Holographic thermal propagator from modularity

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ABSTRACT: It is known that the holographic thermal propagator in 4 spacetime dimensions can be related to the Nekrasov-Shatashvili limit of the Ω-deformed $\mathcal{N} = 2$ supersymmetric SU(2) Yang-Mills theory with $N_f = 4$ hypermultiplets. There are two expansions involved: one is the expansion in small temperature which in the Seiberg-Witten language is equivalent to the semiclassical expansion in inverse powers of the large adjoint vev and the second is the expansion in instanton numbers. Working in the simplified case of zero energy, we find that the latter expansion gives rise to quasi-modular forms which can be resummed as functions of Eisenstein series. The so obtained series in positive powers of small temperature shows clear signs of being asymptotic.

KEYWORDS: AdS-CFT Correspondence, Scale and Conformal Symmetries, Supersymmetric Gauge Theory, Thermal Field Theory

Contents

1 Introduction

There have been various attempts to compute the low temperature expansion of the holographic [1–3] thermal propagator of operators with conformal dimension $\Delta = \nu + 2$ in 4 spacetime dimensions. Differently from 2 spacetime dimensions where the solution is known analytically [4], in 4d a compact analytic form is not known. The above mentioned attempts run from the $\vec{k} = 0$ case [5], large conformal dimension limit [6, 7], to fixed order calculations in the low temperature expansion [8–10]. The method found in [8] seems the most successful among them since it gives a well defined and prescribed ansatz for a given term of the temperature expansion of the propagator. Indeed, the method has been further developed to a very efficient prescription [11, 12]. Notice that these methods give only the stress-tensor sector of the OPE expansion. For the complete full retarded propagator one needs also the double-trace sector, which makes sure that at the temperature T the propagator at time τ is equal to the propagator at time $1/T - \tau$, i.e. it satisfies the KMS relation [13, 14]. To get the contribution of the double-trace sector see for example [8, 9, 12, 15, 16].

Already from the first attempts it was clear that in the 5d bulk the equation to solve is the Heun equation. The constraint on the horizon and the conditions on the boundary need to relate the solutions around different singular points of the Heun equation, i.e. to solve the so called connection problem. How to solve it in general has been shown recently in [17], and used for the thermal propagator in [18]. Although in this way the full propagator is in principle known, it is still written as an infinite sum, where higher terms need more effort to be calculated: it is essentially the calculation of the instanton contribution to the prepotential

F of a Ω-deformed Nekrasov-Shatashvili [19] limit of the $\mathcal{N} = 2$ supersymmetric SU(2) gauge theory with $N_f = 4$ hypermultiplets in the fundamental-antifundamental representation [20], where the masses of the four quarks depend on the shifted scaling dimension $\nu = \Delta - 2$ and the ratio of the energy of the propagator with the temperature ω/T.

Schematically, the holographic retarded thermal propagator is given by [18]

$$G_R \propto \exp\left(-2\partial_\nu H(a, \nu, \omega, q)\right), \tag{1.1}$$

where the power of q counts the instanton number of the instanton prepotential $H(a, \nu, \omega, q)$. The adjoint vev a of the scalar in the vector multiplet is connected to the 3-momentum square $\vec{k}^2$ of the propagator through the Matone relation [21]

$$\vec{k}^2 \propto q\partial_q H(a, \nu, \omega, q) + \ldots, \tag{1.2}$$

where the dots represent a polynomial in a, ν and ω.

We are interested in the propagator of the black brane and hence, we have to fix $q = \exp(-\pi)$ at the end of the calculation. This means that the propagator depends on the energy ω, the momentum square $\vec{k}^2$ and the shifted scaling dimension ν. The temperature dependence enters by replacing the dimensionless energy ω and momentum $\vec{k}$ with

$$\omega \to \omega/T, \quad \vec{k}^2 \to \vec{k}^2/T^2, \tag{1.3}$$

i.e. as a mass unit, so that the low temperature expansion essentially means a large ω and large $\vec{k}^2$ expansion, and, through the Matone relation, it turns out that a is large too. Therefore, crucial to performing a low temperature expansion of the black brane propagator, is the calculation of H in this limit of large $\vec{k}^2$ and ω.

$$H(a, \nu, \omega, q) = \sum_{n=1}^{\infty} q^n \frac{N_n(a, \nu, \omega)}{D_n(a)}. \tag{1.4}$$

There is a well known [22] although cumbersome prescription to get the ratios N_n/D_n. However, in the small temperature expansion we already encounter a problem: both a and ω are large. Complications connected to it will be trivially bypassed by studying the limiting case of $\omega = 0$:

$$H(a, \nu, q) = \sum_{n=1}^{\infty} q^n \frac{N_n(a, \nu)}{D_n(a)}, \tag{1.5}$$

such that we have only to expand in large a. The results will thus not be general, but will nevertheless give quite some information.

The second problem is the infinite sum over powers of q. Since the natural expansion is in large a, we will rewrite the above sum as[1]

$$H(a, \nu, q) = \sum_{n=1}^{\infty} \frac{\alpha_{2n}(q, \nu)}{a^{2n}}. \tag{1.6}$$

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[1]Since only two quark masses are nonzero in the $\omega = 0$ case, only even powers of a appear.

Through explicit computation and general arguments we will find that the prepotential is quasi-modular in $q = \exp(i\pi\tau)$. This property will allow for an efficient resummation in q with low computational cost. Of course it remains an infinite sum over inverse powers of a, but since on dimensional ground $a \propto 1/T$, this expansion is nothing else than the expansion we are looking for. We will be able to write the explicit exact form of $\alpha_{2n}(q, \nu)$ for n not too large (we computed them up to $2n = 58$ but higher coefficients could be easily found). The efficiency seems comparable to other methods [8, 12] but it is done in momentum space and in the $\omega \to 0$ limit instead of the coordinate space and $\vec{k} \to 0$ limit of [12]. This efficiency and the curious quasi-modularity of the propagator[2] are the main results of this paper.

The rest of this paper is organized as follows: in section 2, we review Zamolodchikov's q-recursion and show its importance for the 4-point conformal block of a $2d$ CFT as well as for the instanton part of the Nekrasov partition function. In section 3, we introduce the retarded two-point function of a black brane, which we bring to a compact form for the limit of a large adjoint scalar vev a. This limit corresponds, due to the Matone relation, to the low-temperature limit in which we are interested. In section 4, we explain the S-duality constraints on the prepotential and elaborate on how this leads to quasi-modular forms. We then use this quasi-modularity of the prepotential coefficients in the large a-expansion to calculate them efficiently by matching with Zamolodchikov's q-recursion. Finally, in section 5, we give the low temperature expansion of the propagator. In appendix A, we review the holographic two-point functions for the black brane. In appendix B, we give the Eisenstein series and some of their properties. In appendix C, we present the formulas of α_{2n} up to $2n = 18$, while in appendix D, we give the explicit expansion of the propagator up to T^{40}.

2 Zamolodchikov's *q*-recursion

In this section we rewrite the instantonic part of the Nekrasov-Shatashvili prepotential in terms of the renormalised instanton number q instead of the unrenormalised instanton number t used in [18] applying the AGT corrspondence [23] and the Zamolodchnikov solution [24–27]. This will then be used in the next section.

In general, the 4-point conformal block of a $2d$ CFT can be written as

$$\mathcal{F}(\Delta, \Delta_i, t) = (16q)^{-\alpha^2} t^{Q^2/4 - \Delta_1 - \Delta_2} (1 - t)^{Q^2/4 - \Delta_1 - \Delta_3} \\ \times \theta_3(q)^{3Q^2 - 4(\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4)} \mathcal{H}(q, \Delta, \vec{\mu}), \tag{2.1}$$

where $\Delta_{i=1,2,3,4}$ are the conformal dimensions of the external primary fields at the positions $0, t, 1, \infty$, Δ is the internal (intermediate) conformal dimension and Q is the background charge, related to the central charge of the Virasoro algebra via $c = 1 - 6Q^2$. Additionally, we can parametrize the dimensions as

$$\Delta_i = \frac{Q^2}{4} - \lambda_i^2, \quad \Delta = \frac{Q^2}{4} - \alpha^2, \tag{2.2}$$

where α is the internal momentum that appears in (2.1) and λ_i gives the parameters $\vec{\mu}$ inside $\mathcal{H}$:

$$\mu_1 = \lambda_1 + \lambda_2 + \frac{Q}{2}, \quad \mu_2 = \lambda_1 - \lambda_2 + \frac{Q}{2}, \quad \mu_3 = \lambda_3 + \lambda_4 + \frac{Q}{2}, \quad \mu_4 = \lambda_3 - \lambda_4 + \frac{Q}{2}. \tag{2.3}$$

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[2]The quasi-modularity of the propagator could have been guessed already from [18] and is certainly well known in the $\mathcal{N} = 2$ community.

In the AGT correspondence [23], the central charge of the Liouville theory is related to the Nekrasov deformation parameters ϵ_1, ϵ_2 of the Ω-background in the $4d$ gauge theory as

$$Q = \frac{\epsilon_1 + \epsilon_2}{\sqrt{\epsilon_1 \epsilon_2}} \tag{2.4}$$

and the coupling t is related to the renormalized coupling q via

$$q(t) = \exp\left(-\pi \frac{K(1-t)}{K(t)}\right), \quad K(t) = \frac{1}{2}\int_0^1 \frac{dx}{\sqrt{x(1-x)(1-tx)}}, \tag{2.5}$$

i.e. conversely

$$t(q) = \left(\frac{\theta_2(q)}{\theta_3(q)}\right)^4, \quad \text{where} \quad \theta_2(q) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2}, \quad \theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}. \tag{2.6}$$

In order to compute $\mathcal{H}$, we can use the Zamolodchikov q-recursion [24–26]:

$$\mathcal{H}(q, \Delta, \vec{\mu}) = 1 + \sum_{m,n=1}^{\infty} \frac{q^{mn} R_{m,n}(\vec{\mu})}{\Delta - \Delta_{m,n}} \mathcal{H}(q, \Delta_{m,n} + mn, \vec{\mu}), \tag{2.7}$$

where

$$\Delta_{m,n} = \lambda_{1,1}^2 - \lambda_{m,n}^2, \quad \lambda_{m,n} = \frac{m\epsilon_1 + n\epsilon_2}{2\sqrt{\epsilon_1 \epsilon_2}} \tag{2.8}$$

and

$$R_{m,n}(\vec{\mu}) = \frac{2\prod_{r,s}\prod_{i=1}^4 \left(\mu_i - \frac{Q}{2} - \lambda_{r,s}\right)}{\prod'_{k,l} \lambda_{k,l}} \tag{2.9}$$

with

$$r = -m+1, -m+3, \ldots, m-1$$
$$s = -n+1, -n+3, \ldots, n-1$$
$$k = -m+1, -m+2, \ldots, m-1, m$$
$$l = -n+1, -n+2, \ldots, n-1, n$$

and the prime over the product in the denominator denotes that we have to skip the pairs $(k,l) = (m,n)$ and $(0,0)$.

We are interested in the $\mathcal{N} = 2$ SYM theory with SU(2) gauge group and extra four anti-fundamental hypermultiplets, $f = 4$. For this case, the instanton part of the Nekrasov partition function [22], after applying the AGT conjecture, is given by [23]

$$Z_{\text{inst}}^{(4)}(a, m_i, t, \epsilon_1, \epsilon_2) = t^{\Delta_1 + \Delta_2 - \Delta}(1-t)^{2(\lambda_1 + Q/2)(\lambda_3 + Q/2)} \mathcal{F}(\Delta, \Delta_i, t) \tag{2.10}$$

with $a = \alpha\sqrt{\epsilon_1 \epsilon_2}$ being the vev of the adjoint scalar in the vector multiplet, parameterizing the Coulomb branch and $m_i = (\mu_i - Q/2)\sqrt{\epsilon_1 \epsilon_2}$ being the masses of the four hypermultiplets, which using $\vec{a} = (a_t, a_0, a_1, a_\infty) = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)\sqrt{\epsilon_1 \epsilon_2}$, are given by

$$m_1 = a_t + a_0, \quad m_2 = a_t - a_0, \quad m_3 = a_1 + a_\infty, \quad m_4 = a_1 - a_\infty. \tag{2.11}$$

The instanton part of the Nekrasov partition function, after applying (2.1), reduces to

$$Z_{\text{inst}}^{(4)}(a, m_i, q, \epsilon_1, \epsilon_2) = \left(\frac{t(q)}{16q}\right)^{\alpha^2}(1-t(q))^{\frac{1}{4}\left(Q-\sum_{i=1}^4 \mu_i\right)^2}\theta_3(q)^{2\sum_{i=1}^4 \mu_i(\mu_i-Q)+Q^2}\mathcal{H}(q, \Delta, \vec{\mu})$$

(2.12)

and hence the instanton part of the Nerkrasov partition function in the $\mathcal{N} = 2$ SYM, $N_c = 2, N_f = 4$ theory can be computed with the help of Zamolodchikov's q-recursion.

While the instanton part of the Seiberg-Witten prepotential [20] is given by the limit

$$F_{\text{inst}}^{\text{SW}} = -\lim_{\epsilon_1, \epsilon_2 \to 0}\epsilon_1\epsilon_2 \log Z_{\text{inst}}^{(4)},$$

(2.13)

taking the Nekrasov-Shatashvili (NS) limit $\epsilon_1 = 1, \epsilon_2 \to 0$ [19] gives the instanton part of the NS function

$$F_{\text{inst}}^{\text{NS}} = \lim_{\epsilon_1 \to 1, \epsilon_2 \to 0}\epsilon_1\epsilon_2 \log[(1-t)^{-2(Q/2+\lambda_1)(Q/2+\lambda_3)}Z_{\text{inst}}^{(4)}],$$

(2.14)

where the $(1-t)^{-2(Q/2+\lambda_1)(Q/2+\lambda_3)}$ prefactor is present in order to remove the U(1) factor from the U(2) Nekrasov partition function [23].

We can write the NS free energy in a more explicit way using (2.12) and (2.14), which will be useful later on,[3]

$$F(q, a, \vec{a}) = a^2\left(\log\left(t(q)\right) - \log\left(16q\right)\right) + \left(a_t^2 + a_1^2 - \frac{1}{4}\right)\log\left(1 - t(q)\right)$$
$$+ \left(a_0^2 + a_t^2 + a_1^2 + a_\infty^2 - \frac{1}{4}\right)\log\theta_3^4(q) + H(q, a, \vec{a})$$

(2.15)

where we defined

$$H(q, a, \vec{a}) \equiv \lim_{\epsilon_1 \to 1, \epsilon_2 \to 0}\epsilon_1\epsilon_2 \log \mathcal{H}(q, \Delta, \vec{\mu}).$$

(2.16)

Therefore, we can compute the NS free energy using the Zamolodchikov q-recursion.

3 The retarded Green's function of the black brane

We study a holographic conformal field theory defined on $S^1 \times \mathbb{R}^3$, which is dual to a black brane geometry. The two-point function of a scalar operator in this CFT is obtained by solving the bulk wave equation for its dual scalar field in the black brane background, and then extracting the retarded Green's function.

For a scalar operator of conformal dimension $\Delta = 2 + \nu$, the retarded Green's function is given by the ratio of the response to the source at the AdS boundary. This quantity can be computed explicitly by solving the associated Heun equation, characterized by singular points at $z = 0, t, 1, \infty$ with parameters a_0, a_t, a_1, a_∞, and u [18]:

$$\left[\frac{d^2}{dz^2} + \frac{\frac{1}{4} - a_0^2}{z^2} + \frac{\frac{1}{4} - a_1^2}{(z-1)^2} + \frac{\frac{1}{4} - a_t^2}{(z-t)^2} + \frac{a_0^2 + a_1^2 + a_t^2 - a_\infty^2 - \frac{1}{2}}{z(z-1)} - \frac{(1-t)u}{z(z-1)(z-t)}\right]\chi = 0,$$

(3.1)

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[3]From now on the quantity $F_{\text{inst}}^{\text{NS}}$ will be called simply F.

(for more details see appendix A). The Heun parameters are related to physical quantities of the Minkowski black brane as

$$t = \frac{1}{2}, \ \vec{a} = (a_0, a_t, a_1, a_\infty) = \left(0, \frac{i\omega}{4\pi}, \frac{\nu}{2}, \frac{\omega}{4\pi}\right), \ u = \frac{\omega^2 - 2k^2}{8\pi^2} - \frac{\nu^2}{4}. \tag{3.2}$$

It has been shown [17] that the exact solution to the connection problem of the Heun equation can be expressed through the Nekrasov-Shatashvili prepotential [19] of the $\mathcal{N} = 2$ supersymmetric SU(2) Yang-Mills theory with $N_f = 4$ hypermultiplets, where the masses of the hypermultiplets are combinations of a_0, a_t, a_1, a_∞ (see (2.11)), t is the instanton counting parameter and u parametrizes the moduli space of vacua. The latter is connected with the parameter a corresponding to the vev of the scalar in the vector multiplet via the Matone relation [21]

$$u = -a^2 + a_t^2 + a_0^2 - \frac{1}{4} + t\partial_t F, \tag{3.3}$$

where F is the instanton part of the NS free energy defined in (2.15).

We are interested in the limit where $\omega = 0$ and $|\vec{k}| \to \infty$, i.e. such that when promoted to dimensionful parameters $\omega \to \omega/T$ and $|\vec{k}| \to |\vec{k}|/T$, we are in the zero energy and low temperature limit (low with respect to the three-momentum). In this case, we see that the Matone relation implies $a \to +\infty$.

Following [18], we can write the propagator as (modulo exponentially small terms for the limit we are interested in, $a \to +\infty$)

$$G_R = \pi^{4a_1} e^{-\partial_{a_1} F} \frac{M(a, a_1; a_\infty)}{M(a, -a_1; a_\infty)}, \tag{3.4}$$

where

$$M(\alpha_0, \alpha_1; \alpha_2) = \frac{\Gamma(-2\alpha_1)\Gamma(1 - 2\alpha_0)}{\Gamma(1/2 - \alpha_0 - \alpha_1 + \alpha_2)\Gamma(1/2 - \alpha_0 - \alpha_1 - \alpha_2)}. \tag{3.5}$$

Now, we want to perform a large a expansion of this formula and finally bring the propagator into a compact and useful form for the low temperature expansion. In order to do so, we first need to make use of the gamma function formula in the large a limit, given by [28]

$$\log \Gamma(a + \alpha) = (a + \alpha - 1/2) \log a - a + \frac{1}{2} \log(2\pi) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} B_{n+1}(\alpha)}{n(n+1)a^n} \tag{3.6}$$

with $B_n(\alpha)$ the Bernoulli polynomials, so that (an alternative is to use [29])

$$\frac{\Gamma(a + \alpha)}{\Gamma(a + \beta)} = a^{\alpha - \beta} \exp\left(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n(n+1)a^n} \left(B_{n+1}(\alpha) - B_{n+1}(\beta)\right)\right). \tag{3.7}$$

Note that in the case of $\alpha + \beta = 1$, we have $B_l(1 - \alpha) = (-1)^l B_l(\alpha)$ for any $l > 1$.

Then, we rewrite the Zamolodchikov q-recursion into an explicit form

$$\mathcal{H}(q, \Delta, \vec{\mu}) = 1 + \sum_{\chi=1}^{\infty} c_\chi(a, \vec{\mu}, \epsilon_j) q^\chi, \tag{3.8}$$

where

$$c_\chi(a, \vec{\mu}, \epsilon_j) = \sum_{k=1}^{\chi} \sum_{\sum_{i=1}^{k} m_i n_i = \chi} \frac{\epsilon_1 \epsilon_2}{\left(-a^2 + \left(\frac{m_1 \epsilon_1 + n_1 \epsilon_2}{2}\right)^2\right)} \frac{\prod_{i=1}^{k} R_{m_i n_i}(\vec{\mu})}{\prod_{i=1}^{k-1} \Delta_{m_i n_i} + m_i n_i - \Delta_{m_{i+1} n_{i+1}}}. \quad (3.9)$$

We use this formula to expand (2.16) for large a and obtain

$$H(a, q, \nu) = \sum_{n=1}^{\infty} \frac{\alpha_{2n}(q, \nu)}{a^{2n}}. \quad (3.10)$$

Now, with all of the steps before and using (2.15), we can finally write the propagator in a compact form:

$$\frac{G_R}{G_R^0} = \left(\frac{a^2}{x^2}\right)^\nu \exp\left(-2 \sum_{n=1}^{\infty} \frac{\partial_\nu \tilde{\alpha}_{2n}}{a^{2n}}\right) \quad (3.11)$$

with the conformal propagator being

$$G_R^0 = \frac{\Gamma(-\nu)}{\Gamma(\nu)} \left(\frac{k}{2}\right)^{2\nu}. \quad (3.12)$$

To obtain this formula, we made additionally use of the following:

First we rewrote

$$\frac{B_{2n+1}\left(\frac{1+\nu}{2}\right)}{n(2n+1)} + \partial_\nu \alpha_{2n} = \partial_\nu \tilde{\alpha}_{2n} \quad (3.13)$$

Notice that α_{2n} is a polynomial in q^2 with no constant term, since it is a pure instanton contribution, while $\tilde{\alpha}_{2n}$ has on top of that also a q-independent piece. This rewriting will be useful since $\tilde{\alpha}_{2n}$ will have a simple and computationally useful form, as we will see later.

Second, we defined

$$x^2 \equiv \frac{k^2}{8\pi \Gamma(3/4)^4} \quad (3.14)$$

and then we used that the connection between a and k (resp. x) is the Matone relation

$$x^2 = a^2 - \frac{\nu^2 - 1}{4\pi} - \sum_{n=1}^{\infty} \frac{q \partial_q \tilde{\alpha}_{2n}}{a^{2n}}, \quad (3.15)$$

which by inversion gives

$$a^2 = x^2 + \sum_{n=0}^{\infty} \frac{\beta_{2n}}{x^{2n}}. \quad (3.16)$$

To compute β_{2n} one has to plug in (3.16) into (3.15) and expand in large x. We give few lower terms:

$$\beta_0 = \frac{\nu^2 - 1}{4\pi}, \quad \beta_2 = q \partial_q \tilde{\alpha}_2, \quad \beta_4 = q \partial_q \tilde{\alpha}_4 - \frac{\nu^2 - 1}{4\pi} q \partial_q \tilde{\alpha}_2. \quad (3.17)$$

Note that although we keep the dependence on q (or equivalently t) general in the intermediate steps because of derivative terms, in the final formula (3.11) we set $t = 1/2$, i.e. $q = \exp(-\pi)$.

Our goal is to obtain the low temperature expansion of the propagator, therefore the procedure consists of first calculating $\tilde{\alpha}_{2n}(q, \nu)$, then finding explicitly (3.16), i.e. $a(k)$ through inversion of (3.15), plugging everything in (3.11) and after replacing $|\vec{k}| \to |\vec{k}|/T$, expanding the propagator in positive powers of T.

4 $\mathcal{S}$-duality constraints on the prepotential: quasi-modular forms

In this section, we analyze the emergence of quasi-modular forms in the Nekrasov-Shatashvili limit of the Ω-deformed $\mathcal{N} = 2$, SU(2) gauge theory with $N_f = 4$, arising from the modular properties of the Seiberg-Witten geometry and its quantum deformation [30–32].

The low-energy effective action on the Coulomb branch of the gauge theory, is determined by the prepotential $F_{\mathrm{NS}}(a)$, which can be written as [18]

$$F_{\mathrm{NS}} = -a^2 \log{(t)} + F_{1-\mathrm{loop}} + F, \tag{4.1}$$

with $F_{1-\mathrm{loop}}$ being the perturbative contribution up to 1-loop and F the instanton contribution.

The 1-loop contribution $F_{1-\mathrm{loop}}$ can be expanded in large a as [33]

$$F_{1-\mathrm{loop}} = a^2 \log{(16)} - \frac{\nu^2 - 1}{4} \log{\left(\frac{a^2}{\Lambda^2} \right)} + \sum_{n=1}^{\infty} \frac{f_{2n}}{a^{2n}}, \tag{4.2}$$

while the large a-expansion of the instanton contribution, using eqs. (2.15), (2.6) and (3.10), is given by

$$F = a^2 \log{(t)} - a^2 \log{(16q)} + \frac{\nu^2 - 1}{4} \log{(\theta_4{}^4)} + \sum_{n=1}^{\infty} \frac{\alpha_{2n}}{a^{2n}}. \tag{4.3}$$

The full Nekrasov-Shatashvili prepotential then becomes

$$F_{\mathrm{NS}} = -i\pi\tau a^2 + F_q \tag{4.4}$$

where τ is the gauge coupling in the IR,

$$\tau = \frac{\theta_{\mathrm{IR}}}{\pi} + i\frac{8\pi}{g_{\mathrm{IR}}^2}, \tag{4.5}$$

which is related to the instanton counting parameter as

$$q = e^{i\pi\tau}, \tag{4.6}$$

and

$$F_q = -\frac{\nu^2 - 1}{4} \log{\left(\frac{a^2}{\Lambda^2 \theta_4{}^4} \right)} + \sum_{n=1}^{\infty} \frac{\alpha_{2n} + f_{2n}}{a^{2n}}, \tag{4.7}$$

with an arbitrary scale Λ.

The same theory can be also described using the dual variable

$$a_D = \frac{1}{2\pi i} \frac{\partial F_{\mathrm{NS}}}{\partial a}. \tag{4.8}$$

Now, τ transforms under the modular group, reflecting the electric-magnetic duality symmetries of the Seiberg-Witten, i.e. it transforms under the generators $\mathcal{S}$ and $\mathcal{T}$ of the modular group PSL$(2, \mathbb{Z})$ as

$$\mathcal{S} : \tau \to -\frac{1}{\tau}, \quad \mathcal{T} : \tau \to \tau + 1 \tag{4.9}$$

while a and a_D transform as

$$\mathcal{S}: a \to a_D, \ a_D \to -a, \quad \mathcal{T}: a \to a, \ a_D \to a_D + a. \tag{4.10}$$

The transformation $\mathcal{T}$ on the instaton part of the prepotential $F_{\text{inst}}(a)$ is trivial, since $F_{\text{inst}}(a) = \sum_k q^{2k} F_k(a)$ and $q \to -q$, $a \to a$. So the prepotential transforms as [31]

$$\mathcal{T}: \quad F_{\text{NS}}(a) \to F_{\text{NS}}(a) - i\pi a^2 \tag{4.11}$$

providing $\Lambda^2 \theta_4{}^4$ remains invariant under $\mathcal{T}$.

However, the $\mathcal{S}$-transform is not trivial, it maps the theory with a to the dual theory with a_D and therefore it should map the prepotential to its Legendre transform

$$\mathcal{S}: F_{\text{NS}}(a) \to F_{\text{NS}}(a) - 2\pi i a a_D = F_{\text{NS}}(a) - a\frac{\partial F_{\text{NS}}}{\partial a}. \tag{4.12}$$

This $\mathcal{S}$-duality condition gives a strong constraint on the prepotential and leads to the fact that the prepotential can be written in terms of quasi-modular forms.

To see the modularity directly, let us compute $\alpha_{2n}(q, \nu)$ from (3.10) using (3.8). The explicit formula for α_{2n}, which comes from the large a expansion of H, is given by

$$\alpha_{2n} = -\lim_{\substack{\epsilon_1 \to 1 \\ \epsilon_2 \to 0}} \left[\sum_{m=1}^{n} \frac{(\epsilon_1 \epsilon_2)^{m+1}}{m} \sum_{\substack{\sum_{l=0}^{\infty} k_l = m \\ \sum_{l=0}^{\infty} k_l (l+1) = n}} \frac{m!}{\prod_{l=0}^{\infty} k_l!} \prod_{l=0}^{\infty} \left(\sum_{\chi=1}^{\infty} \tilde{c}_\chi^l q^\chi \right)^{k_l} \right] \tag{4.13}$$

with

$$\tilde{c}_\chi^l = \sum_{k=1}^{\chi} \sum_{\sum_{i=1}^{k} m_i n_i = \chi} \left(\frac{m_1 \epsilon_1 + n_1 \epsilon_2}{2} \right)^{2l} \frac{\prod_{i=1}^{k} R_{m_i n_i}}{\prod_{i=1}^{k-1} \Delta_{m_i n_i} + m_i n_i - \Delta_{m_{i+1} n_{i+1}}}. \tag{4.14}$$

The first few α_{2n}'s are

$$\alpha_2 = -\frac{(\nu^2 - 1)^2}{4} q^2 (1 + 3q^2 + 4q^4 + 7q^6 + 6q^8 + 12q^{10} + 8q^{12} + \ldots),$$

$$\alpha_4 = -\frac{(\nu^2 - 1)^2}{8} q^2 \left(2 + 3q^2(7 - \nu^2) + 8q^4(9 - 2\nu^2) + q^6(191 - 45\nu^2) \right.$$
$$\left. + 12q^8(29 - 8\nu^2) + 36q^{10}(19 - 5\nu^2) + 16q^{12}(61 - 18\nu^2) + \ldots \right)$$

$$\alpha_6 = -\frac{(\nu^2 - 1)^2}{128} q^2 \left(32 + q^2(1301 - 250\nu^2 + 5\nu^4) + 32q^4(329 - 90\nu^2 + 5\nu^4) \right.$$
$$+ q^6(47159 - 14430\nu^2 + 1095\nu^4) + 64q^8(2254 - 758\nu^2 + 67\nu^4)$$
$$\left. + 12q^{10}(31227 - 10790\nu^2 + 1035\nu^4) + 64q^{12}(12469 - 4530\nu^2 + 465\nu^4) + \ldots \right). \tag{4.15}$$

Note, because $\omega = 0$, it follows that $R_{mn} = 0$ for m and n both odd, which implies $\tilde{c}_\chi^l = 0$ for χ odd and hence only even powers of q appear.

Now, we observe that these coefficients together with the 1-loop constant term are quasi-modular forms [34], i.e.

$$\alpha_{2n}(q, \nu) + f_{2n}(\nu) = \sum_{2k+4l+6m=2n} c_{k,l,m}(\nu) E_2{}^k(q) E_4{}^l(q) E_6{}^m(q) \equiv \tilde{\alpha}_{2n}(q, \nu) \tag{4.16}$$

where $E_{2,4,6}(q)$ are the Eisenstein series summarised in appendix B, and that the constant term is given by

$$f_{2n}(\nu) = \frac{B_{2n+2}\left(\frac{1+\nu}{2}\right) - B_{2n+2}}{(2n+1)n(n+1)}, \tag{4.17}$$

where $B_{2n+2} = B_{2n+2}(0)$ are the Bernoulli numbers.[4] Therefore, the definition of $\tilde{\alpha}_{2n}$ in (4.16) matches the condition in (3.13) and $\tilde{\alpha}_{2n}$ is a quasi-modular form of weight $2n$.

So (4.7) can be written as

$$F_q = -\frac{h_0}{2}\log\left(\frac{a^2}{\Lambda^2\theta_4{}^4}\right) + \tilde{H}(q, a, \nu) \tag{4.18}$$

with

$$h_0 = \frac{\nu^2 - 1}{2} \tag{4.19}$$

and

$$\tilde{H}(q, a, \nu) = \sum_{n=1}^{\infty} \frac{\tilde{\alpha}_{2n}(q, \nu)}{a^{2n}}, \tag{4.20}$$

i.e. with exactly the same parameters needed for the propagator (3.11).

The quasi-modularity of $\tilde{\alpha}_{2n}$ comes from the modular anomaly: if $\mathcal{S} : a^2 \to \tau^2 a^2$, i.e. a^2 would transform as a modular form of weight 2, the instanton contribution (minus the constant piece eq. (4.16)) would read

$$\tilde{\alpha}_{2n}(q, \nu) \sim \sum_{4l+6m=2n} c_{0,l,m}(\nu) E_4^l(q) E_6^m(q), \tag{4.21}$$

since any modular form of positive weight can be written with E_4 and E_6 only [34]. However, the above transformation for a is only classical, the correct one is given in eq. (4.10) together with (4.8) and therefore the approximate (4.21) gets replaced by the now exact quasi-modular $\tilde{\alpha}_{2n}$ in (4.16).

Let's now summarise [30, 31] to show that $\mathcal{S}$ duality fixes uniquely the quasi-modular coefficients $c_{k,l,m}$ for $k > 0$ from the knowledge of the modular coefficients $c_{0,l,m}$. On the one side, we have

$$\mathcal{S}\left(F_{\mathrm{NS}}(\tau, a)\right) = F_{\mathrm{NS}}(-1/\tau, a_D), \tag{4.22}$$

while on the other this is just the Legendre transform (4.12). Equating them and using (4.4) brings us to

$$F_q(-1/\tau, a_D) = F_q(\tau, a) - \frac{1}{4\pi i \tau}\left(\frac{\partial F_q(\tau, a)}{\partial a}\right)^2. \tag{4.23}$$

We can rewrite the dual variable (4.8) as

$$a_D = \tau\left(-a + \frac{\delta}{12a}\left(-h_0 + a\frac{\partial\tilde{H}}{\partial a}\right)\right) \tag{4.24}$$

---

[4]Note also that since $q = 0$ gives $\mathcal{H} = 1$, it follows $\alpha_{2n} = 0$ and due to $E_{2,4,6}(q = 0) = 1$ we have $\sum_{2k+4l+6m=2n} c_{k,l,m} = f_{2n}$.

where we introduced

$$\delta = \frac{6}{i\pi\tau}.\tag{4.25}$$

Now, we want to analyze how the operation

$$\tau \to -1/\tau, \quad a \to a_D \tag{4.26}$$

acts on $\tilde{\alpha}_{2n}/a^{2n}$. Due to (4.16) and the behaviour of E_{2k} (see appendix B) this operation is equivalent to

$$E_2 \to E_2 + \delta, \quad a \to a - \frac{\delta}{12a}\left(-h_0 + a\frac{\partial \tilde{H}}{\partial a}\right). \tag{4.27}$$

Taking now that $\Lambda^2\theta_4^4$ transforms as a modular form of weight 2 we can write (4.23) with a little abuse of notation as

$$\tilde{H}\left(E_2 + \delta, a - \frac{\delta}{12a}\left(-h_0 + a\frac{\partial \tilde{H}}{\partial a}\right)\right) - \tilde{H}(E_2, a)$$
$$= h_0 \log\left(1 - \frac{\delta}{12a^2}\left(-h_0 + a\frac{\partial \tilde{H}}{\partial a}\right)\right) - \frac{\delta}{24a^2}\left(-h_0 + a\frac{\partial \tilde{H}}{\partial a}\right)^2. \tag{4.28}$$

This equation is satisfied for arbitrary δ if

$$\frac{\partial \tilde{H}}{\partial E_2} = \frac{1}{24a^2}\left(-h_0 + a\frac{\partial \tilde{H}}{\partial a}\right)^2. \tag{4.29}$$

Expanding $\tilde{H}$ with inverse powers of a^2, gives

$$\frac{\partial \tilde{\alpha}_2}{\partial E_2} = \frac{h_0^2}{24} \tag{4.30}$$

and for any integer $l > 1$,[5]

$$\frac{\partial \tilde{\alpha}_{2l}}{\partial E_2} = \frac{l-1}{6}h_0\tilde{\alpha}_{2(l-1)} + \frac{1}{6}\sum_{i=1}^{l-2} i(l-i-1)\tilde{\alpha}_{2i}\tilde{\alpha}_{2l-2i-2}. \tag{4.31}$$

The procedure to compute $\tilde{\alpha}_{2n}$ (or α_{2n}) is therefore very simple: it is enough to compute a finite number of powers of q for $\tilde{\alpha}_{2n}$ via (4.13) to determine the finite number of coefficients $c_{k,l,m}(\nu)$ using (4.31) for the coefficients with $k \neq 0$ and matching (4.16) to (4.13) for the coefficients with $k = 0$.[6]

---

[5]This coincides with [33], providing for $l > 0$ $h_l = l2^{l+1}\tilde{\alpha}_{2l}$, the masses of the hypermultiplets are taken as $(\nu/2, \nu/2, 0, 0)$, while the parameters of the Ω-deformation are taken $\epsilon_1 = 1$ and $\epsilon_2 = 0$ (Nekrasov-Shatashvili limit).

[6]The number of powers for q that one needs to evaluate with this procedure is very small and hence the computation of $\tilde{\alpha}_{2n}$ is relatively fast. For example, up to $2n = 58$ it is enough to compute only up to q^8 (or even less).

The first few terms are

$$
\tilde{\alpha}_2 = \frac{(\nu^2 - 1)^2}{96}(E_2)
$$

$$
\tilde{\alpha}_4 = \frac{(\nu^2 - 1)^2}{11520}\left(\left(5E_2^2 + E_4\right)\nu^2 \right.
$$
$$
\left. + \left(-5E_2^2 - 13E_4\right)\right)
$$

$$
\tilde{\alpha}_6 = \frac{(\nu^2 - 1)^2}{5806080}\left(\left(175E_2^3 + 84E_4E_2 + 11E_6\right)\nu^4 \right.
$$
$$
- 2\left(175E_2^3 + 588E_4E_2 + 227E_6\right)\nu^2
$$
$$
\left. + \left(175E_2^3 + 1092E_2E_4 + 3323E_6\right)\right)
$$

$$
\tilde{\alpha}_8 = \frac{(\nu^2 - 1)^2}{92897280}\left(\left(245E_2^4 + 196E_4E_2^2 + 44E_6E_2 + 19E_4^2\right)\nu^6 \right.
$$
$$
- 3\left(245E_2^4 + 980E_4E_2^2 + 620E_6E_2 + 339E_4^2\right)\nu^4
$$
$$
+ 3\left(245E_2^4 + 1764E_4E_2^2 + 5036E_6E_2 + 4883E_4^2\right)\nu^2
$$
$$
\left. + \left(-245E_2^4 - 2548E_2^2E_4 - 13292E_2E_6 - 62035E_4^2\right)\right)
$$

(4.32)

and up to $2n = 18$ can be found in appendix C. While we computed up to $2n = 58$ in our analysis, more terms could be easily calculated. The general form is

$$
\tilde{\alpha}_{2n} = (\nu^2 - 1)^2 \sum_{2k+4l+6m=2n} E_2^k E_4^l E_6^m \sum_{s=0}^{n-1} c_{k,l,m,s}\nu^{2s}.
$$

(4.33)

For example, one finds that

$$
c_{3,10,2,16} = \frac{854275683407387318827877078799212899970556476441216410305 2703057689}{13671919114375695071297538063450424596279849133236717355 0080000}
$$

(4.34)

Although this coefficient will not enter the final expression for the propagator, since $E_6(q = \exp(-\pi)) = 0$, it nevertheless shows a generic pattern: it is large. And, what makes it particular interesting, one finds that such numbers increase with n. For example, the numerical coefficients in front of $E_4^n\nu^2$ of $\tilde{\alpha}_{4n}$ are

$$
c_{0,n,0,1} = \left(0.0000868056, 0.00015769, 0.000775588, 0.0131216, 0.581048, 57.3043, \right.
$$
$$
11013.7, 3.76087 \times 10^6, 2.12402 \times 10^9, 1.87667 \times 10^{12}, 2.48054 \times 10^{15},
$$
$$
\left. 4.72847 \times 10^{18}, 1.26077 \times 10^{22}, 4.58196 \times 10^{25}, \dots\right)
$$

(4.35)

or, the coefficients in front of $E_4^n\nu^4$ of $\tilde{\alpha}_{4n}$ are

$$
c_{0,n,0,2} = \left(0., -0.0000109476, -0.0000927139, -0.00186056, -0.0899762, -9.34756, \right.
$$
$$
-1860.93, -651716., -3.7515 \times 10^8, -3.36431 \times 10^{11}, -4.50028 \times 10^{14},
$$
$$
\left. -8.66299 \times 10^{17}, -2.32881 \times 10^{21}, -8.52239 \times 10^{24}, \dots\right).
$$

(4.36)

5 Low temperature expansion of the black brane propagator

In this section, we finally want to compute the low temperature expansion of the black brane propagator, using the results from the previous sections.

First, note that the odd powers of T are always zero, since the dependence on k in the Matone relation is always quadratic. Second, we explicitly checked that the coefficients of T^{4n-2} vanish in all the cases considered. This result is correct on general grounds,[7] but is not obvious from the computational point of view, since all even coefficients appear in the instanton expansion of H. Also, the propagator as a function of a^2, i.e. (3.11) before using (3.15), has all terms proportional to $1/a^{2n}$ with $n \geq 3$ non-zero. Thus, there is a cancellation at work.

However, there is not even a partial cancellation for the terms proportional to T^{4n}: the large magnitude of $\tilde{\alpha}_{2n}$ causes the expansion of the propagator in powers of $\hat{T} \equiv (T/k)^4$ to be asymptotic. As an example, the expansion of the ν^5 term of the propagator is

$$
\left. \frac{G_R}{G_R^0} \right|_{\nu^5} = -12.9879\hat{T} - 68317.4\hat{T}^2 - 1.43349 \times 10^9 \hat{T}^3 - 1.10667 \times 10^{14} \hat{T}^4 \\
- 2.26782 \times 10^{19} \hat{T}^5 - 1.0404 \times 10^{25} \hat{T}^6 - 9.32122 \times 10^{30} \hat{T}^7 + \mathcal{O}\left(\hat{T}^8\right)
\tag{5.1}
$$

while the ν^{10} term gives

$$
\left. \frac{G_R}{G_R^0} \right|_{\nu^{10}} = 84.3425\hat{T}^2 + 1.42249 \times 10^6 \hat{T}^3 + 3.87789 \times 10^{10} \hat{T}^4 + 3.01096 \times 10^{15} \hat{T}^5 \\
+ 6.16188 \times 10^{20} \hat{T}^6 + 2.84528 \times 10^{26} \hat{T}^7 + \mathcal{O}\left(\hat{T}^8\right).
\tag{5.2}
$$

The analytic formula of the low temperature expansion of the propagator is

$$
\frac{G_R}{G_R^0} = 1 + \sum_{l=1}^{10} (-\pi^4)^l C_l \frac{\Gamma[3+\nu]}{\Gamma[-d_l+\nu]} P_l(\nu) \hat{T}^l + \mathcal{O}\left(\hat{T}^{11}\right),
\tag{5.3}
$$

where

$$
d_l = \frac{4l - 1 + (-1)^{l+1}}{2}, \quad C_l = 2^{e_{l,2}} \prod_{p \text{ odd prime}} p^{-e_{l,p}}
\tag{5.4}
$$

with

$$
e_{l,2} = \sum_{k \geq 1} \left(\left\lfloor \frac{2l}{2^k} \right\rfloor - 2 \left\lfloor \frac{l}{2^k} \right\rfloor \right), e_{l,3} = l + \sum_{k \geq 1} \left\lfloor \frac{l}{3^k} \right\rfloor, e_{l,5} = l + \sum_{k \geq 1} \left\lfloor \frac{l}{5^k} \right\rfloor, \\
e_{l,p} = \left\lfloor \frac{l}{\lfloor (p+1)/4 \rfloor} \right\rfloor, \quad p \geq 7.
\tag{5.5}
$$

---

[7]To see it one must either remember that the contributions we are computing come for multi-stress-tensors, or see the Heun equation which explicitly depend only on T^4.

The $P_l(\nu)$ are polynomials with degree $\deg P_l(\nu) = 2\lfloor (3l-2)/2 \rfloor$ and here we give the first few (more can be found in appendix D),

$$
\begin{aligned}
P_1(\nu) &= 1, \\
P_2(\nu) &= 1000 + 324\nu - 62\nu^2 - 9\nu^3 + 7\nu^4, \\
P_3(\nu) &= 4561360 + 4318992\nu + 1416624\nu^2 + 166500\nu^3 + 8535\nu^4 + 5148\nu^5 + 1001\nu^6, \\
P_4(\nu) &= 1701277027200 + 1857907875520\nu + 740735545504\nu^2 + 110670770480\nu^3 \\
&\quad - 3496530760\nu^4 - 1981950060\nu^5 + 97325502\nu^6 + 23044145\nu^7 - 5114365\nu^8 \\
&\quad - 85085\nu^9 + 119119\nu^{10}, \\
P_5(\nu) &= 3370482566457600 + 4815379612782080\nu + 2888959672781312\nu^2 \\
&\quad + 933753289342592\nu^3 + 170641909496096\nu^4 + 16292158804640\nu^5 \\
&\quad + 619660529696\nu^6 + 44793536536\nu^7 + 13694280493\nu^8 + 1029635080\nu^9 \\
&\quad + 41845942\nu^{10} + 20692672\nu^{11} + 2263261\nu^{12}.
\end{aligned}
\tag{5.6}
$$

The asymptotic behavior of the series[8] was also observed in [36] and the series were approximately resummed using a Borel [15] or a mixed Padé-Borel transform [12]. In a similar spirit we will analyse here the low-temperature expansion of the propagator by employing Padé approximants as an alternative resummation technique and compare the result with the numerical solutions of the Heun equation.[9]

The low-temperature expansion of the propagator is an asymptotic series with zero radius of convergence. While truncating this expansion produces a polynomial that gives a reasonable approximation within a narrow domain, it cannot reproduce the correct analytic structure. In contrast, constructing a Padé approximant from the same series yields a rational function that typically provides a more accurate representation over a broader range, since it can capture features such as poles that polynomials miss.

We construct rational approximations of the propagator for $\nu = 0.5$ starting from its low-temperature expansion, where we truncated the series (5.3) at order $d = 7$ in $\hat{T}$, yielding a polynomial expansion. From this truncated series, Padé approximants of order $[n/m]$ with $n + m = d$ of G_R/G_R^0 are defined as rational functions of the form

$$
P_{n,m}(\hat{T}) = \frac{a_0 + a_1\hat{T} + \cdots + a_n\hat{T}^n}{1 + b_1\hat{T} + \cdots + b_m\hat{T}^m},
\tag{5.7}
$$

whose Taylor expansion around $\hat{T} \to 0$ coincides with the truncated series up to $\mathcal{O}(\hat{T}^d)$.

By construction, the Padé approximants reproduce the asymptotic behavior in the limit $\hat{T} \to 0$. To assess their validity at finite $\hat{T}$, we compared $P_{n,m}(\hat{T})$ against the numerical solution of the propagator. We show the matching of the Padé approximant with the truncated perturbative series and the numerical solution in figure 1. The figure illustrates that the Padé approximant coincides with the truncated perturbative series for $0 \leq \hat{T} < \hat{T}_1$, while it also agrees with the numerical solution within a finite interval $\hat{T}_1 \leq \hat{T} \leq \hat{T}_2$. Consequently, the Padé approximant provides the appropriate description for $0 < \hat{T} \leq \hat{T}_2$, whereas the

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[8]For a similar situation in Seiberg-Witten theory see for example [35].

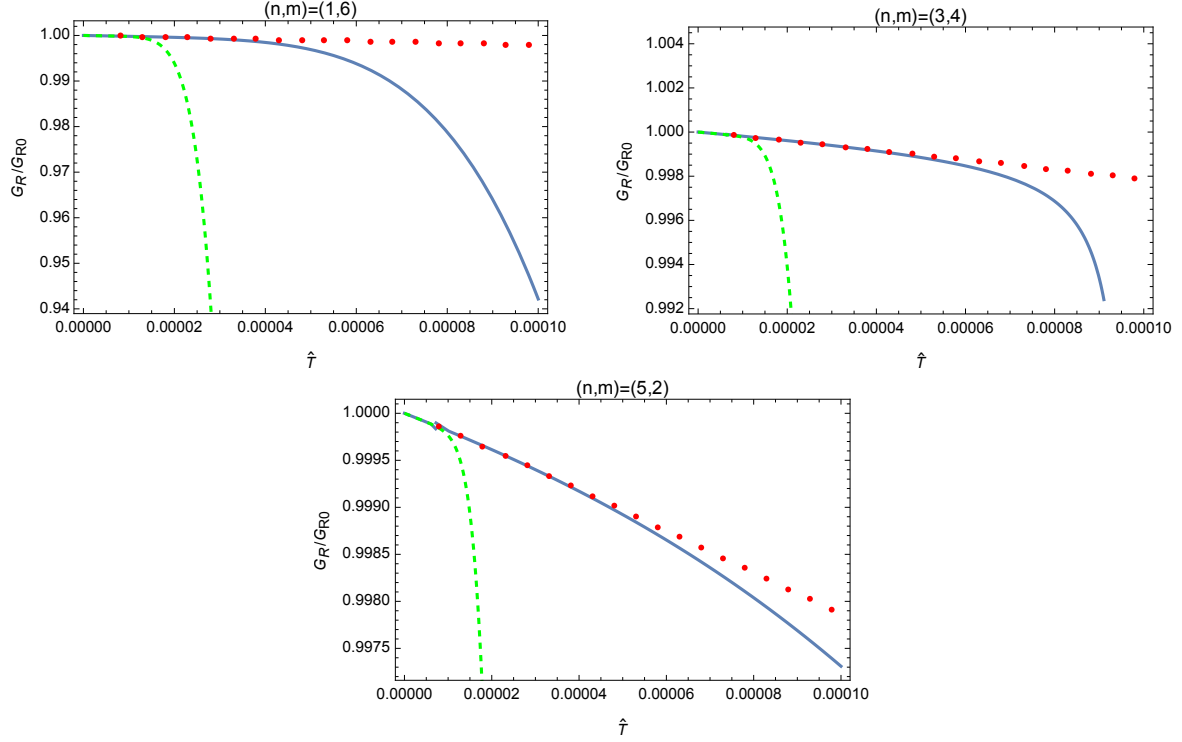[9]We thank Sašo Grozdanov for suggesting it.

Figure 1. For $\nu = 0.5$, we compare the Padé approximant (blue) with the truncated low-temperature expansion of the propagator (dashed, green) as $T \to 0$, and with the numerical solution (dotted, red) within a finite interval. The three panels display Padé approximants of degree $(n, m) = (1, 6)$, $(3, 4)$, and $(5, 2)$, respectively.

truncated perturbative series is valid only for small enough $\hat{T} < \hat{T}_1$ (the full asymptotic series would be valid only strictly at $T = 0$) and only the numerical solution must be employed for $\hat{T} > \hat{T}_2$. The three Padé approximants shown are:

$$P_{1,6}(\hat{T}) = (1 - 8.74285 \times 10^5 \hat{T})(1 - 8.74266 \times 10^5 \hat{T} - 1.59192 \times 10^7 \hat{T}^2 \\ - 4.19358 \times 10^{10} \hat{T}^3 - 6.35033 \times 10^{14} \hat{T}^4 - 3.79428 \times 10^{19} \hat{T}^5 - 4.6629 \times 10^{24} \hat{T}^6)^{-1} \tag{5.8}$$

$$P_{3,4}(\hat{T}) = (1 - 1.27419 \times 10^6 \hat{T} + 1.86395 \times 10^{11} \hat{T}^2 - 1.7809 \times 10^{15} \hat{T}^3) \\ \times (1 - 1.27417 \times 10^6 \hat{T} + 1.86372 \times 10^{11} \hat{T}^2 - 1.77756 \times 10^{15} \hat{T}^3 - 2.43687 \times 10^{16} \hat{T}^4)^{-1} \tag{5.9}$$

$$P_{5,2}(\hat{T}) = (1 - 1.20999 \times 10^6 \hat{T} + 1.48776 \times 10^{11} \hat{T}^2 - 2.65893 \times 10^{12} \hat{T}^3 \\ - 6.32177 \times 10^{15} \hat{T}^4 - 6.06427 \times 10^{19} \hat{T}^5)(1 - 1.20997 \times 10^6 \hat{T} + 1.48754 \times 10^{11} \hat{T}^2)^{-1} \tag{5.10}$$

From figure 1, it appears that the Padé approximant and the numerical solution begin to deviate around $\hat{T} \approx 10^{-4}$. Although extending the convergence interval is beyond the scope of this paper, we verified that higher-order Padé approximants yield a better fit. In particular, convergence tends to be better for odd values of $n + m$. A rough estimate suggests that achieving a satisfactory fit up to $\hat{T} \approx 1$ would require $n + m \approx 1000$.

6 Conclusion

We considered the zero-energy and large-momentum limit (large compared to the temperature) of a holographic conformal field theory on $S^1 \times \mathbb{R}^3$ dual to a black brane. This limit translates due to the Matone relation to a large adjoint scalar vev a in the vector multiplet. Hence, we performed a large a-expansion of the thermal propagator and brought it to a compact form (3.11). To obtain the low-temperature expansion of the propagator, one has to find $a(k)$ through the inverse Matone relation (3.16), plug it into the propagator, expand for large k and finally put $k \to k/T$. However, the final missing ingredient on which the propagator depends on, are the large a expansion coefficients $\alpha_{2n}(q, \nu)$ of the instanton prepotential.

In order to compute these coefficients α_{2n}, we used the S-duality constraints on the instanton prepotential that led to the quasi-modularity of α_{2n} plus a constant piece and matched it with the Zamolodchikov q-recursion. This computational method proved to be efficient, allowing us to obtain α_{2n} up to $2n = 58$, with the potential to easily extend to higher orders.

Finally, we provided the analytic formula of the low-temperature expansion of the propagator up to $\mathcal{O}(T^{44})$ (see (5.3)) and showed numerically that this expansion becomes asymptotic, confirming the results of [36].

There are a few issues that deserve to be further investigated:

- The finite ω terms could be added. There are two technical problems here: as we mentioned before, now in the propagator it is not only the vev a to be large, but also ω. It is not clear what type of expansion should one do. Second, other mass invariants must be included, so that the expansion is now not in powers of the Eisenstein series only, but Jacobi theta functions need to be used making the whole expansion.

- The series obtained seem clearly asymptotic. It would be nice to understand the asymptotic series better doing a full resurgence analysis. The neglected exponentially suppressed terms would here presumably play some role.

- The different approaches of writing the propagator in terms of electric or magnetic variables (a or a_D) should somehow be related to the different representation of the conformal blocks, i.e. the s and t channels.[10]

Acknowledgments

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[10]We thank Oleg Lisovvy for several discussions on similar issues.

A Holographic two-point functions: black brane

Let us consider a holographic $4d$ CFT at finite temperature which is dual to AdS_5 with a black brane [37]. The black brane metric is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\vec{x}^2, \tag{A.1}$$

where the redshift factor is

$$f(r) = r^2 \left(1 - \left(\frac{R_h}{r} \right)^4 \right) \tag{A.2}$$

with AdS radius $L = 1$ and the horizon radius of the black brane R_h. The Hawking temperature of the black brane is then

$$T = \frac{R_h}{\pi}. \tag{A.3}$$

In order to calculate the two-point function of a scalar operator $\mathcal{O}(x)$ with conformal dimension $\Delta = 2 + \nu$ in the $4d$ CFT, we can first relate it to its dual scalar field $\phi(x)$ in the bulk with mass $m = \sqrt{\nu^2 - 4}$ and then solve the wave equation on the black brane background,

$$(\Box - m^2)\phi = 0. \tag{A.4}$$

Since the scalar wave equation in planar AdS_5 commutes with the generators of time translations and the translations along $\vec{x} \in \mathbb{R}^3$, the field can be expanded in a basis of frequency eigenmodes $e^{-i\omega t}$ and momentum eigenmodes $e^{i\vec{k}\vec{x}}$:

$$\phi(t, r, \vec{x}) = \int d\omega \, d^3k \, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \, \varphi_{\omega,\vec{k}}(r). \tag{A.5}$$

Then the wave equation reduces to

$$\left(\frac{1}{r^3}\partial_r(r^3 f(r)\partial_r) + \frac{\omega^2}{f(r)} - \frac{k^2}{r^2} - (\nu^2 - 4) \right) \varphi_{\omega,\vec{k}}(r) = 0 \tag{A.6}$$

and we impose ingoing boundary conditions at the horizon,

$$\varphi_{\omega,\vec{k}}(r) \sim (r - R_h)^{-i\omega/f'(R_h)}, \quad r \to R_h. \tag{A.7}$$

Near the AdS boundary $r \to \infty$ this solution behaves as

$$\varphi_{\omega,\vec{k}}(r) \sim \mathcal{A}(\omega, |\vec{k}|)r^{\nu-2} + \mathcal{B}(\omega, |\vec{k}|)r^{-(\nu+2)}, \tag{A.8}$$

where $\mathcal{A}$ is the source for $\mathcal{O}(x)$ in the boundary CFT and $\mathcal{B}$ is the response which is proportional to the expectation value $\langle \mathcal{O}(x) \rangle$ induced by that source. Therefore, the retarded Green's function, defined as $G_R(t, \vec{x}) = -i\theta(t)\langle [\mathcal{O}(t, \vec{x}), \mathcal{O}(0, 0)] \rangle$, in momentum space is the ratio

$$G_R(\omega, |\vec{k}|) = \frac{\mathcal{B}(\omega, |\vec{k}|)}{\mathcal{A}(\omega, |\vec{k}|)}. \tag{A.9}$$

The connection problem is solving for $\mathcal{A}$ and $\mathcal{B}$ providing the solution satisfies (A.7).

Rewriting the wave equation (A.6) with the new coordinate and field variables,

$$z = \frac{r^2}{r^2 + R_h^2}, \quad \varphi_{\omega,\vec{k}}(r) = \frac{1}{\sqrt{2}}(1-z)^{1/2}\left(\frac{1}{2} - z\right)^{-1/2} z^{-1/2} \chi_{\omega,\vec{k}}(z), \qquad \text{(A.10)}$$

gives the Heun equation (3.1) with parameters (3.2), where $z = t$ is the horizon and $z = 1$ is the boundary. Note that in doing this transformation we absorbed the temperature T in the energy and momentum, i.e. $\vec{k}/T \to \vec{k}$ and $\omega/T \to \omega$, such that ω and $\vec{k}$ in the Heun equation are dimensionless variables.

B Eisenstein series and (quasi-) modular forms

The Eisenstein series are defined by

$$E_{2k}(\tau) = \frac{1}{2\zeta(2k)} \sum_{(m,n)\in\mathbb{Z}^2\backslash\{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}, \qquad \text{(B.1)}$$

which absolutely converges to a holomorphic function of τ in the upper-half plane $\mathbb{H}$.

The Eisenstein series $E_{2k}(\tau)$ for $k > 1$ is a modular form of weight $2k$, i.e. it is PSL$(2,\mathbb{Z})$-covariant, meaning it transforms as

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad E_{2k}(\tau) \to (c\tau + d)^{2k} E_{2k}(\tau), \qquad \text{(B.2)}$$

for $a, b, c, d \in \mathbb{Z}$ with $ad - bc = 1$. The Eisenstein series E_4 and E_6 are the generators of modular forms, meaning that any modular form $M(\tau)$ of weight $2k$ with $k > 1$ can be written as

$$M(\tau) = \sum_{4n+6m=2k} c_{n,m} E_4^n(\tau) E_6^m(\tau) \qquad \text{(B.3)}$$

with $c_{n,m} \in \mathbb{C}$. The Eisenstein series E_2 is not a modular form, it transforms under PSL$(2,\mathbb{Z})$ as

$$E_2(\tau) \to (c\tau + d)^2 E_2(\tau) - \frac{6}{\pi} ic(c\tau + d) \qquad \text{(B.4)}$$

and is therefore a quasi-modular form of weight 2. The ring of quasi-modular forms is given by $\mathbb{C}[E_2, E_4, E_6]$ and therefore any quasi-modular form $Q(\tau)$ of weight $2k$ can be written as

$$Q(\tau) = \sum_{2l+4n+6m=2k} c_{l,n,m} E_2^l(\tau) E_4^n(\tau) E_6^m(\tau) \qquad \text{(B.5)}$$

with $c_{l,n,m} \in \mathbb{C}$.

The Fourier expansion of the first three Eisenstein series in terms of $q = \exp(i\pi\tau)$ is

$$\begin{aligned}
E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty} \sigma_1(n)q^{2n} = 1 - 24\sum_{n=1}^{\infty} \frac{nq^{2n}}{1-q^{2n}}, \\
E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^{2n} = 1 + 240\sum_{n=1}^{\infty} \frac{n^3 q^{2n}}{1-q^{2n}}, \\
E_6(\tau) &= 1 - 504\sum_{n=1}^{\infty} \sigma_5(n)q^{2n} = 1 - 504\sum_{n=1}^{\infty} \frac{n^5 q^{2n}}{1-q^{2n}},
\end{aligned} \qquad \text{(B.6)}$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the divisor function.

The Eisenstein series E_4 and E_6 can be rewritten in terms of theta functions

$$E_4(\tau) = \frac{1}{2}\left(\theta_2(q)^8 + \theta_3(q)^8 + \theta_4(q)^8\right),$$

$$E_6(\tau) = \frac{1}{2}\sqrt{\frac{\left(\theta_2(q)^8 + \theta_3(q)^8 + \theta_4(q)^8\right)^3 - 54(\theta_2(q)\theta_3(q)\theta_4(q))^8}{2}},$$

(B.7)

which on the other hand can be rewritten in terms of the Dedekind eta function,

$$\theta_2(e^{i\tau\pi}) = \frac{2\eta^2(2\tau)}{\eta(\tau)}, \quad \theta_3(e^{i\tau\pi}) = \frac{\eta^5(\tau)}{\eta^2\left(\frac{1}{2}\tau\right)\eta^2(2\tau)}, \quad \theta_4(e^{i\tau\pi}) = \frac{\eta^2\left(\frac{1}{2}\tau\right)}{\eta(\tau)}.$$

(B.8)

The Dedekind eta function has the following special values

$$\eta(i) = \frac{\Gamma\left(\frac{1}{4}\right)}{2\pi^{\frac{3}{4}}}, \quad \eta\left(\frac{1}{2}i\right) = \frac{\Gamma\left(\frac{1}{4}\right)}{2^{\frac{7}{8}}\pi^{\frac{3}{4}}}, \quad \eta(2i) = \frac{\Gamma\left(\frac{1}{4}\right)}{2^{\frac{11}{8}}\pi^{\frac{3}{4}}}.$$

(B.9)

In the final propagator formula, we have to take $t = 1/2$, which means $q = \exp(-\pi)$ and hence $\tau = i$. Therefore, calculating $E_2(i)$ directly and using the special values of the Dedekind eta function for $E_4(i)$ and $E_6(i)$, we obtain

$$E_2(i) = \frac{3}{\pi}, \quad E_4(i) = \frac{3}{4}\frac{\pi^2}{\Gamma[3/4]^8}, \quad E_6(i) = 0.$$

(B.10)

In the propagator formula, we also have to use the derivatives of quasi-modular forms. However, this will again give us quasi-modular forms, since the ring of quasi-modular forms is closed under differentiation, especially, we have

$$q\frac{dE_2}{dq} = \frac{E_2^2 - E_4}{6}, \quad q\frac{dE_4}{dq} = 2\frac{E_2E_4 - E_6}{3}, \quad q\frac{dE_6}{dq} = E_2E_6 - E_4^2.$$

(B.11)

Note that although $E_6 = 0$ for $q = \exp(-\pi)$, we still have to calculate keep it in the calculation of $\tilde{\alpha}_{2n}$, since $q\partial_q E_6 \neq 0$.

C $\tilde{\alpha}_{2n}$ up to $2n = 18$

Here we give the quasi-modular forms $\tilde{\alpha}_{2n}$ up to $2n = 18$ (the first four can be found in (4.32)). For better readability we write

$$\tilde{\alpha}_{2n} = \frac{(\nu^2 - 1)^2}{9 \cdot 2^{n+2}(2n)!!\ n(n+1)} \times \gamma_{2n},$$

(C.1)

with the following γ_{2n}:

$$\gamma_{10} = \frac{1}{14}\Big(6160E_2^3E_4(\nu^2 - 13)(\nu^2 - 1)^3 + 5390E_2^5(\nu^2 - 1)^4$$

(C.2)

$$+165E_2^2E_6(\nu^2 - 1)^2(11\nu^4 - 454\nu^2 + 3323)$$

$$+88E_2E_4^2(17\nu^8 - 896\nu^6 + 13314\nu^4 - 64328\nu^2 + 51893)$$

$$+7E_4E_6(37\nu^8 - 3556\nu^6 + 101454\nu^4 - 1067908\nu^2 + 3734773)\Big)$$

$$\gamma_{12} = \frac{1}{12870}\Big(2477475E_2^4E_4(\nu^2-13)(\nu^2-1)^4 + 1651650E_2^6(\nu^2-1)^5 \tag{C.3}$$

$$+78650E_2^3E_6(\nu^2-1)^3(11\nu^4-454\nu^2+3323)$$

$$+9438E_2^2E_4^2(\nu^2-1)^2(109\nu^6-5463\nu^4+75975\nu^2-312541)$$

$$+4E_6^2(3313\nu^{10}-529637\nu^8+27763210\nu^6-612870058\nu^4+5723061269\nu^2-18823188097)$$

$$+78E_2E_4E_6(4127\nu^{10}-390883\nu^8+11255510\nu^6-124152422\nu^4+506385211\nu^2-393101543)$$

$$+5E_4^3(11405\nu^{10}-1372285\nu^8+57069362\nu^6-1064136746\nu^4+8901924337\nu^2-27206840585)\Big)$$

$$\gamma_{14} = \frac{1}{19305}\Big(20040020E_2^5E_4(\nu^2-13)(\nu^2-1)^5 + 10735725E_2^7(\nu^2-1)^6 \tag{C.4}$$

$$+715715E_2^4E_6(\nu^2-1)^4(11\nu^4-454\nu^2+3323)$$

$$+416416E_2^3E_4^2(\nu^2-1)^3(29\nu^6-1413\nu^4+19335\nu^2-78431)$$

$$+1183E_2^2E_4E_6(\nu^2-1)^2(4369\nu^8-400132\nu^6+11084838\nu^4-114436996\nu^2+394051921)$$

$$+44E_4^2E_6(7907\nu^{12}-1436406\nu^{10}+94161681\nu^8-2973186148\nu^6+47987533077\nu^4$$

$$-371551350726\nu^2+1087272993815)$$

$$+E_2(\nu^2-1)\Big(9E_6^2(43151\nu^{10}-6751659\nu^8+350092950\nu^6-7679205206\nu^4$$

$$+71443709643\nu^2-234419568879)+28E_4^3(62459\nu^{10}-7228363\nu^8+293679134\nu^6$$

$$-5400861542\nu^4+44812474807\nu^2-136264849055)\Big)\Big)$$

$$\gamma_{16} = \frac{1}{765765}\Big(4769524760E_2^6E_4(\nu^2-13)(\nu^2-1)^6 + 2129252125E_2^8(\nu^2-1)^7 \tag{C.5}$$

$$+185825640E_2^5E_6(\nu^2-1)^5(11\nu^4-454\nu^2+3323)$$

$$+92912820E_2^4E_4^2(\nu^2-1)^4(41\nu^6-1947\nu^4+26235\nu^2-104969)$$

$$+1299480E_2^3E_4E_6(\nu^2-1)^3(1537\nu^8-137836\nu^6+3766974\nu^4-38530108\nu^2+131667433)$$

$$+3264E_2E_4^2E_6(\nu^2-1)(109856\nu^{12}-19321257\nu^{10}+1240980357\nu^8-38695339066\nu^6$$

$$+620069070666\nu^4-4782152083053\nu^2+13963525196897)$$

$$+7644E_4E_6^2(2947\nu^{14}-770581\nu^{12}+75452223\nu^{10}-3724724201\nu^8+100656451049\nu^6$$

$$-1478261187903\nu^4+10797325840405\nu^2-30437398423939)$$

$$+65E_4^4(539015\nu^{14}-116540465\nu^{12}+9815323155\nu^{10}-432713630965\nu^8$$

$$+10792646612341\nu^6-150127472234643\nu^4+1058779988148401\nu^2-2917721545074695)$$

$$+68E_2^2(\nu^2-1)^2\Big(98E_4^3(146057\nu^{10}-16299097\nu^8+647502602\nu^6-11746943330\nu^4$$

$$+96678506077\nu^2-292491603509)+15E_6^2(202831\nu^{10}-31105259\nu^8+1596096790\nu^6$$

$$-34792123606\nu^4+322501955723\nu^2-1055677586479)\Big)\Big)$$

$$\gamma_{18} = \frac{1}{43648605}\Big(1800650451600E_2^7E_4(\nu^2-13)(\nu^2-1)^7 + 687748436375E_2^9(\nu^2-1)^8 \tag{C.6}$$

$$+75027102150E_2^6E_6(\nu^2-1)^6(11\nu^4-454\nu^2+3323)$$

$$+138511573200E_2^5E_4^2(\nu^2-1)^5(13\nu^6-603\nu^4+8007\nu^2-31609)$$

$$+224856450E_2^4E_4E_6(\nu^2-1)^4(4853\nu^8-426884\nu^6+11517006\nu^4-116743652\nu^2+395952677)$$

$$+1581408E_2^2E_4^2E_6(\nu^2-1)^2(222386\nu^{12}-37938867\nu^{10}+2389066857\nu^8-73577606926\nu^6$$
$$+1170615608196\nu^4-8992696408623\nu^2+26200748208977)$$
$$+91E_6^3(9098551\nu^{16}-3318610904\nu^{14}+470276428132\nu^{12}-35035373767592\nu^{10}$$
$$+1502228813457130\nu^8-37509603645438056\nu^6+525335751790478884\nu^4$$
$$-3726992758721115032\nu^2+10313477739545468887)$$
$$+105E_4^3E_6(135775123\nu^{16}-40342801304\nu^{14}+4774655759380\nu^{12}-305532216130856\nu^{10}$$
$$+11600359659800626\nu^8-264210820184555816\nu^6+3463480616917047316\nu^4$$
$$-23489837667445792664\nu^2+63057368461013102995)$$
$$+65892E_2^3(\nu^2-1)^3\Big(25E_6^2(84593\nu^{10}-12731221\nu^8+646710122\nu^6-14011129514\nu^4$$
$$+129402887221\nu^2-422586845201)+28E_4^3(370639\nu^{10}-39975359\nu^8+1554065254\nu^6$$
$$-27819098830\nu^4+227110528859\nu^2-683639503363)\Big)$$
$$+171E_2E_4(\nu^2-1)\Big(40E_4^3(9656989\nu^{14}-2008640383\nu^{12}+165036951057\nu^{10}-7164021007451\nu^8$$
$$+177067386656567\nu^6-2450740334488221\nu^4+17238167159967067\nu^2-47436358428035785)$$
$$+7E_6^2(34431581\nu^{14}-8758125803\nu^{12}+843162713889\nu^{10}-41188545488023\nu^8$$
$$+1106248892660407\nu^6-16191938269934529\nu^4+118057239602068235\nu^2$$
$$-332484090889525757)\Big)\Big)$$

D $P_l(\nu)$ up to $l=10$

We give the polynomials $P_l(\nu)$ from the low-temperature expansion of the propagator in (5.3), up to $l=10$ (the first five are already given in (5.6)).

$$P_6(\nu)=1449248566558429728000000+220250806005676345274 8800\nu \tag{D.1}$$
$$+142267967968709020546 3040\nu^2+50205843049277314188 0576\nu^3$$
$$+101304014393631422420992\nu^4+1025036305016952826944 0\nu^5$$
$$+43769734212643252800\nu^6-90640025345532672288\nu^7$$
$$-3799708109870825656\nu^8+632172469553364420\nu^9+2442293730190993 0\nu^{10}$$
$$-4025662418725677\nu^{11}+155598706432911\nu^{12}+17835774403590\nu^{13}$$
$$-5959405472020\nu^{14}-66994788861\nu^{15}+52107058003\nu^{16}$$

$$P_7(\nu)=1298649823891694771673600000+2274588609510698838781440000\nu \tag{D.2}$$
$$+17658707065011142739784 70400\nu^2+799386677935937946478648320\nu^3$$
$$+232650575609538014385083648\nu^4+44976822119182232773974528\nu^5$$
$$+5694023931629785610664704\nu^6+435027736240286972784000\nu^7$$
$$+15510110215415671889456\nu^8+200119127975079578736\nu^9$$
$$+81900351486218166608\nu^{10}+11492453942992100940\nu^{11}$$
$$+442814004978174719\nu^{12}+10126921617057444\nu^{13}+2610383651583347\nu^{14}$$
$$+88059701326740\nu^{15}-216303410323\nu^{16}+2590465169292\nu^{17}+215872097441\nu^{18}$$

$$P_8(\nu) = 3084583371802870011046077256704000000 \tag{D.3}$$
$$+ 5608376377807053382434617084989440000\nu + 45455927305893101539369265365626880 0\nu^2$$
$$+ 216187294110503718808290692414115840\nu^3 + 66552579148342992077152707060881408\nu^4$$
$$+ 13690863858192997167275810885710848\nu^5 + 1841749628537456412420845254915584\nu^6$$
$$+ 142183869814189486712805662557440\nu^7 + 2230691307854165199403125237888\nu^8$$
$$- 64176654415825258342402625107 2\nu^9 - 4385119146329440453820675337 6\nu^{10}$$
$$+ 1597067263815762818232494640\nu^{11} + 227427303719187295114847688\nu^{12}$$
$$- 5209229097118050071989572\nu^{13} - 900630229074100795863726\nu^{14}$$
$$+ 3140474336650660967941 5\nu^{15} + 2489137755514699057953\nu^{16}$$
$$- 1317417827235701721 57\nu^{17} + 10873466630151660369\nu^{18} + 430657623921690165\nu^{19}$$
$$- 189830149584512437\nu^{20} - 2389056502379547\nu^{21} + 796352167459849\nu^{22}$$

$$P_9(\nu) = 601631416585763948100869522036739072000000 \tag{D.4}$$
$$+ 12003076771689673676298611009821541990400 00\nu$$
$$+ 108920770273981108420782275524145849958400 0\nu^2$$
$$+ 59568197623018507787275654268982136504320 0\nu^3$$
$$+ 21897366344335214417992443035606502498304 0\nu^4$$
$$+ 57009114934608848909950136646063506669568\nu^5$$
$$+ 10754735305539390962180572178535251623936\nu^6$$
$$+ 1470705081729382370763534205165493809152\nu^7$$
$$+ 142008862152667112331829277669683137024\nu^8$$
$$+ 8976763775386750149828712979122019328\nu^9$$
$$+ 297707018061182614478218366667667456\nu^{10} + 14143731186395501351246828053393 92\nu^{11}$$
$$+ 29746368561980420381416592750150 4\nu^{12} + 7786374771698800286960412828460 8\nu^{13}$$
$$+ 4422825265001119238391924110016\nu^{14} + 391029320601326438417610969 12\nu^{15}$$
$$+ 266370329897667057439313481 9\nu^{16} + 79751947440381284951576452 8\nu^{17}$$
$$+ 32928608806916832880414356\nu^{18} + 4643639033341259571253 92\nu^{19}$$
$$+ 6987979911961238831755 4\nu^{20} - 2649125946635594458032\nu^{21}$$
$$- 324098494996046534764\nu^{22} + 5778513350441226595 2\nu^{23} + 4213499318030061059\nu^{24}$$

$$P_{10}(\nu) = 5433112703498890574244612008700088837171200000000 0 \tag{D.5}$$
$$+ 111255312027394865271703381539033138577727569920000 0\nu$$
$$+ 103970988935025766767630161939215058903712866304000 0\nu^2$$
$$+ 587720631376690468398528015010800713337425059840000\nu^3$$
$$+ 224183392519402763842077949359384187583550103552000\nu^4$$
$$+ 60810772617745044421683688608513323605738298081280\nu^5$$
$$+ 11997533100179304955183788849171136657427677741056\nu^6$$

$$+17187564422191552344028313093236831774524856565576\nu^7$$

$$+17262343855146205655601201042339601165618078948\nu^8$$

$$+10782885816203721418523648572529952102172620800\nu^9$$

$$+20814466404808599947990019446577246550421120\nu^{10}$$

$$-2703388378149179315380104074806980655934208\nu^{11}$$

$$-216631496882019832804973837864426006398464\nu^{12}$$

$$+2202298964749622030867625963347316000000\nu^{13}$$

$$+68046034576981357551333192714828715449600\nu^{14}$$

$$+12672179123702665717887329599817679456\nu^{15}$$

$$-1782233192840074338417931022749321692\nu^{16}$$

$$-396737797611816495544078224136475100\nu^{17}$$

$$+49234710018788367898621636123209090\nu^{18}+5359332279450555603893265896874415\nu^{19}$$

$$-1354511917336099569714073500029505\nu^{20}+73168038384017400107059627310\nu^{21}$$

$$+21363437832880816724509851968\nu^{22}-38505462660764654340780980700\nu^{23}$$

$$+1231918522152644589733560090\nu^{24}+2652031834321832262792492\nu^{25}$$

$$-9793650439946554376100906\nu^{26}-13457495471856212016340\nu^{27}$$

$$+2297621178121792295472\nu^{28}$$

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