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THE PALETTE INDEX OF SOME CARTESIAN PRODUCTS OF GRAPHS

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Abstract

The palette of a vertex v in a graph G is the set of colors assigned to the edges incident to v. The palette index of G is the minimum number of distinct palettes among the vertices, taken over all proper edge colorings of G. This paper presents results on the palette index of the Cartesian product $G \Box H$, where one of the factor graphs is a path or a cycle. Additionally, it provides exact results and bounds on the palette index of the Cartesian product of two graphs, where one factor graph is isomorphic to a regular or class 1 nearly regular graph.

Keywords: palette index, Cartesian product, regular graph.

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1. Introduction

Let $G = (V, E)$ be a simple connected graph. An *edge-coloring* of G is a map that assigns colors to the edges of G. An edge coloring is *proper* if two incident edges obtain different colors. A k-*edge-coloring* of G is a proper edge-coloring with colors from the set $\{1, \ldots, k\}$. The minimum number of colors required in a proper edge-coloring of a graph G is called the *chromatic index* of G and denoted by $\chi'(G)$.

It is well-known that the chromatic index of a graph G is equal either to Δ or $\Delta + 1$, where Δ denotes its maximum degree; we then say that G is of *class* 1 or *class* 2, respectively.

The *palette* of a vertex $v \in V(G)$ with respect to a proper edge-coloring f of G is the set $P_f(v) = \{f(e) : e \in E(G) \text{ and } e \text{ is incident to } v\}$.

If f is a proper edge-coloring of G and $X \subseteq V(G)$, then $p_f(X)$ is the number of distinct palettes of the vertices of X with respect to f. The *palette index* of a graph G, denoted by $\check{s}(G)$, is the minimal value of $p_f(V(G))$ taken over all proper edge-colorings of G.

The palette index was introduced by Horňák *et al.* in [9], where initial results on the palette index of cubic and complete graphs were provided. Subsequent studies have investigated the palette index for other regular graphs [4, 11]. Further classes of graphs explored in relation to this invariant include trees [2,4,11], complete bipartite graphs [8], multigraphs [1] and Cartesian products. Specifically, [13] presents partial results on the palette index of the Cartesian product of a path and a cycle, while [5] establishes the palette index of the Cartesian product of two paths.

The palette index has also been analyzed in relation to the maximum and minimum degree of a graph [5, 11]. Additionally, it has found applications in modeling the self-assembly of DNA structures with branched junction molecules that possess flexible arms [4].

In this paper, we build upon previous research on the palette index of the Cartesian product of two graphs, focusing on cases where one factor graph is a path or cycle. Additionally, we extend the study to families of Cartesian products where one of the factors is a regular or class 1 "nearly regular" graph.

The next section provides necessary definitions and preliminary results used throughout the paper. Section 3 introduces the class of nearly regular graphs and presents results on the palette index of Cartesian products where one factor is either regular or a class 1 nearly regular graph. Notably, this section establishes that the palette index of a Cartesian product with a class 1 nearly regular graph as one factor is always 2.

Section 4 continues the exploration of the palette index for Cartesian products involving a path or cycle as one factor. In particular, it includes a construction that produces an edge coloring with three palettes for the Cartesian product of two odd cycles.

Finally, Section 5 concludes the paper by applying the results from earlier sections to determine the palette index of Cartesian products where one factor graph is either a cycle or a path and the other is a regular graph.

2. Preliminaries

The *Cartesian product* of graphs G and H is the graph $G \square H$ with vertex set $V(G) \times V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G \square H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, or $x_2 y_2 \in E(H)$ and $x_1 = y_1$. The Cartesian product is clearly commutative.

Let $[n]$ and $[n]_0$ denote the sets $\{1, 2, \ldots, n\}$ and $\{0, 1, \ldots, n-1\}$, respectively.

We will assume in the sequel that $V(P_n) = V(C_n) := [n]_0$ and $V(P_n \Box C_m) = V(C_n \Box C_m) := [n]_0 \times [m]_0$.

Let G be a graph. We define G as *nontrivial* if it contains at least one edge. If $X \subset E(G)$, then the spanning subgraph of G with the edge set $E(G) \setminus X$ is denoted by $G - X$.

A *matching* in a graph G is a subset $M \subseteq E(G)$ such that no two edges in M share a common vertex. Naturally, if f is a proper edge-coloring of G, the set of edges assigned a specific color i under f constitutes a matching in G. A matching M is called *perfect* if every vertex in $V(G)$ is incident to exactly one edge from M.

We begin with the following obvious observation.

Observation 2.1. *If G is a graph such that the degree of every vertex of $V(G)$ is from the set $\{d_1, d_2, \ldots, d_k\}$, then $\check{s}(G) \geq k$.*

It is also not difficult to confirm the following result.

Proposition 2.2. *If M is a perfect matching of a nontrivial graph G, then $\check{s}(G) \leq \check{s}(G - M)$.*

Proof. Let $g : E(G - M) \to C$ be an edge coloring of $G - M$ with $\check{s}(G - M)$ palettes and let $c \notin C$. It is easy to construct the edge coloring h of G, where for every $e \in M$ we set $h(e) := c$, while for every $e' \in E(G) \setminus M$ we set $h(e') := g(e')$. Since for every $v \in V(G)$ we have $P_h(v) = P_g(v) \cup \{c\}$, it follows that $\check{s}(G) \leq \check{s}(G - M)$. ■

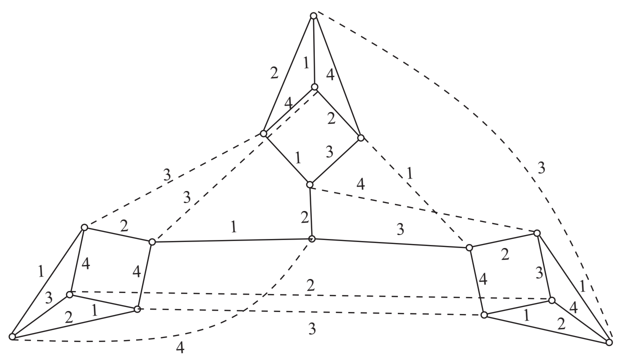

Figure 1. An edge coloring of a 4-regular graph G with a perfect matching M.

With respect to Proposition 2.2, it is worth noting that does not necessarily hold that $\check{s}(G) \geq \check{s}(G - M)$, even if G is a regular graph. Consider, for instance, an edge coloring of the 4-regular graph G shown in Figure 1, where a perfect matching M is indicated by dashed lines. In this case, we observe that $\check{s}(G) = 1$, while $\check{s}(G-M) \neq 1$. This discrepancy arises because $G-M$ is a cubic graph without a perfect matching, and it contains a vertex v such that every edge incident to v is a bridge of $G-M$. Consequently, $\check{s}(G-M) = 4$ (see [11, Proposition 3.3]).

The following two results are shown in [8].

Proposition 2.3. *Let G be an r-regular graph. Then $\chi'(G) = r$ if and only $\check{s}(G) = 1$.*

Proposition 2.4. *If G is a regular graph, then $\check{s}(G) \neq 2$.*

As shown in [10], the Cartesian product of two graphs is class 1 if at least one of the factors is class 1. This result is more formally stated in the next theorem. (Since the proof of the theorem is based on a construction that we will need in the sequel, we stated it explicitly although an analogous approach has been already used in [10].)

Theorem 2.5. *Let G and H be graphs. If G is class 1 nontrivial graph, then $G \Box H$ is class 1.*

Proof. Note first that $\Delta(G \Box H) = \Delta(G) + \Delta(H)$.

Let $g : E(G) \to [\Delta(G)]$ be an edge coloring of G and $h : E(H) \to [\Delta(H)]$ (respectively, $h : E(H) \to [\Delta(H) + 1]$) an edge coloring of H if H is class 1 (respectively, class 2).

We construct an edge coloring f with $\Delta(G) + \Delta(H)$ color as follows.

For every $x_2y_2 \in E(H)$ and every $z_1 \in V(G)$ we set $f((z_1, x_2)(z_1, y_2)) := h(x_2y_2)$. If H is class 1, then for every $x_1y_1 \in E(G)$ and every $z_2 \in V(H)$ we set $f((x_1, z_2)(y_1, z_2)) := g(x_1y_1) + \Delta(H)$. Note that f is clearly a proper edge coloring of $G \Box H$ with $\Delta(G) + \Delta(H)$ colors.

If H is class 2, then we obtain f by choosing first an arbitrary color $c \in [\Delta(G)]$. Then for every $x_1y_1 \in E(G)$ and every $z_2 \in V(H)$ we set

$f((x_1, z_2)(y_1, z_2)) := g(x_1y_1) + \Delta(H)$, if $g(x_1y_1) \neq c$,

$f((x_1, z_2)(y_1, z_2)) := c'$, if $g(x_1y_1) = c$, where $c' \in [\Delta(H) + 1] \setminus P_h(z_2)$.

Note that c' always exists since $|P_h(z_2)| \leq \Delta(H)$.

We can see that f is a proper edge coloring of $G \Box H$ with $\Delta(G) + \Delta(H)$ colors. It follows that $G \Box H$ is class 1. ∎

It will be needed in the sequel that, if in the proof of Theorem 2.5 we construct an edge coloring f by choosing $c = \Delta(G)$, then $f : E(G \Box H) \to [\Delta(G) + \Delta(H)]$ is obtained.

3. Palette Index of Cartesian Products of Regular and Class 1 Nearly Regular Graphs

For graphs G and H, it is shown in [14] that the palette index of $G\Box H$ is bounded above by the product of the palette indices of both factor graphs.

Proposition 3.1. *If G and H are graphs, then $\check{s}(G\Box H) \leq \check{s}(G)\check{s}(H)$.*

Considering Cartesian products of regular graphs, the following observation from [4] is noteworthy.

Proposition 3.2. *If G is an r-regular graph then $\check{s}(G) \leq r+1$.*

We will show in the sequel that for the Cartesian products of regular and some related graphs the above upper bounds can be significantly improved.

Notice that the Cartesian product of two regular graphs is clearly a regular graph. Thus, Theorem 2.5 and Proposition 2.3 yield the following corollary.

Corollary 3.3. *Let G and H be regular graphs. If G is class 1 nontrivial graph, then $\check{s}(G\Box H) = 1$.*

Let G' be a connected r-regular class 1 nontrivial graph and G a spanning subgraph of G'. We say that G is a *class* 1 *nearly regular graph* (*derived from* G') or shortly NRG if G' admits a perfect matching M such that $G' - M$ is class 1 and $G = G' - X$, where $\emptyset \neq X \subset M$.

Let $g : E(G) \to [r]$ be an edge coloring of a graph G and let $C_1^g, \ldots, C_r^g$ be the corresponding color classes. Note that if G is NRG derived from an r-regular graph G', then for every $j \in [r]$ there exists an r-edge coloring g of G', such that $G = G' - X$, where $X \subset C_j^g$.

Theorem 3.4. *Let H be a connected regular graph. If G is* NRG, then $\check{s}(G\Box H) = 2$.

Proof. Since G is not a regular graph, we have $\check{s}(G\Box H) \geq 2$. We will construct a proper edge coloring f of $G\Box H$ with two distinct palettes.

Suppose that H is a r'-regular graph, while G is derived from a class 1 r-regular graph G'. It follows that there exists an edge coloring $g : E(G') \to [r]$ of G' such that $G = G' - X$, where $X \subset C_r^g$.

Remind that by Corollary 3.3, it holds that $G'\Box H$ is class 1.

Let $h : E(H) \to [r']$ (respectively, $h : E(H) \to [r'+1]$) be an edge coloring of H if H is class 1 (respectively, H is class 2). Since G' is class 1, we can construct an $(r+r')$-edge coloring f of $G'\Box H$ as we shown in the proof of Theorem 2.5. That is to say, if H is class 2, then for every $x_1y_1 \in E(G)$ and every $z_2 \in E(H)$ we set

$f((x_1, z_2)(y_1, z_2)) := g(x_1y_1) + r' + 1$, if $g(x_1y_1) \neq r$,

$$f((x_1,z_2)(y_1,z_2)) := c', \text{ where } c' \in [r'+1] \setminus P_h(z_2), \text{ if } g(x_1y_1) = r.$$

It follows that the palette of every vertex of $G'\square H$ with respect to f equals $[r+r']$. Moreover, f restricted to $G\square H$ admits two palettes

$[r+r']$, for every vertex of degree $r+r'$; and

$[r+r'-1]$, for every vertex of degree $r+r'-1$, i.e., a vertex incident to an edge of X in $G'\square H$.

Since we showed that for every connected regular graph H and class 1 nearly regular graph G we can always found an edge coloring of $G\square H$ with two palettes, it follows that $\check{s}(G\square H) = 2$. ■

If a connected regular graph G is class 1, then notice that $G - e$ is NRG for every $e \in E(G)$. This observation provides the following corollary to Theorem 3.4.

Corollary 3.5. *Let H and G be connected regular graphs. If G is a class* 1 *nontrivial graph and $e \in E(G)$, then $\check{s}((G-e)\square H) = 2$.*

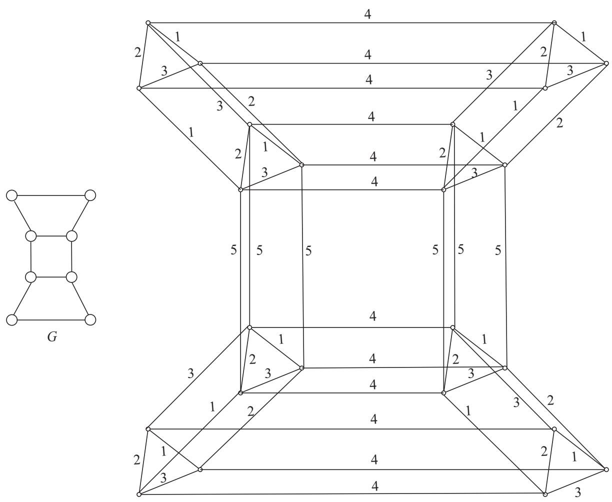

Figure 2. An edge coloring of a subgraph of $G\square C_3$ with 2 palettes, where G is an NRG derived from Q_3.

A little more involved application of Theorem 3.4 considers the well known class of hypercube graphs known as r-cubes. Remind that the vertex set of the *r-cube* Q_r consists of all r-tuples $b_1 \cdots b_r$, $b_i \in \{0,1\}$. Two vertices of Q_r are adjacent if corresponding r-tuples differ in precisely one coordinate. Note that $Q_1 = K_2$, while for $r \geq 2$ we have $Q_r = Q_{r-1}\square K_2$. It is not difficult to see that Q_r is class 1.

A subgraph H of a graph G is *isometric* if $d_H(u,v) = d_G(u,v)$ for any pair of vertices u and v from H. Isometric subgraphs of hypercubes are called *partial cubes*.

Let $\alpha : V(G) \to V(Q_r)$ be an isometric embedding of G into the r-cube, i.e., for every $u, v \in V(G)$ we have $d_G(u,v) = d_{Q_r}(\alpha(u), \alpha(v))$. We will denote the i-th coordinate of α with α_i, i.e., $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$.

By definition, end-vertices of an arbitrary edge of Q_r differ exactly in coordinate i for some $i \in [r]$. Let G be a partial cube with an isometric embedding $\alpha : V(G) \to V(Q_r)$. The set of all edges $uv \in E(G)$ satisfying the condition $\alpha_i(u) \neq \alpha_i(v)$ is denoted by E_i, more formally: $E_i = \{uv : uv \in E(G), \alpha_i(u) \neq \alpha_i(v)\}$. (These sets are known as classes of the equivalence relation Θ, see [7] for the details.)

Clearly, the sets $E_1, E_2, \ldots, E_r$ partition the set of edges of G. Moreover, the function $f : E(G) \to [r]$, where $f(e) = i$ for every $e \in E_i$, is a proper edge coloring of G. Thus, it is not difficult to see that the following result holds.

Proposition 3.6. *Let G be a partial cube with an isometric embedding $\alpha : V(G) \to V(Q_r)$, $r \geq 2$, and H be a regular graph. If $X \subset E_i$, $i \in [r]$, where $E_i = \{uv : uv \in E(G), \alpha_i(u) \neq \alpha_i(v)\}$, then*

$$\check{s}((G - X) \Box H) = 2.$$

As an example to Proposition 3.6 consider an edge coloring of a subgraph of $Q_3 \Box C_3$ isomorphic to $G \Box C_3$, where G is an NRG derived from Q_3. The edge coloring f of $G \Box C_3$ is constructed following the argument of the proof of Theorem 3.4, where $h : E(C_3) \to [3]$ and $g : E(Q_3) \to \{4, 5, 6\}$.

4. Palette Index of Cartesian Products with a Path and Cycle

To establish the palette index of a path, note that P_n admits a perfect matching if and only if n is even.

Proposition 4.1. *Let $n \geq 3$. Then*

$$\check{s}(P_n) = \begin{cases} 2, & n \text{ even}, \\ 3, & n \text{ odd}. \end{cases}$$

The palette index of a cycle follows from Propositions 2.3 and 3.2.

Proposition 4.2. *Let $n \geq 3$. Then*

$$\check{s}(C_n) = \begin{cases} 1, & n \text{ even}, \\ 3, & n \text{ odd}. \end{cases}$$

The palette index of the Cartesian product of two paths is presented as follows (see [5]).

Theorem 4.3. *Let $s, t \geq 2$. Then*

$$\check{s}(P_s \square P_t) = \begin{cases} 1, & s = t = 2, \\ 2, & \min(s,t) = 2,\ \max(s,t) \geq 3, \\ 3, & s,t \geq 3 \text{ and } s \cdot t \text{ is even}, \\ 5, & s,t \geq 3 \text{ and } s \cdot t \text{ is odd}. \end{cases}$$

The palette index of the Cartesian product of a path and cycle is studied in [13] where the following partial result is presented.

Proposition 4.4. *Let $s, t \geq 3$.*

(i) *If s and t are both odd, then $\check{s}(C_s \square P_t) = 4$.*

(ii) *If s is even, then $\check{s}(C_s \square P_t) = 2$.*

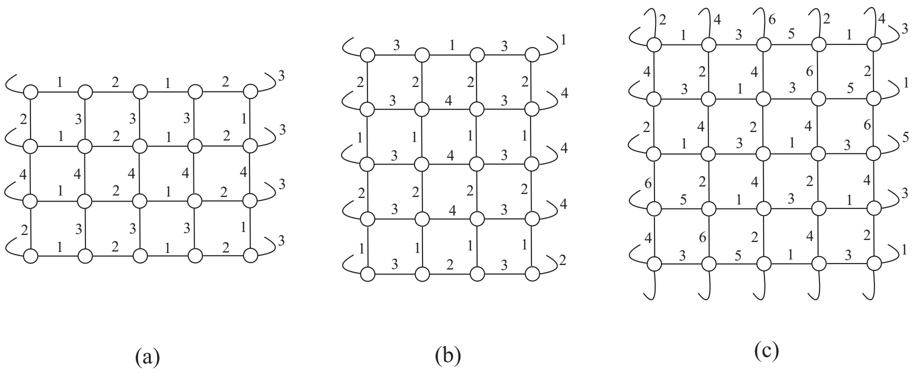

Figure 3. Edge colorings of: (a) $C_5 \square P_4$ with two palettes, (b) $C_4 \square P_5$ with two palettes (c) $C_5 \square C_5$ with three palettes.

Theorem 4.5. *Let $s, t \geq 3$. Then*

$$\check{s}(C_s \square P_t) = \begin{cases} 4, & s \text{ and } t \text{ are both odd}, \\ 2, & \text{otherwise}. \end{cases}$$

Proof. Clearly, $\check{s}(C_s \square P_t) \geq 2$ for every s and t. With respect to Proposition 4.4, we have to confirm the result for every even t, i.e., to show that for every even integer t and every s there exists an edge coloring of $C_s \square P_t$ with two palettes. Since the existence of a proper construction clearly follows from Corollary 3.5 (see an example in the left-hand side of Figure 3), the proof is completed. ■

Notice also an example of an edge coloring of $C_4 \square P_5$ with two palettes in Figure 3(b) which can be generalized as we will show in Section 5.

In the rest of this section we consider value of the palette index of Cartesian products of two cycles.

Let G be a graph. An *even cycle decomposition* of G of size k is a partition $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \ldots, \mathcal{E}_{k-1}\}$ of the edge-set of G such that the edges of $\mathcal{E}_i$ compose disjoint even cycles of G. The studies presented in this section were inspired by the results of Bonvicini and Mazzuoccolo [3], who showed that if a 4-regular graph G admits palette index 3, then G has an even cycle decomposition of size 3 or an even 2-factor.

Let $s \geq t \geq 3$ be integers and $j \in [s]_0$, $k \in [t]_0$. Let us define two types of vertical edges of $C_s \square C_t$:

$v^+_{j,k} := (j,k)(j,(k+1) \bmod t)$ (an "ascending" vertical edge),

$v^-_{j,k} := (j,k)(j,(k-1) \bmod t)$ (a "descending" vertical edge);

and a horizontal edge

$h_{j,k} := (j,k)((j+1) \bmod s, k)$.

The vertex (j,k) is called the *initial vertex* of $v^+_{j,k}$, $v^-_{j,k}$ and $h_{j,k}$. The other end-vertex (i.e., not initial) of a vertical or horizontal edge is called the *terminal vertex.*

Let $\ell := \frac{s-t}{2} \bmod t$ and $h := \left\lfloor \frac{s-t}{2t} \right\rfloor$. We will construct a partition of the edge set of $C_s \square C_t$ based on the following sets

$$Z^{s,t}_i = Z^{s,t}_{i,1} \cup Z^{s,t}_{i,2} \cup Z^{s,t}_{i,3},\ i \in [t]_0$$

such that

$$Z^{s,t}_{i,1} = \left\{ v^+_{j,i+j}, h_{j,i+j+1} \,|\, j \in [\ell]_0 \right\},$$

$$Z^{s,t}_{i,2} = \left\{ v^-_{j+\ell,i-j+\ell}, h_{j+\ell,i-j+\ell-1} \,|\, j \in [\ell]_0 \right\},$$

$$Z^{s,t}_{i,3} = \left\{ v^-_{j+2\ell,i-j}, h_{j+2\ell,i-j-1} \,|\, j \in [t(2h+1)]_0 \right\},$$

where additions and subtraction in the second coordinate are performed modulo t.

Consider for example $Z^{13,5}_2$ depicted in Figure 4, where the edges of this set are drawn with dashed lines. Note that the initial vertex with the smallest first coordinate of $Z^{13,5}_{2,1}$ is $(0,2)$, while its counterparts in $Z^{13,5}_{2,2}$ and $Z^{13,5}_{2,3}$ are $(4,1)$ and $(8,2)$, respectively.

Proposition 4.6. *If $s \geq t \geq 3$ are odd integers, then $\{Z^{s,t}_0, Z^{s,t}_1, \ldots, Z^{s,t}_{t-1}\}$ partition the set of edges of $C_s \square C_t$.*

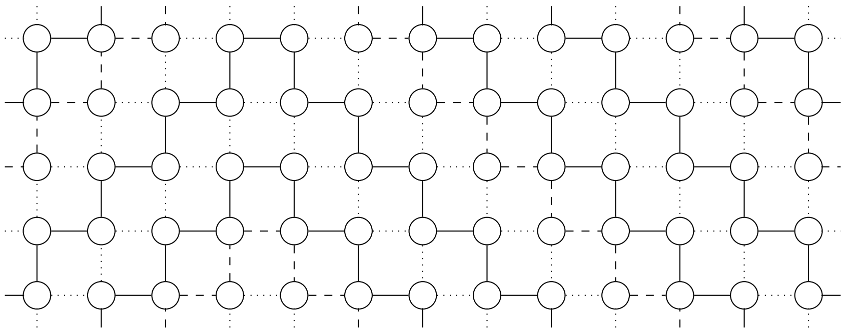

Figure 4. An even cycle decomposition of $C_{13}\Box C_5$.

Proof. Note that the first coordinates of vertices from the set $Z_i^{s,t}$ are pairwise distinct.

We first show that for every $i \neq k$ we have $Z_i^{s,t} \cap Z_k^{s,t} = \emptyset$. To confirm this, notice that for every j and every i,k, $i \neq k$, we have $v^+_{j,i+j} \neq v^+_{j,k+j}$, $v^-_{j+\ell,i-j+\ell} \neq v^-_{j+\ell,k-j+\ell}$, $v^-_{j+2\ell,i-j} \neq v^-_{j+2\ell,k-j}$, $h_{j,i+j+1} \neq h_{j,k+j+1}$, $h_{j+\ell,i-j+\ell-1} \neq h_{j+\ell,k-j+\ell-1}$, $h_{j+2\ell,i-j-1} \neq h_{j+2\ell,k-j-1}$, i.e., the initial vertices of $Z_i^{s,t}$ and $Z_k^{s,t}$ do not coincide. It follows that $Z_i^{s,t} \cap Z_k^{s,t} = \emptyset$.

Since $|Z_i^{s,t}| = 4\ell + 2t(2h+1)$, $\ell = \frac{s-t}{2} \bmod t$ and $h = \left\lfloor \frac{s-t}{2t} \right\rfloor$, we obtain $|Z_i^{s,t}| = 2s$. It follows that

$$\left| \bigcup_{i=0}^{t-1} Z_i^{s,t} \right| = \sum_{i=0}^{t-1} \left| Z_i^{s,t} \right| = 2st = |E(C_s \Box C_t)|.$$

This assertion completes the proof. ■

Proposition 4.7. *Let $s \geq t \geq 3$ be odd integers. If $\mathcal{E}_j = \bigcup_{i \in [t]_0, i \equiv j \pmod 3} Z_i^{s,t}$, then $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ is an even cycle decomposition of $C_s \Box C_t$.*

Proof. Note first that in the set $Z_i^{s,t}$ the following pairs of edges are incident:

- for every $j \in [\ell]_0$: $v^+_{j,i+j}$ and $h_{j,i+j+1}$ (edges in $Z_{i,1}^{s,t}$); $v^-_{j+\ell,i-j+\ell}$ and $h_{j+\ell,i-j+\ell-1}$ (edges in $Z_{i,2}^{s,t}$);
- for every $j \in [t(2h+1)]_0$: $v^-_{j+2\ell,i-j}$ and $h_{j+2\ell,i-j-1}$ (edges in $Z_{i,3}^{s,t}$);
- for every $j \in [\ell-1]_0$: $h_{j,i+j+1}$ and $v^+_{j+1,i+j+1}$ (edges in $Z_{i,1}^{s,t}$); $h_{j+\ell,i-j+\ell-1}$ and $v^-_{j+\ell+1,i-j+\ell+1l}$ (edges in $Z_{i,2}^{s,t}$);
- for every $j \in [t(2h+1)]_0$: $h_{j+2\ell,i-j-1}$ and $v^-_{j+2\ell+1,i-j+1}$ (edges in $Z_{i,3}^{s,t}$).

Moreover, $h_{\ell-1,i+\ell}$ is incident to $v^-_{\ell,i+\ell}$, $h_{2\ell-1,i}$ is incident to $v^-_{2\ell,i}$, and $h_{s-1,i}$ is incident to $v^-_{0,i}$.

It follows that edges of $Z_i^{s,t}$ compose a circle in $C_s \square C_t$. As we can see in the proof of Proposition 4.6, this cycle is of length $2s$. Furthermore, if $i \equiv k \equiv j \pmod 3$ and $i \neq k$, then every initial vertex of an edge of $Z_i^{s,t}$ and every initial vertex of an edge of $Z_k^{s,t}$ are at distance at least 2. Hence, the edges of $\mathcal{E}_j$ compose disjoint even cycles of $C_s \square C_t$. From Proposition 4.6 now it follows that $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ is an even cycle decomposition of $C_s \square C_t$. ∎

An even cycle decomposition of $C_{13} \square C_5$ is depicted in Figure 4.

Theorem 4.8. *Let $s \geq t \geq 3$. Then*

$$\check{s}(C_s \square C_t) = \begin{cases} 1, & s \text{ or } t \text{ is even}, \\ 3, & s \text{ and } t \text{ are both odd}. \end{cases}$$

Proof. For s or t even, the theorem follows from Corollary 3.3 since a cycle of even length is clearly class 1.

If s and t are both odd, then $C_s \square C_t$ is class 2 and by Proposition 2.4 we have $\check{s}(C_s \square C_t) \geq 3$. As shown in [3], the existence of an even cycle decomposition of size 3 in $C_n \square C_m$ implies that $C_n \square C_m$ admits the palette index 3.

It is shown in Proposition 4.7 that $\mathcal{E} = \{\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2\}$ is an even cycle decomposition of $C_s \square C_t$ of size 3, where $\mathcal{E}_j = \bigcup_{i \in [t]_0, i \equiv j \pmod 3} Z_i^{s,t}$. We now construct the edge coloring $c : E(C_s \square C_t) \to [6]$ as follows

$$c(e) = \begin{cases} 2j+1, & e \text{ is a horizontal edge in } \mathcal{E}_j, \\ 2j+2, & e \text{ is a vertical edge in } \mathcal{E}_j. \end{cases}$$

By the definition of the set $Z_i^{s,t}$, every terminal vertex of an ascending vertical edge of $Z_i^{s,t}$ equals the initial vertex of an ascending vertical edge of $Z_{i+1}^{s,t}$, while every initial vertex of a descending vertical edge of $Z_i^{s,t}$ corresponds to the terminal vertex of a descending vertical edge of $Z_{i+1}^{s,t}$ (addition modulo 3).

It follows that the palette of a vertex of $C_s \square C_t$ with respect to c is either $\{1,2,3,4\}$, $\{1,2,5,6\}$ or $\{3,4,5,6\}$. This assertion completes the proof. ∎

An example of a proper edge coloring with three palettes of $C_5 \square C_5$ is depicted in Figure 3.

5. Paths, Cycles and Regular Graphs

In this section, we show that general upper bounds on the palette index of a Cartesian product can be significantly improved when one factor graph is a cycle or path and the other is a regular graph.

Theorem 5.1. *Let G be a nontrivial regular graph. If $s \geq 3$, then*

(i) $\check{s}(C_s \square G) \leq \check{s}(G) + 2$.

Moreover,

(ii) *if s is even or G is class 1, then $\check{s}(C_s \square G) = 1$,*

(iii) *if G is a class 2 cubic graph with a perfect matching and s is odd, then $\check{s}(C_s \square G) \in \{1, 3\}$.*

Proof. If s is even or G is class 1, then one of the factor graphs is class 1 and (ii) follows from Corollary 3.3.

To prove (i), suppose then that s is odd and G a class 2 r-regular graph. Let $g : E(G) \to [r+1]$ be a proper edge coloring of G and let h be a proper edge coloring of G with $\check{s}(G)$ palettes such that $h(v) \notin \{r+2, r+3\}$ for every $v \in V(G)$. We will construct a proper edge coloring f of $C_s \square G$ for $u, v \in V(G)$ and $i, j \in [s]_0$ as follows

$$f((u,i)(v,j)) = \begin{cases} g(uv), & uv \in E(G) \text{ and } i = j \neq s-1, \\ h(uv), & uv \in E(G) \text{ and } i = j = s-1, \\ c, & u = v,\ i \in [s-2]_0 \text{ is even and } j = i+1, \\ & \text{where } c \in [r+1] \setminus P_g(v), \\ r+2, & u = v,\ i \in [s-2] \text{ is odd and } j = i+1, \\ r+3, & u = v,\ i = s-1 \text{ and } j = 0. \end{cases}$$

Since we can see that

– for every $i \in [s-2]$ and every $v \in V(G)$ we have $P_f((v,i)) = [r+2]$,

– for every $v \in V(G)$ we have $P_f((v,0)) = [r+1] \cup \{r+3\}$,

– for every $v \in V(G)$ we have $P_f((v,s-1)) = P_h(v) \cup \{r+2, r+3\}$,

case (i) is settled.

To prove (iii), let G denote a class 2 cubic graph with a perfect matching M and F_M the corresponding 1-factor of G, i.e., a 1-regular spanning subgraph of G. Note that $E(C_s \square F_M)$ is a perfect matching of $C_s \square G$ and $C_s \square G - E(C_s \square F_M) = C_s \square (G - M)$.

Since G is class 2, G is not bipartite. Moreover, since $G - M$ is 2-regular, its connected component are cycles with at least one of them having odd length. Thus, $C_s \square (G - M)$ is a graph whose connected components are Cartesian products of two cycles. Remind from Theorem 4.8 that the palette index of the Cartesian product of two cycles is either 1 (at least one of them is of even length) or 3 (both of them are of odd length). Therefore, $\check{s}(C_s \square (G - M)) = 3$. By Proposition 2.2, we have $\check{s}(C_s \square G) \leq \check{s}(C_s \square (G - M)) = 3$. Finally, Proposition 2.4 completes the proof. ■

As an example of edge colorings provided by Theorem 5.1(ii), observe Figure 5, where an edge coloring of the Cartesian product of the Petersen graph and the triangle with three palettes is partially depicted. (For clarity, the labeling of the "inner" product of C_3 and C_5 with colors 1, 3, and 5 is omitted.)

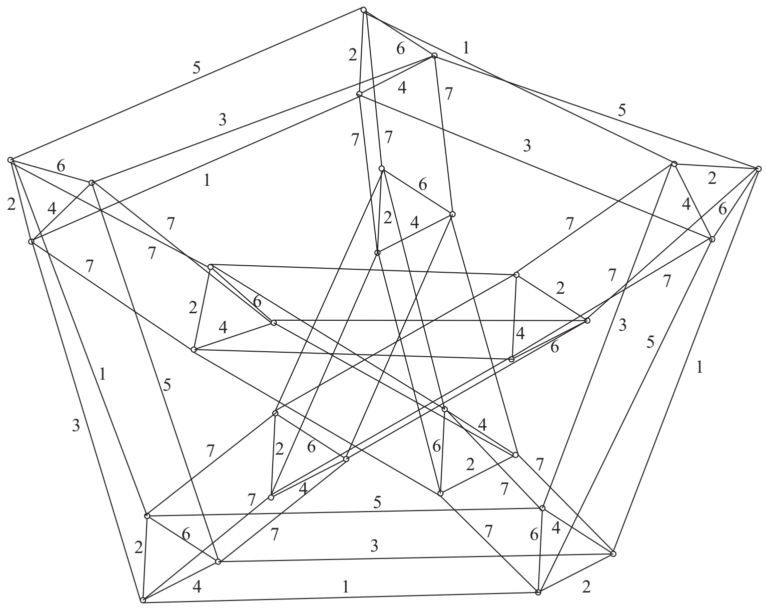

Figure 5. A (partial) edge coloring of the Cartesian product of the Petersen graph and triangle with three palettes.

Theorem 5.2. *Let G be a nontrivial regular graph. If $s \geq 3$, then*

(i) $\check{s}(P_s \square G) \leq \check{s}(G) + 2$.

Moreover,

(ii) *if s even or G is class* 1, *then* $\check{s}(P_s \square G) = 2$,

(iii) *if G is a class 2 cubic graph with a perfect matching and s is odd, then* $\check{s}(P_s \square G) \in \{2, 3, 4\}$.

Proof. Clearly, $\check{s}(P_s \square G) \geq 2$.

If s is even, then P_s is an NRG and (ii) follows from Theorem 3.4. If G is class 1 and s is odd, we construct a suitable edge coloring of $P_s \square G$ in the sequel.

Let g, g' and g'' be proper edge colorings of G with exactly one palette, such that for every $v \in V(G)$ we have $1, 2 \notin P_g(v))$, $P_{g'}(v) = \{2\} \cup (P_g(v) \setminus \{c\})$ and $P_{g''}(v) = \{1\} \cup (P_g(v) \setminus \{c\})$, where c is an arbitrary color of g. We construct a proper edge coloring f of $P_s \square G$ with two distinct palettes as follows

– for every odd $i \in [s-2]$ and every $v \in V(G)$ we set $f((v,i)(v,i-1)) = 1$ and $f((v,i)(v,i+1)) = 2$,

– for every $i \in [s-2]$ and every $uv \in E(G)$ we set $f((u,i)(v,i)) = g(uv)$,

– for every $v \in V(G)$ we set $f((u,0)(v,0)) = g'(uv)$ and $f((u,s-1)(v,s-1)) = g''(uv)$.

For example consider an edge coloring of $P_5 \square C_4$ depicted in Figure 3(b). Note that g, g', and g'' are edge colorings of C_4 with one palette, where for every $v \in V(C_4)$ we have $P_g(v) = \{3,4\}$, $P_{g'}(v) = \{2,3\}$ and $P_{g''}(v) = \{1,3\}$.

Since we can see that for every $i \in [s-2]$ and every $v \in V(G)$ we have $P_f((v,i)) = P_g(v) \cup \{1,2\}$, while for every $v \in V(G)$ we have $P_f((v,0)) = P_f((v,s-1)) = (P_g(v) \setminus \{c\}) \cup \{1,2\}$, case (ii) is settled.

To prove (i), consider again a proper edge coloring f of $C_s \square G$ constructed in the proof of Theorem 5.1, where s is odd and G a class 2 r-regular graph. It is not difficult to see that f restricted to $P_s \square G$ for every $i \in [s-2]$ and every $v \in V(G)$ implies $P_f((v,i)) = [r+2]$, while for every $v \in V(G)$ we have $P_f((v,0)) = [r+1]$ and $P_f((v,s-1)) = P_h((v)) \cup \{r+2\}$. This argument settles the proof of case (i).

To prove (iii), let G be a cubic graph with a perfect matching M, and let F_M be the corresponding 1-factor of G. Analogously to the proof of Theorem 5.1(iii), we notice that $P_s \square (G-M)$ is a graph whose connected components are Cartesian products of a path and a cycle, such that at least one of the factors is induced on an odd number of vertices.

By Theorem 4.5, the palette index of the Cartesian product of a path and a cycle is either 2 (if at least one of the factors has an even number of vertices) or 4 (if both factors have an odd number of vertices). Thus, we obtain that $\check{s}(P_s \square (G-M)) = 4$.

By Proposition 2.2, we have $\check{s}(P_s \square G) \leq \check{s}(P_s \square (G-M)) = 4$. It follows that $\check{s}(P_s \square G) \in \{2,3,4\}$, and the proof is completed. ■

Note that the bound provided by Theorem 5.1(i) improves the bounds from Propositions 3.1 and 3.2. For example, let G be isomorphic to K_7. From [9], we know that $\check{s}(K_7) = 3$, which implies $\check{s}(C_s \square K_7) \leq 5$. In contrast, Propositions 3.1 and 3.2 provide the weaker bound $\check{s}(C_s \square K_7) \leq 9$. However, it remains unknown whether the bounds given by Theorem 5.1(i) and Theorem 5.2(i) are sharp.

Regarding the lower bound on $\check{s}(G \square H)$, it is worth noting that, in general, it does not depend on $\check{s}(G)$ and $\check{s}(H)$. Specifically, if G and H are regular graphs that both contain a perfect matching, then $\check{s}(G \square H) = 1$, even if $\check{s}(G) > 1$ and $\check{s}(H) > 1$ (see [12, Theorem 2.2]).

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