



Monotonicity properties for joint and generalized spectral radius and their essential versions of weighted geometric symmetrizations

Katarina Bogdanović and Aljoša Peperko 

Abstract. We prove new monotonicity properties for joint and generalized spectral radius and their essential versions of weighted geometric symmetrizations of bounded sets of positive kernel operators on L^2 . To our knowledge, several proved properties are new even in the finite dimensional case.

Mathematics Subject Classification. 47A10, 47B65, 47B34, 15A42, 15A60, 15B48.

Keywords. Weighted geometric symmetrizations, bounded sets of operators, positive kernel operators, joint and generalized spectral radius and their essential versions.

1. Introduction

Let $A = [a_{ij}]$ be an entrywise nonnegative $n \times n$ matrix and let $S(A) = [\sqrt{a_{ij}a_{ji}}]$ be its geometric symmetrization. In [39], Schwenk proved the inequality

$$r(S(A)) \leq r(A), \quad (1)$$

for the spectral radius $r(\cdot)$ by using graph-theoretical methods. In [16], Elsner, Johnson and Dias Da Silva proved that the inequality

$$r(A_1^{(\alpha_1)} \circ A_2^{(\alpha_2)} \circ \dots \circ A_m^{(\alpha_m)}) \leq r(A_1)^{\alpha_1} r(A_2)^{\alpha_2} \dots r(A_m)^{\alpha_m} \quad (2)$$

Katarina Bogdanović and Aljoša Peperko contributed equally to this work.

for Hadamard weighted geometric mean holds for nonnegative $n \times n$ matrices $A_1, A_2, \dots, A_m$ and nonnegative numbers $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\sum_{j=1}^m \alpha_j \geq 1$. Here $A^{(\alpha)} = [a_{ij}^\alpha]$ denotes the Hadamard (Schur) power of A and $A \circ B = [a_{ij} b_{ij}]$ denotes the Hadamard (Schur) product of matrices A and B . Clearly, (2) generalizes (1), since $S(A) = A^{(\frac{1}{2})} \circ (A^T)^{(\frac{1}{2})}$. Let us point out that inequality (2) can straightforwardly be deduced from an earlier result by Kingman [21] and that (1) is a special case of earlier results by Karlin and Ost [20, Theorem 2.1 and Remark 1] and that the case $\sum_{j=1}^m \alpha_j = 1$ of (2) was already obtained in [20, Remark 1] (in [20] these results were applied in the context of finite stationary Markov chains). Since then inequalities and equalities on Hadamard weighted geometric means and weighted geometric symmetrizations received a lot of attention and have been applied in a variety of contexts (see e.g. [3, 5–8, 11–13, 15, 22, 28, 29, 31–35] and also a more detailed list of references in [8]).

In [11], Drnovšek proved that in the case when $\sum_{j=1}^m \alpha_j = 1$, inequality (2) holds also for positive compact operators on Banach function spaces. In [13], Drnovšek and the second author of the current article established that the compactness assumption can be removed and that analogous results hold also for operator norm and also for numerical radius on L^2 . Further they proved additional results for products of Hadamard weighted geometric means (see Theorem 1 below). In [28], the second author also showed that analogous results also hold for Hausdorff measure of non-compactness and for essential spectral radius on suitable Banach functions spaces (including L^2 , see Theorem 1 below). In [13] and [28], also generalizations of inequality (1) for products and sums of geometric symmetrizations of positive kernel operators were proved (see inequalities (26) and (27) below).

In [35], Shen and Huang studied weighted geometric symmetrizations $S_\alpha(A) = [a_{ij}^\alpha a_{ji}^{1-\alpha}]$ for $\alpha \in [0, 1]$ and for nonnegative $n \times n$ matrices. They showed that for a given square nonnegative matrix A the function $\alpha \mapsto r(S_\alpha(A))$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$ ([35, Theorem 3.3]). They also proved an analogous result for the operator (largest singular value) norm ([35, Theorem 2.3]). In [6, Theorem 2.7], we obtained an analogous result for the spectral radius, essential spectral radius, operator norm, Hausdorff measure of non-compactness and numerical radius of weighted geometric symmetrizations of a given positive kernel operator on L^2 . In [8] we further extended a technique of Shen and Huang to obtain additional results. For instance, it was proved in [8, Corollary 3.3] that also the function $\alpha \mapsto r(S_\alpha(A_1)S_\alpha(A_2))$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$, where A_1 and A_2 are positive kernel operators on L^2 (and that the analogue of this result holds also for the essential spectral radius). In the current article we generalize the results of [8] to the setting of bounded sets of positive kernel operators on L^2 by establishing new results for joint and generalized spectral radius, for essential versions of joint and generalized spectral radius and for

operator norm, Hausdorff measure of non-compactness and numerical radius of such sets. Perhaps the trickiest part of the argument that our results from [8] can be generalized to this more general context is the proof of inclusions (32) and (33) in the proof of Theorem 8.

The rest of the article is organized in the following way. In Section 2 we recall some definitions and results that will be needed in our proofs and in Section 3 we prove our results. The main results of this article are Theorems 8 and 14.

2. Preliminaries

Let μ be a σ -finite positive measure on a σ -algebra $\mathcal{M}$ of subsets of a non-void set X . Let $M(X, \mu)$ be the vector space of all equivalence classes of (almost everywhere equal) complex measurable functions on X . A Banach space $L \subseteq M(X, \mu)$ is called a *Banach function space* if $f \in L$, $g \in M(X, \mu)$, and $|g| \leq |f|$ imply that $g \in L$ and $\|g\| \leq \|f\|$. Throughout the article, it is assumed that X is the carrier of L , that is, there is no subset Y of X of strictly positive measure with the property that $f = 0$ a.e. on Y for all $f \in L$ (see [41]).

Standard examples of Banach function spaces are Euclidean spaces, $L^p(X, \mu)$ spaces for $1 \leq p \leq \infty$, the space c_0 of all null convergent sequences (equipped with the usual norms and the counting measure) and other less known examples such as Orlicz, Lorentz, Marcinkiewicz and more general rearrangement-invariant spaces (see e.g. [4, 9, 19] and the references cited there), which are important e.g. in interpolation theory and in the theory of partial differential equations. Recall that the cartesian product $L = E \times F$ of Banach function spaces is again a Banach function space, equipped with the norm $\|(f, g)\|_L = \max\{\|f\|_E, \|g\|_F\}$.

If $\{f_n\}_{n \in \mathbb{N}} \subset M(X, \mu)$ is a decreasing real valued sequence and $f = \inf\{f_n \in M(X, \mu) : n \in \mathbb{N}\}$, then we write $f_n \downarrow f$. A Banach function space L has an *order continuous norm*, if $0 \leq f_n \downarrow 0$ implies $\|f_n\|_L \rightarrow 0$ as $n \rightarrow \infty$. It is well known that spaces $L^p(X, \mu)$, $1 \leq p < \infty$, have order continuous norm. Moreover, the norm of any reflexive Banach function space is order continuous.

In particular, we are interested in Banach function spaces L such that L and its Banach dual space L^* have order continuous norms. Examples of such spaces are $L^p(X, \mu)$, $1 < p < \infty$, while the space $L = c_0$ is an example of a non-reflexive Banach sequence space, such that L and $L^* = l^1$ have order continuous norms.

By an *operator* on a Banach function space L we always mean a linear operator on L . An operator K on L is said to be *positive* if it maps nonnegative functions to nonnegative ones, i.e., $KL_+ \subset L_+$, where L_+ denotes the positive cone $L_+ = \{f \in L : f \geq 0 \text{ a.e.}\}$. Given operators K and H on L , we write $K \geq H$ if the operator $K - H$ is positive.

Recall that a positive operator K is always bounded, i.e., its operator norm

$$\|K\| = \sup\{\|Kf\|_L : f \in L, \|f\|_L \leq 1\} = \sup\{\|Kf\|_L : f \in L_+, \|f\|_L \leq 1\} \quad (3)$$

is finite (the second equality in (3) follows from $|Kf| \leq K|f|$ for $f \in L$). Also, its spectral radius $r(K)$ is always contained in the spectrum.

In the special case $L = L^2(X, \mu)$ we can define the *numerical radius* $w(K)$ of a bounded operator K on $L^2(X, \mu)$ by

$$w(K) = \sup\{|\langle Kf, f \rangle| : f \in L^2(X, \mu), \|f\|_2 = 1\}.$$

If, in addition, K is positive, then it is easy to prove that

$$w(K) = \sup\{\langle Kf, f \rangle : f \in L^2(X, \mu)_+, \|f\|_2 = 1\}.$$

From this it follows easily that $w(K) \leq w(H)$ for all positive operators K and H on $L^2(X, \mu)$ with $K \leq H$.

An operator K on a Banach function space L is called a *kernel operator* if there exists a $\mu \times \mu$ -measurable function $k(x, y)$ on $X \times X$ such that, for all $f \in L$ and for almost all $x \in X$,

$$\int_X |k(x, y)f(y)| d\mu(y) < \infty \quad \text{and} \quad (Kf)(x) = \int_X k(x, y)f(y) d\mu(y).$$

One can check that a kernel operator K is positive iff its kernel k is non-negative almost everywhere.

Let L be a Banach function space such that L and L^* have order continuous norms and let K and H be positive kernel operators on L . By $\gamma(K)$ we denote the Hausdorff measure of non-compactness of K , i.e.,

$$\gamma(K) = \inf\{\delta > 0 : \text{there is a finite } M \subset L \text{ such that } K(D_L) \subset M + \delta D_L\},$$

where $D_L = \{f \in L : \|f\|_L \leq 1\}$. Then $\gamma(K) \leq \|K\|$, $\gamma(K+H) \leq \gamma(K) + \gamma(H)$, $\gamma(KH) \leq \gamma(K)\gamma(H)$ and $\gamma(\alpha K) = \alpha\gamma(K)$ for $\alpha \geq 0$. Also $0 \leq K \leq H$ implies $\gamma(K) \leq \gamma(H)$ (see e.g. [23, Corollary 4.3.7 and Corollary 3.7.3]). Let $r_{ess}(K)$ denote the essential spectral radius of K , i.e., the spectral radius of the Calkin image of K in the Calkin algebra. Then

$$r_{ess}(K) = \lim_{j \rightarrow \infty} \gamma(K^j)^{1/j} = \inf_{j \in \mathbb{N}} \gamma(K^j)^{1/j} \quad (4)$$

and $r_{ess}(K) \leq \gamma(K)$. Recall that if $L = L^2(X, \mu)$, then $\gamma(K^*) = \gamma(K)$ and $r_{ess}(K^*) = r_{ess}(K)$, where K^* denotes the adjoint of K (see e.g. [23, Proposition 4.3.3, Theorems 4.3.6 and 4.3.13 and Corollary 3.7.3], [27, Theorem 1], [22]). Note that equalities (4) and $r_{ess}(K^*) = r_{ess}(K)$ are valid for any bounded operator K on a given complex Banach space L (see e.g. [23, Theorem 4.3.13 and Proposition 4.3.11], [27, Theorem 1]).

It is well-known that kernel operators play an important, often even central, role in a variety of applications from differential and integro-differential

equations, problems from physics (in particular from thermodynamics), engineering, statistical and economic models, etc (see e.g. [18,31] and the references cited there). For the theory of Banach function spaces and more general Banach lattices we refer the reader to the books [1,2,4,23,41].

Let Σ be a bounded set of bounded operators on a complex Banach space L . For $m \geq 1$, let

$$\Sigma^m = \{A_1 A_2 \cdots A_m : A_i \in \Sigma\}.$$

The generalized spectral radius of Σ is defined by

$$r(\Sigma) = \limsup_{m \rightarrow \infty} \left[\sup_{A \in \Sigma^m} r(A) \right]^{1/m} \quad (5)$$

and is equal to

$$r(\Sigma) = \sup_{m \in \mathbb{N}} \left[\sup_{A \in \Sigma^m} r(A) \right]^{1/m}.$$

The joint spectral radius of Σ is defined by

$$\hat{r}(\Sigma) = \lim_{m \rightarrow \infty} \left[\sup_{A \in \Sigma^m} \|A\| \right]^{1/m}. \quad (6)$$

Similarly, the generalized essential spectral radius of Σ is defined by

$$r_{ess}(\Sigma) = \limsup_{m \rightarrow \infty} \left[\sup_{A \in \Sigma^m} r_{ess}(A) \right]^{1/m} \quad (7)$$

and is equal to

$$r_{ess}(\Sigma) = \sup_{m \in \mathbb{N}} \left[\sup_{A \in \Sigma^m} r_{ess}(A) \right]^{1/m}.$$

The joint essential spectral radius of Σ is defined by

$$\hat{r}_{ess}(\Sigma) = \lim_{m \rightarrow \infty} \left[\sup_{A \in \Sigma^m} \gamma(A) \right]^{1/m}. \quad (8)$$

It is well known that $r(\Sigma) = \hat{r}(\Sigma)$ for a precompact nonempty set Σ of compact operators on L (see e.g. [24,37,38]), in particular for a bounded set of complex $n \times n$ matrices (see e.g. [10,14,26,36] and the references cited there). This equality is called the Berger-Wang formula or also the generalized spectral radius theorem. It is known that also the generalized Berger-Wang formula holds, i.e. that for any precompact nonempty set Σ of bounded operators on L we have

$$\hat{r}(\Sigma) = \max\{r(\Sigma), \hat{r}_{ess}(\Sigma)\}$$

(see e.g. [24,37,38]). Observe also that it was proved in [24] that in the definition of $\hat{r}_{ess}(\Sigma)$ one may replace the Hausdorff measure of noncompactness by several other seminorms, for instance it may be replaced by the essential norm.

In general $r(\Sigma)$ and $\hat{r}(\Sigma)$ may differ even in the case of a bounded set Σ of compact positive operators on L (see [36] or also [30]).

Also, in [17] the reader can find an example of two positive non-compact weighted shifts A and B on $L = l^2$ such that $r(\{A, B\}) = 0 < \hat{r}(\{A, B\})$. As already noted in [37] also $r_{ess}(\Sigma)$ and $\hat{r}_{ess}(\Sigma)$ may in general be different.

The theory of the generalized and the joint spectral radius has many important applications for instance to discrete and differential inclusions, wavelets, invariant subspace theory (see e.g. [10, 37, 38, 40] and the references cited there). In particular, $\hat{r}(\Sigma)$ plays a central role in determining stability in convergence properties of discrete and differential inclusions. In this theory the quantity $\log \hat{r}(\Sigma)$ is known as the maximal Lyapunov exponent (see e.g. [40]).

We have the following well known facts that hold for all $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}\}$:

$$\rho(\Sigma^m) = \rho(\Sigma)^m \text{ and } \rho(\Psi\Sigma) = \rho(\Sigma\Psi) \quad (9)$$

where $\Psi\Sigma = \{AB : A \in \Psi, B \in \Sigma\}$ and $m \in \mathbb{N}$.

Let K and H be positive kernel operators on a Banach function space L with kernels k and h respectively, and $\alpha \geq 0$. The *Hadamard (or Schur) product* $K \circ H$ of K and H is the kernel operator with kernel equal to $k(x, y)h(x, y)$ at point $(x, y) \in X \times X$ which can be defined (in general) only on some order ideal of L . Similarly, the *Hadamard (or Schur) power* $K^{(\alpha)}$ of K is the kernel operator with kernel equal to $(k(x, y))^\alpha$ at point $(x, y) \in X \times X$ which can be defined only on some order ideal of L .

Let $K_1, \dots, K_m$ be positive kernel operators on a Banach function space L , and $\alpha_1, \dots, \alpha_m$ nonnegative numbers such that $\sum_{j=1}^m \alpha_j = 1$. Then the *Hadamard weighted geometric mean* $K = K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \dots \circ K_m^{(\alpha_m)}$ of the operators $K_1, \dots, K_m$ is a positive kernel operator defined on the whole space L , since $K \leq \alpha_1 K_1 + \alpha_2 K_2 + \dots + \alpha_m K_m$ by the inequality between the weighted arithmetic and geometric means.

Let us recall the following result, which was proved in [13, Theorem 2.2] and [28, Theorem 5.1 and Example 3.7] (see also e.g. [30, Theorem 2.1]).

Theorem 1. *Let $\{K_{ij}\}_{i=1, j=1}^{n, m}$ be positive kernel operators on a Banach function space L and $\alpha_1, \alpha_2, \dots, \alpha_m$ nonnegative numbers.*

If $\sum_{j=1}^m \alpha_j = 1$, then the positive kernel operator

$$K := \left(K_{11}^{(\alpha_1)} \circ \dots \circ K_{1m}^{(\alpha_m)}\right) \dots \left(K_{n1}^{(\alpha_1)} \circ \dots \circ K_{nm}^{(\alpha_m)}\right) \quad (10)$$

satisfies the following inequalities

$$K \leq (K_{11} \dots K_{n1})^{(\alpha_1)} \circ \dots \circ (K_{1m} \dots K_{nm})^{(\alpha_m)}, \quad (11)$$

$$\begin{aligned} \|K\| &\leq \left\| (K_{11} \dots K_{n1})^{(\alpha_1)} \circ \dots \circ (K_{1m} \dots K_{nm})^{(\alpha_m)} \right\| \\ &\leq \|K_{11} \dots K_{n1}\|^{\alpha_1} \dots \|K_{1m} \dots K_{nm}\|^{\alpha_m} \end{aligned} \quad (12)$$

$$\begin{aligned} r(K) &\leq r\left((K_{11} \dots K_{n1})^{(\alpha_1)} \circ \dots \circ (K_{1m} \dots A_{nm})^{(\alpha_m)}\right) \\ &\leq r(K_{11} \dots K_{n1})^{\alpha_1} \dots r(K_{1m} \dots K_{nm})^{\alpha_m}. \end{aligned} \quad (13)$$

If, in addition, L and L^* have order continuous norms, then

$$\begin{aligned}\gamma(K) &\leq \gamma\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\ &\leq \gamma(K_{11} \cdots K_{n1})^{\alpha_1} \cdots \gamma(K_{1m} \cdots K_{nm})^{\alpha_m},\end{aligned}\quad (14)$$

$$\begin{aligned}r_{ess}(K) &\leq r_{ess}\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\ &\leq r_{ess}(K_{11} \cdots K_{n1})^{\alpha_1} \cdots r_{ess}(K_{1m} \cdots K_{nm})^{\alpha_m}.\end{aligned}\quad (15)$$

If, in addition, $L = L^2(X, \mu)$, then

$$\begin{aligned}w(K) &\leq w\left((K_{11} \cdots K_{n1})^{(\alpha_1)} \circ \cdots \circ (K_{1m} \cdots K_{nm})^{(\alpha_m)}\right) \\ &\leq w(K_{11} \cdots K_{n1})^{\alpha_1} \cdots w(K_{1m} \cdots K_{nm})^{\alpha_m}.\end{aligned}\quad (16)$$

The following result is a special case of Theorem 1.

Theorem 2. Let $K_1, \dots, K_m$ be positive kernel operators on a Banach function space L and $\alpha_1, \dots, \alpha_m$ nonnegative numbers.

If $\sum_{j=1}^m \alpha_j = 1$, then

$$\|K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}\| \leq \|K_1\|^{\alpha_1} \|K_2\|^{\alpha_2} \cdots \|K_m\|^{\alpha_m} \quad (17)$$

and

$$r(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq r(K_1)^{\alpha_1} r(K_2)^{\alpha_2} \cdots r(K_m)^{\alpha_m}. \quad (18)$$

If, in addition, L and L^* have order continuous norms, then

$$\gamma(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq \gamma(K_1)^{\alpha_1} \gamma(K_2)^{\alpha_2} \cdots \gamma(K_m)^{\alpha_m} \quad (19)$$

and

$$r_{ess}(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq r_{ess}(K_1)^{\alpha_1} r_{ess}(K_2)^{\alpha_2} \cdots r_{ess}(K_m)^{\alpha_m}. \quad (20)$$

If, in addition, $L = L^2(X, \mu)$, then

$$w(K_1^{(\alpha_1)} \circ K_2^{(\alpha_2)} \circ \cdots \circ K_m^{(\alpha_m)}) \leq w(K_1)^{\alpha_1} w(K_2)^{\alpha_2} \cdots w(K_m)^{\alpha_m}. \quad (21)$$

Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha_1, \dots, \alpha_m$ be positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then the bounded set of positive kernel operators on L , defined by

$$\Psi_1^{(\alpha_1)} \circ \cdots \circ \Psi_m^{(\alpha_m)} = \{A_1^{(\alpha_1)} \circ \cdots \circ A_m^{(\alpha_m)} : A_1 \in \Psi_1, \dots, A_m \in \Psi_m\},$$

is called the *weighted Hadamard (Schur) geometric mean* of sets $\Psi_1, \dots, \Psi_m$.

The set $\Psi_1^{(\frac{1}{m})} \circ \cdots \circ \Psi_m^{(\frac{1}{m})}$ is called the *Hadamard (Schur) geometric mean* of sets $\Psi_1, \dots, \Psi_m$.

We will need the following well-known inequalities (see e.g. [25]). For nonnegative measurable functions and for nonnegative numbers α and β such that $\alpha + \beta \geq 1$ we have

$$f_1^\alpha g_1^\beta + \cdots + f_m^\alpha g_m^\beta \leq (f_1 + \cdots + f_m)^\alpha (g_1 + \cdots + g_m)^\beta. \quad (22)$$

More generally, for nonnegative measurable functions $\{f_{ij}\}_{i=1,j=1}^{n,m}$ and for non-negative numbers $\alpha_j, j = 1, \dots, m$, such that $\sum_{j=1}^m \alpha_j \geq 1$ we have

$$(f_{11}^{\alpha_1} \cdots f_{1m}^{\alpha_m}) + \cdots + (f_{n1}^{\alpha_1} \cdots f_{nm}^{\alpha_m}) \leq (f_{11} + \cdots + f_{n1})^{\alpha_1} \cdots (f_{1m} + \cdots + f_{nm})^{\alpha_m}. \quad (23)$$

The following result was established in [7, Theorems 3.2(i) and 3.6(i)] by applying Theorem 1 and (23).

Theorem 3. Let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of positive kernel operators on a Banach function space L , let $\rho \in \{r, \hat{r}\}$ and assume that $\alpha_1, \dots, \alpha_m$ are positive numbers such that $\sum_{i=1}^m \alpha_i = 1$.

Then

$$\begin{aligned} & \rho \left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)} \right) \cdots \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)} \right) \right) \\ & \leq \rho \left((\Psi_{11} \cdots \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} \cdots \Psi_{km})^{(\alpha_m)} \right) \\ & \leq \rho (\Psi_{11} \cdots \Psi_{k1})^{\alpha_1} \cdots \rho (\Psi_{1m} \cdots \Psi_{km})^{\alpha_m} \end{aligned} \quad (24)$$

and

$$\begin{aligned} & \rho \left(\left(\Psi_{11}^{(\alpha_1)} \circ \cdots \circ \Psi_{1m}^{(\alpha_m)} \right) + \cdots + \left(\Psi_{k1}^{(\alpha_1)} \circ \cdots \circ \Psi_{km}^{(\alpha_m)} \right) \right) \\ & \leq \rho \left((\Psi_{11} + \cdots + \Psi_{k1})^{(\alpha_1)} \circ \cdots \circ (\Psi_{1m} + \cdots + \Psi_{km})^{(\alpha_m)} \right) \\ & \leq \rho (\Psi_{11} + \cdots + \Psi_{k1})^{\alpha_1} \cdots \rho (\Psi_{1m} + \cdots + \Psi_{km})^{\alpha_m}. \end{aligned} \quad (25)$$

If, in addition, L and L^* have order continuous norms, then (24) and (25) hold also for $\rho \in \{r_{ess}, \hat{r}_{ess}\}$.

Let K be a positive kernel operator on $L = L^2(X, \mu)$ with a kernel k and let $\alpha \in [0, 1]$. Denote by $S_\alpha(K) = K^{(\alpha)} \circ (K^*)^{(1-\alpha)}$ a positive kernel operator on L with a kernel $s_\alpha(k)(x, y) = k^\alpha(x, y)k^{1-\alpha}(y, x)$. Note that $S(K) = S_{\frac{1}{2}}(K)$ is a geometric symmetrization of K , which is a selfadjoint and positive kernel operator on $L^2(X, \mu)$ with a kernel $\sqrt{k(x, y)k(y, x)}$. Let $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$. It was proved in [6, Proposition 2.2 (19), (20)] that

$$\begin{aligned} & \rho(S_\alpha(K_1) \cdots S_\alpha(K_n)) \\ & \leq \rho \left((K_1 \cdots K_n)^{(\alpha)} \circ ((K_n \cdots K_1)^*)^{(1-\alpha)} \right) \leq \rho(K_1 \cdots K_n)^\alpha \rho(K_n \cdots K_1)^{1-\alpha} \end{aligned} \quad (26)$$

and

$$\rho(S_\alpha(K_1) + \cdots + S_\alpha(K_n)) \leq \rho(K_1 + \cdots + K_n). \quad (27)$$

The following two results were proved in [8, Theorems 3.1 and 3.5].

Theorem 4. Let $K_1, \dots, K_n$ be positive kernel operators on $L = L^2(X, \mu)$.

For $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$ define $\rho_n : [0, 1] \rightarrow [0, \infty)$ by

$$\rho_n(\alpha) = \sqrt{\rho(S_\alpha(K_1)S_\alpha(K_2) \cdots S_\alpha(K_n))\rho(S_\alpha(K_n)S_\alpha(K_{n-1}) \cdots S_\alpha(K_1))}.$$

Then ρ_n is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

In particular, $\rho_n(\alpha) \geq \rho_n(\frac{1}{2})$ for each $\alpha \in [0, 1]$.

Theorem 5. Let K_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$ be positive kernel operators on $L = L^2(X, \mu)$.

For $\rho \in \{r, r_{ess}, \gamma, \|\cdot\|, w\}$ define $\bar{\rho}_n : [0, 1] \rightarrow [0, \infty)$ by

$$\bar{\rho}_n(\alpha) = (\rho((S_\alpha(K_{11}) + \dots + S_\alpha(K_{1m})) \cdots (S_\alpha(K_{n1}) + \dots + S_\alpha(K_{nm}))))^{\frac{1}{2}} \\ \times (\rho((S_\alpha(K_{n1}) + \dots + S_\alpha(K_{nm})) \cdots (S_\alpha(K_{11}) + \dots + S_\alpha(K_{1m}))))^{\frac{1}{2}}.$$

Then

$$\bar{\rho}_n(\alpha) \leq \rho((K_{11} + \dots + K_{1m}) \cdots (K_{n1} + \dots + K_{nm}))^{\frac{1}{2}} \\ \times \rho((K_{n1} + \dots + K_{nm}) \cdots (K_{11} + \dots + K_{1m}))^{\frac{1}{2}} \quad (28)$$

for each $\alpha \in [0, 1]$.

Moreover, $\bar{\rho}_n$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

In particular, $\bar{\rho}_n(\alpha) \geq \bar{\rho}_n(\frac{1}{2})$ for each $\alpha \in [0, 1]$.

In the next section we generalize the above two results to the setting of bounded sets of positive kernel operators on $L^2(X, \mu)$ (Theorems 8 and 14 below).

3. Results

In this section we extend Theorems 4 and 5 to the setting of bounded sets of positive kernel operators on $L^2(X, \mu)$.

Let Ψ be a bounded set of positive kernel operators on $L = L^2(X, \mu)$ and $\alpha \in [0, 1]$. Denote by Ψ^* and $S_\alpha(\Psi)$ bounded sets of positive kernel operators on L defined by $\Psi^* = \{A^* : A \in \Psi\}$ and $S_\alpha(\Psi) = \Psi^{(\alpha)} \circ (\Psi^*)^{(1-\alpha)} = \{A^{(\alpha)} \circ (B^*)^{(1-\alpha)} : A, B \in \Psi\}$. Note that $S_\alpha(\Psi)^* = S_\alpha(\Psi^*) = \{(A^*)^{(\alpha)} \circ B^{(1-\alpha)} : A, B \in \Psi\}$.

Let $\Psi, \Psi_1, \dots, \Psi_n$ be bounded sets of positive kernel operators on $L = L^2(X, \mu)$, $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}\}$ and $\alpha \in [0, 1]$. In [7, Proposition 4.2] the following generalizations of (26) and (27) were proved

$$\rho(S_\alpha(\Psi_1) \cdots S_\alpha(\Psi_n)) \leq \rho\left((\Psi_1 \cdots \Psi_n)^{(\alpha)} \circ ((\Psi_n \cdots \Psi_1)^*)^{(1-\alpha)}\right) \\ \leq \rho(\Psi_1 \cdots \Psi_n)^\alpha \rho(\Psi_n \cdots \Psi_1)^{1-\alpha} \quad (29)$$

and

$$\rho(S_\alpha(\Psi_1) + \dots + S_\alpha(\Psi_m)) \leq \rho(S_\alpha(\Psi_1 + \dots + \Psi_m)) \\ \leq \rho(\Psi_1 + \dots + \Psi_m). \quad (30)$$

In particular ([7, Proposition 4.2 (58)]),

$$\rho(S_\alpha(\Psi)) \leq \rho(\Psi). \quad (31)$$

Let us denote $\|\Psi\| = \sup_{A \in \Psi} \|A\|$, $\gamma(\Psi) = \sup_{A \in \Psi} \gamma(A)$ and $w(\Psi) = \sup_{A \in \Psi} w(A)$.

We show below in Corollary 7 that (29) and (30) hold also for all $\rho \in \{\|\cdot\|, \gamma, w\}$. To do this we first point out the following analogue of Theorem 3 for all $\rho \in \{\|\cdot\|, \gamma, w\}$ that easily follows from Theorem 1 and (23).

Corollary 6. *Let $\{\Psi_{ij}\}_{i=1,j=1}^{k,m}$ be bounded sets of positive kernel operators on a Banach function space L and assume that $\alpha_1, \dots, \alpha_m$ are positive numbers such that $\sum_{i=1}^m \alpha_i = 1$. Then (24) and (25) hold for $\rho = \|\cdot\|$.*

If, in addition, L and L^ have order continuous norms, then (24) and (25) hold also for $\rho = \gamma$.*

If, in addition, $L = L^2(X, \mu)$, then (24) and (25) hold also for $\rho = w$.

Proof. Inequalities (24) in all three cases follow from Theorem 1, while inequalities (25) follow from (23), monotonicity of ρ and from Theorem 2. $\square$

Remark 1. Let $\mathcal{L}$ denote the set of all Banach sequence spaces L such that all standard vectors are included in L and have norm 1 (for precise definitions see e.g. [7], [22], [32]). If $\alpha_1, \dots, \alpha_m$ are positive numbers such that $\sum_{i=1}^m \alpha_i \geq 1$, then for $\rho = \|\cdot\|$ inequalities (24) and (25) hold also for bounded sets of nonnegative matrices that define positive operators on $L \in \mathcal{L}$ by [22, Theorem 2.1 (ii)]. If, in addition, L and L^* have order continuous norms then the same conclusion holds also for $\rho = \gamma$ by [22, Theorem 3.5]. However, if in addition, $L = l^2$, then the same conclusion does not hold for $\rho = w$ (see e.g. [13, Example 3.2]).

The following result establishes an analogue of (29) and (30) for all $\rho \in \{\|\cdot\|, \gamma, w\}$ and it extends [6, Proposition 2.2].

Corollary 7. *Let $\Psi_1, \dots, \Psi_m$ be bounded sets of positive kernel operators on $L^2(X, \mu)$ and let $\alpha \in [0, 1]$. Then (29) and (30) hold for all $\rho \in \{\|\cdot\|, \gamma, w\}$.*

Proof. Let $\rho \in \{\|\cdot\|, \gamma, w\}$. By Corollary 6 we have

$$\begin{aligned} \rho(S_\alpha(\Psi_1) \cdots S_\alpha(\Psi_m)) &= \rho\left((\Psi_1^{(\alpha)} \circ (\Psi_1^*)^{(1-\alpha)}) \cdots (\Psi_m^{(\alpha)} \circ (\Psi_m^*)^{(1-\alpha)})\right) \\ &\leq \rho\left((\Psi_1 \cdots \Psi_m)^{(\alpha)} \circ ((\Psi_m \cdots \Psi_1)^*)^{(1-\alpha)}\right) \\ &\leq \rho(\Psi_1 \cdots \Psi_m)^\alpha \rho((\Psi_m \cdots \Psi_1)^*)^{1-\alpha} = \rho(\Psi_1 \cdots \Psi_m)^\alpha \rho(\Psi_m \cdots \Psi_1)^{1-\alpha}, \end{aligned}$$

where the last equality follows from the fact that $\rho(\Psi) = \rho(\Psi^*)$. This completes the proof of (29) for $\rho \in \{\|\cdot\|, \gamma, w\}$.

The inequalities in (30) are proved in similar way by applying (25). $\square$

The following results generalizes Theorem 4.

Theorem 8. *Let $\Psi_1, \dots, \Psi_n$ be bounded sets of positive kernel operators on $L = L^2(X, \mu)$ and $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. Define $\rho_n : [0, 1] \rightarrow [0, \infty)$ by*

$$\rho_n(\alpha) = \sqrt{\rho(S_\alpha(\Psi_1)S_\alpha(\Psi_2) \cdots S_\alpha(\Psi_n))\rho(S_\alpha(\Psi_n)S_\alpha(\Psi_{n-1}) \cdots S_\alpha(\Psi_1))}.$$

Then ρ_n is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

In particular, $\rho_n(\alpha) \geq \rho_n(\frac{1}{2})$ for each $\alpha \in [0, 1]$.

Proof. Let $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$ and $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$. Then $\alpha \in (0, 1)$ and

$$S_{\alpha_2}(\Psi) \subset S_\alpha(S_{\alpha_1}(\Psi)) \quad (32)$$

for every bounded set of positive kernel operators on $L = L^2(X, \mu)$. Indeed, for $A^{(\alpha_2)} \circ (B^*)^{(1-\alpha_2)} \in S_{\alpha_2}(\Psi)$ where $A, B \in \Psi$ we have

$$\begin{aligned} A^{(\alpha_2)} \circ (B^*)^{(1-\alpha_2)} &= A^{(\alpha_1 + (1-\alpha_1)(1-\alpha))} \circ (B^*)^{(\alpha(1-\alpha_1) + \alpha_1(1-\alpha))} \\ &= (A^{(\alpha_1)} \circ (B^*)^{(1-\alpha_1)})^{(\alpha)} \circ ((B^*)^{(\alpha_1)} \circ A^{(1-\alpha_1)})^{(1-\alpha)} \in S_\alpha(S_{\alpha_1}(\Psi)), \end{aligned}$$

since $A^{(\alpha_1)} \circ (B^*)^{(1-\alpha_1)} \in S_{\alpha_1}(\Psi)$ and $(B^*)^{(\alpha_1)} \circ A^{(1-\alpha_1)} \in S_{\alpha_1}(\Psi^*) = S_{\alpha_1}(\Psi)^*$. From (32) and (29) it follows that

$$\begin{aligned} \rho_n(\alpha_2) &= \sqrt{\rho(S_{\alpha_2}(\Psi_1) \cdots S_{\alpha_2}(\Psi_n)) \rho(S_{\alpha_2}(\Psi_n) \cdots S_{\alpha_2}(\Psi_1))} \\ &\leq \sqrt{\rho(S_\alpha(S_{\alpha_1}(\Psi_1)) \cdots S_\alpha(S_{\alpha_1}(\Psi_n))) \rho(S_\alpha(S_{\alpha_1}(\Psi_n)) \cdots S_\alpha(S_{\alpha_1}(\Psi_1)))} \\ &\leq \sqrt{\rho((S_{\alpha_1}(\Psi_1) \cdots S_{\alpha_1}(\Psi_n))^\alpha \rho((S_{\alpha_1}(\Psi_n) \cdots S_{\alpha_1}(\Psi_1))^{1-\alpha}) \times} \\ &\quad \sqrt{\rho((S_{\alpha_1}(\Psi_n) \cdots S_{\alpha_1}(\Psi_1))^\alpha \rho((S_{\alpha_1}(\Psi_1) \cdots S_{\alpha_1}(\Psi_n))^{1-\alpha})} \\ &= \sqrt{\rho(S_{\alpha_1}(\Psi_1) \cdots S_{\alpha_1}(\Psi_n)) \rho(S_{\alpha_1}(\Psi_n) \cdots S_{\alpha_1}(\Psi_1))} = \rho_n(\alpha_1), \end{aligned}$$

which proves that ρ_n is a decreasing function on $[0, \frac{1}{2}]$.

To prove the case $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$, let $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$. Then $\alpha \in (0, 1)$ and

$$S_{\alpha_1}(\Psi) \subset S_\alpha(S_{\alpha_2}(\Psi)) \quad (33)$$

for every bounded set Ψ of positive kernel operators on $L = L^2(X, \mu)$. Indeed, let $C^{(\alpha_1)} \circ (D^*)^{(1-\alpha_1)} \in S_{\alpha_1}(\Psi)$, where $C, D \in \Psi$. Then

$$\begin{aligned} C^{(\alpha_1)} \circ (D^*)^{(1-\alpha_1)} &= C^{(\alpha_2 + (1-\alpha_2)(1-\alpha))} \circ (D^*)^{(\alpha(1-\alpha_2) + \alpha_2(1-\alpha))} \\ &= (C^{(\alpha_2)} \circ (D^*)^{(1-\alpha_2)})^{(\alpha)} \circ ((D^*)^{(\alpha_2)} \circ C^{(1-\alpha_2)})^{(1-\alpha)} \in S_\alpha(S_{\alpha_2}(\Psi)), \end{aligned}$$

since $C^{(\alpha_2)} \circ (D^*)^{(1-\alpha_2)} \in S_{\alpha_2}(\Psi)$ and $(D^*)^{(\alpha_2)} \circ C^{(1-\alpha_2)} \in S_{\alpha_2}(\Psi^*) = S_{\alpha_2}(\Psi)^*$. Applying (33) and (29) we obtain

$$\begin{aligned} \rho_n(\alpha_1) &= \sqrt{\rho(S_{\alpha_1}(\Psi_1) \cdots S_{\alpha_1}(\Psi_n)) \rho(S_{\alpha_1}(\Psi_n) \cdots S_{\alpha_1}(\Psi_1))} \\ &\leq \sqrt{\rho(S_\alpha(S_{\alpha_2}(\Psi_1)) \cdots S_\alpha(S_{\alpha_2}(\Psi_n))) \rho(S_\alpha(S_{\alpha_2}(\Psi_n)) \cdots S_\alpha(S_{\alpha_2}(\Psi_1)))} \\ &\leq \sqrt{\rho((S_{\alpha_2}(\Psi_1) \cdots S_{\alpha_2}(\Psi_n))^\alpha \rho((S_{\alpha_2}(\Psi_n) \cdots S_{\alpha_2}(\Psi_1))^{1-\alpha}) \times} \\ &\quad \sqrt{\rho((S_{\alpha_2}(\Psi_n) \cdots S_{\alpha_2}(\Psi_1))^\alpha \rho((S_{\alpha_2}(\Psi_1) \cdots S_{\alpha_2}(\Psi_n))^{1-\alpha})} \\ &= \sqrt{\rho(S_{\alpha_2}(\Psi_1) \cdots S_{\alpha_2}(\Psi_n)) \rho(S_{\alpha_2}(\Psi_n) \cdots S_{\alpha_2}(\Psi_1))} = \rho_n(\alpha_2), \end{aligned}$$

which completes the proof. $\square$

The following corollary generalizes [6, Theorem 2.7] ([7, Corollary 3.2]).

Corollary 9. *Let Ψ be a bounded set of positive kernel operators on $L = L^2(X, \mu)$ and $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. Then the function $\rho_1 : [0, 1] \rightarrow [0, \infty)$, defined by $\rho_1(\alpha) = \rho(S_\alpha(\Psi))$, is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.*

In particular, $\rho(S_\alpha(\Psi)) \geq \rho(S(\Psi))$ for each $\alpha \in [0, 1]$.

The following corollary generalizes [8, Corollary 3.3].

Corollary 10. *Let Ψ_1 and Ψ_2 be bounded sets of positive kernel operators on $L = L^2(X, \mu)$ and let $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. Then the function $\rho_2 : [0, 1] \rightarrow [0, \infty)$, defined by $\rho_2(\alpha) = \rho(S_\alpha(\Psi_1)S_\alpha(\Psi_2))$, is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.*

In particular, $\rho(S_\alpha(\Psi_1)S_\alpha(\Psi_2)) \geq \rho(S(\Psi_1)S(\Psi_2))$ for each $\alpha \in [0, 1]$.

Next we extend [8, Proposition 3.4].

Proposition 11. *Let $\Psi_1, \dots, \Psi_n$ be bounded sets of positive kernel operators on $L = L^2(X, \mu)$ and $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. Then the function $\widetilde{\rho}_n : [0, 1] \rightarrow [0, \infty)$, defined by $\widetilde{\rho}_n(\alpha) = \rho(S_\alpha(\Psi_1) + \dots + S_\alpha(\Psi_n))$, is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.*

Proof. Let $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$ and $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$. Then $\alpha \in (0, 1)$ and $S_{\alpha_2}(\Psi) \subset S_\alpha(S_{\alpha_1}(\Psi))$ by (32). Applying (30) we obtain

$$\begin{aligned} \widetilde{\rho}_n(\alpha_2) &= \rho(S_{\alpha_2}(\Psi_1) + \dots + S_{\alpha_2}(\Psi_n)) \leq \rho(S_\alpha(S_{\alpha_1}(\Psi_1)) + \dots + S_\alpha(S_{\alpha_1}(\Psi_n))) \\ &\leq \rho(S_{\alpha_1}(\Psi_1) + \dots + S_{\alpha_1}(\Psi_n)) = \widetilde{\rho}_n(\alpha_1), \end{aligned}$$

which proves that $\widetilde{\rho}_n$ is decreasing on $[0, \frac{1}{2}]$. For $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$ let $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$. It follows that $\alpha \in (0, 1)$ and by (33) we have $S_{\alpha_1}(\Psi) \subset S_\alpha(S_{\alpha_2}(\Psi))$ for every bounded set of positive kernel operators on $L = L^2(X, \mu)$. By (30) it follows

$$\begin{aligned} \widetilde{\rho}_n(\alpha_1) &= \rho(S_{\alpha_1}(\Psi_1) + \dots + S_{\alpha_1}(\Psi_n)) \leq \rho(S_\alpha(S_{\alpha_2}(\Psi_1)) + \dots + S_\alpha(S_{\alpha_2}(\Psi_n))) \\ &\leq \rho(S_{\alpha_2}(\Psi_1) + \dots + S_{\alpha_2}(\Psi_n)) = \widetilde{\rho}_n(\alpha_2), \end{aligned}$$

which completes the proof. $\square$

We will need a special case of the following result (Corollary 13 below).

Lemma 12. *Let Ψ_{ij} and Σ_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$ be bounded sets of positive kernel operators on a Banach function space L and let $\alpha \in [0, 1]$. If $\rho \in \{r, \hat{r}, \|\cdot\|\}$, then*

$$\begin{aligned} &\rho\left(\left((\Psi_{11}^{(\alpha)} \circ \Sigma_{11}^{(1-\alpha)}) + \dots + (\Psi_{1m}^{(\alpha)} \circ \Sigma_{1m}^{(1-\alpha)})\right)\right. \\ &\quad \left.\dots \left((\Psi_{n1}^{(\alpha)} \circ \Sigma_{n1}^{(1-\alpha)}) + \dots + (\Psi_{nm}^{(\alpha)} \circ \Sigma_{nm}^{(1-\alpha)})\right)\right) \\ &\leq \rho\left(\left((\Psi_{11} + \dots + \Psi_{1m})^{(\alpha)} \circ (\Sigma_{11} + \dots + \Sigma_{1m})^{(1-\alpha)}\right)\right) \dots \end{aligned}$$

$$\dots \left((\Psi_{n1} + \dots + \Psi_{nm})^{(\alpha)} \circ (\Sigma_{n1} + \dots + \Sigma_{nm})^{(1-\alpha)} \right). \quad (34)$$

If, in addition, L and L^* have order continuous norms, then (34) holds also for $\rho \in \{r_{ess}, \hat{r}_{ess}, \gamma\}$.

If, in addition, $L = L^2(X, \mu)$, then (34) holds also for $\rho = w$.

Proof. Let $\rho \in \{r, \hat{r}, \|\cdot\|\}$. Let $l \in \mathbb{N}$ and

$$A \in \left(\left((\Psi_{11}^{(\alpha)} \circ \Sigma_{11}^{(1-\alpha)}) + \dots + (\Psi_{1m}^{(\alpha)} \circ \Sigma_{1m}^{(1-\alpha)}) \right) \dots \left((\Psi_{n1}^{(\alpha)} \circ \Sigma_{n1}^{(1-\alpha)}) + \dots + (\Psi_{nm}^{(\alpha)} \circ \Sigma_{nm}^{(1-\alpha)}) \right) \right)^l$$

Then $A = A_1 \cdots A_l$ and for each $k = 1, \dots, l$ we have

$$A_k = (A_{k11}^{(\alpha)} \circ B_{k11}^{(1-\alpha)} + \dots + A_{k1m}^{(\alpha)} \circ B_{k1m}^{(1-\alpha)}) \cdots (A_{kn1}^{(\alpha)} \circ B_{kn1}^{(1-\alpha)} + \dots + A_{knm}^{(\alpha)} \circ B_{knm}^{(1-\alpha)})$$

where $A_{kij} \in \Psi_{ij}$, $B_{kij} \in \Sigma_{ij}$ for $k = 1, \dots, l$, $i = 1, \dots, n$ and $j = 1, \dots, m$.

Then by (23) we have

$$A_k \leq C_k = ((A_{k11} + \dots + A_{k1m})^{(\alpha)} \circ (B_{k11} + \dots + B_{k1m})^{(1-\alpha)}) \cdots ((A_{kn1} + \dots + A_{knm})^{(\alpha)} \circ (B_{kn1} + \dots + B_{knm})^{(1-\alpha)})$$

for each $k = 1, \dots, l$ and

$$C_k \in \left((\Psi_{11} + \dots + \Psi_{1m})^{(\alpha)} \circ (\Sigma_{11} + \dots + \Sigma_{1m})^{(1-\alpha)} \right) \cdots \left((\Psi_{n1} + \dots + \Psi_{nm})^{(\alpha)} \circ (\Sigma_{n1} + \dots + \Sigma_{nm})^{(1-\alpha)} \right).$$

Therefore for $C = C_1 \cdots C_l$ we have $A \leq C$ and

$$C \in \left((\Psi_{11} + \dots + \Psi_{1m})^{(\alpha)} \circ (\Sigma_{11} + \dots + \Sigma_{1m})^{(1-\alpha)} \right) \cdots \left((\Psi_{n1} + \dots + \Psi_{nm})^{(\alpha)} \circ (\Sigma_{n1} + \dots + \Sigma_{nm})^{(1-\alpha)} \right)^l.$$

By monotonicity of spectral radius and operator norm and definitions, the inequality (34) for $\rho \in \{r, \hat{r}\}$ follows. The case $\rho = \|\cdot\|$ is proved similarly by taking $l = 1$ in the proof above. The remaining cases follow in a similar manner. $\square$

Corollary 13. Let Ψ_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$ be bounded sets of positive kernel operators on $L^2(X, \mu)$ and $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. If $\alpha \in [0, 1]$ then

$$\begin{aligned} & \rho((S_\alpha(\Psi_{11}) + \dots + S_\alpha(\Psi_{1m})) \cdots (S_\alpha(\Psi_{n1}) + \dots + S_\alpha(\Psi_{nm}))) \\ & \leq \rho(S_\alpha(\Psi_{11} + \dots + \Psi_{1m}) \cdots S_\alpha(\Psi_{n1} + \dots + \Psi_{nm})) \end{aligned} \quad (35)$$

Now we are in position to generalize Theorem 5 to the setting of bounded sets of positive kernel operators on $L^2(X, \mu)$.

Theorem 14. Let Ψ_{ij} for $i = 1, \dots, n$ and $j = 1, \dots, m$ be bounded sets of positive kernel operators on $L^2(X, \mu)$ and $\rho \in \{r, \hat{r}, r_{ess}, \hat{r}_{ess}, \|\cdot\|, \gamma, w\}$. Define $\bar{\rho}_n : [0, 1] \rightarrow [0, \infty)$ by

$$\bar{\rho}_n(\alpha) = (\rho((S_\alpha(\Psi_{11}) + \dots + S_\alpha(\Psi_{1m})) \cdots (S_\alpha(\Psi_{n1}) + \dots + S_\alpha(\Psi_{nm}))))^{\frac{1}{2}} \times \\ (\rho((S_\alpha(\Psi_{n1}) + \dots + S_\alpha(\Psi_{nm})) \cdots (S_\alpha(\Psi_{11}) + \dots + S_\alpha(\Psi_{1m}))))^{\frac{1}{2}}.$$

Then

$$\bar{\rho}_n(\alpha) \leq \rho((\Psi_{11} + \dots + \Psi_{1m}) \cdots (\Psi_{n1} + \dots + \Psi_{nm}))^{\frac{1}{2}} \times \quad (36) \\ \rho((\Psi_{n1} + \dots + \Psi_{nm}) \cdots (\Psi_{11} + \dots + \Psi_{1m}))^{\frac{1}{2}}$$

for each $\alpha \in [0, 1]$. Moreover, $\bar{\rho}_n$ is decreasing on $[0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, 1]$.

Proof. By double application of (35) and then of (29) we obtain

$$\bar{\rho}_n(\alpha) \leq \rho(S_\alpha(\Psi_{11} + \dots + \Psi_{1m}) \cdots S_\alpha(\Psi_{n1} + \dots + \Psi_{nm}))^{1/2} \times \\ \rho(S_\alpha(\Psi_{n1} + \dots + \Psi_{nm}) \cdots S_\alpha(\Psi_{11} + \dots + \Psi_{1m}))^{1/2} \\ \leq \rho((\Psi_{11} + \dots + \Psi_{1m}) \cdots (\Psi_{n1} + \dots + \Psi_{nm}))^{\frac{\alpha}{2}} \times \\ \rho((\Psi_{n1} + \dots + \Psi_{nm}) \cdots (\Psi_{11} + \dots + \Psi_{1m}))^{\frac{1-\alpha}{2}} \times \\ \rho((\Psi_{n1} + \dots + \Psi_{nm}) \cdots (\Psi_{11} + \dots + \Psi_{1m}))^{\frac{\alpha}{2}} \times \\ \rho((\Psi_{11} + \dots + \Psi_{1m}) \cdots (\Psi_{n1} + \dots + \Psi_{nm}))^{\frac{1-\alpha}{2}} \\ = \rho((\Psi_{11} + \dots + \Psi_{1m}) \cdots (\Psi_{n1} + \dots + \Psi_{nm}))^{\frac{1}{2}} \times \\ \rho((\Psi_{n1} + \dots + \Psi_{nm}) \cdots (\Psi_{11} + \dots + \Psi_{1m}))^{\frac{1}{2}},$$

which proves (36). Let $0 \leq \alpha_1 < \alpha_2 \leq \frac{1}{2}$. For $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_1 - 1}$ we have $\alpha \in (0, 1)$ and $S_{\alpha_2}(K) \subset S_\alpha(S_{\alpha_1}(K))$. Then by (36)

$$\bar{\rho}_n(\alpha_2) \leq \\ (\rho((S_\alpha(S_{\alpha_1}(\Psi_{11})) + \dots + S_\alpha(S_{\alpha_1}(\Psi_{1m}))) \cdots (S_\alpha(S_{\alpha_1}(\Psi_{n1})) + \dots + S_\alpha(S_{\alpha_1}(\Psi_{nm}))))^{\frac{1}{2}} \times \\ (\rho((S_\alpha(S_{\alpha_1}(\Psi_{n1})) + \dots + S_\alpha(S_{\alpha_1}(\Psi_{nm}))) \cdots (S_\alpha(S_{\alpha_1}(\Psi_{11})) + \dots + S_\alpha(S_{\alpha_1}(\Psi_{1m}))))^{\frac{1}{2}} \\ \leq (\rho((S_{\alpha_1}(\Psi_{11}) + \dots + S_{\alpha_1}(\Psi_{1m})) \cdots (S_{\alpha_1}(\Psi_{n1}) + \dots + S_{\alpha_1}(\Psi_{nm}))))^{\frac{1}{4}} \times \\ (\rho((S_{\alpha_1}(\Psi_{n1}) + \dots + S_{\alpha_1}(\Psi_{nm})) \cdots (S_{\alpha_1}(\Psi_{11}) + \dots + S_{\alpha_1}(\Psi_{1m}))))^{\frac{1}{4}} \times \\ (\rho((S_{\alpha_1}(\Psi_{n1}) + \dots + S_{\alpha_1}(\Psi_{nm})) \cdots (S_{\alpha_1}(\Psi_{11}) + \dots + S_{\alpha_1}(\Psi_{1m}))))^{\frac{1}{4}} \times \\ (\rho((S_{\alpha_1}(\Psi_{11}) + \dots + S_{\alpha_1}(\Psi_{1m})) \cdots (S_{\alpha_1}(\Psi_{n1}) + \dots + S_{\alpha_1}(\Psi_{nm}))))^{\frac{1}{4}} = \bar{\rho}_n(\alpha_1),$$

which proves that $\bar{\rho}_n$ is decreasing on $[0, \frac{1}{2}]$.

To prove that $\bar{\rho}_n$ is increasing on $[\frac{1}{2}, 1]$ let $\frac{1}{2} \leq \alpha_1 < \alpha_2 \leq 1$. For $\alpha = \frac{\alpha_1 + \alpha_2 - 1}{2\alpha_2 - 1}$ we have $\alpha \in (0, 1)$ and $S_{\alpha_1}(K) \subset S_\alpha(S_{\alpha_2}(K))$. Similarly as above it follows from (36) that $\bar{\rho}_n(\alpha_1) \leq \bar{\rho}_n(\alpha_2)$, which completes the proof. $\square$

Acknowledgements

The first author thanks the colleagues and staff at the Faculty of Mechanical Engineering, University of Ljubljana and at the Institute of Mathematics, Physics and Mechanics for their hospitality during the research stay in Slovenia. The second author is also a member of the University of Ljubljana, KTT21 project group (grant SN-ZRD/22-27/0510).

Funding The first author acknowledges a partial support of COST Short Term Scientific Mission program (action CA18232) and the Slovenian Research and Innovation Agency (grants P1-0222 and P1-0288).

The second author acknowledges a partial support of the Slovenian Research and Innovation Agency (grants P1-0222 and J2-2512).

Data Availability Statement There is no additional data for this article.

Declarations

Competing Interests The authors have no relevant financial or non-financial interests to disclose.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

- [1] Abramovich, Y.A., Aliprantis, C.D.: An invitation to operator theory. American Mathematical Society, Providence (2002)
- [2] Aliprantis, C.D., Burkinshaw, O.: Positive operators, Reprint of the 1985 original. Springer, Dordrecht (2006)
- [3] Audenaert, K.M.R.: Spectral radius of Hadamard product versus conventional product for non-negative matrices. *Linear Algebra Appl.* **432**, 366–368 (2010)
- [4] Bennett, C., Sharpley, R.: Interpolation of Operators. Academic Press Inc, Orlando (1988)
- [5] Bogdanović, K.: Inequalities on the generalized and the joint spectral radius of bounded sets of positive operators on Banach function and sequence spaces, (2023), [arXiv:2311.02444v1](https://arxiv.org/abs/2311.02444)

- [6] Bogdanović, K., Peperko, A.: Hadamard weighted geometric mean inequalities for the spectral and essential spectral radius of positive operators on Banach function and sequence spaces, *Positivity* **26**, Article number: 25 (2022)
- [7] Bogdanović, K., Peperko, A.: Inequalities and equalities on the joint and generalized spectral and essential spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators. *Linear Multilinear Algebra* **71**(17), 2839–2857 (2023). <https://doi.org/10.1080/03081087.2022.2121369>
- [8] Bogdanović, K., Peperko, A., Monotonicity properties of weighted geometric symmetrizations, *Journal of Mathematical Inequalities*,: Vol. 18. Num. **4**, 1535–1546 (2024). <https://doi.org/10.7153/jmi-2024-18-87>
- [9] Curbera, G.P., Ricker, W.J.: Compactness properties of Sobolev imbeddings for rearrangement invariant norms. *Transactions AMS* **359**, 1471–1484 (2007)
- [10] Dai, X.: Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices. *J. Math. Anal. Appl.* **379**, 827–833 (2011)
- [11] Drnovšek, R.: Spectral inequalities for compact integral operators on Banach function spaces. *Math. Proc. Camb. Phil. Soc.* **112**, 589–598 (1992)
- [12] Drnovšek, R.: Sequences of bounds for the spectral radius of a positive operator. *Linear Algebra Appl.* **574**, 40–45 (2019)
- [13] Drnovšek, R., Peperko, A.: Inequalities for the Hadamard weighted geometric mean of positive kernel operators on Banach function spaces. *Positivity* **10**, 613–626 (2006)
- [14] Elsner, L.: The generalized spectral radius theorem: An analytic-geometric proof. *Linear Algebra Appl.* **220**, 151–159 (1995)
- [15] Elsner, L., Hershkowitz, D., Pinkus, A.: Functional inequalities for spectral radii of nonnegative matrices. *Linear Algebra Appl.* **129**, 103–130 (1990)
- [16] Elsner, L., Johnson, C.R., Dias Da Silva, J.A.: The Perron root of a weighted geometric mean of nonnegative matrices. *Linear Multilinear Algebra* **24**, 1–13 (1988)
- [17] Guinand, P.S.: On quasinilpotent semigroup of operators. *Proc. Amer. Math. Soc.* **86**, 485–486 (1982)
- [18] Jörgens, K.: *Linear integral operators, Surveys and Reference Works in Mathematics* **7**, Pitman Press, (1982)
- [19] Kazhikhov, A.V., Mamontov, A.E.: Transport equations and Orlicz spaces, 535–544. In: Jeltsch R., Fey M. (eds) *Hyperbolic Problems: Theory, Numerics, Applications*. International Series of Numerical Mathematics, vol 130. Birkhäuser, Basel, (1999)
- [20] Karlin, S., Ost, F.: Some monotonicity properties of Schur powers of matrices and related inequalities. *Linear Algebra Appl.* **68**, 47–65 (1985)
- [21] Kingman, J.F.C.: A convexity property of positive matrices, *Quart. J. Math. Oxford Ser.* **12**(2) 283–284 (1961)
- [22] Lins, B., Peperko, A., Inequalities on the essential joint and essential generalized spectral radius, *Journal of Mathematical Inequalities*,: Vol. 18. Num. **4**, 1489–1514 (2024)
- [23] Meyer-Nieberg, P.: *Banach lattices*. Springer-Verlag, Berlin (1991)

- [24] Morris, I.D.: The generalized Berger-Wang formula and the spectral radius of linear cocycles. *J. Funct. Anal.* **262**, 811–824 (2012)
- [25] Mitrinović, D.S.: *Analitic Inequalities*. Springer Verlag, Berlin Heidelberg New York (1970)
- [26] Müller, V., Peperko, A.: Generalized spectral radius and its max algebra version. *Linear Algebra Appl.* **439**, 1006–1016 (2013)
- [27] Nussbaum, R.D.: The radius of the essential spectrum. *Duke Math. J.* **37**, 473–478 (1970)
- [28] Peperko, A.: Inequalities for the spectral radius of non-negative functions. *Positivity* **13**, 255–272 (2009)
- [29] Peperko, A.: Bounds on the generalized and the joint spectral radius of Hadamard products of bounded sets of positive operators on sequence spaces. *Linear Algebra Appl.* **437**, 189–201 (2012)
- [30] Peperko, A.: Bounds on the joint and generalized spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators. *Linear Algebra Appl.* **533**, 418–427 (2017)
- [31] Peperko, A.: Inequalities on the joint and generalized spectral and essential spectral radius of the Hadamard geometric mean of bounded sets of positive kernel operators. *Linear Multilinear Algebra* **67**(11), 2159–2172 (2019)
- [32] Peperko, A.: Inequalities for the spectral radius and essential spectral radius of positive operators on Banach sequence spaces. *Positivity* **25**(4), 1659–1675 (2021)
- [33] Schep, A.R.: Bounds on the spectral radius of Hadamard products of positive operators on l_p -spaces, *Electronic J. Linear. Algebra* **22**, 443–447 (2011)
- [34] Schep, A.R.: Corrigendum for Bounds on the spectral radius of Hadamard products of positive operators on l_p -spaces, *preprint at Research gate: Corrigendum-Hadamard* (2011)
- [35] Shen, S.-Q., Huang, T.-Z.: Several inequalities for the largest singular value and spectral radius of matrices. *Mathematical Inequalities and Applications* **10**, 713–722 (2007)
- [36] Shih, M.-H., Wu, J.-W., Pang, C.-T.: Asymptotic stability and generalized Gelfand spectral radius formula. *Linear Alg. Appl.* **252**, 61–70 (1997)
- [37] Shulman, V.S., Turovskii, Yu.V.: Joint spectral radius, operator semigroups and a problem of W.Wojtyński, *J. Funct. Anal.* **177**, 383–441 (2000)
- [38] Shulman, V.S., Turovskii, Y.V.: Topological radicals and joint spectral radius. *Funct. Anal. Appl.* **46**(4), 287–304 (2012)
- [39] Schwenk, A.J.: Tight bounds on the spectral radius of asymmetric nonnegative matrices. *Linear Algebra Appl.* **75**, 257–265 (1986)
- [40] Wirth, F.: The generalized spectral radius and extremal norms. *Linear Algebra Appl.* **342**, 17–40 (2002)
- [41] Zaanen, A.C.: *Riesz Spaces II*. North Holland, Amsterdam (1983)

Katarina Bogdanović
Faculty of Mathematics
University of Belgrade
Studentski trg 16
Belgrade SRB-11000
Serbia
e-mail: katarinabgd77@gmail.com

Aljoša Peperko
Faculty of Mechanical Engineering
University of Ljubljana
Aškerčeva 6
Ljubljana SI-1000
Slovenia

and

Institute of Mathematics, Physics and Mechanics
Jadranska 19
Ljubljana SI-1000
Slovenia
e-mail: aljosa.peperko@fs.uni-lj.si

Received: July 1, 2025.

Accepted: January 9, 2026.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.