Iterative construction of conserved quantities in dissipative nearly integrable systems

Iris Ulčakar[1,2] and Zala Lenarčič[1]

[1] *Jožef Stefan Institute, 1000 Ljubljana, Slovenia*

[2] *University of Ljubljana, Faculty for physics and mathematics, 1000 Ljubljana, Slovenia*

Integrable systems offer rare examples of solvable many-body problems in the quantum world. Due to the fine-tuned structure, their realization in nature and experiment is never completely accurate, therefore effects of integrability are observed only transiently. One way to overcome this limitation is to weakly couple nearly integrable systems to baths and driving: these will stabilize integrable effects up to arbitrary time and encode them in the stationary state approximated by a generalized Gibbs ensemble. However, the description of such driven dissipative nearly integrable models is challenging and no exact analytical methods have been proposed so far. Here, we develop an iterative scheme in which integrability breaking perturbations (baths) determine the conserved quantities that play the leading role in a highly efficient truncated generalized Gibbs ensemble description. Our scheme paves the way for easier calculations in thermodynamically large systems and can be used to construct unknown conserved quantities.

Integrable models have played a paramount role in our understanding of nonequilibrium dynamics because, in some cases, their dynamics can be followed exactly. A modern milestones descriptions of integrable models has been the observation that the steady states reached after a sudden excitation are locally describable with a generalized Gibbs ensemble (GGE) [1–5]. This observation was a natural generalization of the equilibration of thermalizing generic models, where the steady state is locally described with a thermal Gibbs ensemble [6, 7]. The applicability of generalized Gibbs ensembles on a finite timescale was first established experimentally with cold atoms [8]. Recently, also spatially inhomogeneous dynamics of integrable systems was formulated using a local GGE description and generalized hydrodynamics [9–11], receiving experimental confirmations as well [12–16].

Due to their fine-tuned interactions, integrable models cannot be exactly realized in nature. On short to intermediate timescales, integrable models are realizable with quantum simulators [17–19] and can approximately describe some materials [20–22]. On longer timescales, additional Hamiltonian terms, such as longer range interaction in trapped ions experiment, coupling to phonons in solid state experiment, trapping potential, transverse couplings, and loss of atoms in cold atomic setups prevent integrable effects from persisting up to arbitrary times [12, 13, 23, 24]. The only way to sustain integrable effects in nearly integrable systems up to arbitrary times is to drive them out of equilibrium [25–29]. To prevent heating due to driving, such systems must also be weakly open. Numerical evidence shows that then the time evolution [25] as well as the steady state [26–29] is again approximately described by a GGE. The equations of motion and the steady-state values for the Lagrange parameters associated with the conserved quantities entering the GGE are given deterministically by the integrability breaking perturbations, i.e., by the drive and coupling to the baths. This opens the possibility for GGE engineering [28] and stabilizing potentially technologically useful phenomena. For example, in spin chain materials, approximately described by the Heisenberg model, efficient spin and energy pumping could be realized [26]. So far, time-dependent GGE descriptions have also been used to describe the effect of particle loss in cold atoms [30, 31] and non-interacting systems coupled to baths [28, 29, 32–35].

While the expanded role of GGE as a description of weakly open, nearly integrable systems is fundamentally important, in this case, concrete calculations of dynamics and steady states are much more demanding than in isolated systems. Weak integrability breaking, which causes non-elastic scatterings and a slow reshuffling of quasiparticle content, makes the usual quasiparticle treatment of integrable models much harder; for interacting systems analytically (probably) impossible [30, 31]. One possible simplification is to approximate the GGE with macroscopically many Lagrange parameters with its truncated version. Such an approximation has been used in the context of isolated integrable systems [4, 36–38], as well as in driven dissipative, nearly integrable systems [25, 26, 28].

We propose a new iterative scheme, which adds the leading conserved quantities to the truncated GGE iteratively, as suggested by the driving and dissipation itself. Here, leading means having the main contribution to the GGE. We examine the convergence to the exact result and show that a good approximation is typically achieved within a few steps. Meaning, we have to find a solution of a few coupled equations for a few leading conserved quantities, instead of considering extensively many coupled equations for all conserved quantities. The only input for the method is the basis from which iterative conserved quantities are constructed.

Setup. We consider driven dissipative, nearly integrable setups with the dominant unitary dynamics given by an integrable Hamiltonian H_0. Weak driving and dissipation could be due to a Floquet unitary drive and coupling to a (thermal) bath; however, for simplicity, we will consider weak coupling to non-thermal Markovian baths, whose

action is described by Lindblad operators L_i. As pointed out previously [25–29], non-static integrability breaking perturbations should stabilize a GGE generically, and the formalism described below can treat them all. The Liouville equation gives the dynamics of the density matrix operator,

$$\hat{\mathcal{L}}\rho = -i[H_0,\rho] + \hat{\mathcal{D}}\rho,\ \hat{\mathcal{D}}\rho = \epsilon\sum_i L_i\rho L_i^\dagger - \frac{1}{2}\{L_i^\dagger L_i,\rho\}, \tag{1}$$

where $\epsilon \ll 1$ is the strength of the coupling to baths. We will consider homogeneous coupling to baths, where Lindblad operators L_i of the same form act on every site i.

Perturbatively, the zeroth order approximation to the steady state is of a diagonal form in terms of eigenstates of H_0, $H_0|m\rangle = E_m|m\rangle$,

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\rho = \sum_m a_m |m\rangle\langle m| \equiv \rho_D. \tag{2}$$

Weights a_m are obtained from the kernel of the dissipator projected on the diagonal subspace [27, 39], $\sum_n D_{mn}a_n = 0$, $D_{mn} = \langle m|(\hat{\mathcal{D}}|n\rangle\langle n|)|m\rangle$. If the dissipator preserves a symmetry of the Hamiltonian, eigenstates can be taken within the symmetry sector with a unique steady state.

As suggested in our previous works [25–28], for integrable H_0, the zeroth order diagonal ensemble ρ_D should be thermodynamically equivalent to a generalized Gibbs ensemble,

$$\lim_{\epsilon\to 0}\lim_{t\to\infty}\lim_{L\to\infty}\rho = \frac{e^{-\sum_m\lambda_m C_m}}{\mathrm{Tr}[e^{-\sum_m\lambda_m C_m}]} \equiv \rho_{\boldsymbol{\lambda}}, \tag{3}$$

where C_m are the (quasi-)local conserved quantities of the underlying integrable model, $[C_m, H_0] = 0$, and λ_m the associated Lagrange multipliers. In the steady state, the latter are determined by the stationarity conditions for all conserved quantities

$$\langle\dot{C}_{m'}\rangle = \mathrm{Tr}\left[C_{m'}\hat{\mathcal{D}}\frac{e^{-\sum_m\lambda_m C_m}}{\mathrm{Tr}[e^{-\sum_m\lambda_m C_m}]}\right] \stackrel{!}{=} 0 \quad \forall m'. \tag{4}$$

I.e., one must find the set of λ_m for which the flow of all conserved quantities is zero ($\stackrel{!}{=}0$). This equation is very instructive: (i) it tells us that the form of integrability breaking dissipators (Lindblad operators) will determine the λ_m values, and (ii) in order to find λ_m, one must solve a set of coupled non-linear equations. To reduce the complexity of step (ii), an approximate description in terms of a truncated GGE (tGGE) with a finite number of included conserved quantities has been used [25, 26, 28]. In that case, the expectation values of included conserved charges and of local operators constituting them were well captured. However, other local observables showed stronger deviations, particularly those overlapping with quasi-local conserved operators. To partially mend for that, the diagonal part of latter observables was included in the tGGE.

Also the dynamics towards the steady state can be approximated with a time-dependent GGE [25]. The equation of motion for $\lambda_m(t)$ is derived by the use of super-projector $\hat{P}$ onto slow modes, which are for the GGE Ansatz naturally given by the operators $\frac{\partial\rho_{\boldsymbol{\lambda}}}{\partial\lambda_m}$ tangential to the GGE manifold,

$$\hat{P}X = -\sum_{m,n}\frac{\partial\rho_{\boldsymbol{\lambda}}}{\partial\lambda_m}(\chi^{-1})_{m,n}\mathrm{Tr}[C_n X]. \tag{5}$$

Here, $\chi_{m,n} = -\mathrm{Tr}[C_m\frac{\partial\rho_{\boldsymbol{\lambda}}}{\partial\lambda_n}] = \langle C_m C_n\rangle_{\rho_{\boldsymbol{\lambda}}} - \langle C_m\rangle_{\rho_{\boldsymbol{\lambda}}}\langle C_n\rangle_{\rho_{\boldsymbol{\lambda}}}$ is the $\{m,n\}$ entry of matrix χ and $\langle O\rangle_{\rho_{\boldsymbol{\lambda}}} = \mathrm{Tr}[\rho_{\boldsymbol{\lambda}}O]$. Applying the super-projector to the slow dynamics on the GGE manifold,

$$\hat{P}\dot{\rho}_{\boldsymbol{\lambda}} = -\sum_{m,n}\frac{\partial\rho_{\boldsymbol{\lambda}}}{\partial\lambda_m}(\chi^{-1})_{m,n}\mathrm{Tr}[C_n\hat{\mathcal{D}}\rho_{\boldsymbol{\lambda}}] = \sum_m\frac{\partial\rho_{\boldsymbol{\lambda}}}{\partial\lambda_m}\dot{\lambda}_m \tag{6}$$

gives the rate of change for the Lagrange multiplier associated with C_m,

$$\dot{\lambda}_m = -\sum_n(\chi^{-1})_{m,n}\mathrm{Tr}[C_n\hat{\mathcal{D}}\rho_{\boldsymbol{\lambda}}]. \tag{7}$$

In the super-projector language it is given by the flow along the corresponding tangential direction. Here, the initial conditions $\lambda_m(0)$ are given by the initial state, as in the prethermal state [40, 41].

Iteratively constructed truncated GGE. We use the above super-projector technique to iteratively add the leading conserved quantities to a truncated description of the steady state for a given dissipator $\hat{\mathcal{D}}$. Given that the steady state Lagrange parameters are selected by the dissipator, Eq. (4), we will, in the first place, use $\hat{\mathcal{D}}$ to select the conserved quantities that we include in a truncated GGE Ansatz. If such an iterative truncated description converges to the exact one quickly, the procedure reduces the number of conditions (4) that need to be solved.

In the procedure, we iteratively construct conserved quantities $\tilde{C}_k$ from the user-defined operator basis Q_m. The latter should ideally be the set of all known (quasi)-local conserved quantities $Q_m = C_m$ of the integrable model H_0, but one can also restrict it to contain only some of them. If all conserved quantities are not known or are hard to work with, one can use the basis with projectors $Q_m = |m\rangle\langle m|$ onto all eigenstates of H_0 within the symmetry sector with a unique steady state. However, this introduces certain finite size effects that we discuss in the next section.

The iterative procedure has the following steps:

Step 0: Start with a thermal state $\rho_{\tilde{\boldsymbol{\lambda}}}^{(0)} = e^{-\tilde{\lambda}_0^{(0)}H_0}/Z$ and find $\tilde{\lambda}_0^{(0)}$ from condition (4), $\mathrm{Tr}[H_0\hat{\mathcal{D}}e^{-\tilde{\lambda}_0^{(0)}H_0}] \stackrel{!}{=} 0$.

Step 1: Add the first iterative conserved quantity of the form

$$\tilde{C}_1 = \mathcal{N}_1^{-1} \sum_m w_m^{(1)} Q_m,$$
$$w_m^{(1)} = -\sum_n (\chi_{(0)}^{-1})_{mn} \mathrm{Tr}[Q_n \hat{\mathcal{D}} \rho_{\tilde{\boldsymbol{\lambda}}}^{(0)}], \tag{8}$$

where, according to Eq. (7), weights $w_m^{(1)}$ are given by the flows along additional directions $\frac{\partial \rho_{\boldsymbol{\lambda}}}{\partial \lambda_m}|_{\rho_{\tilde{\boldsymbol{\lambda}}}^{(0)}}$ when we allow for a GGE manifold that is not one dimensional (thermal) as in Step 0, but is spanned by additional basis Q_m conserved quantities. A new direction $\frac{\partial \rho_{\boldsymbol{\lambda}}}{\partial \lambda_m}|_{\rho_{\tilde{\boldsymbol{\lambda}}}^{(0)}}$ that causes a stronger correction to the existing solution is more important and should be weighted by a stronger bias $w_m^{(1)}$. In the end, we find $\{\tilde{\lambda}_0^{(1)}, \tilde{\lambda}_1^{(1)}\}$ for $\rho_{\tilde{\boldsymbol{\lambda}}}^{(1)} \propto e^{-\tilde{\lambda}_0^{(1)} H_0 - \tilde{\lambda}_1^{(1)} \tilde{C}_1}$ from the condition (4) for H_0 and $\tilde{C}_1$.

Step k: Add kth iterative conserved quantity

$$\tilde{C}_k = \mathcal{N}_k^{-1} \sum_m w_m^{(k)} Q_m,$$
$$w_m^{(k)} = -\sum_n \left(\chi_{(k-1)}^{-1}\right)_{mn} \mathrm{Tr}[Q_n \hat{\mathcal{D}} \rho_{\tilde{\boldsymbol{\lambda}}}^{(k-1)}] \tag{9}$$

and find $\{\tilde{\lambda}_{k'}^{(k)}\}$ for $\rho_{\tilde{\boldsymbol{\lambda}}}^{(k)} \propto e^{-\sum_{k'=0}^{k} \tilde{\lambda}_{k'}^{(k)} \tilde{C}_{k'}}$ from the set of $k+1$ conditions (4) for $\{\tilde{C}_{k'}\}_{k'=0}^{k}$, where we denote the $\tilde{C}_0 = H_0$. Normalization $\mathcal{N}_k$ can be absorbed into the corresponding Lagrange parameter. However, it can also be chosen such that $\tilde{C}_k$ scales as an extensive operator. The susceptibility matrix, $(\chi_{(k)})_{m,n} = \langle Q_m Q_n \rangle_{\rho_{\tilde{\boldsymbol{\lambda}}}^{(k)}} - \langle Q_m \rangle_{\rho_{\tilde{\boldsymbol{\lambda}}}^{(k)}} \langle Q_n \rangle_{\rho_{\tilde{\boldsymbol{\lambda}}}^{(k)}}$, must be evaluated in each iterative step. Matrix $\chi_{(k)}$ is not invertible for the non-local basis with $Q_m = |m\rangle\langle m|$, however, one can regularize it as explained in the Supplementary material (SM), [42]. In case of (additional) unitary (Floquet) perturbations, the iterative procedure can be generalized to cover those as well [42].

Results. First, to quantify the (finite-size) error of different (truncated) GGE descriptions on length-scale ℓ, we use the distance between density matrices [37]

$$d(\rho_1, \rho_2) = \sqrt{\frac{\mathrm{Tr}[(\rho_1 - \rho_2)^2]}{\mathrm{Tr}[\rho_1^2] + \mathrm{Tr}[\rho_2^2]}} \tag{10}$$

and compare reduced density matrices on ℓ consecutive sites for different (truncated) GGE descriptions $\rho_{\boldsymbol{\lambda},\ell}$ and the diagonal solution $\rho_{D,\ell}$, Eq. (2). As noted in Ref. [37], the distance between two reduced space GGEs scales as $d(\rho_{\boldsymbol{\lambda},\ell}, \rho_{\boldsymbol{\lambda}',\ell}) \sim \ell \sum_i |\lambda_i - \lambda_i'|$ for large enough ℓ. Also for ρ_D close to a GGE, we expect $d(\rho_{\boldsymbol{\lambda},\ell}, \rho_{D,\ell})$ to scale with ℓ for large enough ℓ.

In Fig. 1, we show the scaled distances $d(\rho_{\boldsymbol{\lambda},\ell}, \rho_{D,\ell})/\ell$ for our first example, the transverse field Ising model

$$H_0 = \sum_i J \sigma_i^z \sigma_{i+1}^z + h_x \sigma_i^x, \tag{11}$$

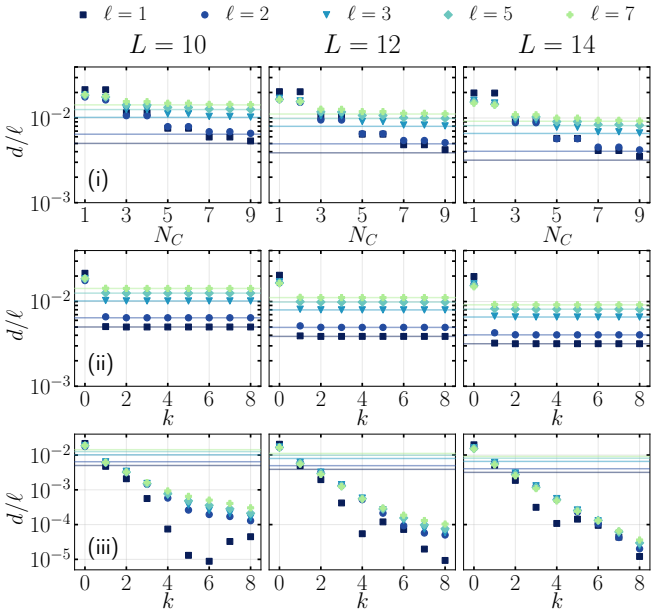

FIG. 1. Scaled distances $d(\rho_{\boldsymbol{\lambda},\ell}, \rho_{D,\ell})/\ell$, Eq. (10), between the reduced density matrices of support ℓ for the diagonal steady state ensemble, Eq. (2), and different (truncated) generalized Gibbs ensembles. Solid lines correspond to the best GGEs with all $N_C^{\mathrm{all}} = 2L - 2$ local conserved quantities on a given system size. From top to bottom, we consider tGGEs with: (i) N_C local conserved quantities $C_m, m = 0, ..., N_C - 1$, (ii) iterative scheme after the kth step for the basis $Q_m = C_m$, $m = 1, ..., N_C^{\mathrm{all}} - 1$, (iii) iterative scheme after the kth step for basis $Q_m = |m\rangle\langle m|$. Parameters: Ising model with $h_x = 0.6$, $J = 1$, system sizes $L = 10, 12, 14$.

which is a paradigmatic non-interacting integrable model. It preserves a series of local extensive operators C_m [43], see [42]. We choose the following Lindblad operators of a rather general form, with $a = 0.2$,

$$L_i = S_i^+ S_{i+1}^- + i S_{i+1}^- S_{i+2}^+ + a \sigma_i^x \sigma_{i+1}^z, \tag{12}$$

which stabilize a non-trivial steady state. Such structured dissipators could be realized with trapped ions simulators [28, 44] or superconducting circuits [45]. Moreover, we stress that our algorithm is generic and applicable to any non-hermitian Lindblad operators leading to (within the symmetry sector) a unique, non-trivial steady state.

Solid lines on all Fig. 1 panels denote the saturated distances $d_{s,\ell}$ for the best GGE solution on a given system size L, including all $N_C^{\mathrm{all}} = 2L - 2$ local conserved quantities on that system size. The solution is obtained by solving extensively many $N_C^{\mathrm{all}} \sim 2L$ coupled Eq. (4). As L is increased, saturated $d_{s,\ell}$ are decreased because in the thermodynamic limit, the complete GGE and the diagonal solution are equivalent, $\lim_{L\to\infty} d_{s,\ell} \to 0$. Finite $d_{s,\ell}$ are just a finite-size effect. (i) Traditional tGGE: In the first row, we show the convergence to $d_{s,\ell}$ for a tGGE based on $N_C \leq N_C^{\mathrm{all}}$ most local conserved quantities

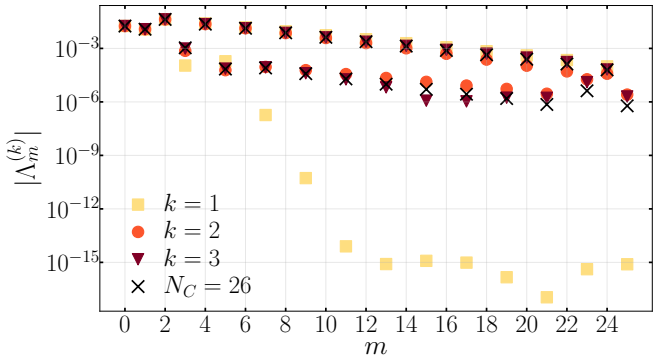

FIG. 2. Interpreting the iterative procedure using the basis with all local conserved quantities $Q_m = C_m, m = 1, ..., 2L-3$. The weights $\Lambda_m^{(k)} = |\sum_{k'=1}^{k} \tilde{\lambda}_{k'}^{(k)} w_m^{(k')}/\mathcal{N}_{k'}|$ at each basis element Q_m after $k = 1, 2, 3$ iterative steps show which basis elements are more important. The exact weights are shown with crosses. Parameters: transverse field Ising model with $h_x = 0.6$ and $J = 1$ on $L = 14$ sites.

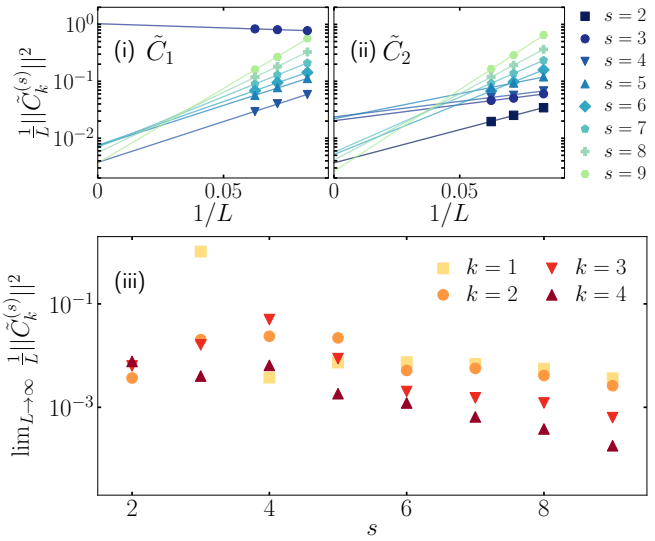

FIG. 3. Nature of iterative conserved quantities, constructed in the basis of projectors on eigenstates, $Q_m = |m\rangle\langle m|$. (i,ii) Finite size scaling of norms $\frac{1}{L}||\tilde{C}_k^{(s)}||^2$ for the part $\tilde{C}_k^{(s)}$ of $k = 1, 2$ iterative conserved operator $\tilde{C}_k = \sum_s \tilde{C}_k^{(s)}$ that acts non-trivially only on support s. (iii) Thermodynamic norms $\lim_{L\to\infty} \frac{1}{L}||\tilde{C}_k^{(s)}||^2$ indicate that iterative conserved operators $\tilde{C}_k$ have a quasi-local nature. Parameters: Heisenberg model on $L = 12, 14, 16$.

$C_m, m = 0, \dots N_C - 1$, where C_0 is the Hamiltonian. To find the solution, N_C coupled Eqs. (4) are solved. As N_C is increased, a better description with smaller distances d is obtained, but convergence is rather slow. (ii) Iterative tGGE with local basis: In the middle row, we show the convergence of $d(\rho^{(k)}_{\tilde{\boldsymbol{\lambda}},\ell}, \rho_{D,\ell})/\ell$ with respect to the number of iterative steps k for the iterative scheme using the basis of *all* local conserved quantities $Q_m = C_m$, excluding the Hamiltonian. The saturated distances $d_{s,\ell}$ (solid lines) are approached after only $k = 1$ iterative step. The advantage is that we now solve for $k + 1 = 2$ instead of extensively many $N_C^{\text{all}} \sim 2L$ conditions (4) in order to find the steady state. (iii) Iterative tGGE with non-local basis: In the last row, we present the convergence of the iterative scheme in the non-local basis $Q_m = |m\rangle\langle m|$ of projectors onto the eigenstates of H_0. The saturated distances $d_{s,\ell}$ (solid lines) are met after only $k = 1$ iterative step. Further reduction of d is largely due to non-local contributions in the non-local basis Q_m, which inherently share finite size effects of the diagonal solution ρ_D, Eq. (2). In that sense, ρ_D and the iterative solution in the diagonal basis, contain thermodynamically irrelevant information. We should note also that the iterative procedure in the non-local basis cannot be more efficient than finding ρ_D. However, its usefulness is in the interpretability, which is achieved by analyzing the structure of the leading conserved quantities. Below, we discuss the advantages of our iterative procedure in both bases.

When all (quasi-)local conserved quantities are known, the advantage of the iterative procedure performed in their basis is in efficiently interpreting the steady state by establishing the weights $\Lambda_m^{(k)}$ at different basis elements $Q_m = C_m$, in Fig. 2 shown for $k = 1, 2, 3$ iterative steps [46]. Weights $\Lambda_m^{(k)}$ reveal that conserved quantities C_{2n}, which are even under the parity transformation, are more important for our example and are well estimated after a single $k = 1$ iterative step. In the next iterative steps, smaller weights at less important odd conserved quantities C_{2n-1} are also captured. To further illustrate the fast convergence, we compare the iterative results to the asymptotic weights (absolute values of Lagrange parameters $|\lambda_m|$ denoted by crosses), obtained by solving extensively many conditions (4) for all local conserved quantities.

If conserved quantities are unknown or hard to work with, one should perform the iterative procedure in the non-local diagonal basis $Q_m = |m\rangle\langle m|$. The usefulness is again in the interpretability, obtained by analyzing the structure of leading iterative conserved quantities. Here, we perform it for the Heisenberg model,

$$H_0 = \sum_i \sigma_i^x \sigma_{i+1}^x + \sigma_i^y \sigma_{i+1}^y + \sigma_i^z \sigma_{i+1}^z, \qquad (13)$$

known to have additional exotic quasi-local conserved quantities [2, 47, 48]. We again use Lindblad operators Eq. (12), now with $a = 0$ so that we can work in a single (largest) $S^z = 0$ magnetization sector. An analysis similar to Fig. 1 is performed in the SM, here we focus on the interpretability. In order to assess the nature (i.e. locality) of iterative conserved quantities, we extract the norm $\lim_{L\to\infty} \frac{1}{L}||\tilde{C}_k^{(s)}||^2$ of the part of iterative conserved quantities that acts non-trivially on s consecutive sites, $\tilde{C}_k = \sum_s \tilde{C}_k^{(s)}$. Operator $\tilde{C}_k^{(s)}$ is obtained by

summing the overlaps of $\tilde{C}_k$ with all Pauli strings acting non-trivially between s sites. In Fig. 3(i,ii), we plot the norms $\frac{1}{L}||\tilde{C}_k^{(s)}||^2$ for $k = 1, 2$ and different supports s on given system sizes $L = 12, 14, 16$ [49]. For small L, norms even increase with s. Only when we extrapolate the finite size result to the thermodynamic limit via $1/L$ scaling, Fig. 3(i,ii), for large enough s, norms are decaying exponentially with s, Fig. 3(iii), as expected for quasi-local conserved operators [47, 48]. Thus, only once we remove the finite-size non-local contributions, inherent to the non-local basis, we can conclude that our iterative conserved quantities are a quasi-local superposition of local [50] and quasi-local conserved quantities of the Heisenberg model [47, 48].

Conclusions and Outlook. In this Letter, we propose an iterative construction of conserved quantities of leading importance to describe nearly integrable, driven dissipative systems. Our approach is motivated by the fact that in such setups, the integrability breaking perturbations (couplings to baths and drives) determine the Lagrange parameters of a GGE approximation to the steady state. Here, we use the dissipator to select the combination of conserved quantities contributing significantly to the truncated GGE description. Such a physically motivated construction of truncated GGEs reduces the complexity of calculating steady state parameters, i.e., the Lagrange parameters. Namely, instead of solving extensively many coupled conditions (4) for all (quasi)local conserved quantities, we need to solve for just a few. A precise number is model and precision-dependent but is generally expected to be $\mathcal{O}(1)$ and low.

A clear usefulness of our approach that we already showcased here is in the interpretability: (i) If working in the basis of all local conserved operators, their weights reveal which of them are important for given Lindblads. (ii) If all (quasi-)local conserved quantities are not known or hard to work with, the iterative procedure can be performed in the non-local basis of projectors on eigenstates. The (quasi-)local structure of leading iterative conserved quantities can be analyzed a posteriori, giving the information about potentially unknown conserved quantities of the unperturbed model [51].

When evaluating the actual reduction of complexity, one should also consider the complexity of evaluating the rate equations (4) and building the conserved quantities, Eq. (9). Our current study used exact diagonalization, at a gain of thorough benchmarking against the diagonal ensemble, but at the loss of diagonalization itself representing the bottleneck of the procedure. At least for non-interacting many-body integrable systems, one can evaluate Eq. (4) and construct iterative conserved quantities (9) in the basis of mode occupation operators with polynomial complexity in system size, in which case our method reduces the exponent and simplifies thermodynamically large calculations; see comment [52] for scaling arguments. For interacting integrable models, Eqs.(4,9) could be evaluated using partition function approach [53, 54]. However, we leave exploring thermodynamic aspects for non-interacting and interacting models to a future study.

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