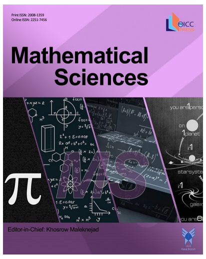

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Fixed cost allocation with a minimum distance to fair allocation in Fuzzy Data Envelopment Analysis

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Original Research

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Abstract:

Introduction: Resource allocation in Data Envelopment Analysis (DEA) has been extensively studied, yet most works focus on redistributing available resources rather than allocating unavoidable fixed costs among decision-making units (DMUs). This study addresses the important problem of fixed cost allocation, aiming to ensure that inefficient DMUs can become efficient while keeping allocations as fair as possible, thereby providing a practical decision-making tool in real-world contexts such as banking and manufacturing. We propose a new linear DEA-based model that allocates fixed costs so as to transform an inefficient DMU into an efficient one, with the objective of minimizing the deviation from fair allocation. The model is then generalized to a fuzzy environment by incorporating triangular fuzzy numbers for inputs, outputs, and costs, and validated using benchmark datasets from Cook and Kress and Wang et al. The results demonstrate that the proposed models can successfully enhance the efficiency of targeted DMUs while producing allocations close to fairness, and the fuzzy extension proves robust in handling imprecise data. The key novelties of this research are (i) introducing a linear efficiency-improving allocation model with minimum distance to fairness, (ii) extending the allocation problem to fuzzy DEA by considering fuzzy costs alongside fuzzy inputs and outputs, and (iii) showing that this integrated framework has not been addressed before in the literature, thereby offering a novel, practical, and equitable approach for fixed cost allocation in DEA.

Keywords: Data Envelopment Analysis; Efficiency; Allocation; Fair allocation; Triangular fuzzy number

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1. Introduction

The data envelopment analysis technique is utilized to evaluate the relative efficiency of a set of decision-making units. This issue was published in 1978 by Edward Rhoods in collaboration with Charnes and Cooper in a paper called CCR. In 1984, Charnes, Cooper and Banker developed the CCR method that led to the BCC model [1]. However, in this model, the returns to scale is variable. In addition to the models presented in the CCR and BCC articles, other basic models such as additive model, multiplicative model were presented in data envelopment analysis.

The data envelopment analysis has been studied in diverse fields such as sensitivity analysis in DEA. Sensitivity analysis of data envelopment analysis models is very important. The first paper on sensitivity analysis in data envelopment analysis was proposed by Charnes et al. in 1985 [2]. Subsequently, many articles have been presented in this regard, but the examined issue in this paper is a modification in the number of indexes .For example, in many of these issues, there are fixed costs that are imposed on decision-makers. The cost imposed on the decision-making units can be considered as a new input, which can maintain or modify the relative efficiency measurement of them. In any case, the allocation of a new cost to units should be fair.

In 1999, Cook and Cross [3] introduced the two axioms of invariance-efficiency and Pareto-minimality and studied the input in the fixed resource allocation problem. The method they presented was based on these two principles and the fairness of resource allocation was considered. Cook and Kress [3] then proposed a method by converting the input conditions to output. After that, the proposed method of Cook and Zhou [4] was generalized from the CCR model to the BCC model. Lin [5] proved that by adding certain constraints by the Cook and Zhou method no optimal solution is obtained and to solve this problem and achieve a possible allocation, he added specific goals to his model. In 2003, Beasley [6] introduced a nonlinear method that maximizes the average performance of all DMUs. The problem with this method was that in some cases it was not possible. Hence, Amir Teymouri and Kordrostami [7] introduced a new method based on DEA as well as fixed resource allocation.

After that, Jahanshahloo et al. [8] used another method using CSW concept and unprincipled efficiency to allocate fixed resources. Jahanshahloo et al. [8] showed that the Beasley method [6], which was impossible by Amirteimoori and Kordrostami [7], is always possible. After that Li et al. [9] expanded the work done by Jahanshahloo et al. [8].

Jahanshahloo et al. [8] and L Li et al. [9] determined a unique allocation program based on common weights and the principles of non-change of efficiency.

Lin and Chen [10] studied a situation where the allocated cost is a complement to the original inputs. After that, a lot of research was done on allocation. Yu et al. [11], Zhu et al. [12] and Li et al. [13] extended the fixed cost allocation problem to network situations by considering the internal two-stage processes, all three methods are implemented under the efficiency-maximization assumption. Li et al ([14, 15, 16]), An et al [17] and Hosseinzadeh Lotfi et al. [18] Looked at the allocation issue from a different perspective. They proposed a new method in which each DMU unit first determines the minimum and maximum amount that it can contribute to the payment of fixed costs to its own and other units' returns. Then a convex combination of minimum and maximum values is considered as the amount of cost allocated to each unit. Among the recent researches in this field, farzipoor saeni et al. [19], Feng and Romas [20], Gupta et al. [21] and Torres et al. [22] can be mentioned.

Since in most real-life problems data may be imprecise thus, the data are considered as fuzzy. The theory of fuzzy sets in comparison with Aristotle's logic, which requires accurate and quantitative data, was first introduced in 1965 by Iranian scientist Lotfi Asgar zadeh [23]. Since the presentation of this theory till now, it has been expanded to a large extent and has been applied in various fields. Using fuzzy logic, it is possible to formulate mathematically most of the imprecise content and variables (see Zadeh [24]) and provide a framework for making decisions in uncertainty conditions. Many studies have been done in this regard. For example, references [25, 26, 27, 28, 29]. . ..

In this paper, the performance status of decision making units after the introduction of a new resource and its allocation between units is examined. But the important point in allocating the new resource is that this allocation should be done in such a way that, firstly, the intended unit achieves maximum efficiency and, secondly, this allocation has a minimum distance from the fair allocation. We then assume that the inputs and outputs as well as the added source for allocation between units are all fuzzy. We now generalize the proposed model for new resource allocation in fuzzy mode. In this paper, we use the models provided by Wang et al. [29] to evaluate the performance of units with fuzzy data. We are now looking for an allocation of the new source that firstly maximizes the efficiency of DMU_o and secondly minimizes the resulting allocation distance relative to the fair allocation. In most of the papers presented so far, the existing inputs have been reallocated between units to achieve the desired conditions. But in this article, a new source has entered the problem. Also in the case of fuzzy data, in addition to the inputs and outputs being considered fuzzy, the new source is also assumed to be fuzzy. The above can be the advantages of this article over the series of articles in this field.

Research Gap and Contribution: Most studies on cost allocation in DEA focus on redistributing existing resources, often addressing either efficiency or fairness in isolation. However, they do not simultaneously ensure that an inefficient DMU becomes efficient while keeping the allocation close to fairness, and they rarely consider fuzzy data environments.

This study fills the gap by proposing a **linear DEA-based model** that converts a selected inefficient DMU into an efficient one with minimum deviation from fair allocation, and by **extending the model to a fuzzy setting** where inputs, outputs, and costs are represented as triangular fuzzy numbers. To our knowledge, this is the first work to integrate efficiency improvement, fairness, and fuzzy cost allocation in a unified framework.

This article is divided into the following sections: In the section 2, we have an overview of the required backgrounds (basic DEA models, DEA models in fuzzy environment, fuzzy DEA models based upon fuzzy arith-

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metic and fair allocation). In section 3, a model is presented that aims to find an allocation of a fixed cost so that in addition to turn an inefficient unit into an efficient one, it has a minimum distance from the fair allocation. The proposed model is then generalized to fuzzy mode. In Section 4, two examples of the application of the models in Section 3 are presented. Finally, in Section 5, we conclude about this paper.

2. background

2.1 Basic models of DEA

For a long time, evaluation of a DMU has been considered as a complex problem especially when it includes multiple inputs and multiple outputs in such a way a set of weights has to be specified to aggregate the outputs and inputs distinctly to form a ratio as the efficiency. To this end, DEA method is suggested, which permits every DMU to choose their best weights whereas requiring the resulted ratio of the aggregated outputs to the aggregated inputs of all DMUs to be less than or equal to 1. CCR model is a linear programming (LP) based method proposed by Charnes et al. [30].

The evaluated entity efficiency of the CCR model can be calculated by a ratio of the weighted output to the weighted input provided that the ratio for every entity is not bigger than 1. The mathematical description is given below.

$$
\begin{aligned}
\text{Maximize} \quad & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}} \\
S.t. \quad & \frac{\sum_{r=1}^{s} u_r y_r}{\sum_{i=1}^{m} v_i x_{ij}} \leq 1 \;,\quad j = 1, ..., n \\
& v_i, u_r \geq \varepsilon \;,\; i = 1, ..., m \;,\; r = 1, ..., s
\end{aligned}
\tag{1}
$$

Where n is the number of decision making units and DMU_o is the decision making unit which is evaluated by model (1). x_{ij} for $i = 1, \ldots, m$ and y_{rj} for $r = 1, \ldots, s$ is (i)th input and (r)th output of DMU_j, respectively for $j = 1, .., n$. All inputs and outputs are non-negative and at least one of the components of each of them is positive.

Using Charnes-Cooper transformations [31], model (1) becomes the following linear model:

$$
\begin{aligned}
\text{Maximize} \quad & \sum_{r=1}^{s} u_r y_{ro} \\
S.t \quad & \sum_{r=1}^{s} u_r y_r - \sum_{i=1}^{m} v_i x_{ij} \leq 0 \;,\; j = 1, ..., n \\
& \sum_{i=1}^{m} v_i x_{io} = 1 \\
& v_i, u_r \geq \varepsilon \;,\; i = 1, ..., m \;,\; r = 1, ..., s
\end{aligned}
\tag{2}
$$

It is obvious that in model (2) DMUo is efficient if and only if $\sum_{r=1}^{s} u_r^* y_{ro} = 1$ else it is inefficient.

2.2 Fuzzy Theory

A fuzzy number is a set of ordered pairs as $\tilde{A}(x) = \{(x, \mu_{\tilde{A}}(x))\}$in which $\mu_{\tilde{A}}(x)$ is called membership function of this fuzzy number [7]. A Fuzzy Number $\tilde{A}$ is called L-R fuzzy number if there exist functions L (for Left), R (for Right), and scalars $\alpha 0, \beta >$, such that

$$
\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m-x}{\alpha}\right) & x \leq m \\ R\left(\frac{x-m}{\beta}\right) & x \geq m \end{cases}
$$

R and L are non-increasing function from R^+ to $[0, 1]$ and this fuzzy number is expressed as (m, α, β). The real number m is a mean value $\tilde{A}$ and α and β are sequentially its Left and Right expanse.

If $L(x) = R(x) = \begin{cases} 1 - x & 0 \leq x \leq 1 \\ 0 & otherwise \end{cases}$ then $\tilde{A}$ is a triangular fuzzy number. Each triangular fuzzy number is expressed as (x^l, x^m, x^u) which imply x^m is the main value $(\mu_{\tilde{A}}(x^m) = 1)$ and x^l is the pessimistic value $(\mu_{\tilde{A}}(x^l) = 0)$ and $(\mu_{\tilde{A}}(x^u) = 0)$ is the optimistic value.

A fuzzy set $\tilde{A} = (x^l, x^m, x^u)$ is a generalized Left Right Fuzzy Number (LRFN) of Dubois and Prade [32] if its membership function satisfies the following:

$$
\mu_{\tilde{A}}(x) = \begin{cases} L\left(\frac{m_1 - x}{m_1 - l}\right) & l \leq x \leq m_1 \\ 1 & m_1 \leq x \leq m_2 \\ R\left(\frac{x - m_2}{u - m_2}\right) & m_2 \leq x \leq u \\ 0 & otherwise \end{cases}
$$

Where L and R are strictly decreasing functions defined on $[0, 1]$ and satisfying the conditions:

$$
\begin{aligned}
L(x) = R(x) = 1, \quad x \leq 0 \\
L(x) = R(x) = 0, \quad x \leq 1
\end{aligned}
$$

For $m_1 = m_2$, we have the classical definition of (LRFN) of Dubois and Prade [33].Trapezoidal Fuzzy Numbers (TrFN) are special cases of Generalized Left Right Fuzzy Numbers(GLRFN) with $L(x) = R(x) = 1 - x$.

2.3 DEA models in fuzzy environment

In customary data envelopment analysis, the values of inputs and outputs with the exact values are determined by experts. However in imprecise environments the experts' premise of precise data is very unreal. So, the experts decide to consider DEA and its models in fuzzy environments [34, 35, 32, 27, 28, 36, 29]. . . . Suppose n DMUs with m inputs and s outputs are evaluated. Consider the vectors related to inputs and outputs of DMU_j for $j = 1, \ldots, n$ are respectively in the form of $\widetilde{X}_j = (\tilde{x}_{1j}, \ldots, \tilde{x}_{mj})$ and $\widetilde{Y}_j = (\tilde{y}_{1j}, \ldots, \tilde{y}_{sj})$. Furthermore suppose $\tilde{x}_{ij}$ and $\tilde{y}_{rj}$ are a triangular fuzzy number as $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ and $\tilde{y}_{rj} = (y^l_{rj}, y^m_{rj}, y^u_{rj})$.

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If we consider model (1) with fuzzy inputs and outputs, we will have:

$$\text{Maximize} \quad \frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \tag{3}$$

$$S.t. \quad \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij}} \leq 1 \ , \ j = 1, ..., n$$

$$v_i, u_r \geq \varepsilon \ , \ i = 1, ..., m \ , \ r = 1, ..., s$$

Where $v_i (i = 1, \ldots, m)$ and $u_r (r = 1, \ldots, s)$ are the weights related with input i and output r, respectively.

So far various attributes have been presented for ranking fuzzy numbers. One of them is suggested by Ramik and Rimanek [37]for ranking the triangular fuzzy numbers. They suggested the Definition as follow:

Definition 2.1 Given two triangular fuzzy numbers $\tilde{A} = (A^l, A^m, A^u)$and $\tilde{B} = (B^l, B^m, B^u)$, $\tilde{A} \leq \tilde{B}$ is described by the following inequalities

$$\tilde{A} \leq \tilde{B} \Leftrightarrow A^l \leq B^l, A^m \leq B^m, A^u \leq B^u \tag{4}$$

Definition 2.2 Addition, subtraction, triangular fuzzy numbers are defined as follows [38].
a) $\tilde{x} + \tilde{y} = (x^l + y^l, x^m + y^m, x^u + y^u)$
b) $\tilde{x} - \tilde{y} = (x^l - y^l, x^m - y^m, x^u - y^u)$
Let $\tilde{x}, \tilde{y}$be two positive triangular fuzzy numbers then multiplication, and division of them are defined as follows:
c) $\tilde{x} \times \tilde{y} \approx (x^l y^l, x^m y^m, x^u y^u)$
d) $\frac{\tilde{x}}{\tilde{y}} \approx (\frac{x^l}{y^u}, \frac{x^{\ m}}{y^m}, \frac{x^u}{y^l})$

2.4 Fuzzy DEA models based upon fuzzy arithmetic

There are several papers in which DEA models are developed based on fuzzy arithmetic logic. In this paper, we use the models presented by Wang et al [29]. In their paper, like the current paper, fuzzy numbers are assumed to be triangular fuzzy numbers. Suppose there are n DMUs to be evaluated, each with m non-negative fuzzy inputs $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ for $i = 1, \ldots, m$; $j = 1, \ldots, n$ and s non-negative fuzzy outputs $\tilde{y}_{ij} = (y^l_{ij}, y^m_{ij}, y^u_{ij})$ for $r = 1, \ldots, s$; $j = 1, \ldots, n$. Crisp input or output data can be seen as a special case of triangular fuzzy input or output. In this case $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ such a way that $x^l_{ij} = x^m_{ij} = x^u_{ij}$ or $\tilde{y}_{ij} = (y^l_{ij}, y^m_{ij}, y^u_{ij})$ with $y^l_{ij} = y^m_{ij} = y^u_{ij}$.

In fuzzy data mode, efficiency of DMU$_o$ can be defined as follows:

$$\tilde{\theta}_o = \frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io}} \tag{5}$$

Where v_i for $i = 1, \ldots, m$ and u_r for $r = 1, \ldots, s$ are sequentially the weights assigned to the inputs and outputs. $\tilde{\theta}_o$ is introduced as a fuzzy efficiency which is a fuzzy number.

The fuzzy efficiency defined in Eq. (5) according to fuzzy arithmetic can be expressed as follows:

$$\begin{aligned} \tilde{\theta}_o &= \frac{\sum_{r=1}^{s} u_r \left(y^l_{ro}, y^m_{ro}, y^u_{ro}\right)}{\sum_{i=1}^{m} v_i \left(x^l_{io}, x^m_{io}, x^u_{io}\right)} \\ &= \frac{\left[\sum_{r=1}^{s} u_r y^l_{ro}, \sum_{r=1}^{s} u_r y^m_{ro}, \sum_{r=1}^{s} u_r y^u_{ro}\right]}{\left[\sum_{i=1}^{m} v_i x^l_{io}, \sum_{i=1}^{m} v_i x^m_{io}, \sum_{i=1}^{m} v_i x^u_{io}\right]} \\ &\simeq \left[\frac{\sum_{r=1}^{s} u_r y^l_{ro}}{\sum_{i=1}^{m} v_i x^u_{io}}, \frac{\sum_{r=1}^{s} u_r y^m_{ro}}{\sum_{i=1}^{m} v_i x^m_{io}}, \frac{\sum_{r=1}^{s} u_r y^u_{ro}}{\sum_{i=1}^{m} v_i x^l_{io}}\right] \end{aligned} \tag{6}$$

Therefore, the following fuzzy DEA model is made to measure the fuzzy efficiency of DMU$_o$:

$$\text{Maximize} \quad \tilde{\theta}_O \simeq [\theta^l_O, \theta^m_O, \theta^u_O] \tag{7}$$

$$= \left[\frac{\sum_{r=1}^{s} u_r y^l_{ro}}{\sum_{i=1}^{m} v_i x^u_{io}}, \frac{\sum_{r=1}^{s} u_r y^m_{ro}}{\sum_{i=1}^{m} v_i x^m_{io}}, \frac{\sum_{r=1}^{s} u_r y^u_{ro}}{\sum_{i=1}^{m} v_i x^l_{io}}\right]$$

$$s.t \quad \tilde{\theta}_j \approx [\theta^l_j, \theta^m_j, \theta^u_j]$$

$$= \left[\frac{\sum_{r=1}^{s} u_r y^l_{rj}}{\sum_{i=1}^{m} v_i x^u_{ij}}, \frac{\sum_{r=1}^{s} u_r y^m_{rj}}{\sum_{i=1}^{m} v_i x^m_{ij}}, \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij}}\right] \leq 1,$$

$$j = 1, \ldots, n$$

$$u_r \ , \ v_i \geq \varepsilon \ , \ r = 1, ..., s \ , \ i = 1, ..., m.$$

DMU$_o$ is the unit under evaluation. In model (7), as long as $\theta^u_j \leq 1$ is satisfied, then $\theta^l_j \leq 1$ and $\theta^m_j \leq 1$ will be automatically satisfied. Model (7) can therefore be simplified as:

$$\text{Maximize} \quad \tilde{\theta}_O \simeq [\theta^l_O, \theta^m_O, \theta^u_O] \tag{8}$$

$$= \left[\frac{\sum_{r=1}^{s} u_r y^l_{ro}}{\sum_{i=1}^{m} v_i x^u_{io}}, \frac{\sum_{r=1}^{s} u_r y^m_{ro}}{\sum_{i=1}^{m} v_i x^m_{io}}, \frac{V \sum_{r=1}^{s} u_r y^u_{ro}}{\sum_{i=1}^{m} v_i x^l_{io}}\right]$$

$$s.t \quad \theta^u_j = \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij}} \leq 1 \qquad j = 1, ..., n$$

$$u_r, v_i \geq \varepsilon, r = 1, ..., s, i = 1, ..., m.$$

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By solving the following three fractional programming models the best possible values of θ^l_j, θ^m_j and θ^u_j can be obtained:

$$\text{Maximize} \quad \theta^l_O = \frac{\sum_{r=1}^{s} u_r y^l_{ro}}{\sum_{i=1}^{m} v_i x^u_{io}} \tag{9}$$
$$s.t \quad \theta^u_j = \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij}} \leq 1, \quad j = 1,...,n$$
$$u_r \ , \ v_i \geq \varepsilon \ , \ r = 1,...,s \ , \ i = 1,...,m.$$

$$\text{Maximize} \quad \theta^m_O = \frac{\sum_{r=1}^{s} u_r y^m_{ro}}{\sum_{i=1}^{m} v_i x^m_{io}} \tag{10}$$
$$s.t \quad \theta^u_j = \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij}} \leq 1 \ , \quad j = 1,...,n$$
$$u_r \ , \ v_i \geq \varepsilon \ , \ r = 1,...,s \ , \ i = 1,...,m.$$

$$\text{Maximize} \quad \theta^u_O = \frac{\sum_{r=1}^{s} u_r y^u_{ro}}{\sum_{i=1}^{m} v_i x^l_{io}} \tag{11}$$
$$s.t \quad \theta^u_j = \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij}} \leq 1 \ , \ j = 1,...,n$$
$$u_r \ , v_i \geq \varepsilon \ , \ r = 1,...,s \ , \ i = 1,...,m.$$

Due to the sameness of the feasible region of models (9), (10) and (11), the inequality $\theta^l_o \leq \theta^m_o \leq \theta^u_o$ is clearly established. These three models can be transformed into three linear programming (LP) models as below:

$$\text{Maximize} \quad \theta^l_O = \sum_{r=1}^{s} u_r y^l_{ro} \tag{12}$$
$$s.t \quad \sum_{r=1}^{s} u_r y^u_{rj} - \sum_{i=1}^{m} v_i x^l_{ij} \leq 0, \ j = 1,...,n$$
$$\sum_{i=1}^{m} v_i x^u_{io} = 1$$
$$u_r \ , v_i \geq \varepsilon \ , \ r = 1,...,s \ , i = 1,...,m.$$

$$\text{Maximize} \quad \theta^m_O = \sum_{r=1}^{s} u_r y^m_{ro} \tag{13}$$
$$s.t \quad \sum_{r=1}^{s} u_r y^u_{rj} - \sum_{i=1}^{m} v_i x^l_{ij} \leq 0 \ , \ j = 1,...,n$$
$$\sum_{i=1}^{m} v_i x^m_{io} = 1$$
$$u_r \ , \ v_i \geq \varepsilon \ , \ r = 1,...,s \ , \ i = 1,...,m.$$

$$\text{Maximize} \quad \theta^u_O = \sum_{r=1}^{s} u_r y^u_{ro} \tag{14}$$
$$s.t \quad \sum_{r=1}^{s} u_r y^u_{rj} - \sum_{i=1}^{m} v_i x^l_{ij} \leq 0 \ , \ j = 1,...,n$$
$$\sum_{i=1}^{m} v_i x^l_{io} = 1$$
$$u_r \ , \ v_i \geq \varepsilon \ , \ r = 1,...,s \ , \ i = 1,...,m.$$

After solving models (12), (13) and (14), it is obvious that the efficiency of DMU_O will be obtained as $\tilde{\theta}_o \approx (\theta^l_o, \theta^m_o, \theta^u_o)$.

2.5 Fair allocation

Suppose n decision making units with m inputs and s outputs are evaluated. Furthermore consider x_{ij} is i'th input of DMU_j, for $i = 1, \dots, m$; $j = 1, \dots, n$. Also consider y_{rj} is r'th output of DMU_j, for $r = 1, \dots, s$; $j = 1, \dots, n$. Now a new source comes into play. Fair allocation of the new source between units is calculated by the following formula [39]:

$$r_j = R \times \frac{\sum_{i=1}^{m} x_{ij}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x_{iq}} \quad j = 1, \dots, n \tag{15}$$

Where R is available amount of the new source and r_j is DMU'_j share from the new source for $j = 1, \dots, n$.

Now Suppose there are n DMU_s to be evaluated, each with m fuzzy inputs s $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ for $i = 1, \dots, m$; $j = 1, \dots, n$ and s fuzzy outputs $\tilde{y}_{rj} = (y^l_{rj}, y^m_{rj}, y^u_{rj})$ for $r = 1, \dots, m$; $j = 1, \dots, n$. By extending Formula (15), we will have:

$$r^l_j = R^l \times \frac{\sum_{i=1}^{m} x^l_{ij}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x^l_{iq}} \quad j = 1, \dots, n \tag{15.1}$$

$$r^m_j = R^m \times \frac{\sum_{i=1}^{m} x^m_{ij}}{\sum_{q=1}^{n} \sum_{i=1}^{m} x^m_{iq}} \quad j = 1, \dots, n \tag{15.2}$$

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$$r_j^u = R^u \times \frac{\sum_{i=1}^{m} x_{ij}^u}{\sum_{q=1}^{n}\sum_{i=1}^{m} x_{iq}^u} \qquad j = 1, \ldots, n \qquad (15.3)$$

Where $\tilde{R} = (R^l, R^m, R^u)$ is available amount of the new fuzzy source.

3. Proposed model

Consider a bank manager with multiple branches. Suppose some branches are rated as efficient and some as inefficient with current inputs and outputs. A common cost is now injected into all branches. The manager of the bank in question intends to divide the fixed cost among the branches in such a way that one of the inefficient branches becomes efficient, in addition to the fact that the allocation made has a minimum distance from the fair allocation.

Jahanshahloo et al. [39] proposed a formula to fair allocation a new resource between DMUs. In the current paper, we allocate the new resource between units in such a way that in addition to the decision making unit reaching its maximum efficiency, this allocation has a minimum distance ($\| \ \|_\infty$) from the fair allocation. Here, the desired allocation and the fair allocation are considered in the form of two vectors, and their distance is measured using $\| \ \|_\infty$. So the main idea of this article is to add a new resource and allocate it among the decision making units to meet the favorable conditions. The meaning of favorable conditions is that:

1. The allocation should be done in such a way that the considered unit achieves maximum efficiency in the presence of the new source.
2. The allocation distance obtained from the fair allocation should be minimized.

The advantage of this paper is that the presented models are linear. Another advantage of this article is that in both crisp and fuzzy environments, it uses simultaneous fair allocation and maximizing the efficiency of the desired unit. This is despite the fact that, as can be seen in Table 1, none of the new researches in the field of resource allocation have focused on fuzzy data. So this section consists of 2 parts. In the first part, we assume that the values of inputs and outputs, as well as the new source that we intend to allocate, are all crisp. In the second part, we extend the models presented in the first part to fuzzy mode.

3.1 New resource allocation method

Suppose n decision-making units with m inputs and s outputs are evaluated. A new resource is now allocated between units. It can be easily shown that since the number of variables in fractional model (model (1)) increases after the addition of a new source, the value of the objective function, which is in fact the efficiency of DMU_o, will not decrease. Therefore, it is clear that after the introduction of the new source into the model, efficient units will remain efficient. But in the case of inefficient units, a significant improvement in their efficiency may occur with the introduction of a new source. Now suppose DMU_o is inefficient. (Model (16)) is the fractional model for evaluation DMU_o in presence of the new source. Suppose k is an existing value of the new source. Also, v_{m+1} is the related weight to this source. a_j is also the new source share for DMU_j for $j = 1, \ldots, n$.

$$\text{Maximize} \quad \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io} + v_{m+1} a_o}$$

$$S.t. \quad \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij} + v_{m+1} a_j} \le 1, \qquad (16.1)$$

$$j = 1, \ldots, n$$

$$\sum_{j=1}^{n} a_j = k \qquad (16.2)$$

$$v_i \ , \ u_r \ge \varepsilon \ , \ i = 1, \ldots, m \ , \ r = 1, \ldots, s$$

$$v_{m+1} \ge \varepsilon$$

$$a_j \ge 0 \ , \ j = 1, \ldots, n$$

In Theorem 3.1 we prove that the optimal value of the objective function of model 1 is equal to 1. In other words, this proves that a new resource can be allocated between units in such a way that an inefficient unit becomes efficient.

Theorem 3.1 For each inefficient unit, there is an allocation of new resource that can convert it to an efficient unit.

Proof. Suppose DMU_o is evaluated as inefficient by (model (1)) considering m inputs and s outputs. Also assume that $u^* = (u_1^*, \ldots, u_s^*)$, $v^* = (v_1^*, \ldots, v_m^*)$ is the optimal solution for model (1). Now considering these assumptions, we construct a feasible solution with a value of the objective function equal to 1 for Model (16).

$$u_r = \frac{u_r^*}{\sum_{r=1}^{s} u_r^* y_{ro}} \ge \varepsilon \ , \ r = 1, \ldots, s$$

$$v_i = \frac{v_i^*}{\sum_{i=1}^{m} v_i^* x_{io}} \ge \varepsilon \ , \ i = 1, \ldots, m$$

$$a_j = \frac{k}{n-1} \ge \varepsilon \ , \ j = 1, \ldots, n; j \ne o$$

$$a_o = 0$$

Using the above definitions, the value of the objective function of Model (16) is obtained 1 as follows:

$$\frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io} + v_{m+1} a_o} = \frac{\sum_{r=1}^{s} \frac{ur^*}{\sum_{r=1}^{s} u_r^* y_{ro} y_{ro}}}{\sum_{i=1}^{m} \frac{v_i^*}{\sum_{i=1}^{m} v_i^* x_{io} x_{io} + v_{m+1}(0)}}$$

$$= \frac{\frac{1}{\sum_{r=1}^{s} u_r^* y_{ro}} \sum_{r=1}^{s} u_r^* y_{ro}}{\frac{1}{\sum_{i=1}^{m} v_i^* x_{io}} \sum_{i=1}^{m} v_i^* x_{io} + 0} = \frac{1}{1} = 1$$

Constraint (16.2) is also established as follows:

$$\sum_{j=1}^{n} a_j = \sum_{\substack{j=1 \\ j \neq o}}^{n} a_j + a_o = \sum_{\substack{j=1 \\ j \neq o}}^{n} \frac{k}{n-1} + 0 = (n-1)\left(\frac{k}{n-1}\right) = k$$

Now we need to get v_{m+1} in such a way that constraint (16.1) is established. Given that the value of the objective function is equal to 1, we conclude that the constraint (1.1) for $j = o$ is binding. So it is sufficient to check the constraint (16.1) for $j \neq o$.

$$\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij} + v_{m+1} a_j} \leq 1 \quad (j \neq o)$$

so

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} \leq v_{m+1} a_j \quad (j \neq o)$$

Then

$$\sum_{r=1}^{s} \frac{u_r^*}{\sum_{r=1}^{s} u_r^* y_{r0}} y_{rj} - \sum_{i=1}^{m} \frac{v_i^*}{\sum_{i=1}^{s} v_i^* x_{i0}} x_{ij} \leq v_{m+1} a_j \quad (j \neq o)$$

afterwards

$$\sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}} - \sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} \leq v_{m+1} \frac{k}{n-1} \quad (j \neq o)$$

According to that $\frac{n-1}{k} > 0$ we will have:

$$\left(\frac{n-1}{k}\right) \sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}} - \sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} \leq v_{m+1} \quad (j \neq o)$$

as a result

$$v_{m+1} = \frac{n-1}{k} \max_{j \neq o} \left\{ \sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}} - \sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} \right\}$$

It is necessary to explain that if $\max_{j \neq o} \left\{ \sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}} - \sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} \right\}$ Thus

$$\frac{\sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}}}{\sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}}} < 1 \quad j \neq o$$

So for every v_{m+1} and $a_j \geq 0$, the following inequality holds:

$$\frac{\sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}}}{\sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} + v_{m+1} a_j} < 1 \quad j \neq o$$

As a result, constraint (16.1) is satisfied again. Therefore, we consider v_{m+1} as follows:

$$v_{m+1} = \max \left\{ \varepsilon \frac{n-1}{k} \max_{j \neq o} \left\{ \sum_{r=1}^{s} \frac{u_r^* y_{rj}}{\sum_{r=1}^{s} u_r^* y_{ro}} - \sum_{i=1}^{m} \frac{v_i^* x_{ij}}{\sum_{i=1}^{m} v_i^* x_{io}} \right\} \right\} > 0$$

Which ε is a non-Archimedean number. In this way, a feasible solution with a value of objective function equal to 1 for model (16) is obtained:

$$u_r = \frac{u_r^*}{\sum_{r=1}^{s} u_r^* y_{ro}} \geq \varepsilon \quad r = 1, \ldots, s$$

$$v_i = \frac{v_i^*}{\sum_{i=1}^{m} v_i^* x_{io}} \geq \varepsilon \quad i = 1, \ldots, m$$

$$a_j = \frac{k}{n-1} \geq 0 \quad j = 1, \ldots, m$$

$$a_0 = 0$$

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$$v_{m+1} = \max\left\{\varepsilon\frac{n-1}{k}\max_{j\neq o}\left\{\sum_{r=1}^{s}\frac{u_r^* y_{rj}}{\sum_{r=1}^{s}u_r^* y_{ro}} - \sum_{i=1}^{m}\frac{v_i^* x_{ij}}{\sum_{i=1}^{m}v_i^* x_{io}}\right\}\right\} > 0$$

Given that the value of the objective function for this feasible solution is equal to 1, we conclude that this solution is actually the optimal solution of model (16). Therefore we got an allocation of the new resource that turned DMU_o into an efficient unit. ■

Now we try to linearize model (16.2). Due to the fact that $v^*_{m+1} > 0$, we can divide the form and denominator of the objective function as well as the constraint (16.1) by $v_{m+1} > 0$. Then, using Charnes and Cooper transformation [31], model (16.2) turns in to the following linear model:

$$\begin{aligned}
&\text{Maximize} \quad \sum_{r=1}^{s} u_r y_{ro} \qquad (17)\\
&S.t. \quad \sum_{i=1}^{m} v_i x_{io} + a_o = 1\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - a_j \le 0 \quad j = 1,\dots,n\\
&\sum_{j=1}^{n} a_j = k\\
&v_i, u_r \ge \varepsilon, \quad i = 1,\dots,m, \quad r = 1,\dots,s\\
&a_j \ge 0, \quad j = 1,\dots,n
\end{aligned}$$

The advantage of Model (17) over Model (16.1) is that it is linear, in addition to having fewer variables. Given the equivalence of models (16.1) and (17) and using theorem 1, we can get the optimal value of the objective function model (17) equals to 1. Therefore, we use Model (17) to find an allocation that turns an inefficient unit into an efficient one, after the introduction of the new source. However, the allocation obtained by Model (17) may be unfair. That is, a large share of the new resource may be allocated to certain units, despite the fact that these units may be small. Also a small share of the new source may be allocated to large units. In order to solve this problem, we change the model (17) in such a way that in addition to making DMU_o efficient, it minimizes the desired allocation distance from the fair allocation. The desired allocation distance with fair allocation is measured by $\| \|_\infty$. The following model is presented for this purpose:

$$\begin{aligned}
&\text{Minimize} \quad \left(\text{Maximize}\left\{\left|a_j - r_j\right| \;;\; j = 1,\dots,n\right\}\right)\\
&s.t. \quad \sum_{r=1}^{s} u_r y_{ro} - \sum_{i=1}^{m} v_i x_{io} - a_o = 0 \qquad (18)\\
&\sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} v_i x_{ij} - a_j \le 0\,,\; j = 1,\dots,n\\
&\sum_{j=1}^{n} a_j = k\\
&v_i, u_r \ge \varepsilon, \quad i = 1,\dots,m, \quad r = 1,\dots,s\\
&a_j \ge 0, \quad j = 1,\dots,n
\end{aligned}$$

Where r_j is DMU_j share of the fair allocation derived by Formula (15). Constraint (18) requires DMU_o to be efficient. The other constraints of this model are in fact the same constraints of the Model (17). Therefore, considering the relationship between models (17) and (18), clearly every optimal solution of model (17) is a feasible solution of model (18). This is stated in Theorem 3.2.

Theorem 3.2 *Model (18) is feasible.*

Proof: According to Theorem 3.1, any optimal solution of Model (17) is a feasible solution for Model (18). ■

To solve the model (18), suppose that $\alpha = \max\left\{|a_j - r_j| \;;\; j = 1,\dots,n\right\}$, Therefore, we will have the following n inequalities:

$$\alpha \ge |a_j - r_j| \;;\; j = 1,\dots,n$$

as a result

$$-\alpha \le a_j - r_j \le \alpha \;;\; j = 1,\dots,n$$

The above inequalities can be written as follows:

$$a_j - \alpha \le r_j \;;\; j = 1,\dots,n$$
$$a_j + \alpha \ge r_j \;;\; j = 1,\dots,n$$

Therefore, considering the objective function of model (18) as (min α) and adding the above ($2n$) constraints, model (18) can be solved as a linear programming problem.

3.2 New resource allocation on fuzzy data

Many real problems in this world do not have crisp data. This prompted us to extend the model presented in the previous section (model (18)) to fuzzy data mode. In this section we assume that all inputs and outputs as well as the new source added to the problem are fuzzy triangular numbers. Suppose n decision making units are evaluated with m and s inputs and outputs, respectively. In this article, we consider fuzzy numbers to be triangular. Thus, suppose $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ and $\tilde{y}_{rj} = (ys^l_{rj}, y^m_{rj}, y^u_{rj})$ are the ith input and the rth output of DMU_j, respectively for $j = 1,\dots,n$. Here, the models presented in [29] are used to evaluate the efficiency of units. Therefore, the efficiency of each unit will be obtained as a triangular fuzzy number. Suppose $\tilde{\theta}^*_o = [\theta^{*l}_O, \theta^{*m}_O, \theta^{*u}_O]$ is the efficiency obtained for DMU_o. But the reason for using the Wang et al model is that the same production possibility set is used to calculate $\theta^{*l}_O, \theta^{*m}_O$ and θ^{*u}_O.

Now a new fuzzy source comes into play. The purpose is to investigate the effect of adding a new resource on efficiency of DMU_o. As mentioned in section 3.1, after adding a new source to the problem, the efficiency of any of decision making units will not decrease. Therefore, if $\tilde{\eta}^*_o = [\eta^{*l}_O, \eta^{*m}_O, \eta^{*u}_O]$ is the efficiency of DMU_o after adding the new source, then the following inequality will be satisfied.

$$\tilde{\eta}^*_o \ge \tilde{\theta}^*_o \iff [\eta^{*l}_O, \eta^{*m}_O, \eta^{*u}_O] \ge [\theta^{*l}_O, \theta^{*m}_O, \theta^{*u}_O] \iff \begin{cases} \eta^{*l}_O \ge \theta^{*l}_O \\ \eta^{*m}_O \ge \theta^{*m}_O \\ \eta^{*u}_O \ge \theta^{*u}_O \end{cases}$$

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We are now looking for an allocation of the new source that firstly maximizes the efficiency of DMU_o and secondly minimizes the resulting allocation distance relative to the fair allocation. To get the maximum efficiency of DMU_o after adding a new source to the problem, we solve the following fractional model:

$$\begin{aligned}
\text{Maximize} \quad & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{ro}}{\sum_{i=1}^{m} v_i \tilde{x}_{io} + v_{m+1}\tilde{a}_o} \\
S.t. \quad & \frac{\sum_{r=1}^{s} u_r \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i \tilde{x}_{ij} + v_{m+1}\tilde{a}_j} \le 1, \quad j = 1,\dots,n \\
& \sum_{j=1}^{n} \tilde{a}_j = \tilde{k} \\
& v_i \ , \ u_r \ge \varepsilon \ , \quad i = 1,\dots,m \ , \ r = 1,\dots,s \\
& v_{m+1} \ge \varepsilon \\
& \tilde{a}_j \ge (0,0,0) \ , \ j = 1,\dots,n
\end{aligned} \tag{19}$$

Where$\tilde{k} = (k^l, k^m, k^u)$ is the existing amount of the new fuzzy source added to the problem and $\tilde{a}_j = (a^l_j, a^m_j, a^u_j)$ is the amount of that resource that is assigned to DMU_j . To solve the above fuzzy model, we use the method of [29]. So the following three crisp models must be solved. These models are solved for each of the units of interest to the manager, which can certainly be any of the decision-making units. Of course, considering that the efficient units will remain efficient after adding fixed cost to the system, these models are specially considered for inefficient units

$$\begin{aligned}
\eta_0^{*l} = \quad \text{Maximize} \quad & \frac{\sum_{r=1}^{s} u_r y^l_{ro}}{\sum_{i=1}^{m} v_i x^u_{io} + v_{m+1} a^u_o} \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij} + v_{m+1} a^l_j} \le 1, \ j = 1,\dots,n \\
& \sum_{j=1}^{n} a^l_j = k^l \\
& \sum_{j=1}^{n} a^m_j = k^m \\
& \sum_{j=1}^{n} a^u_j = k^u \\
& 0 \le a^l_j \le a^m_j \le a^u_j \ , \ j = 1,\dots,n \\
& v_i, u_r \ge \varepsilon, \ i = 1,\dots,m, \ r = 1,\dots,s \\
& v_{m+1} \ge \varepsilon
\end{aligned} \tag{20}$$

$$\begin{aligned}
\eta_0^{*m} = \quad \text{Maximize} \quad & \frac{\sum_{r=1}^{s} u_r y^m_{ro}}{\sum_{i=1}^{m} v_i x^m_{io} + v_{m+1} a^m_o} \\
s.t. \quad & \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij} + v_{m+1} a^l_j} \le 1 \ , \ j = 1,\dots,n \\
& \sum_{j=1}^{n} a^l_j = k^l \\
& \sum_{j=1}^{n} a^m_j = k^m \\
& \sum_{j=1}^{n} a^u_j = k^u \\
& 0 \le a^l_j \le a^m_j \le a^u_j \ , \ j = 1,\dots,n \\
& v_i, u_r \ge \varepsilon, i = 1,\dots,m, r = 1,\dots,s \\
& v_{m+1} \ge \varepsilon
\end{aligned} \tag{21}$$

$$\begin{aligned}
\eta_0^{*u} = \quad \text{Maximize} \quad & \frac{\sum_{r=1}^{s} u_r y^u_{ro}}{\sum_{i=1}^{m} v_i x^l_{io} + vs_{m+1} a^l_o} \\
S.t. \quad & \frac{\sum_{r=1}^{s} u_r y^u_{rj}}{\sum_{i=1}^{m} v_i x^l_{ij} + v_{m+1} a^l_j} \le 1, \ j = 1,\dots,n \\
& \sum_{j=1}^{n} a^l_j = k^l \\
& \sum_{j=1}^{n} a^m_j = k^m \\
& \sum_{j=1}^{n} a^u_j = k^u \\
& 0 \le a^l_j \le a^m_j \le a^u_j, j = 1,\dots,n \\
& v_i, u_r \ge \varepsilon, i = 1,\dots,m, r = 1,\dots,s \\
& v_{m+1} \ge \varepsilon
\end{aligned} \tag{22}$$

Theorem 3.3 a) Models (20), (21) and (22) are feasible.

b) Consider n decision making units with m inputs and s outputs. Also assume $\tilde{x}_{ij} = (x^l_{ij}, x^m_{ij}, x^u_{ij})$ and $\tilde{y}_{rj} = (y^l_{rj}, y^m_{rj}, y^u_{rj})$ are the ith input and the rth output of DMU_j, respectively for $j = 1,\dots,n$. Suppose $\tilde{\theta}^*_o = [\theta^{*l}_O, \theta^{*m}_O, \theta^{*u}_O]$ is efficiency of DMU_o obtained by solving models (12), (13) and (14). A new fuzzy source now enters the problem as $\tilde{k} = (k^l, k^m, k^u)$, and models (20),

(21) and (22) are solved in presence of the new source in order to evaluation DMU$_o$. Assume that the $\tilde{\eta}^*_o = [\eta^{*l}_O, \eta^{*m}_O, \eta^{*u}_O]$ is efficiency of DMU$_o$ in the presence of the new source obtained by solving models (20) ,(21) and (22) then $[\eta^{*l}_O, \eta^{*m}_O, \eta^{*u}_O] \geq [\theta^{*l}_O, \theta^{*m}_O, \theta^{*u}_O]$.

c) With the same assumptions of part (b),The optimal value of the objective function of model (22) is equal to one. that's mean $\eta^{*u}_O = 1$.

Proof. Suppose (u^*, v^*) is the optimal solution of model (14). The following solution would be a feasible solution of Models (20), (21) and (22):

$$v_i = \frac{v_i^*}{\sum_{i=1}^m v_i^* x_{io}} \geq \varepsilon \quad , \quad i = 1, \dots, m$$

$$u_r = \frac{u_r^*}{\sum_{r=1}^s u_r^* y_{ro}} \geq \varepsilon \quad , \quad r = 1, \dots, s$$

$$a_j^l = \frac{k^l}{n-1} \geq 0 \quad , \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_j^m = \frac{k^m}{n-1} \geq 0 \quad , \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_j^u = \frac{k^u}{n-1} \geq 0 \quad , \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_o^l = a_o^m = a_o^u = 0$$

$$v_{m+1} = \max \left\{ \varepsilon, \frac{n-1}{k^l} \max_{j \neq o} \left\{ \sum_{r=1}^s \frac{u_r^* y_{rj}^u}{\sum_{r=1}^s u_r^* y_{ro}^u} - \sum_{i=1}^m \frac{v_i^* x_{ij}^l}{\sum_{i=1}^m v_i^* x_{io}^l} \right\} \right\} > 0$$

It should be noted that the method of obtaining the above solution is similar to Theorem 3.1. Also the value of the objective function of the model (22) for the above answer is equal to 1. as a result $\eta^{*u}_O = 1$. Thus, parts (a) and (c) of the theorem were proved. Now let's prove part (b). According to $\eta^{*u}_O = 1$, it is clear that $\eta^{*u}_O \geq \theta^{*u}_O$. Now suppose (u^*, v^*) is the optimal solution of model (12). The following solution is feasible for model (20):

$$u_r = u_r^* \geq \varepsilon \quad r = 1, \dots, s$$

$$v_i = v_i^* \geq \varepsilon \quad i = 1, \dots, m$$

$$a_j^l = \frac{k^l}{n-1} \geq 0 \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_j^m = \frac{k^m}{n-1} \geq 0 \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_j^u = \frac{k^u}{n-1} \geq 0 \quad j = 1, \dots, n \; ; \; j \neq o$$

$$a_o^l = a_o^m = a_o^u = 0$$

$$v_{m+1} = \varepsilon > 0$$

Where ε is a non-Archimedean number. The value of the objective function of model (20) for above solution is equal to the optimal value of the objective function of θ^{*l}_O. As a result $\eta^{*l}_O \geq \theta^{*l}_O$. The same can be said about the relationship between models (13) and (21), therefore $\eta^{*m}_O \geq \theta^{*m}_O$. Thus $[\eta^{*l}_O, \eta^{*m}_O, \eta^{*u}_O] \geq [\theta^{*l}_O, \theta^{*m}_O, \theta^{*u}_O]$. ■

Models (20), (21) and (22) are nonlinear. To linearize these models, we divide the face and denominator of the objective functions and fractional constraints by v_{m+1}, then use the Charnes-Cooper transformation. In this way, 3 linear models are obtained as follows:

$$\begin{aligned} \eta_0^{*l} = \quad & \text{Maximize} \quad \sum_{r=1}^s u_r y_{ro}^l \\ & S.t. \quad \sum_{i=1}^m v_i x_{io}^u + a_o^u = 1 \\ & \sum_{r=1}^s u_r y_{rj}^u - \sum_{i=1}^m v_i x_{ij}^l - a_j^l \leq 0 \, , \, j = 1, \dots, n \\ & \sum_{j=1}^n a_j^l = k^l \\ & \sum_{j=1}^n a_j^m = k^m \\ & \sum_{j=1}^n a_j^u = k^u \\ & 0 \leq a_j^l \leq a_j^m \leq a_j^u, j = 1, \dots, n \\ & v_i, u_r \geq \varepsilon, i = 1, \dots, m, r = 1, \dots, s \end{aligned} \tag{23}$$

$$\begin{aligned} \eta_0^{*m} = \quad & \text{Maximize} \quad \sum_{r=1}^s u_r y_{ro}^m \\ & S.t. \quad \sum_{i=1}^m v_i x_{io}^m + a_o^m = 1 \\ & \sum_{r=1}^s u_r y_{rj}^u - \sum_{i=1}^m v_i x_{ij}^l - a_j^l \leq 0, j = 1, \dots, n \\ & \sum_{j=1}^n a_j^l = k^l \\ & \sum_{j=1}^n a_j^m = k^m \\ & \sum_{j=1}^n a_j^u = k^u \\ & 0 \leq a_j^l \leq a_j^m \leq a_j^u, j = 1, \dots, n \\ & v_i, u_r \geq \varepsilon \; i = 1, \dots, m, r = 1, \dots, s \end{aligned} \tag{24}$$

$$\begin{aligned} \eta_0^{*u} = \quad & \text{Maximize} \quad \sum_{r=1}^s u_r y_{ro}^u \\ & S.t. \quad \sum_{i=1}^m v_i x_{io}^l + a_o^l = 1 \\ & \sum_{r=1}^s u_r y_{rj}^u - \sum_{i=1}^m v_i x_{ij}^l - a_j^l \leq 0, j = 1, \dots, n \end{aligned} \tag{25}$$

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$$\sum_{j=1}^{n} a_j^l = k^l$$
$$\sum_{j=1}^{n} a_j^m = k^m$$
$$\sum_{j=1}^{n} a_j^u = k^u$$
$$0 \le a_j^l \le a_j^m \le a_j^u \ \ , j = 1, \dots, n$$
$$v_i, u_r \ge \varepsilon, i = 1, \dots, m, r = 1, \dots, s$$

Similar to that described in Section 3.1, the allocations obtained by models (23), (24), and (25) may be far from fair allocation. Therefore, models are presented here, minimize the resulting allocation distance to fair allocation in addition to maximizing the efficiency of the decision making unit in question.

$$\text{Minimiz} \quad \left(\text{Maximize}\left\{\left|a_j^l - r_j^l\right|, \left|a_j^m - r_j^m\right|, \left|a_j^u - r_j^u\right|; j = 1, \dots, n\right\}\right) \tag{26}$$
$$S.t. \quad \sum_{r=1}^{s} u_r y_{ro}^l - (\eta_0^{*l}) \sum_{i=1}^{m} v_i x_{io}^u - \eta_0^{*l} a_o^u = 0$$
$$\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - a_j^l \le 0, j = 1, \dots, n$$
$$\sum_{j=1}^{n} a_j^l = k^l$$
$$\sum_{j=1}^{n} a_j^m = k^m$$
$$\sum_{j=1}^{n} a_j^u = k^u$$
$$0 \le a_j^l \le a_j^m \le a_j^u \ \ , j = 1, \dots, n$$
$$v_i, u_r \ge \varepsilon \quad i = 1, \dots, m \ , \ \ r = 1, \dots, s$$

Model (26) can be converted into linear models similar to the method that explained before, for model (18). Suppose that $\alpha = \max\{|a_j^l - r_j^l|, |a_j^m - r_j^m|, |a_j^u - r_j^u| \ ; \ j = 1, \dots, n\}$. Therefore, we will have the following $3 \times n$ inequalities:

$$\alpha \ge |a_j^l - r_j^l| \ \ ; \ \ j = 1, \dots, n$$
$$\alpha \ge |a_j^m - r_j^m| \ \ ; \ \ j = 1, \dots, n$$
$$\alpha \ge |a_j^u - r_j^u| \ \ ; \ \ j = 1, \dots, n$$

as a result

$$-\alpha \le a_j^i - r_j^i \le \alpha; \qquad i = l, m, u; \quad j = 1, \dots, n$$

The above inequalities can be written as follows:

$$a_j^i - \alpha \le r_j^i; \qquad i = l, m, u; \quad j = 1, \dots, n$$
$$a_j^i + \alpha \ge r_j^i; \qquad i = l, m, u; \quad j = 1, \dots, n$$

Therefore, considering the objective function of model (26) as (min α) and adding the above (6*n) constraints, model (26) can be solved as a linear programming problem. In the same way, models (27) and (28) can also be converted into linear programming models.

The first constraint of model (26) implies: $\eta_0^l = \eta_0^{*l}$, because:

$$\sum_{r=1}^{s} u_r y_{ro}^l - (\eta_0^{*l}) \sum_{i=1}^{m} v_i x_{io}^u - \eta_0^{*l} a_o^u = 0$$
$$\Leftrightarrow \sum_{r=1}^{s} u_r y_{ro}^l = (\eta_0^{*l}) \sum_{i=1}^{m} v_i x_{io}^u + \eta_0^{*l} a_o^u$$

$$\sum_{r=1}^{s} u_r y_{ro}^l = (\eta_0^{*l})(\sum_{i=1}^{m} v_i x_{io}^u + a_o^u)$$
$$\Leftrightarrow \frac{\sum_{r=1}^{s} u_r y_{ro}^l}{\sum_{i=1}^{m} v_i x_{io}^u + a_o^u} = \eta_0^{*l} \Leftrightarrow \eta_0^l = \eta_0^{*l}$$

The other constraints of this model (26) are the same as the constraints of model (23). Therefore, by solving model (26), we can find an allocation of the new source that in addition to maximizing η_0^l, the distance (using $\| \ \|_\infty$.) of the allocation obtained from the fair allocation is minimized. Similarly, models (27) and (28) are presented in order to obtain allocations with a minimum distance from the fair allocation that lead to maximum efficiency (η_0^{*m} and $\eta_0^{*u} = 1$).

$$\text{Minimize} \quad \left(\text{Maximize}\left\{\left|a_j^l - r_j^l\right|, \left|a_j^m - r_j^m\right|, \left|a_j^u - r_j^u\right| \ ; \ j = 1, \dots, n\right\}\right) \tag{27}$$
$$S.t. \quad \sum_{r=1}^{s} u_r y_{ro}^m - (\eta_0^{*m}) \sum_{i=1}^{m} v_i x_{io}^m - \eta_0^{*m} a_o^m = 0$$
$$\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - a_j^l \le 0 \ \ , j = 1, \dots, n$$
$$\sum_{j=1}^{n} a_j^l = k^l$$
$$\sum_{j=1}^{n} a_j^m = k^m$$
$$\sum_{j=1}^{n} a_j^u = k^u$$
$$0 \le a_j^l \le a_j^m \le a_j^u, j = 1, \dots, n$$
$$v_i, u_r \ge \varepsilon \ i = 1, \dots, m \ , \ \ r = 1, \dots, s$$

$$\text{Minimize} \quad \left(\text{Maximize}\left\{\left|a_j^l - r_j^l\right|, \left|a_j^m - r_j^m\right|, \left|a_j^u - r_j^u\right| \ ; \ j = 1, \dots, n\right\}\right) \tag{28}$$
$$S.t. \quad \sum_{r=1}^{s} u_r y_{ro}^u - (1) \sum_{i=1}^{m} v_i x_{io}^l - 1 a_o^l = 0$$
$$\sum_{r=1}^{s} u_r y_{rj}^u - \sum_{i=1}^{m} v_i x_{ij}^l - a_j^l \le 0 \ , j = 1, \dots, n$$

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$$\sum_{j=1}^{n} a_j^l = k^l$$
$$\sum_{j=1}^{n} a_j^m = ks^m$$
$$\sum_{j=1}^{n} a_j^u = k^u$$
$$0 \le a_j^l \le a_j^m \le a_j^u \,, j = 1, \ldots, n$$
$$v_i, u_r \ge \varepsilon \; i = 1, \ldots, m \,, \; r = 1, \ldots, s$$

Similar to model (26), Models (27) and (28) can also be converted into linear programming. Obviously, in model (28), 1 is placed instead of η_o^{*u}.

Therefore, our main models in this article are models (26), (27) and (28). Of course, it should be noted that to solve models (26) and (27), solving models (23) and (24) is necessary to obtain the values of η_o^{*l} and η_o^{*m}. In the next section, we will show the application of the above-mentioned models by providing examples.

4. Illustrative Examples

In this section, by providing two examples, we show the application of the models presented in the previous section. In first example, 12 units with 3 inputs and 2 outputs (Table 1) are considered. This data has already been used in [32, 3, 4, 40, 39]. The data of second example (Table 3) are taken from [29].This data is related to a performance assessment problem in China where eight manufacturing enterprises (DMU_s) are to be evaluated in terms of two inputs and two outputs.

Example 4.1 In this example, we evaluate the performance of the units in Table 1 in the presence of 3 inputs and 2 outputs mentioned. As we can see in the last column of Table 1, DMU_4, DMU_5, DMU_8, DMU_9 and DMU_{12} are efficient.

A fixed cost is now added to the problem in the form of a new input. Like Beasley [7], consider the fixed cost to be 100. Fair allocation of fixed cost mentioned is shown in Table 2 (by formula (15). We now use model (18) to obtain the fixed cost allocation (=100). As discussed in detail in Section 3, the resulting allocation is such that it has the minimum possible distance from a fair allocation and can convert a particular inefficient unit into an efficient unit. We solve model (18) for DMU_o=DMU_1, DMU_{11}. The results of solving model (18) for DMU_1 and DMU_{11} are shown in Table 2. After allocating the fixed cost, the efficiency of all units is calculated again and the results are given in Table 2. In the last row of this table, the minimum distance of the obtained allocations to the fair allocation (the optimal value of the objective function of model (18)) is given. It should be noted that the executions in this table are rounded to 2 decimal places. From the results of this table, it is clear that, as we expected, after the new allocation, the intended unit has become efficient and the efficiency of other units has not decreased. But the other result is that the allocation distance obtained from the fair allocation for DMU_o=DMU_1 (1.14) is less than the obtained allocation distance relative to the fair allocation for DMU_o=DMU_{11} (3.69). This shows that in order to make DMU_1 efficient, we need to violate fairness more than to make DMU_{11} efficient.

Example 4.2 The data in Table 3 are taken from [29]. This data is related to a performance assessment problem in China where eight manufacturing enterprises (DMUs) are to be evaluated in terms of two inputs and two outputs. Table 4 shows the efficiency of these units obtained using Models (12), (13) and (14).

Now the new fuzzy source $\tilde{R} = (R^l, R^m, R^u) = (15000, 16000, 16500)$ enters the problem. Now, using models (23), (24) and (25), the new source is allocated between units in such a way that lower bound, middle value and upper bound of $\text{DMU}_o^{\prime}$s efficiency is maximized, respectively. The results of solving models (23), (24) and (25) are given in Table 5. It should be noted that the performance measures in Table 5 are cut from the seventh decimal point onwards. Because if they were rounded up, models (26), (27) and (28) would be infeasible. Looking closely at this table, we find that the efficiency measures are greater than or equal to the efficiency measures in Table 4, in addition to the fact that the upper bound of the efficiency size is 1 for all decision making units due to the allocation of new source. But as mentioned in Section 4, the allocation obtained to achieve maximum efficiency may be a long way from a fair allocation. For example, allocating the new resource that results in maximum upper bound efficiency for DMU H ($\eta_8^{*u} = 1$) is shown in Table 6, which is a long way from fair allocation. The fair allocation obtained from Formula (15.1), (15.2) and (15.3) is given in the Table 7. Now we solve model 28 to obtain the allocation with minimum distance from the fair allocation to achieve $\eta_8^{*u} = 1$. The result is as follows: The distance of allocation obtained in Table 8 to the fair allocation is 109. Now it is time to solve models (26), (27) and (28) for all of the DMUs. These models give us allocations of the new source that, in addition to achieving maximum efficiency, have a minimum distance from the fair allocation. Results of solving the mentioned models are shown in Table 9. Table 9 shows the minimum allocation distance obtained by solving models (26), (27) and (28) that resulted in maximum (lower bound, middle value or upper bound) of $\text{DMU}_o^{\prime}s$ efficiency: In Table 9, the decimal part of the numbers is omitted. The results of Table 9 show:

The minimum allocation distance relative to the fair allocation for unit B (related to $\eta_o^l = \eta_o^{*l}$) is zero. It can be seen in Table 4, Unit B has the maximum lower bound efficiency before the new resource is added.

1. The minimum allocation distance relative to the fair allocation for unit B (related to $\eta_o^m = \eta_o^{*m}$) is zero. It can be seen in Table 4, Unit B has the maximum middle value efficiency before the new resource is added.

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Table 1. TABLE I Data of example1

DMU	Input1	Input2	Input3	Output1	Output2	Original CCR efficiency
DMU_1	350	39	9	67	751	0.7567
DMU_2	298	26	8	73	611	0.9230
DMU_3	422	31	7	75	584	0.7470
DMU_4	281	16	9	70	665	1.0000
DMU_5	301	16	6	75	445	1.0000
DMU_6	360	29	17	83	1070	0.9612
DMU_7	540	18	10	72	457	0.8604
DMU_8	276	33	5	78	590	1.0000
DMU_9	323	25	5	75	1074	1.0000
DMU_{10}	444	64	6	74	1072	0.8318
DMU_{11}	323	25	5	25	350	0.3333
DMU_{12}	444	64	6	104	1199	1.0000

Table 2. Fair allocation and result of solving model (18) for DMU_o =DMU_1, DMU_{11}

		Solving model (18) for DMU_o=DMU_1		Solving model (18) DMU_o=DMU_{11}	
DMU	Fair allocation	allocation	Efficiency after allocation	Allocation	Efficiency after allocation
DMU_1	8.22	7.08	1.0000	8.09	1.0000
DMU_2	6.86	7.61	0.9500	7.10	1.0000
DMU_3	9.50	8.36	0.8800	6.93	1.0000
DMU_4	6.32	7.46	1.0000	7.46	1.0000
DMU_5	6.67	7.07	1.0000	5.78	1.0000
DMU_6	8.39	9.07	1.0000	11.18	1.0000
DMU_7	11.73	10.59	0.8600	8.04	0.8600
DMU_8	6.49	7.63	1.0000	7.07	1.0000
DMU_9	7.29	8.43	1.0000	10.98	1.0000
DMU_{10}	10.62	9.48	0.8900	10.94	1.0000
DMU_{11}	7.29	6.15	0.4500	3.60	1.0000
DMU_{12}	10.62	11.06	1.0000	12.83	1.0000
Min distance from fair allocation:		1.14		3.69	

Table 3. Data of example 2 [36]

Enterprises	Inputs		Outputs	
(DMU_s)	MC	NOE	GOV	PQ
A	(2120, 2170, 2210)	1870	(14500, 14790, 14860)	(3.1, 4.1, 4.9)
B	(1420, 1460, 1500)	1340	(12470, 12720, 12790)	(1.2, 2.1, 3.0)
C	(2510, 2570, 2610)	2360	(17900, 18260, 18400)	(3.3, 4.3, 5.0)
D	(2300, 2350, 2400)	2020	(14970, 15270, 15400)	(2.7, 3.7, 4.6)
E	(1480, 1520, 1560)	1550	(13980, 14260, 14330)	(1.0, 1.8, 2.7)
F	(1990, 2030, 2100)	1760	(14030, 14310, 14400)	(1.6, 2.6, 3.6)
G	(2200, 2260, 2300)	1980	(16540, 16870, 17000)	(2.4, 3.4, 4.4)
H	(2400, 2460, 2520)	2250	(17600, 17960, 18100)	(2.6, 3.6, 4.6)

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Table 4. Results of solving models (12), (13) and (14)

Enterprises (DMU_s)	Lower bound	Middle value	Upper bound
A	0.8124	0.9033	1.0000
B	0.9750	0.9945	1.0000
C	0.7946	0.8122	0.9045
D	0.7764	0.8050	0.9070
E	0.9603	0.9872	1.0000
F	0.8352	0.8518	0.8852
G	0.8752	0.8927	0.9457
H	0.8195	0.8363	0.8864

Table 5. Results of solving models (23), (24) and (25)

DMU	η_o^{*l}	η_o^{*m}	η_o^{*u}
A	0.9758	0.9953	1.0000
B	0.9750	0.9945	1.0000
C	0.9728	0.9924	1.0000
D	0.9721	0.9916	1.0000
E	0.9756	0.9951	1.0000
F	0.9743	0.9937	1.0000
G	0.9729	0.9924	1.0000
H	0.9724	0.9923	1.0000

Table 6. allocating the new resource that results in maximum upper bound efficiency for DMU_H

DMU_j	a_j^l	a_j^m	a_j^u
A	14999.687431	15999.687431	16499.687431
B	0.114963	0.114963	0.114963
C	0.000000	0.000000	0.000000
D	0.000000	0.000000	0.000000
E	0.175046	0.175046	0.175046
F	0.000000	0.000000	0.000000
G	0.022560	0.022560	0.022560
H	0.000000	0.000000	0.000000

Table 7. Fair allocation by formula (15.1), (15.2) and (15.3)

DMU	r_j^l	r_j^m	r_j^u
A	1896.988906	2023.161189	2082.276523
B	1312.202853	1402.190923	1449.427776
C	2315.372425	2468.85759	2536.498608
D	2053.882726	2188.419405	2255.799567
E	1440.570523	1537.402191	1587.225487
F	1782.884311	1897.965571	1969.996907
G	1987.321712	2123.317684	2184.348902
H	2210.776545	2358.685446	2434.42623

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Table 8. Result of solving model (28) for $\eta_8^{*u} = 1$

DMU_j	a_j^l	a_j^m	a_j^u
A	2006.672045	2006.672045	2006.672045
B	1421.885992	1421.885992	1421.885992
C	2217.096469	2432.213859	2601.993067
D	2139.119182	2146.116428	2146.116428
E	1427.297189	1647.085330	1696.908626
F	1673.201172	2007.648710	2007.648710
G	2013.634545	2013.634545	2294.032041
H	2101.093406	2324.743091	2324.743091

Table 9. Result of solving model (26), (27) and (28) for all of the DMUs

DMU		A	B	C	D
Minimum allocation distance to fair allocation	$\eta_o^l = \eta_o^{*l}$	303	0	333	411
	$\eta_o^m = \eta_o^{*m}$	259	0	271	344
	$\eta_o^u = \eta_o^{*u}$	0	0	104	102
DMU		E	F	G	H
Minimum allocation distance to fair allocation	$\eta_o^l = \eta_o^{*l}$	73	260	217	271
	$\eta_o^m = \eta_o^{*m}$	48	224	189	238
	$\eta_o^u = \eta_o^{*u}$	0	117	58	109

2. The minimum allocation distance relative to the fair allocation for units A, B and E (related to $\eta_o^u = \eta_o^{*u}$) is zero. It can be seen in Table 1, Units A, B and E have the maximum upper bound efficiency before the new resource is added.

5. Comparative Analysis With Existing Approaches

Figure 1 compares the fair allocation with the allocations produced by the proposed model when targeting (i) DMU_1 and (ii) DMU_{11} for efficiency improvement. The grouped bars show how the proposed allocations deviate from fairness across all 12 DMU_s. Consistent with the results, the L1 distance from fairness is 1.14 (target=DMU_1) and 3.69 (target=DMU_{11}), while both targeted units become efficient and other units do not lose efficiency.

Figure 3 summarizes the minimum distance to fair allocation achieved by the three fuzzy variants—maximizing lower bound (Model 26), middle value (Model 27), and upper bound (Model 28)—for DMUs A–H. This highlights the fairness–efficiency trade-off across targets; e.g., B attains zero distance for LB and MV cases, reflecting that it already had high baseline bounds. Figure 2 plots the post-allocation efficiencies for both targeting strategies. As reported in our paper, targeting DMU1 and DMU11 each ensures DMUo becomes efficient (θ=1); non-targets maintain or improve efficiency, aligning with the linear model's guarantee.

Table 10 (summary).

- Cook & Kress (1999) emphasize fairness via axioms (efficiency-invariance, Pareto-minimality) but do not guarantee making a chosen inefficient DMU efficient.
- Cook & Zhu (2005) distribute shared costs fairly via DEA but are not designed to force a specific DMU to efficiency.
- Beasley (2003) maximizes average performance but may suffer infeasibility and lacks explicit fairness-distance control.
- Jahanshahloo et al. (2004; 2017) develop common-weights / efficiency-invariance schemes; strong on fairness invariance, not DMU-targeted.
- Lin & Chen (2017) treat fixed cost as a complement to inputs; fairness distance not optimized.
- Network/two-stage works (e.g., Yu et al., Li et al., Zhu et al.) extend allocation to internal processes; they balance network fairness/efficiency but do not minimize fairness distance nor target a specific DMU's efficiency.
- Game-theoretic allocations (e.g., Shapley/Nucleolus in Feng & Ramos, 2024) are fairness-principled but not integrated with DEA's targeted efficiency guarantee.
- This work (proposed) uniquely guarantees an inefficient DMU becomes efficient and minimizes distance to fair allocation in both crisp and fuzzy contexts, with linear solvability; the main limitation is that it targets one DMU per solve, requiring multiple solves for multiple targets.

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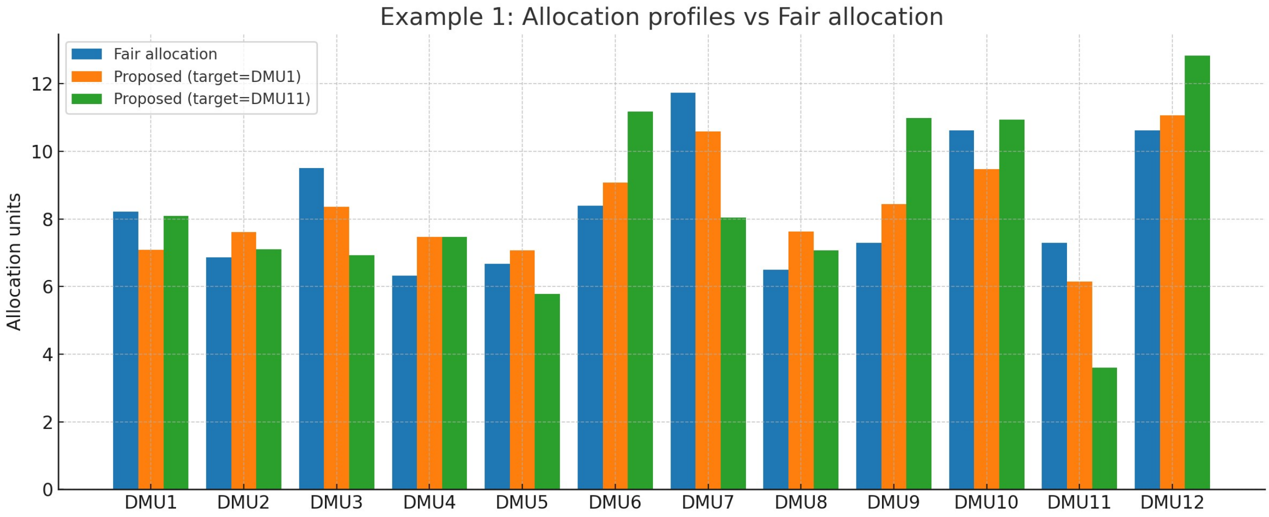

Figure 1. Allocations at different DMUs

Table 10. contrasts our framework with representative approaches in the literature on key capabilities—targeted efficiency guarantee, fairness control, explicit distance minimization, fuzzy support, and linearity—and notes typical limitations

Approach	Guarantees targeted DMU becomes efficient?	Fairness criterion controlled?	Minimizes distance to fair allocation?	Supports fuzzy data?	Linearity / LP?	Notable limitations
Cook & Kress (1999) [3]	No	Yes (axioms)	No (implicit only)	No	Yes	Focus on fairness; does not guarantee inefficient DMU becomes efficient
Cook & Zhu (2005) [4]	No	Yes	No	No	Yes	Shared costs via DEA; not designed to force specific DMU efficiency
Beasley (2003) [6]	No	Indirect	No	No	Nonlinear	May face infeasibility; optimizes mean performance not fairness distance
Jahanshahloo et al. (2004/2017) [39, 8]	No	Yes (common weights)	No	No	Yes	Emphasizes efficiency invariance; not DMU-targeted
Lin & Chen (2017) [10]	No	Complement input	No	No	DEA-based	Treats fixed cost as input complement; no fairness-distance control
Yu et al. (2016) [11]	No	Yes (network)	No	No	DEA-based	Network focus; no DMU-targeted efficiency guarantee
Li et al. (2018) [13]-[16]	No	Various	No	No	Mixed	Cross-efficiency/game variants; not fairness-distance minimizing
Feng & Ramos (2024) [20]	No	Yes (Shapley/Nucleolus)	Partially	No	Varies	Game allocations; not integrated with DEA efficiency targets
This work (proposed)	Yes (guaranteed)	Yes	Yes (explicit L1)	Yes (triangular fuzzy)	Yes (linearized)	One DMU targeted per solve (solve multiple times for many)

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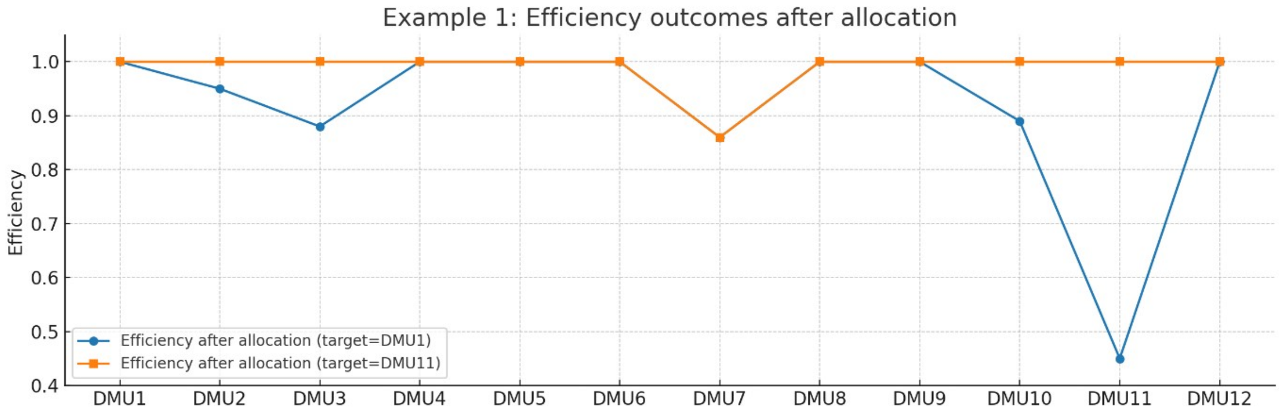

Figure 2. Efficiency of DMUs after allocations

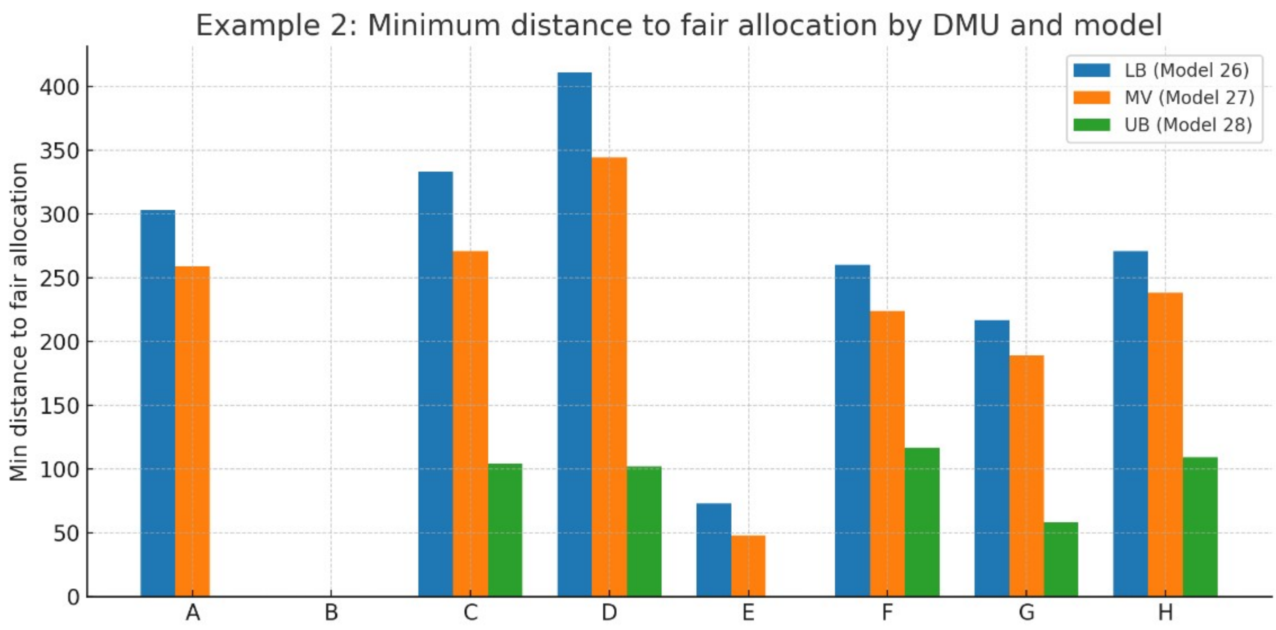

Figure 3. Minimum distances to fair allocations

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6. Conclusion

In many papers, fixed cost allocation between units has been discussed. In each of the papers presented in this regard, the authors pursue a specific purpose of this allocation. In this paper, we want this allocation to be such that a specific inefficient unit becomes efficient. The reason for this idea may be that a central decision maker may pay special attention to some inefficient units in such a way that he wants to turn them into efficient units. In addition, in order to have an allocation as fair as possible, the objective function of the model is such that the resulting allocation has the minimum possible distance from the fair allocation. In the following, due to the widespread use of fuzzy data in applied Sciences, the proposed model is generalized to the case where inputs and outputs are fuzzy. The advantage of the models presented in this article is their linearity, which makes them much easier to use. In addition, no articles on the subject have been published so far.

But the disadvantage of the method presented in this paper is that the proposed model turns only one inefficient unit into an efficient one. So the proposed model must be solved separately for each inefficient unit and each time it will provide a different allocation than before. Therefore an interesting idea that could be the subject of future research is to generalize the models in which several inefficient units become efficient at the same time.

Authors contributions
All the authors have participated sufficiently in the intellectual content, conception and design of this work or the analysis and interpretation of the data (when applicable), as well as the writing of the manuscript.

Availability of data and materials
The data that support the findings of this study are available from the corresponding author, upon reasonable request.

Conflict of interests
The author declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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