we provide the details only for the construction corresponding the domain D_n. Since the domain D_n is two-connected, we first explain how we choose continuous family of arcs connecting the boundary of bK_n and bK_{n+1}, corresponding to the arcs $\gamma(n, \{\phi_1, \ldots, \phi_k\})$ in the proof of Theorem 3.2. We can choose pairwise disjoint smooth arcs γ_n^p and γ_n^q in $K_{n+1} \setminus (\operatorname{Int} K_n \cup \gamma_n)$ which intersect bK_n and bK_{n+1} transversally at their endpoints such that the endpoints of γ_n^p are $p_n^{k_n}$ and $p_{n+1}^{k_{n+1}}$, and the endpoints of γ_n^q are $q_n^{k_n-1}$ and $q_{n+1}^{k_{n+1}}$, see the left part of Figure 3.

Then the domain D_n is the union of two closed simply connected domains D_n^+ and D_n^- with the arcs γ_n, γ_n^p and γ_n^q as their common boundary. There is a diffeomorphism Ψ_n^+ from D_n^+ to the convex hull C of points $(2, 0)$, $(2, 1)$, $(1, 2)$, $(-1, 2)$, $(-2, 1)$, $(-2, 0)$ in $\mathbb{R}^2$ that maps $q_{n+1}^{k_{n+1}}$ to $(-2, 0)$, $q_n^{k_n-1}$ to $(-2, 1)$, $p_n^{k_n-1}$ to $(-1, 2)$, $q_n^{k_n}$ to $(1, 2)$, $p_n^{k_n}$ to $(2, 1)$, and $p_{n+1}^{k_{n+1}}$ to $(2, 0)$ (and similarly for D_n^-). The vertical segments in C provide arcs in D_n^+. More precisely, for any $\phi \in (0, \pi)$ the points $\varphi_n^{k_n-1}(\phi)$, $\varphi_n^{k_n}(\phi)$ from bK_n are mapped to points (x', y'), (x'', y'') for some $x' \in (-2, -1)$, $x'' \in (1, 2)$ and $y', y'' > 0$ on the beveled edges of C. Then segments from (x', y') to $(x', 0)$ and (x'', y'') to $(x'', 0)$ mapped back to D_n^+ by $(\Psi_n^+)^{-1}$ give the required arcs. This construction gives a Runge family, and the proof is reduced to the proof in the noncritical case: First we extend the maps $F_{n,1}, F_{n,2}$ continuously to maps from $B \times (K_n \cup \gamma_n \cup \gamma_n^p \cup \gamma_n^q)$ such that the image of $B \times (\gamma_n \cup \gamma_n^p \cup \gamma_n^q)$ lies in $\{z \in \mathbb{C} : \Re z > n\}$, and that the image of $B \times \{p_{n+1}^{k_{n+1}}\}$ and $B \times \{q_{n+1}^{k_{n+1}}\}$ lies in $\{z \in \mathbb{C} : \Re z > n + 1\}$. The continuous extension to $B \times (\gamma_n \cup \gamma_n^p \cup \gamma_n^q)$ with the image in $\{z \in \mathbb{C} : \Re z > n\}$ is possible by the property $(d)_n$ and since the former set is contractible. Now the construction can proceed similarly to the construction in the regular case, and here we explain the main differences: In the noncritical case, the functions $\phi_{n,j}$ determined boundary arcs $(\alpha_{\text{odd}})_B$ and $(\alpha_{\text{even}})_B$ such that $\Re F_{n,1} > n$ on $(\alpha_{\text{odd}})_B$, and $\Re F_{n,2} > n$ on $(\alpha_{\text{even}})_B$. In the critical case, we start at the point $p_n^{k_n}$ on bK_n and move in the positive direction along k_n-th component of bK_n; we first get some arcs with alternating parity until we reach the point $\varphi_n^{k_n}(\phi_{n,2m_n^{k_n}-1}^{k_n}(b))$. These arcs determine the domains in D_n^+ that belong to D_{odd}, D_{even} as before. Observe that for all $b \in B$ and $x \in \varphi_n^{k_n}([\phi_{n,2m_n^{k_n}-1}^{k_n}(b), \pi]) \cup \gamma_n \cup \varphi_n^{k_n-1}([0, \phi_{n,2}^{k_n-1}(b)]) =: I^+(b)$ we have $\Re F_{n,1}(b, x) > n$. Therefore, the set I^+ can be seen as a part of $(\alpha_{\text{odd}})_B$, and the corresponding domain as a part of D_{odd}. As we move further along the boundary of the $(k_n - 1)$-th component of bK_n in the positive direction, from $\varphi_n^{k_n-1}(\phi_{n,2}^{k_n-1}(b))$ to $\varphi_n^{k_n-1}(\phi_{n,2l_n^{k_n-1}}^{k_n-1}(b))$ we obtain alternating arcs and domains as before, first we get some from D_n^+ and then some in D_n^-. Similarly to the above, we have for all $b \in B$ and $x \in \varphi_n^{k_n-1}([\phi_{n,2l_n^{k_n-1}}^{k_n-1}(b), 2\pi]) \cup \gamma_n \cup \varphi_n^{k_n}([\pi, \phi_{n,2m_n^{k_n}+1}^{k_n}(b)]) =: I^-(b)$ that $\Re F_{n,2}(b, x) > n$, and the set I^- can be viewed as a part of $(\alpha_{\text{even}})_B$, and the corresponding domain as a part of D_{even}. As we move further along the boundary of the k_n-th component of bK_n, we get some arcs with alternating parity until we reach the starting point $p_n^{k_n}$. Then we proceed with the proof as in the noncritical case. On the right part of Figure 3, we denoted the arcs in $(\alpha_{\text{odd}})_B$ darker than the arcs in $(\alpha_{\text{even}})_B$.

In the second case, the endpoints of the arc γ_n lie on the same component of bK_n. In this case, the number of components of bK_{n+1} is one greater than the number of components of bK_n. By rearranging the notation, we may assume that the endpoints of γ_n are $p_n^{k_n}$ and $q_n^{k_n}$. The set $K_{n+1} \setminus \text{Int}\, K_n$ is a union of a two connected domain D_n in X, and perhaps a finite number of annuli, where the set $\gamma_n \cup (bK_n \cap D_n)$ separates the other boundary components of D_n. We can choose smooth arcs $\gamma_n'^p$, $\gamma_n'^q$, $\gamma_n''^p$, $\gamma_n''^q$ in $K_{n+1} \setminus \text{Int}\, K_n$ which intersect bK_n and bK_{n+1} transversally and only at their endpoints, such that the endpoints of $\gamma_n'^p$ are $p_n^{k_n}$ and $p_{n+1}^{k_{n+1}-1}$, the endpoints of $\gamma_n'^q$

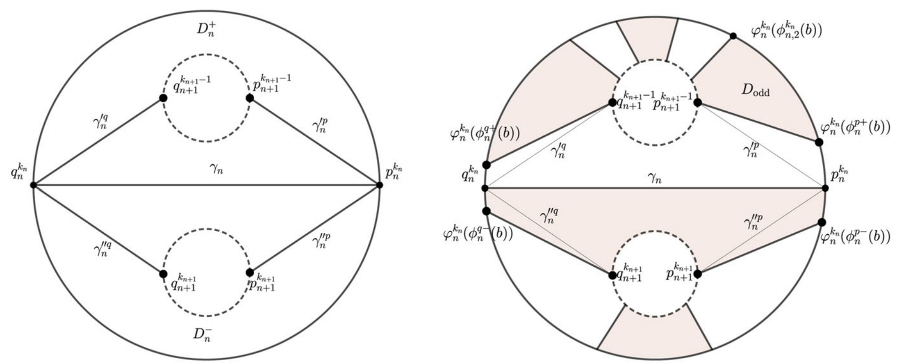

Fig. 4 Critical case 2

are $q_n^{k_n}$ and $q_{n+1}^{k_{n+1}-1}$, the endpoints of $\gamma_n''^p$ are $p_n^{k_n}$ and $p_{n+1}^{k_{n+1}}$, the endpoints of $\gamma_n''^q$ are $q_n^{k_n}$ and $q_{n+1}^{k_{n+1}}$. Furthermore, we can achieve that arcs $\gamma_n, \gamma_n'^p, \gamma_n'^q, \gamma_n''^p, \gamma_n''^q$ intersect pairwise at most at their endpoints. We denote by D_n^+ the simply connected component of the set $D_n \setminus (\gamma_n'^p \cup \gamma_n'^q)$. Assume that D_n^+ contains $\varphi_n^{k_n}((0,\pi))$, the other case is symmetrical. Let D_n^- be the simply connected component of the set $D_n \setminus (\gamma_n''^p \cup \gamma_n''^q)$ which contains $\varphi_n^{k_n}((\pi, 2\pi))$. See the left part of Figure 4.

There is a diffeomorphism ψ_n^+ from $\bar{D}_n^+$ (and ψ_n^- from $\bar{D}_n^-$) to the unit square $[0,1] \times [0,1]$ in $\mathbb{R}^2$ that maps the arcs $\gamma_n'^p, \gamma_n'^q$ ($\gamma_n''^p, \gamma_n''^q$) to the vertical edges, and the arc $\varphi_n^{k_n}((0,\pi))$, ($\varphi_n^{k_n}((\pi, 2\pi))$) to the upper edge of the square. By the properties $(c)_n - (d)_n$ there are continuous functions $\phi_n^{p\pm}, \phi_n^{q\pm} : B \to (0, 2\pi)$ such that for each $b \in B$ we have $\phi_n^{p+}(b) \in (0, \phi_{n,2}^{k_n}(b))$, $\phi_n^{p-}(b) \in (\phi_{n,2l_n^{k_n}}^{k_n}(b), 2\pi)$, $\phi_n^{q+}(b) \in (\phi_{2m_n^{k_n}-1}^{k_n}(b), \pi)$, $\phi_n^{q-}(b) \in (\pi, \phi_{2m_n^{k_n}+1}^{k_n}(b))$ and such that the restrictions of the maps $F_{n,1}, F_{n,2}$ to the arcs $\varphi_n^{k_n}([0, \phi_n^{p+}(b)])$, $\varphi_n^{k_n}([\phi_n^{p-}(b), 2\pi])$, $\varphi_n^{k_n}([\phi_n^{q+}(b), \pi])$ and $\varphi_n^{k_n}([\pi, \phi_n^{q-}(b)])$ map into $\{z \in \mathbb{C} : \Re z > n\}$. Note that in this step we added 4 functions to the family $\phi_{n,i}^{k_n}$. For every $b \in B$ and every $t \in [0,1]$ we get a segment from $(t, 0)$ to $((1-t)\psi_n^+(\varphi_n^{k_n}(\phi_n^{p+}(b))) + t\psi_n^+(\varphi_n^{k_n}(\phi_n^{q+}(b))), 1)$ in the unit square, and by pushing back with $(\psi_n^+)^{-1}$ we obtain a family of arcs in $\bar{D}_n^+$ that correspond to the union of radial line segments (and similarly for $\bar{D}_n^-$). In particular, for $t = 0$ we get arc from $\varphi_n^{k_n}(\phi_n^{p+}(b))$ to $p_{n+1}^{k_{n+1}-1}$, and for $t = 1$ we get the arc from $\varphi_n^{k_n}(\phi_n^{q+}(b))$ to $q_{n+1}^{k_{n+1}-1}$. Next, we explain how we divide the domain D_n into domains D_{odd} and D_{even} which reduces the proof to the proof in the noncritical case (see the right part of Figure 4).

We start with the point $\varphi_n^{k_n}(\phi_n^{p+}(b))$ and move in the positive direction along k_n-th component of bK_n. We get arcs with alternating parity until we reach the point $\varphi_n^{k_n}(\phi_n^{q+}(b))$ determining the domains which alternately belong to $D_{\text{odd}}, D_{\text{even}}$. For $b \in B$ and $x \in \varphi_n^{k_n}([\phi_n^{q+}(b), \pi]) \cup \gamma_n \cup \varphi_n^{k_n}([0, \phi_n^{p+}(b)]) =: I^+(b)$ it holds that $\Re F_{n,2}(b, x) > n$, thus, I^+ can be taken as a part of α_{even} and the corresponding domain as a part of D_{even}. For $b \in B$ and $x \in \varphi_n^{k_n}([\pi, \phi_n^{q-}(b)]) \cup \gamma_n \cup \varphi_n^{k_n}([\phi_n^{p-}(b), 2\pi]) =:$

$I^-(b)$ we have that $\Re F_{n,1}(b,x) > n$, thus, I^- can be taken as a part of α_{odd} and the corresponding domain as a part of D_{odd}. From the point $\varphi_n^{k_n}(\phi_n^{q-}(b))$ we move in the positive direction along bK_n until we reach the point $\varphi_n^{k_n}(\phi_n^{p-}(b))$ and again the points $\varphi_n^{k_n}(\phi_{n,i}^{k_n}(b))$ determine the arcs with alternating parity. Again, this reduces the construction to the noncritical case, which completes the proof. □

Proof of Theorem 1.3 In the proof of Theorem 1.1, we constructed a continuous map $F : B \times X \to \mathbb{C}^2$ such that for every $b \in B$ the map $F(b,\cdot) : (X, J_b) \to \mathbb{C}^2$ is proper holomorphic, and, furthermore, for every $b \in B$, $\max\{\Re F_1(b,\cdot), \Re F_2(b,\cdot)\}$ goes to infinity as we leave any compact set of X, which implies that the map $(\Re F_1, \Re F_2)(b,\cdot) : (X, J_b) \to \mathbb{R}^2$ is proper harmonic. □

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