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On 2-domination and 2-rainbow domination of cylindrical graphs

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Abstract
Cylindrical graphs and torus grid graphs are naturally constructed from subgraphs of the infinite grid by certain identifications of boundary vertices. Considering various domination type problems, it is usually possible to find an optimal solution on the infinite grid. To the contrary, exact values of invariants for the cylindrical and torus grid graphs are typically only known for special subfamilies, and are in general hard to compute. The 2-domination and 2-rainbow domination of cylindrical graphs is studied, and some new formulae and improved bounds are reported. We also consider weak 2-domination and singleton rainbow domination.

Keywords 2-domination · Weak 2-domination · Singleton 2-rainbow domination · Cylindrical graphs

Mathematics Subject Classification 305C69 · 05C76

1 Introduction

Domination of graphs is a popular topic in graph theory (Haynes et al. 1998). There are many variations of the basic domination, see e.g. Haynes et al. (2020) Here we are interested in 2-domination and in rainbow domination, more precisely in 2-rainbow, singleton 2-rainbow domination, and weak 2-domination. Two among them are well-known and extensively studied in the literature. The k-rainbow domination problem was initially explored in Brešar et al. (2008), for a recent survey see Brešar (2020). The 2-domination is a special case of k-domination, a variation that has been studied in the literature for a long time (Fink and Jacobson 1985).

As it is well known that the domination problems mentioned above are NP-hard in general (Brešar 2020; Martínez et al. 2022), it interesting to study classes of graphs for which efficient algorithms may exist. In some cases, it is even possible to find exact formulae for infinite families of graphs. An example of such family are the cylindrical graphs $P_m \square C_n$ that have recently enjoyed significant interest. Recent work regarding the domination of cylindrical

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graphs includes the following. Improved lower bounds for basic domination problem were provided recently in Guichard (2024). The best so far lower bound for the 2-domination number of cylindrical graphs is proven in Martínez et al. (2022). The computation of the 2-domination number of cylindrical graphs for arbitrary paths and small cylinders was addressed in Garzón et al. (2022a), and for cylinders with small paths and arbitrary cycles in Garzón et al. (2022b).

In this paper we prove a lower bound for the weak 2-domination number that is at the same time a lower bound for some other varieties including the 2-domination number. Our proof is much shorter than the proof in Martínez et al. (2022) and generalizes the result as it is valid for larger domain. We also give a construction that provides an upper bound for the singleton 2-rainbow domination. These observations imply some new bounds for several varieties of domination and, in some cases, provide exact values.

The rest of the paper is organized as follows. In the next section, we list our results. Sections 3 and 4 provide formal definitions and proofs. In the last section, we discuss some challenges for future woek.

2 The results

Recently, Martínez et al. (2022) have provided a lower bound for 2-domination of the Cartesian product of a path P_m, $m \geq 16$, and a cycle C_n, $n \geq 16$:

$$\gamma_2(P_m \Box C_n) \geq \frac{(m+2)n}{3} \tag{1}$$

In the same paper, they note that for $m \geq 8$ and $n \geq 3$, $n \equiv 0 (\mathrm{mod} 3)$, it holds

$$\gamma_2(P_m \Box C_n) = \frac{(m+2)n}{3} \tag{2}$$

Here we will provide an alternative proof of (1) that at the same time generalizes the result. Our proof is valid for smaller m, i.e. $m \geq 4$, and it also holds for some other closely related variations of the 2-domination. We will use the following inequalities

$$\gamma_{w2}(G) \leq \gamma_{r2}(G) \leq \tilde{\gamma}_{r2}(G) \qquad \text{and} \qquad \gamma_{w2}(G) \leq \gamma_2(G) \leq \tilde{\gamma}_{r2}(G) \tag{3}$$

where $\gamma_{w2}(G)$, $\gamma_2(G)$, $\gamma_{r2}(G)$, and $\tilde{\gamma}_{r2}(G)$ denote the weak 2-domination number, 2-domination number, 2-rainbow domination number, and singleton 2-rainbow domination number, respectively. The r-rainbow domination and weak 2-domination were first studied in Brešar et al. (2008), singleton domination was defined in Erveš and Žerovnik (2021), while the k-domination appears in the literature much earlier (Fink and Jacobson 1985). For definitions see Sect. 3.

In this paper we provide a constructive proof of the upper bound for singleton 2-rainbow domination (see Proposition 7)

$$\tilde{\gamma}_{r2}(P_m \Box C_n) \leq (m+2)\left\lfloor \frac{n}{3} \right\rfloor + (n^2 \bmod 3)m. \tag{4}$$

On the other hand, a lower bound (5) for weak 2-domination is given in Proposition 12

$$\gamma_{w2}(P_m \Box C_n) \geq \frac{(m+2)n}{3}. \tag{5}$$

These two bounds, together with the inequalities among the domination varieties (3) give us improved upper and lower bounds for cylindrical graphs. In some cases, exact values are obtained (see Theorem 1).

Theorem 1 *For $m \geq 4$ and $n \equiv 0 \pmod 3$,*

$$\gamma_{w2}(P_m \Box C_n) = \gamma_2(P_m \Box C_n) = \gamma_{r2}(P_m \Box C_n) = \tilde{\gamma}_{r2}(P_m \Box C_n) = \frac{(m+2)n}{3}. \tag{6}$$

Proof The lower and upper bounds proven by Propositions 12 and 7 are equal when $n \equiv 0 \pmod 3$. Using the inequalities (3), the result follows. □

The bounds are summarized in the next theorems. We write them as separate assertions to emphasize the possible independent importance of each of them.

Theorem 2 *For $m \geq 4$ and $n \geq 3$. Then for the 2-domination mumber of cylindrical graph $\gamma_2(P_m \Box C_n)$ we have*

$$\frac{(m+2)n}{3} \leq \gamma_2(P_m \Box C_n) \leq (m+2)\left\lfloor \frac{n}{3} \right\rfloor + \begin{cases} 0 & \text{if } n \bmod 3 = 0 \\ m & \text{if } n \bmod 3 \neq 0 \end{cases}$$

Theorem 3 *For $m \geq 4$ and $n \geq 3$. Then for the 2-rainbow domination mumber of cylindrical graph $\gamma_2(P_m \Box C_n)$ we have*

$$\frac{(m+2)n}{3} \leq \gamma_{r2}(P_m \Box C_n) \leq (m+2)\left\lfloor \frac{n}{3} \right\rfloor + \begin{cases} 0 & \text{if } n \bmod 3 = 0 \\ m & \text{if } n \bmod 3 \neq 0 \end{cases}$$

Theorem 4 *For $m \geq 4$ and $n \geq 3$. Then for the singleton 2-rainbow domination mumber of cylindrical graph $\tilde{\gamma}_2(P_m \Box C_n)$ we have*

$$\frac{(m+2)n}{3} \leq \tilde{\gamma}_{r2}(P_m \Box C_n) \leq (m+2)\left\lfloor \frac{n}{3} \right\rfloor + \begin{cases} 0 & \text{if } n \bmod 3 = 0 \\ m & \text{if } n \bmod 3 \neq 0 \end{cases}$$

Theorem 5 *For $m \geq 4$ and $n \geq 3$. Then for the weak 2-domination mumber of cylindrical graph $\gamma_{w2}(P_m \Box C_n)$ we have*

$$\frac{(m+2)n}{3} \leq \gamma_{w2}(P_m \Box C_n) \leq (m+2)\left\lfloor \frac{n}{3} \right\rfloor + \begin{cases} 0 & \text{if } n \bmod 3 = 0 \\ m & \text{if } n \bmod 3 \neq 0 \end{cases}$$

3 Preliminaries

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. Edges are pairs of vertices. As usual, the edge $\{i, j\} \in E(G)$ is shortly denoted by ij.

The Cartesian product of two graphs, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other.

The Cartesian product of graphs is one of the standard graph products (Hammack et al. 2011). The Cartesian product is commutative. In other words: $G \Box H$ is isomorphic to $H \Box G$.

In Cartesian product, copies of each factor appear as subgraphs and are called layers. In $G \Box H$, there are $|V(G)|$ isomorphic copies of H and $|V(H)|$ isomorphic copies of G.

The path on m vertices P_m is usually defined as the graph on vertices $V(P_m) = \{i \mid 0 \leq i < m\}$ and the set of edges $E(P_m) = \{ij \mid 0 \leq i < m-1, j = i+1\}$. The cycle on n vertices C_n is usually defined as the graph on vertices $V(C_n) = \{i \mid 0 \leq i < n\}$ and the set of edges $E(C_n) = \{ij \mid 0 \leq i < n, j \equiv i+1 \pmod n\}$. Here we are mainly interested in *cylindrical graphs* that are Cartesian products of a path and a cycle, $P_m \Box C_n$.

A set of vertices S is a *dominating set* of G if every vertex in the complement $V(G) \setminus S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the *domination number*, $\gamma(G)$. There are many variations of the basic domination, see e.g. Haynes et al. (2020). Here we are interested in 2-domination and in rainbow domination, more precisely in 2-rainbow, singleton 2-rainbow domination and weak 2-domination.

3.1 Variations of domination

A *2-dominating set* D is a set of vertices such that each vertex not in D has at least two neighbors in D. The *2-domination number* $\gamma_2(G)$ is the number of vertices in a smallest 2-dominating set for G.

A *k-rainbow dominating function* (kRDF) f of G assigns subsets of $\{1, 2, \ldots, k\}$ to vertices, such that for vertex v with $f(v) = \emptyset$, $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$. The *weight* $w(f)$ of kRDF f is $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is the *k-rainbow domination number* denoted by $\gamma_{rk}(G)$.

An interesting special case are kRD functions that assign only singletons and empty sets. Such functions are called *singleton kRD functions* (SkRDF) and the minimal weight obtained when considering only SkRD functions is the *singleton k-rainbow domination number*, $\tilde{\gamma}_{rt}$ (Erveš and Žerovnik 2021). Clearly, $\gamma_{rk} \leq \tilde{\gamma}_{rk}$.

Weak 2-domination was introduced in Brešar et al. (2008) as an auxiliary notion in the study of 2-rainbow domination on trees. A function $g : V(G) \to \{0, 1, 2\}$ is called a *weak 2-domination* function (W2DF) if it has the following property. For any vertex with $g(v) = 0$ it holds $\sum_{u \in N(v)} g(u) \geq 2$. The *weight* of g is $w(g) = \sum_{v \in V} g(v)$ and the *weak 2-domination number* of G, $\gamma_{w2}(G)$, is the minimum weight over all W2DF of G. Obviously, any 2-rainbow domination function f gives rise to a weak 2-domination function g defined as $g(v) = |f(v)|$. In words, $g(v)$ is the cardinality of the set of colors assigned by f. Thus clearly $\gamma_{w2} \leq \gamma_{r2}$.

Furthermore, it is straightforward to see that weak 2-domination is also a relaxation of 2-domination as any 2-domination function is also a weak 2-domination function. On the other hand, a weak 2-domination function is a 2-domination function only if $f(v) \leq 1$ for all $v \in V$. Consequently, $\gamma_{w2} \leq \gamma_2$.

Similarly, any singleton 2-rainbow domination function is a 2-domination function, while the opposite is not true in general, so we have $\gamma_2 \leq \tilde{\gamma}_{r2}$ (Erveš and Žerovnik 2021).

Summarizing, we have the inequalities (3)

$$\gamma_{w2}(G) \leq \gamma_{r2}(G) \leq \tilde{\gamma}_{r2}(G) \quad \text{and} \quad \gamma_{w2}(G) \leq \gamma_2(G) \leq \tilde{\gamma}_{r2}(G).$$

4 Proofs

4.1 Basic construction - upper bound for singleton rainbow domination

Recall from Brezovnik et al. (2024) that a 2RDF of the *grid graphs* may be based on one of the patterns of infinite grid that are given below. By grid graphs we mean the graphs that are constructed by taking a finite induced subgraph of the infinite grid, and then identify some pairs of the vertices on the boundary. Examples include, besides the cylindrical graphs that we study here also Cartesian products of cycles (that are examples of toroidal graphs) Brezovnik et al. (2024) and graph bundles of cycles over cycles Brezovnik et al. (2024). The products of paths are obtained without any identifications needed.

In order to outline the two patterns, let us start with two functions defined on $\mathbb{Z} \times \mathbb{Z}$,

$$f_1(i,j) = \begin{cases} 0, & i \not\equiv j \bmod 3 \\ 1 + (j \bmod 2), & i \equiv j \bmod 3 \end{cases} \tag{7}$$

and

$$f_2(i,j) = \begin{cases} 0, & i \not\equiv j \bmod 3 \\ 1 + (i \bmod 2), & i \equiv j \bmod 3 \end{cases}. \tag{8}$$

Define the coordinate system, for example with

$$\begin{bmatrix} \dots & \dots & \dots & \dots \\ \dots & (0,0) & (0,1) & \dots \\ \dots & (1,0) & (1,1) & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{9}$$

to obtain either

$$f_1 = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \dots \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{10}$$

or

$$f_2 = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{11}$$

It is straightforward to see that both patterns can be seen as a tilling with quadrilaterals of size 3×6 (or, 6×3),

$$\left[\begin{array}{c|cccccc|ccc}
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \dots \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \dots \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \dots \\ \hline
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \dots \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 \dots \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \dots \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right]. \tag{12}$$

and

$$\left[\begin{array}{c|ccc|ccc|ccc|c}
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline
\dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\
\dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \dots \\
\dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\
\dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\
\dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{array}\right], \tag{13}$$

illustrating the well-known fact Stępień et al. (2015) that $\gamma_{r2}(C_m \Box C_n) = \frac{1}{3}mn$ when m, n are multiples of 3, where at least one must also be a multiple of 6.

Here we will consider the products of a path over a cycle, so we start with subpatterns with $m + 2$ rows because columns will be used to define the assignment on the copies of the path. At first, think of a row corresponding to an infinite cycle as the second "factor". For example

$$\left[\begin{array}{c|ccc|ccc|ccc|cr}
\dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots & (i=-1) \\ \hline
\dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \dots & (i=0) \\
\dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & (i=1) \\
\dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots & \\
\dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\dots & \star & \star & \star & \star & \star & \star & \star & \star & \star & \dots & (i=m-1) \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & (i=m)
\end{array}\right]. \tag{14}$$

Observe that all vertices, with exception of some in the first and the last row, are dominated. The next operation (see (15)), i.e. merging rows 0 and –1, gives rise to an assignment in which all vertices at coordinates $(0, \star)$ are 2-rainbow dominated.

$$\left[\begin{array}{c|ccc|ccc|ccc|cr}
\dots & 1\downarrow & \dots & \dots & 1\downarrow & \dots & \dots & 1\downarrow & \dots & \dots & \dots & (i=-1) \\ \hline
\dots & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 & 0 & \dots & (i=0) \\
\dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots & (i=1) \\
\dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots & \\
\dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\dots & \star & \star & \star & \star & \star & \star & \star & \star & \star & \dots & (i=m-1) \\ \hline
\dots & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \uparrow & \dots & (i=m)
\end{array}\right]. \tag{15}$$

Clearly, doing the same with rows $m-1$ and m results in an assignment in which all positions in row $m-1$ get dominated from neighbors in rows $m-1$ and $m-2$.

Now consider application of the last pattern to domination of $P_m \Box C_n$. Clearly, if n mod $3 = 0$, the pattern defines a 2RDF, and we have $\gamma_{r2}(P_m \Box C_n) \leq \frac{(m+2)n}{3}$. Otherwise, when n mod $3 \neq 0$, we need to add some colors to obtain a 2RDF which gives us the upper bound, as we do in the proof of the following lemma.

Lemma 6 *For $m, n \geq 1$ we have*

$$\tilde{\gamma}_{r2}(P_m \Box C_n) \leq (m+2)\left\lceil \frac{n}{3} \right\rceil \tag{16}$$

Proof Start with the infinite pattern (15) based on f_2, and observe that there are $m+2$ colors used in any 3 consecutive columns. This holds for any $m \geq 1$. If n mod $3 = 0$, then clearly we have a 2RDF, thus $\gamma_{r2}(P_m \Box C_n) \leq (m+2)\frac{n}{3}$.

If n mod $3 \neq 0$, then simply taking the assignment based n consecutive columns clearly does not give a 2RDF. We distingush two cases.

Case 1. If n mod $3 = 2$, then start with the pattern based on f_2 and define a new function F on columns $0 \leq j \leq n-1$ and rows $0 \leq i \leq m-1$ as follows:

$$F(i,j) = \begin{cases} f_2(i,j) & \text{if } 1 \leq i \leq m-2 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(0,j), f_2(-1,j)\} & \text{if } i = 0 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(m-1,j), f_2(m,j)\} & \text{if } i = m-1 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(i,n-1), f_2(i,0)\} & \text{if } 0 \leq i \leq m-1 \text{ and } j = n-1 \end{cases} \tag{17}$$

In definition of F, the first row clearly just copies the assignment f_2. The second and third row of definition (17) formally express the merging explained above, recall (15). Observe that, for example in rows 0 and –1 in pattern (15) at least one of $f_2(0, j)$ and $f_2(-1, j)$ iz zero. So "max" is equivalent to union of colors, and, at the same time, we see that the union is never a set of two colors.

The last row in (17) merges columns $n-1$ and 0 (which is the same as column n). Again, the maximum is a way of expressing the fact that the nonzero element in column n is moved to the column $n-1$.

Summarizing, we conclude that F defines a singleton 2RDF of $P_m \Box C_n$ using at most $\frac{(m+2)(n+1)}{3} = (m+2)\left\lceil \frac{n}{3} \right\rceil$ colors.

Case 2. If n mod $3 = 1$, then start with the pattern based on f_2 and define a new function F on columns $0 \leq j \leq n-1$ and rows $0 \leq i \leq m-1$ as follows:

$$F(i,j) = \begin{cases} f_2(i,j) & \text{if } 1 \leq i \leq m-2 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(0,j), f_2(-1,j)\} & \text{if } i = 0 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(m-1,j), f_2(m,j)\} & \text{if } i = m-1 \text{ and } 0 \leq j \leq n-2 \\ \max\{f_2(i,n-1), f_2(i,0), f_2(i,+1)\} & \text{if } 0 \leq i \leq m-1 \text{ and } j = n-1 \end{cases} \tag{18}$$

As in the previous case, the first part of definition defines the S2RDF on columns for $0 \leq j \leq n-2$. The last row in (18) merges columns $n-1$ and $n = 0$ and $n+1 = 1$. The pattern assures that the maximum is a way of expressing the fact that in fixed row, exactly one of the elements in columns $n-1$, n and $n+1$ is not zero. Thus the maximum is again equivalent to the union and F provides a singleton 2RDF of $P_m \Box C_n$ with at most $\frac{(m+2)(n+2)}{3} = (m+2)\left\lceil \frac{n}{3} \right\rceil$ colors. □

In elaboration of the two cases in Lemma 6 we have used the pattern on columns from 0 to n. However, we can improve the upper bound slightly if we choose the columns more carefully.

An alternative is to start with $3\left\lceil\frac{n}{3}\right\rceil$ columns, and pick three consecutive columns. The three chosen columns are altered by removing one or two elements in each row to obtain a table with one or two columns less. We will show that the $m+2$ colors used in the three consecutive columns can be replaced by at most m colors.

Proposition 7 *For $m, n \geq 1$ we have*

$$\tilde{\gamma}_{r2}(P_m \square C_n) \leq (m+2)\left\lfloor\frac{n}{3}\right\rfloor + \begin{cases} 0 & \text{if } n \bmod 3 = 0 \\ m & \text{if } n \bmod 3 \neq 0 \end{cases} = (m+2)\left\lfloor\frac{n}{3}\right\rfloor + (n^2 \bmod 3)m$$

Proof First, let $n \bmod 3 = 2$. Consider the pattern with m rows and $3\left\lceil\frac{n}{3}\right\rceil$ columns Choose three consecutive columns and delete one of the zeros. The choice of the position of zero to be deleted is defined by the following simple rules. In row 0, the deleted 0 must be in the same column as the nonzero element in row -1. Similarly for the rows $m-1$ and m, the nonzero element in row m defines the position of the deleted zero in row $m-1$. In all other rows the choice is arbitrary. One possibility is indicated below ($1\times$ and $0\times$ stand for deleted elements).

$$\left[\begin{array}{c|ccc|ccc|ccc|cr}
\dots & 1 & 0 & 0 & 1\times & 0 & 0 & 1 & 0 & 0 & \dots & (i=-1) \\
\hline
\dots & 0 & 2 & 0 & 0\times & 2 & 0 & 0 & 2 & 0 & \dots & (i=0) \\
\dots & 0 & 0 & 1 & 0\times & 0 & 1 & 0 & 0 & 1 & \dots & (i=1) \\
\dots & 2 & 0 & 0 & 2 & 0 & 0\times & 2 & 0 & 0 & \dots & \\
\dots & 0 & 1 & 0 & 0 & 1 & 0\times & 0 & 1 & 0 & \dots & \\
\dots & 0 & 0 & 2 & 0\times & 0 & 2 & 0 & 0 & 2 & \dots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\
\dots & \star & \star & \star & \star & \star & \star & \star & \star & \star & \dots & (i=m-1) \\
\hline
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & (i=m)
\end{array}\right]. \tag{19}$$

Observe that this defines a S2RDF on $n = 3\left\lceil\frac{n}{3}\right\rceil - 1$ columns.

When $n \bmod 3 = 1$, delete the remaining zeros in the chosen columns. It is straightforward to check that the resulting table corresponds to S2RD function $n = 3\left\lceil\frac{n}{3}\right\rceil - 2$ columns. □

Remark The upper bounds were obtained by a rather rough construction. However, on the other hand, it seems that there is no obvious way of improvement. It also does not appear to be easy to improve the bound if we drop the rainbow coloring condition, i.e. if we just look for 2-domination functions.

4.2 Lower bound for weak 2-rainbow domination

For convenience, we introduce some more notation. For $0 \leq i \leq m-1$, the set of vertices $\mathcal{C}^i = \{(i,0), (i,1), (i,2), \cdots, (i, n-1)\}$ is called the i-th row of $P_m \square C_n$.

Recall that a weak 2-domination function is a function $f : V \to \{0, 1, 2\}$, where for vertices with $f(v) = 0$ we have $\sum_{u \in N(v)} f(u) \geq 2$. In words, a vertex may be assigned one or two colors, and the uncolored vertices must have at least two colors in the neighborhood. (We use term colors although we do not distinguish among them, just count.)

Let f be a 2RDF of $P_m \square C_n$ and $s_i = \sum_{x \in \mathcal{C}^i} f(x)$. (We write briefly $f(v)$ instead of $f(i, j)$ where no confusion is possible.) Note that s_i is a sum of contributions of vertices with $f(v) = 1$ and $f(v) = 2$. The sequence $(s_0, s_1, \cdots, s_{m-1})$, is called the 2RDF sequence that corresponds to f.

Lemma 8 *Let f be a $\gamma_{w2}\,(P_m \Box C_n)$-function. Then we have, for $1 \le i \le m-2$,*

$$s_{i-1} + s_{i+1} \ge 2n - 4s_i, .$$

Proof Note that at most s_i vertices of the row $\mathcal{C}^i$ are colored (equality holds in the case when all $f(v) = 1$). Other (uncolored) vertices in $\mathcal{C}^i$, at least $n - s_i$ of them, have a total demand at least $2(m - s_i)$. Since at most $2s_i$ of this demand can be fulfilled by the colored vertices of $\mathcal{C}^i$, we must have at least $2m - 2s_i - 2s_i$ colors in the neighborhood of $\mathcal{C}^i$. Hence, $s_{i-1} + s_{i+1} \ge 2m - 4s_i$, as claimed. □

Lemma 9 *Let f be a $\gamma_{w2}\,(P_m \Box C_n)$-function. We can assume (a) $s_0 \le \frac{2n}{3}$ and (b) $s_{m-1} \le \frac{2n}{3}$.*

Proof We prove (a), the reasoning for (b) is analogous. If $s_0 > \frac{2n}{3}$, then we can define a new W2RDF as follows. Recall that $\gamma_{w2}(C_n) = \frac{2n}{3}$, so we can find a vertex with $|f(0,k)| > 0$ so that the function $\tilde{f}$ defined as $\tilde{f}(0,k) = f(0,k) - 1$ and $\tilde{f}(x) = f(x)$ for all vertices $x \ne (0,k)$ still dominates all vertices in $\mathcal{C}^0$. Define $\tilde{f}(1,k) = \min\{f(1,k) + 1, 2\}$ to ensure that the vertex $(1,k)$ remains dominated. Note that, in case already $f(1,k) = 2$, we have contradiction as f was not a $\gamma_{w2}\,(P_m \Box C_n)$-function. Otherwise, we have got a new function of the same weight that dominates the graph. Repeat the process until we end with a W2RDF with $s_0 = \frac{2n}{3}$. □

Lemma 10 *Let f be a $\gamma_{w2}\,(P_m \Box C_n)$-function. Then $s_0 + s_1 \ge n$. Assume $s_0 \le \frac{2n}{3}$ and write $s_0 = \frac{2n}{3} - \Delta$ for some $\Delta \ge 0$. Then, $s_1 \ge \frac{n}{3} + 4\Delta$.*

Proof Say $s_{-1} = 0$ and use Lemma 8, to obtain $s_1 \ge 2n - 4s_0$. Obviously, if $s_0 = \frac{2n}{3}$, then $s_1 \ge 2n - 4s_0 \ge \frac{n}{3}$.

Now assume $s_0 < \frac{2n}{3}$ and write $s_0 = \frac{2n}{3} - \Delta$, hence $s_1 \ge 2n - 4(\frac{2n}{3} - \Delta) = 2n - \frac{8n}{3} + 4\Delta = \frac{n}{3} + 4\Delta$. So $s_0 + s_1 \ge n$. □

The proof of the next proposition is based on idea of discharging. We follow the ideas of Shao et al. (2019) and Brezovnik et al. (2024).

We define a discharging rule in which the rows with sufficiently large s_i give half of their overweight to one or both of the neighboring columns. More precisely, let f' be a new function on the vertex set of $P_m \Box C_n$ that assigns a positive real number to each vertex. Note that f' need not be a W2RDF because noninteger weights are allowed. Denote by $s'_i = \sum_{x \in \mathcal{P}^i} f'(x)$ and let $(s'_1, s'_2, \ldots, s'_m)$ be the sequence corresponding to f'.

We aim to obtain $s'_i \ge \frac{n}{3}$ for $0 < i < m-1$, $s'_0 \ge \frac{2n}{3}$ and $s'_{m-1} \ge \frac{2n}{3}$. Indeed, it turns out that this can be done by one round of discharging from the s_i corresponding to an arbitrary W2RDF. Roughly speaking, the overloaded rows give half of their overweights to the two neighbors whenever applicable. There are obviously some exceptions needed around the first and the last row.

Formally, the discharging rules are as follows:

1. For $1 < i < m-2$: If $s_i > \frac{n}{3}$ then set $s'_i = \frac{n}{3}$. If $s_i \le \frac{n}{3}$, then
   (a) if $s_{i-1} > \frac{n}{3}$ and $s_{i+1} > \frac{n}{3}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{3}) + \frac{1}{2}(s_{i+1} - \frac{n}{3})$,
   (b) if $s_{i-1} > \frac{n}{3}$ and $s_{i+1} < \frac{n}{3}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - \frac{n}{3})$,
   (c) if $s_{i-1} < \frac{n}{3}$ and $s_{i+1} > \frac{n}{3}$, then $s'_i = s_i + \frac{1}{2}(s_{i+1} - \frac{n}{3})$.
2. if $s_0 > \frac{2n}{3}$ then set $s'_0 = \frac{2n}{3}$ and otherwise, if $s_0 < \frac{2n}{3}$ then $s'_0 = s_0 + \frac{1}{2}(s_1 - \frac{n}{3})$.
3. if $s_1 > \frac{n}{3}$ then set $s'_1 = \frac{n}{3}$ and otherwise, $s_1 \le \frac{n}{3}$, then

(a) if $s_0 > \frac{2n}{3}$ and $s_2 > \frac{n}{3}$, then $s_1' = s_1 + (s_0 - \frac{2n}{3}) + \frac{1}{2}(s_2 - \frac{n}{3})$,
(b) if $s_0 > \frac{2n}{3}$ and $s_2 < \frac{n}{3}$, then $s_1' = s_1 + (s_0 - \frac{2n}{3})$,
(c) if $s_0 < \frac{2n}{3}$ and $s_2 > \frac{n}{3}$, then $s_1' = s_1 + \frac{1}{2}(s_2 - \frac{n}{3})$.

4. if $s_{m-1} > \frac{2n}{3}$ then set $s_{m-1}' = \frac{2n}{3}$ and otherwise, if $s_{m-1} \le \frac{2n}{3}$ then $s_{m-1}' = s_{m-1} + \frac{1}{2}(s_{m-2} - \frac{n}{3})$.
5. if $s_{m-2} > \frac{n}{3}$ then set $s_{m-2}' = \frac{n}{3}$ and otherwise, if $s_{m-2} \le \frac{n}{3}$, then

(a) if $s_{m-1} > \frac{2n}{3}$ and $s_{m-3} > \frac{n}{3}$, then $s_{m-2}' = s_{m-2} + (s_{m-1} - \frac{2n}{3}) + \frac{1}{2}(s_{m-1} - \frac{n}{3})$,
(b) if $s_{m-1} > \frac{2n}{3}$ and $s_{m-3} < \frac{n}{3}$, then $s_{m-2}' = s_{m-2} + (s_{m-1} - \frac{2n}{3})$,
(c) if $s_{m-1} < \frac{2n}{3}$ and $s_{m-3} > \frac{n}{3}$, then $s_{m-2}' = s_{m-2} + \frac{1}{2}(s_{m-1} - \frac{n}{3})$.

Lemma 11 *Let $m \ge 4$, $n \ge 3$, and f be a $\gamma_{w2}(P_m \Box C_n)$-function. Then $\sum_{i=0}^{m-1} s_i \ge \sum_{i=0}^{m-1} s_i' = \frac{(m+2)n}{3}$.*

Proof Clearly, if $s_0 \ge \frac{2n}{3}$, $s_{m-1} \ge \frac{2n}{3}$, and $s_i \ge \frac{n}{3}$ for $0 < i < m-1$, $\sum_{i=0}^{m-1} s_i \ge \frac{(m+2)n}{3}$, and there is nothing to prove.

Observe that it is not possible to have three consecutive s_i with $s_i < \frac{n}{3}$. Indeed, from Lemma 8 we have $s_{i-1} + s_i + s_{i+1} \ge 2n - 3s_i > n$. Below we also assume $s_0 \le \frac{2n}{3}$ and $s_{m-1} \le \frac{2n}{3}$, recalling Lemma 9.

First consider the inner rows, $1 < i < m-2$. We claim that $s_i' \ge \frac{n}{3}$. If $s_i \ge \frac{n}{3}$ then $s_i' = \frac{n}{3}$ by discharging rule 1. If $s_i < \frac{n}{3}$ then by the observation above, we can not have $s_{i-1} < \frac{n}{3}$ and $s_{i+1} < \frac{n}{3}$. Let us distinguish the other three possibilities.

- If $s_{i-1} \ge \frac{n}{3}$ and $s_{i+1} \ge \frac{n}{3}$, then by Lemma 8, $s_{i-1} + s_{i+1} \ge 2n - 4s_i$. Write $s_i = \frac{n}{3} - \Delta$, $s_{i-1} = \frac{n}{3} + \Delta_1$, $s_{i+1} = \frac{n}{3} + \Delta_2$, $s_{i-1} + s_{i+1} = \frac{n}{3} + \Delta_1 + \frac{n}{3} + \Delta_2 \ge 2n - 4(\frac{n}{3} - \Delta) = \frac{2n}{3} + 4\Delta$. so $\Delta_1 + \Delta_2 \ge 4\Delta$ and hence, by rule 1(a),

$$s_i' = s_i + \frac{n}{3} - \Delta + \frac{1}{2}\Delta_1 + \frac{1}{2}\Delta_2 > \frac{n}{3}.$$

- Second, assume $s_{i-1} \ge \frac{n}{3}$ and $s_{i+1} < \frac{n}{3}$. Then by Lemma 8, $s_{i-1} \ge 2n - 4s_i - s_{i+1}$. Write $s_i = \frac{n}{3} - \Delta$, hence $s_{i-1} \ge 2n - 4s_i - s_{i+1} = \frac{2n}{3} + 4\Delta - s_{i+1} > \frac{n}{3} + 4\Delta$ and by rule 1(b),

$$s_i' = s_i + \frac{1}{2}\left(s_{i-1} - \frac{n}{3}\right) = \frac{n}{3} - \Delta + \frac{1}{2}(4\Delta) > \frac{n}{3}.$$

- Finaly, assume $s_{i-1} < \frac{n}{3}$ and $s_{i+1} \ge \frac{n}{3}$ and proceed analogously as in the previous case.

It remains to check the four rows $i = 0, 1, m-2$, and $m-1$.

- If $s_0 < \frac{2n}{3}$ then by Lemma 10, $s_1 \ge \frac{n}{3} + 4\Delta$, where $\Delta = \frac{2n}{3} - s_0$, and, after recharging

$$s_0' = s_0 + 2\Delta \ge \frac{2n}{3}.$$

- If $s_1 < \frac{n}{3}$ then, using Lemma 10, we have $s_0 + s_1 \ge n$ and by rule 3,

$$s_1' \ge s_1 + \left(s_0 - \frac{2n}{3}\right) \ge n - \frac{2n}{3} = \frac{n}{3}.$$

- Similarly for $i = m - 2$ and $m - 1$.

□

Proposition 12 *For* $m \geq 4$, *and* $n \geq 3$, $\gamma_{w2}(P_m \Box C_n) \geq \frac{(m+2)n}{3}$.

Proof Follows directly from Lemma 11. □

5 Conclusions

As already mentioned, we do not see a straightforward avenue that would give an improvement of the construction that leads to the general upper bound in our theorems. We conjecture that, at least in some cases, the upper bound is indeed the exact value of at least some of the variations of domination studied here. The challenge which may be an interesting research question is to prove this conjecture. At least for small enough m, the algebraic approach KlavŽar and Žerovnik (1996) (see also Martínez et al. 2022; Garzón et al. 2022b) may be the tool of choice. For small n, the algebraic approach may be even more useful as it is not neccessary to store the whole auxiliary matrix in the memory, since it is enough to work with only one row of the matrix, as observed in Pavlič and Žerovnik (2013), Žerovnik (2006).

Another interesting research task may be to handle the twisted products, i.e. the graph bundles Pisanski et al. (1983) that are obtained from the infinite grid by nontrivial identifications. When generalizing the cylindrical graphs to bundles, the only nontrivial case is to take the path as the fibre, and use the only nontrivial automorphism of the path, that is reflection, to obtain bands that are discretizations of the Moebius band. It is obvious that the lower bound proven here applies, and we conjecture that the lower bound will be strict in this case, similarly as it holds for graph bundles of cycles over cycles Brezovnik et al. (2024).

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Declarations

Conflict of interest The author declares he has no interests in relation to contents of this article.

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