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Degree-balanced decompositions of cubic graphs

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ABSTRACT

We show that every cubic graph on n vertices contains a spanning subgraph in which the number of vertices of each degree deviates from $\frac{n}{4}$ by at most $\frac{1}{2}$, up to three exceptions. This resolves the conjecture of Alon and Wei (2023) for cubic graphs.

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1. Introduction

In this paper, we only consider simple graphs (i.e., graphs with no loops and no parallel edges). Given a graph G and a nonnegative integer k , we denote by $m(G, k)$ the number of vertices of degree k in G , and by $m(G)$ the maximum number of vertices of the same degree in G .

Recently, Alon and Wei [1] considered the problem of searching for a spanning subgraph H of a d -regular graph G , for which $m(H)$ is the smallest possible. Clearly, the lower bound for this value is $m(H) \geq \lceil \frac{n}{d+1} \rceil$. The authors of [1] suspect that the best general upper bound for such $m(H)$ is however very close to the same quantity. More specifically, they proposed the following two conjectures.

Conjecture 1.1 (Alon & Wei [1]). *Every d -regular graph G on n vertices contains a spanning subgraph H such that for every k , $0 \leq k \leq d$,*

$$\left| m(H, k) - \frac{n}{d+1} \right| \leq 2.$$

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Conjecture 1.2 (Alon & Wei [1]). Every graph G on n vertices and minimum degree δ contains a spanning subgraph H satisfying

$$m(H) \leq \frac{n}{\delta + 1} + 2.$$

These conjectures thus address the quest for a highly irregular subgraph H in a given graph G , that is a subgraph with degrees nearly as diversified as possible, almost as much as degrees in the host graph G permit. Alon and Wei [1] managed to prove the conjectures up to constant multiplicative factors. Namely, they assured the existence of spanning subgraphs H with $m(H) \leq 8\frac{n}{d} + 2$ and $m(H) \leq 16\frac{n}{\delta} + 4$ in all n -vertex d -regular graphs and graphs with minimum degree $\delta > 0$, respectively. They also confirmed [Conjecture 1.1](#) asymptotically for small enough d compared to n , showing that each n -vertex d -regular graph with $d = o((n/\log n)^{1/3})$ contains a spanning subgraph H with $m(H, k) = (1 + o(1))\frac{n}{d+1}$, for every $0 \leq k \leq d$. Later, Fox, Luo, and Pham [2] improved this result, by significantly extending the range of d for which its statement holds, i.e., towards all $d = o(n/(\log n)^{12})$. Finally, Ma and Xie [3] provided the first bound for [Conjecture 1.1](#) independent of n .

Theorem 1.3 (Ma & Xie [3]). Every d -regular multigraph G on n vertices contains a spanning subgraph H such that for every k , $0 \leq k \leq d$,

$$\left| m(H, k) - \frac{n}{d + 1} \right| \leq 2d^2.$$

For general n -vertex graphs with $\delta > 0$, Alon and Wei [1] provided spanning subgraphs H with $m(H) \leq (1 + o(1))\lceil \frac{n}{\delta+1} \rceil + 2$ and confirmed [Conjecture 1.2](#) up to additive factor 1 for sufficiently large n with $\delta^{1.24} \geq n$.

Apart from Ma and Xie, who used a novel deterministic approach to prove their result, the above-mentioned asymptotic approximations of [Conjectures 1.1](#) and [1.2](#) were based on an analysis of a randomized procedure of choosing H , exploited earlier, e.g., in [4–7] to investigate a related concept, the so-called irregularity strength of graphs. Within this problem, instead of looking for a subgraph with limited frequency of every potential degree, one strives to assure at most one vertex of each degree. This is achieved either by multiplying the edges of the host graph G or equivalently weighting its edges with positive integers and considering weighted degrees. The least k admitting such a weighting with the maximum weight k is called the *irregularity strength* of G and denoted $s(G)$. This notion was introduced in [8] in the 80 s, along with the key open problem of the related field from [9], which asserts that $s(G) \leq \frac{n}{d} + C$ for every d -regular graph G of order n , that is just a constant above a trivial lower bound. See, e.g., [6] for an extensive list of references to papers devoted to studying this conjecture. These, as well as the main objective of our paper, are related with problems concerning the so-called degree-constrained subgraphs, see, e.g., [10] (and also [11,12]), whose results were in particular exploited by Alon and Wei in [1]. In the same paper, they also established several more direct relations between the irregularity strength of graphs and the existence of almost irregular subgraphs we focus on.

In this paper, instead of asymptotics, we investigate exact bounds for 3-regular (i.e., *cubic*) graphs. In particular, we prove that [Conjecture 1.1](#) itself holds for such graphs, taking the first modest step towards solving the conjecture in its literal form. It was Alon and Wei [1] who observed that one cannot replace the constant 2 in [Conjecture 1.1](#) with 1, as exemplifies, e.g., the graph comprised of two components isomorphic to the cycle of length 4. They however suspected that such a strengthening might be possible for every d up to a limited number of small exceptions. Such strengthening of [Conjecture 1.1](#) for $d = 3$ is exactly implied by our main result below, which is even slightly stronger, and in face of the mentioned trivial lower bound is optimal for all cubic graphs.

Theorem 1.4. Every cubic graph G on n vertices, not isomorphic to K_4 , $K_{3,3}$, or $3K_4$, contains a spanning subgraph H such that for every k , $0 \leq k \leq 3$,

$$m(H, k) \in \left\{ \left\lfloor \frac{n}{d + 1} \right\rfloor, \left\lceil \frac{n}{d + 1} \right\rceil \right\}.$$

We note that [Conjecture 1.1](#) for the case $d = 3$ was independently confirmed by Ma and Xie [3], who employed a refinement of their local adjustments method. Our result differs from theirs in two aspects. We exploit an entirely different technique, which is strictly constructive and yields a straightforward algorithm directly generating the desired subgraph. But more importantly, our result for cubic graphs is stronger than the one in [3], which basically confirms [Conjecture 1.1](#) for this graph class, i.e., implies an upper bound of 2 for the achievable maximum deviation of a subgraph degree frequency from $n/4$. In turn, we provide a complete solution for this invariant, determining the exact value of achievable maximum deviation for every cubic graph, which, apart from a few exceptional cases, equals 0 or $1/2$.

2. Proof of [Theorem 1.4](#)

In this section, we prove the main result of the paper. We first present some additional terminology and auxiliary observations.

We say that a d -regular graph G is $(n_d, n_{d-1}, \dots, n_0)$ -decomposable if there exists a spanning subgraph H of G such that $n_i = m(H, i)$, for every $i \in \{0, \dots, d\}$.

Since in the complement \bar{H} of H the degree of a vertex of degree k from H equals $d - k$, the following observation immediately follows.

Observation 1. *If a d -regular graph is $(n_d, n_{d-1}, \dots, n_0)$ -decomposable, then it is also (n_0, n_1, \dots, n_d) -decomposable.*

Given two subsets of vertices X and Y in a graph $G = (V, E)$, by $e(X, Y)$ and $e(X)$, we denote the number of edges having one end vertex in X and the other in Y , and having both endvertices in X , respectively. By $\text{out}(X)$ we denote the number of edges joining X with $V \setminus X$ in G . Moreover, given an edge-coloring of G , by $e_c(X, Y)$ and $e_c(X)$ we denote the number of edges of color c having one end vertex in X and the other in Y , and having both endvertices in X , respectively. For a vertex v , we additionally denote by $N_c(v)$ the set of neighbors u of v in G such that uv is colored c .

2.1. Connected graphs

In order to handle later graphs with multiple components we first prove a strengthening of [Theorem 1.4](#) in the connected case, implying the existence of specific types of decompositions.

Lemma 2.1. *Let G be a connected cubic graph on n vertices. The following statements hold:*

- (i) if $n = 4t$ and G is not isomorphic to K_4 , then G is (t, t, t, t) -decomposable;
- (ii) if $n = 4t$, then G is $(t - 1, t - 1, t + 1, t + 1)$ -decomposable;
- (iii) if $n = 4t + 2$ and G is not isomorphic to $K_{3,3}$, then G is $(t, t + 1, t, t + 1)$ -decomposable;
- (iv) if $n = 4t + 2$, then G is $(t - 1, t, t + 1, t + 2)$ -decomposable.

Proof. Let G be a connected cubic graph and let $n = |V(G)|$. We will show that for each of the statements (i)-(iv), we can construct a spanning subgraph H of G with a suitable number of vertices of each degree. In particular, we will color the edges of G with colors 0 and 1, and the edges of color 1 will represent the edges of H .

We consider the cases with $t = 1$ separately. Suppose first that $t = 1$ and $n = 4t$. Then G is isomorphic to K_4 and the graph $H = 2K_1 \cup K_2$ realizes the statement (ii). Suppose now that $t = 1$ and $n = 4t + 2$. There are two cubic graphs on 6 vertices: $K_{3,3}$ and the 3-prism (i.e., the complement of C_6). If G is isomorphic to $K_{3,3}$, then the graph $H = 3K_1 \cup P_3$ realizes the statement (iv). Otherwise G is isomorphic to the 3-prism. The statement (iii) is realized by the graph H being isomorphic to a triangle with one pendant edge and two isolated vertices, and the statement (iv) is realized by the graph $H = 3K_1 \cup P_3$.

So, we may assume that $t \geq 2$ and thus $n \geq 8$. We first color all the edges of G with 0, and in the following steps, we carefully recolor some of them with color 1. We call a vertex with k incident edges of color 1 a k -vertex, and we denote the set of all k -vertices by V_k ; note that after recoloring

edges with 0 or 1, we always update the sets V_i accordingly. Note also that, by the Handshaking Lemma, the sizes of V_3 and V_1 always have the same parity. By n_i we denote the target number of i -vertices in H .

Our argument is divided into several natural stages.

Stage 1: Determining 3-vertices. We choose the vertices for V_3 one by one; namely, we start by finding a shortest cycle $C = v_1v_2 \dots v_g$ in G , where g is the length of C .

We first claim that $g \leq n/2$. Suppose to the contrary that $g > n/2$. Since $n \geq 8$, then $g \geq 5$. By the minimality of C , it has no chords and no two vertices of C can have a common neighbor outside C . Hence, each of the g vertices on C has its unique neighbor not in C , and thus $2g \leq n$, a contradiction.

We continue by coloring all edges incident to v_1 with 1, and then proceed with recoloring the edges incident with the vertices v_2, v_3 , and so on along C until $|V_3| = n_3$ or $V_3 = V(C)$. Note that since G is not isomorphic to K_4 and C is a shortest cycle, at every step exactly one vertex is introduced to V_3 .

Now, if $g < n_3$, then we continue recoloring to 1 all edges incident with any vertex adjacent to a vertex from V_3 which is not adjacent to any vertex from V_2 , until $|V_3| = n_3$. Note that we can always choose such a vertex v . Indeed, this is obvious if $e_0(V_2, V_1 \cup V_0) > 0$. Otherwise, as $|V_3| + |V_2| < n_3 + n_2 < n$ and G is connected, there is a neighbor $v \in V_1$ of some vertex in V_3 ; since $e_0(V_2, V_1 \cup V_0) = 0$, v has no neighbors in V_2 . Therefore, in each step, the size of V_3 increases by 1 and we can achieve the target size of V_3 .

The choice of the vertices for V_3 guarantees that the graph induced by V_3 is connected. This means that $e(V_3) \geq n_3 - 1$, and if $g \leq n_3$, then there is a cycle in V_3 , giving $e(V_3) \geq n_3$.

We now compute the number of edges $\text{out}(V_3)$ with exactly one end vertex in V_3 . By counting the half-edges incident with the vertices in V_3 , we infer

$$\text{out}(V_3) = 3n_3 - 2e(V_3) \leq 3n_3 - 2(n_3 - 1) = n_3 + 2.$$

Therefore, after completing the set V_3 , we have that

$$2|V_2| + |V_1| = \text{out}(V_3) \leq n_3 + 2.$$

It follows that

$$|V_2| \leq n_2 \quad \text{and} \quad |V_1| \leq n_3 + 2 \leq n_1 + 2. \tag{1}$$

Since in the cases (ii) and (iv), we have $n_1 = n_3 + 2$, this implies that in cases (ii) and (iv), $|V_1| \leq n_1$. Moreover, if $g \leq n_3 + 1$, then all the edges of C are colored with 1, and thus $|V_1| \leq n_3 \leq n_1$. **Stage**

2: Determining 2-vertices. We continue by completing the set V_2 , which may already contain some vertices from the previous stage. First, we define three rules for choosing 2-vertices.

- (R₁) If $|V_2| < n_2 - 1$ and there is a vertex $v \in V_1 \cap N_1(V_2 \cup V_3)$ adjacent to a vertex $u \in V_1$ (hence, uv has color 0), then we color uv by 1; thus increasing the size of V_2 by 2 and decreasing the size of V_1 by 2.
- (R₂) If $|V_2| < n_2$ and there is a vertex $v \in V_1$ with a neighbor $u \in V_0$, then we color uv by 1; thus increasing the size of V_2 by 1, decreasing the size of V_0 by 1, and retaining the size of V_1 .
- (R₃) If $|V_2| < n_2$, $|V_1| < n_1$, and there is a vertex $v \in V_0$ with two neighbors $u, w \in V_0$, then we color vu and vw with 1; thus increasing the size of V_2 by 1, increasing the size of V_1 by 2, and decreasing the size of V_0 by 3.

Next, until $|V_2| = n_2$, we repeatedly apply the rule R_1 if possible, and otherwise the rule R_2 if possible. Moreover, when R_2 is applied, we prefer coloring an edge of C if possible.

We claim that if $|V_2| < n_2$ and we cannot apply rules R_1 or R_2 , then by (1), we have that $|V_1| \leq n_1$. Recall that in the cases (ii) and (iv), as well as when $g \leq n_3 + 1$, this holds even at the beginning of Stage 2, while applying R_1 or R_2 does not increase $|V_1|$. In the remaining two cases, (i) and (iii), we have $n_3 + n_2 = n/2 \geq g$. Moreover, we also have that $g > n_3 + 1$. In this case, we have $|V_2| = 0$ at the beginning of Stage 2. Suppose that we cannot apply R_1 until we apply R_3 or $|V_2| = n_2$. Then,

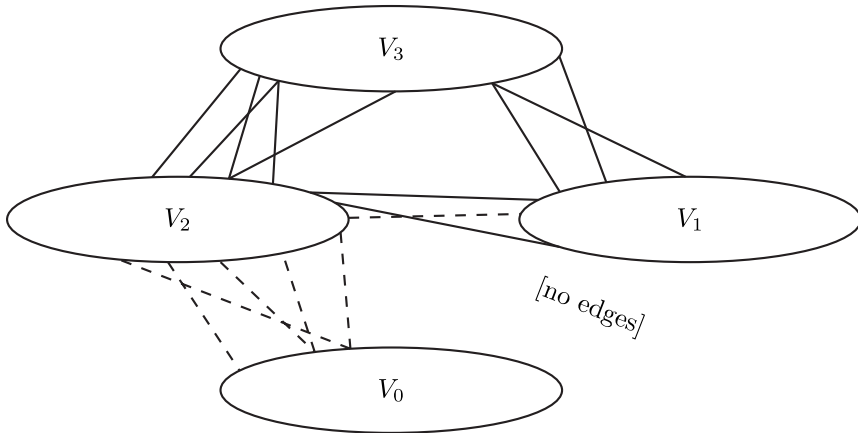


Fig. 1. The graph G in the case when R_1, R_2 do not apply, i.e. $e(V_1, V_0) = 0$. The edges of color 1 and 0 are depicted solid and dashed, respectively.

by the rule, we apply R_2 and color the edge in C first. This process will be repeated at most $g - 2$ times. After that, $|V_2| = g - |V_3| - 2 \leq n_2 - 2$, and there is only one edge in C of color 0 incident to two vertices in V_1 . Hence, we apply R_1 on this edge.

Suppose now that we cannot apply rules R_1 or R_2 , and we still have $|V_2| < n_2$. We will show that we can apply R_3 , meaning also that we retain $|V_1| \leq n_1$ (recall that the size of V_1 increases by 2 at every application of R_3).

Since R_1 and R_2 cannot be applied, we have that there is no edge between V_1 and V_0 (the rule R_2 does not apply), and that either $e_0(V_1) = e(V_1) = 0$ or $|V_2| = n_2 - 1$ (the rule R_1 does not apply). We will show by contradiction that in any case, we can apply the rule R_3 . So, suppose that R_3 cannot be applied. We consider two cases.

Suppose first that $e(V_1, V_0) = 0$ and $e(V_1) = 0$ (see Fig. 1). Then, since G is connected and thus there is at least one edge between V_2 and V_0 , and for each vertex $v \in V_1$, we have $N_0(v) \subseteq V_2$, it follows that $2|V_1| < |V_2| < n_2$. Now, if there is a vertex $v \in V_0$ with two neighbors $u, w \in V_0$, then we must have $|V_1| = n_1$. But in this case, $2n_1 < n_2$, which means that $t < 1$, a contradiction. So, every vertex in V_0 has at most one neighbor from V_0 and therefore, $e(V_0, V_2) \geq 2|V_0|$. By the construction, we also have that $|V_0| > n_0$ and $|V_2| - 2|V_1| \geq e(V_0, V_2)$, which altogether gives

$$n_2 > |V_2| \geq 2|V_1| + 2|V_0| \geq 2(n_0 + 1),$$

a contradiction.

Second, suppose that $e(V_1, V_0) = 0$ and $|V_2| = n_2 - 1$ (see Fig. 1).

We first argue that $e(V_1) \leq 2$. Indeed, if $e(V_1) \geq 1$ when $|V_2| \leq n_2 - 2$, we would apply R_1 . Otherwise, when $|V_2| \leq n_2 - 2$, we have that $e(V_1) = 0$ and either the rule R_2 is applied, meaning that by coloring the edge vu , at most 2 edges between vertices in V_1 are introduced (in the case when u had three neighbors from V_1 before coloring uv), or the rule R_3 is applied and at most 1 edge between vertices in V_1 is introduced (in the case when v and w are adjacent).

Now, if every vertex in V_0 has at most one neighbor from V_0 , then we again have $e(V_0, V_2) \geq 2|V_0| \geq 2(n_0 + 1)$, and moreover, since $|V_2| \geq e(V_0, V_2)$, we infer that

$$n_2 > |V_2| \geq 2(n_0 + 1),$$

a contradiction.

Therefore, there is at least one vertex $v \in V_0$ with two neighbors $u, w \in V_0$ and $|V_1| = n_1$ (recall that by the parity, if $|V_1| < n_1$, then also $|V_1| \leq n_1 - 2$). By the connectivity of G , we have $|V_2| > e_0(V_2, V_1)$, and since $e(V_1) \leq 2$, we have $e_0(V_2, V_1) \geq 2(n_1 - 2)$. Thus,

$$n_2 - 1 = |V_2| > e_0(V_2, V_1) \geq 2(n_1 - 2),$$

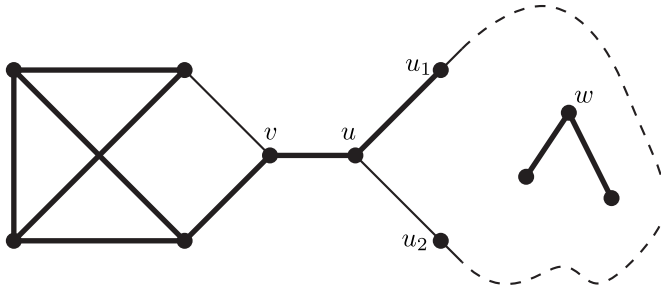


Fig. 2. The graph G in the case (iii) when $t = 3$ (where neighbors of w may coincide with u_2 or neighbors of u_1, u_2 other than u). The bold edges depict the edges of color 1.

and hence

$$n_2 > 2n_1 - 3. \tag{2}$$

In the cases of small n , we do not reach a contradiction, and thus we need to show separately that every cubic graph with the given properties admits a suitable decomposition. We analyze (2) for each of the four theorem's statements separately (recall that $t \geq 2$):

- (i) $t > 2t - 3 \Rightarrow t < 3$. However, this means that $t = 2$, and since $|V_1| \leq n_1$ once we potentially need to use R3, there will be no vertex $v \in V_0$ with three neighbors in V_1 . Thus, $e(V_1) \leq 1$ and $t > 2t - 1$, implying that $t < 1$.
- (ii) $t - 1 > 2(t + 1) - 3 \Rightarrow t < 0$, hence, this case is irrelevant.
- (iii) $t + 1 > 2t - 3 \Rightarrow t < 4$. In the case with $t = 2$, we argue similarly as in (i), and infer that $t < 2$. So, we only need to consider the case with $t = 3$. This means that we need to find a (3, 4, 3, 4)-decomposition of G .

First, we claim that $e(V_1) = 2$. Indeed, if $e(V_1) \leq 1$, then from $n_2 - 1 > 2(n_1 - 1)$, it follows that $t < 2$, a contradiction.

Next, we already have that $|V_3| = |V_2| = |V_1| = 3$, and so $|V_0| = 5$. Moreover, since $e(V_1, V_0) = 0$ and $e(V_1) = 2$, we have that $e_0(V_2, V_1) = 2$, and consequently $e(V_2, V_0) = 1$. Therefore, the five vertices of V_0 must induce a subgraph H' of G isomorphic to K_4 with one subdivided edge, where the unique vertex of degree 2, call it v , is adjacent to a vertex u from V_2 . Let u_1 and u_2 be the two neighbors of u , distinct from v . Since G is cubic with 14 vertices, there is also a vertex w not adjacent with any vertex from V_0 or with the vertices u, u_1 . Now, we recolor the edges of G using color 1 for all the edges of H' except one edge incident with v , the edges uv, uu_1 , and two edges incident with w . We color the remaining edges with 0 (see Fig. 2). This gives a spanning subgraph of G with three 3-vertices, four 2-vertices, three 1-vertices, and four 0-vertices as required, and thus G has a (3, 4, 3, 4)-decomposition.

- (iv) $t > 2(t + 1) - 3 \Rightarrow t < 1$, hence, this case is irrelevant.

This means that we can always attain the target number of vertices in V_2 by using the three rules and that in all the cases $|V_1| \leq n_1$.

Stage 3: Determining 1-vertices. This stage is realized only if $|V_1| < n_1$, and so, by the parity, $|V_1| \leq n_1 - 2$. Again, we add 1-vertices iteratively without changing 3- and 2-vertices. This means that, in every step, we increase $|V_1|$ by 2 by coloring an edge between two 0-vertices with 1.

Clearly, if there is an edge between two 0-vertices, then we can increase $|V_1|$. Thus, we only need to consider the situation when $e(V_0) = 0$. In such a case, since $|V_3| = n_3, |V_2| = n_2$, and $|V_1| \leq n_1 - 2$, we have that $|V_0| \geq n_0 + 2$, and so

$$e(V_0, V_1 \cup V_2) = 3|V_0| \geq 3(n_0 + 2).$$

On the other hand,

$$e(V_0, V_1 \cup V_2) \leq n_2 + 2|V_1| \leq n_2 + 2(n_1 - 2).$$

It thus follows that

$$3n_0 + 6 \leq n_2 + 2n_1 - 4,$$

which leads to a contradiction, as in all cases, $n_0 \geq n/4$, $n_1, n_2 \leq n/4 + 1$.

Therefore, we can always construct the sets V_i , for $i \in \{0, 1, 2, 3\}$ with the given sizes; this completes the proof. \square

2.2. Non-connected graphs

Next, we prove that the decomposition properties guaranteed by [Lemma 2.1](#) hold for all cubic graphs, also those with more components, with two additional exceptions.

Theorem 2.2. *Let G be a cubic graph on n vertices. The following statements hold:*

- (i) if $n = 4t$ and G is not isomorphic to K_4 and $3K_4$, then G is (t, t, t, t) -decomposable;
- (ii) if $n = 4t$ and G is not isomorphic to $2K_4$, then G is $(t - 1, t - 1, t + 1, t + 1)$ -decomposable;
- (iii) if $n = 4t + 2$ and G is not isomorphic to $K_{3,3}$, then G is $(t, t + 1, t, t + 1)$ -decomposable;
- (iv) if $n = 4t + 2$, then G is $(t - 1, t, t + 1, t + 2)$ -decomposable.

Proof. By [Lemma 2.1](#), we may assume that G has at least two connected components.

Let G be the minimal counterexample to the theorem with respect to the number of connected components. We consider two cases.

Case 1. At least one component of G , we call it H , is not isomorphic to K_4 nor $K_{3,3}$. Let $n_1 = |V(G - H)|$ and $n_2 = |V(H)|$. We consider two subcases regarding the number of vertices of G .

(a) Suppose that $n = 4t$.

If $n_1 = 4t_1$, for some integer t_1 , then $n_2 = 4t_2$, for some integer t_2 , and so $t = t_1 + t_2$. Suppose first that $G - H$ is not isomorphic to $2K_4$. Then, by the minimality of G , $G - H$ is $(t_1 - 1, t_1 - 1, t_1 + 1, t_1 + 1)$ -decomposable and H is (using also [Observation 1](#)) $(t_2 + 1, t_2 + 1, t_2 - 1, t_2 - 1)$ -decomposable, which implies that G is $(t_1 + t_2, t_1 + t_2, t_1 + t_2, t_1 + t_2)$ -decomposable, and hence (t, t, t, t) -decomposable; this realizes statement (i). The statement (ii) is realized, since H is also (t_2, t_2, t_2, t_2) -decomposable, implying that G is $(t_1 - 1 + t_2, t_1 - 1 + t_2, t_1 + 1 + t_2, t_1 + 1 + t_2)$ -decomposable, hence $(t - 1, t - 1, t + 1, t + 1)$ -decomposable. If $G - H$ is isomorphic to $2K_4$, then $t_1 = 2$ and we use the fact that $2K_4$ is $(2, 2, 2, 2)$ -decomposable, which settles both the case (i) and the case (ii), since H is (t_2, t_2, t_2, t_2) -decomposable and $(t_2 - 1, t_2 - 1, t_2 + 1, t_2 + 1)$ -decomposable, respectively.

Otherwise, $n_1 = 4t_1 + 2$, for some integer t_1 , and $n_2 = 4t_2 + 2$, for some integer t_2 , and so $t = t_1 + t_2 + 1$. Now, we use that $G - H$ is $(t_1 - 1, t_1, t_1 + 1, t_1 + 2)$ -decomposable. We realize the statement (i) by using that H is $(t_2 + 2, t_2 + 1, t_2, t_2 - 1)$ -decomposable, which gives that G is $(t_1 + t_2 + 1, t_1 + t_2 + 1, t_1 + t_2 + 1, t_1 + t_2 + 1)$ -decomposable, and hence (t, t, t, t) -decomposable. To realize the statement (ii), we make use of the fact that H is $(t_2 + 1, t_2, t_2 + 1, t_2)$ -decomposable, which gives that G is $(t_1 + t_2, t_1 + t_2, t_1 + t_2 + 2, t_1 + t_2 + 2)$ -decomposable, and hence $(t - 1, t - 1, t + 1, t + 1)$ -decomposable.

(b) Suppose that $n = 4t + 2$.

If $n_1 = 4t_1$, for some integer t_1 , then $n_2 = 4t_2 + 2$, for some integer t_2 , and so $t = t_1 + t_2$. Again, suppose first that $G - H$ is not isomorphic to $2K_4$. Then, we exploit the fact that $G - H$ is $(t_1 - 1, t_1 - 1, t_1 + 1, t_1 + 1)$ -decomposable. Since H is $(t_2 + 2, t_2 + 1, t_2, t_2 - 1)$ -decomposable, it follows that G is $(t_1 + t_2 + 1, t_1 + t_2, t_1 + t_2 + 1, t_1 + t_2)$ -decomposable, and thus $(t + 1, t, t + 1, t)$ -decomposable, which, using [Observation 1](#), realizes the statement (iii). The statement (iv) is realized by the fact that H is $(t_2, t_2 + 1, t_2, t_2 + 1)$ -decomposable, implying that G is $(t_1 + t_2 - 1, t_1 + t_2, t_1 + t_2 + 1, t_1 + t_2 + 2)$ -decomposable, and hence $(t - 1, t, t + 1, t + 2)$ -decomposable. If $G - H$ is isomorphic to $2K_4$, then $t_1 = 2$ and we again use the fact that $2K_4$ is $(2, 2, 2, 2)$ -decomposable, settling the cases (iii) and (iv) due to

$(t_2, t_2 + 1, t_2, t_2 + 1)$ -decomposability and $(t_2 - 1, t_2, t_2 + 1, t_2 + 2)$ -decomposability of H , respectively.

If $n_1 = 4t_1 + 2$, for some integer t_1 , then $n_2 = 4t_2$, for some integer t_2 , and again $t = t_1 + t_2$. We use the fact that $G - H$ is $(t_1 - 1, t_1, t_1 + 1, t_1 + 2)$ -decomposable. Since H is $(t_2 + 1, t_2 + 1, t_2 - 1, t_2 - 1)$ -decomposable, we have that G is $(t_1 + t_2, t_1 + t_2 + 1, t_1 + t_2, t_1 + t_2 + 1)$ -decomposable, and hence $(t, t + 1, t, t + 1)$ -decomposable, realizing the statement (iii). Since H is (t_2, t_2, t_2, t_2) -decomposable, we have that G is $(t_1 + t_2 - 1, t_1 + t_2, t_1 + t_2 + 1, t_1 + t_2 + 2)$ -decomposable, and hence $(t - 1, t, t + 1, t + 2)$ -decomposable, realizing the statement (iv).

Case 2. Every component of G is isomorphic to K_4 or $K_{3,3}$. Note first that by [Observation 1](#), we have that any graph H comprised of two isomorphic connected components H' , where H' admits an (a, b, c, d) -decomposition such that $a + d = b + c$, admits a (t', t', t', t') -decomposition, where $t' = \frac{|V(H)|}{4}$; we call such a decomposition *perfectly balanced*. As certificate lists of admissible decompositions below, pairs of K_4 and of $K_{3,3}$ thus admit perfectly balanced decompositions.

Next, we list possible decompositions of K_4 and $K_{3,3}$, which will be used in combinations for realizations of target decompositions of G . It is easy to verify that K_4 is (a, b, c, d) -decomposable, for every

$$(a, b, c, d) \in \{(0, 0, 0, 4), (0, 0, 2, 2), (0, 0, 4, 0), (0, 1, 2, 1), (0, 2, 2, 0), (0, 3, 0, 1)\} .$$

For $K_{3,3}$, we have that it is (a, b, c, d) -decomposable, for every

$$(a, b, c, d) \in \{(0, 0, 0, 6), (0, 0, 2, 4), (0, 0, 4, 2), (0, 0, 6, 0), (0, 1, 2, 3), (0, 1, 4, 1), (0, 2, 2, 2), (0, 3, 2, 1), (0, 4, 0, 2), (0, 4, 2, 0), (1, 0, 3, 2), (1, 1, 3, 1)\} .$$

Now, let k and ℓ be the number of connected components of G isomorphic to K_4 and $K_{3,3}$, respectively. We consider the cases regarding the parity of k and ℓ .

(a) Suppose that k and ℓ are both even. Then the statement (i) is realized by the remark above that every pair of isomorphic components of G admits a perfectly balanced decomposition.

To show that the statement (ii) can be realized, we consider two cases regarding ℓ . Suppose first that $\ell > 0$. Then, we split G in two parts, the first part being $H_1 = kK_4 \cup (\ell - 2)K_{3,3}$ and hence having a perfectly balanced decomposition, and the second part being $H_2 = 2K_{3,3}$. For the two copies of $K_{3,3}$ we use a $(2, 2, 2, 0)$ -decomposition and a $(0, 0, 2, 4)$ -decomposition. This altogether gives a $(t - 1, t - 1, t + 1, t + 1)$ -decomposition of G .

Suppose now that $\ell = 0$ and thus $k > 2$. We again split G in two parts, the first part being $H_1 = (k - 4)K_4$, which has a perfectly balanced decomposition, and the second part being $H_2 = 4K_4$, which has a $(3, 3, 5, 5)$ -decomposition (exploiting a $(2, 2, 0, 0)$ -decomposition, a $(1, 0, 3, 0)$ -decomposition, a $(0, 1, 2, 1)$ -decomposition, and a $(0, 0, 0, 4)$ -decomposition of K_4).

(b) Suppose that k and ℓ are both odd. Again, we split G ; namely, into the graph $H_1 = (k - 1)K_4 \cup (\ell - 1)K_{3,3}$ and the graph $H_2 = K_4 \cup K_{3,3}$. The graph H_1 admits a perfectly balanced decomposition, and for the graph H_2 , in order to realize the statement (iii), we use a $(2, 2, 0, 0)$ -decomposition of K_4 and a $(0, 1, 2, 3)$ -decomposition of $K_{3,3}$, obtaining a $(t, t + 1, t, t + 1)$ -decomposition of G . To realize the statement (iv), for H_2 , we use a $(0, 0, 0, 4)$ -decomposition of K_4 and a $(1, 2, 3, 0)$ -decomposition of $K_{3,3}$, obtaining a $(t - 1, t, t + 1, t + 2)$ -decomposition of G .

(c) Suppose that k is even and ℓ is odd. Note first that the statement (iv) is trivially realized, since $K_{3,3}$ admits a $(0, 1, 2, 3)$ -decomposition, and we obtain a desired decomposition of G by means of a perfectly balanced decomposition of $kK_4 \cup (\ell - 1)K_{3,3}$ and a $(0, 1, 2, 3)$ -decomposition of one $K_{3,3}$.

Hence, we only need to realize the statement (iii). We consider two subcases regarding ℓ . Suppose first that $\ell > 1$. Then, $\ell \geq 3$ and we split G into $H_1 = kK_4 \cup (\ell - 3)K_{3,3}$ and $H_2 = 3K_{3,3}$. For H_1 , we exploit a perfectly balanced decomposition, and for the three components of H_2 , we make use of the fact that $K_{3,3}$ is $(0, 0, 2, 4)$ -decomposable, $(2, 3, 0, 1)$ -decomposable, and $(2, 2, 2, 0)$ -decomposable, yielding a $(4, 5, 4, 5)$ -decomposition for H_2 .

Suppose now that $\ell = 1$. Then, $k \geq 2$. We split G in $H_1 = (k - 2)K_4$ and $H_2 = 2K_4 \cup K_{3,3}$. Again, for H_1 , we use a perfectly balanced decomposition, and for H_2 , a $(0, 0, 0, 4)$ -decomposition and a $(1, 2, 1, 0)$ -decomposition of K_4 , and a $(2, 2, 2, 0)$ -decomposition of $K_{3,3}$, giving a $(3, 4, 3, 4)$ -decomposition for H_2 .

- (d) Suppose that k is odd and ℓ is even. In this case, by the perfectly balanced decomposition of pairs of isomorphic components of G and the fact that K_4 admits a $(0, 0, 2, 2)$ -decomposition, the statement (ii) is realized.

We will show that except in the case with $k = 3$ and $\ell = 0$, we can always realize also the statement (i).

Suppose first that $\ell = 0$. Then $k \geq 5$. We split G into $H_1 = (k - 5)K_4$ and $H_2 = 5K_4$. For H_1 , we use a perfectly balanced decomposition, and for the five components of H_2 , we use a $(0, 0, 0, 4)$ -decomposition, a $(0, 1, 2, 1)$ -decomposition, a $(1, 0, 3, 0)$ -decomposition, and twice a $(2, 2, 0, 0)$ -decomposition, yielding a $(5, 5, 5, 5)$ -decomposition of H_2 .

Suppose now that $\ell \geq 2$. We split G in $H_1 = (k - 1)K_4 \cup (\ell - 2)K_{3,3}$ and $H_2 = K_4 \cup 2K_{3,3}$. Again, for H_1 , we use a perfectly balanced decomposition. For the unique K_4 component of H_2 , we use a $(0, 0, 0, 4)$ -decomposition, and for the two $K_{3,3}$ components, we use twice a $(2, 2, 2, 0)$ -decomposition, hence obtaining a $(4, 4, 4, 4)$ -decomposition of H_2 .

This completes the proof. \square

It is easy to see that the four listed exceptions indeed do not admit the required decompositions.

3. Concluding remarks

As $2K_4$ admits a perfectly balanced decomposition, [Theorem 2.2](#) immediately implies [Theorem 1.4](#). It is straightforward to verify that out of the remaining exceptional cubic graphs: K_4 , $3K_4$, $K_{3,3}$, the last one imposes the largest maximum deviation of $m(H, k)$ from $n/4$, i.e. $3/2$ (and 1 for the remaining two graphs). Thus, [Theorem 2.2](#) implies also the following corollary confirming [Conjecture 1.1](#) for cubic graphs.

Corollary 3.1. *Every cubic graph G on n vertices contains a spanning subgraph H such that for every k , $0 \leq k \leq 3$,*

$$\left| m(H, k) - \frac{n}{4} \right| \leq \frac{1}{2},$$

unless G is isomorphic to K_4 , $3K_4$ or $K_{3,3}$, for which $\max_{0 \leq k \leq 3} |m(H, k) - \frac{n}{4}| \leq \frac{3}{2}$.

Alon and Wei [1] observed that the Handsaking Lemma immediately implies that $\max_{0 \leq k \leq d} |m(H, k) - \frac{n}{d+1}|$ must be at least 1 for any spanning subgraph H of a d -regular graph G of order n with $\frac{n}{d+1}$ and $\lceil \frac{d}{2} \rceil$ being both odd integers. They however admitted that possibly the upper bound of 1 can be provided for all d -regular graphs of sufficiently large order. We specify this suspicion within the following bold conjecture, which additionally asserts that the condition above, mentioned by Alon and Wei, is the only reason for achieving the bound 1 for large enough graphs.

Conjecture 3.2. *For every d there is a finite family \mathcal{D}_d of exceptional graphs such that every d -regular graph G of order n which does not belong to \mathcal{D}_d contains a spanning subgraph H such that for every k , $0 \leq k \leq d$,*

- if $\frac{n}{d+1}, \lceil \frac{d}{2} \rceil$ are odd integers, then $|m(H, k) - \frac{n}{d+1}| \leq 1$;
- otherwise, $|m(H, k) - \frac{n}{d+1}| \leq \frac{d}{d+1}$ for even d , and $|m(H, k) - \frac{n}{d+1}| \leq \frac{d-1}{d+1}$ for odd d .

The asserted stronger bound for odd d again refers to the Handshaking Lemma. Note that our [Corollary 3.1](#) proves [Conjecture 3.2](#) for cubic graphs. We however have no idea how to adapt our approach in the case of $d \geq 4$ in order to prove [Conjecture 3.2](#) or the original [Conjecture 1.1](#). We reckon even the case of $d = 4$ constitutes an interesting open problem.

It is trivial to observe that [Conjecture 3.2](#) holds for $d = 1$. For consistency, we thus conclude this paper by sketching an easy argument that it also holds in the case of 2-regular graphs, with $\mathcal{D}_2 = \{2C_3, 2C_4\}$.

Observation 2. Every 2-regular graph G of order n not isomorphic to $2C_3$ or $2C_4$ contains a spanning subgraph H such that for every $k \in \{0, 1, 2\}$,

- if $\frac{n}{3}$ is an odd integer, then $|m(H, k) - \frac{n}{3}| \leq 1$;
- otherwise, $|m(H, k) - \frac{n}{3}| \leq \frac{2}{3}$.

Proof (Sketch). It is straightforward to verify the assertion for $n \leq 8$. In the remaining cases it is easy to note that one may effortlessly find a subgraph H' of G with any fixed number $n_2 \leq n - 4$ of 2-vertices and at most 4 vertices of degree 1 (basically by including in H' all consecutive edges around subsequent cycles making up G as long as necessary, with one possibly indispensable jump to a different component at the end). We then may easily supplement H' with isolated edges to obtain a necessary (even) number of 1-vertices in the resulting H . \square

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