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The 2-rainbow domination number of Cartesian bundles over cycles

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Abstract
A k-rainbow dominating function (kRDF) f of G assigns subsets of $\{1, 2, \ldots, k\}$ to vertices, such that for vertex v with $f(v) = \emptyset$, $\bigcup_{u \in N(v)} f(u) = \{1, 2, \ldots, k\}$. The weight $w(f)$ of kRDF f is $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is the k-rainbow domination number denoted by $\gamma_{rk}(G)$. This paper focuses on the 2-rainbow domination number of Cartesian graph bundles of cycles over cycles, extending recent results for Cartesian product of cycles. Exact values are given for certain infinite families, and tight lower and upper bounds are established for general case.

Keywords 2-rainbow domination · Domination number · Graph bundles

Mathematics Subject Classification 05C69 · 05C76

1 Introduction

Graph bundles (Pisanski and Vrabec 1982; Pisanski et al. 1983) are generalization of graph products and covering graphs. It is less known that some other well-known interconnection network topologies are Cartesian graph bundles, such as twisted

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hypercubes (Cull and Larson 1995; Efe 1991) and multiplicative circulant graphs (Stojmenović 1997). For example, some graph bundles have lower diameters in comparison to products (Banič and Žerovnik 2010; Erveš and Žerovnik 2015), and were indeed used as a supercomputer architecture in the early years (Barnes et al. 1968). Several graph invariants were studied on graph bundles, including domination number (Zmazek and Žerovnik 2006) and chromatic number (Klavžar and Mohar 1995).

The k-rainbow domination problem was initially explored in Brešar et al. (2008), and it has garnered significant interest (Brešar 2020). Notably, in Brešar and Kraner Šumenjak (2007), the authors established the equivalence between 2-rainbow domination and conventional domination in the prism $G\square K_2$, along with proving the NP-completeness of deciding whether a graph possesses a 2-rainbow dominating function of a specific weight. Additionally, in Brešar and Rall (2015), the authors provided characterizations for pairs of graphs G and H such that $\gamma(G\square H) = \min\{\gamma(G)|V(G)|, \gamma(H)|V(H)|\}$, using the concept of k-rainbow domination in their proofs.

A significant volume of research has focused on examining 2-rainbow domination within generalized Petersen graphs, as exemplified in Brezovnik et al. (2023); Tong et al. (2009); Xu (2009); Erveš and Žerovnik (2021). In recent years, the field of 2-rainbow domination and its variations has expanded even further. For instance, Meybodi et al. (2021) explored k-rainbow domination in graphs with bounded treewidth and Kuzman investigated k-rainbow domination of regular graphs (Kuzman 2020). Kim, in (2021), studied k-rainbow domination of middle graphs. An independent variant of k-rainbow domination on lexicographic products of graphs was examined closely in Brezovnik and Kraner Šumenjak (2019). Lastly, Kosari and Asgharsharghi (2022) conducted research on the ℓ-distance k-rainbow domination number of graphs.

In this paper, we study 2-rainbow domination number of Cartesian graph bundles of cycles over cycles. The analysis generalizes the results on the 2-rainbow domination number of Cartesian product of cycles (Brezovnik et al. 2024).

Let us first recall some basic definitions. Cartesian graph bundles are a generalization of Cartesian graph products.

Let B and G be graphs and Aut(G) be the set of automorphisms of G. To any ordered pair of adjacent vertices $u, v \in V(B)$ we assign an automorphism of G, formally $\varphi : E(B) \to \mathrm{Aut}(G)$. For brevity, we will write $\varphi(u, v) = \varphi_{u,v}$ and assume that $\varphi_{v,u} = \varphi_{u,v}^{-1}$ for any $u, v \in V(B)$.

Now we construct the graph X as follows. The vertex set of X is the Cartesian product of vertex sets, $V(X) = V(B) \times V(G)$. The edges of X are given by the rule: the vertices (b_1, g_1) and (b_2, g_2) are adjacent in X if either ($b_1 = b_2$ and $g_1g_2 \in E(G)$) or ($b_1b_2 \in E(B)$ and $g_2 = \varphi_{b_1,b_2}(g_1)$). We call X a *Cartesian graph bundle* with base B and fibre G and write $X = B\square^{\varphi}G$.

Our main results are summarized in the next two theorems. Before writing the theorems let us recall some basic facts. It is well known (Klavžar and Mohar 1995) that graph bundles are localy products, i.e. two adjacent fibres induce a product of fibre with K_2. This local product naturally defines an isomorphism between two copies of the fibre, or, equivalently, an automorphism of the fibre. Furthermore, for bundles over cycles that are not products it is possible to define a labeling such that these local isomorphisms (all but one) are identities. If the fibre is a cycle, then the nontrivial automorphism can be either a shift or a reflection. For details and formal definitions, see the next sections.

1. The first theorem is a summary of lower and upper bounds for bundles with cyclic ℓ-shifts. More precisely, the bundles $C_n \Box^{\varphi} C_m$ where $\varphi_{n-1,0} = \sigma_\ell$ and $\varphi_{i,i+1} = id$ for all other i.

Theorem 1 *Let $m \geq 3$ and $n \geq 6$, and let $\varphi_{n-1,0} = \sigma_\ell$ be a cyclic shift and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\left\lfloor \frac{m}{3} \right\rfloor n \leq \left(\left\lfloor \frac{m}{3} \right\rfloor + \alpha \right) n \leq \gamma_{r2}(C_n \Box^{\varphi} C_m) \leq \left\lceil \frac{m}{3} \right\rceil (n + \beta) \leq \left\lceil \frac{m}{3} \right\rceil (n + 2),$$

where

$$\alpha = \begin{cases} 0, & m \equiv 0 \bmod 3 \\ \frac{1}{2}, & m \equiv 1 \bmod 3 \\ 1, & m \equiv 2 \bmod 3 \end{cases},$$

and

$$\beta = \begin{cases} 0, & n + \ell \equiv 0 \bmod 3, \text{ where } n \text{ is even} \\ 1, & n + \ell \equiv 1, 2, 5 \bmod 6 \\ 2, & n + \ell \equiv 4 \bmod 6 \\ & \text{or } n + \ell \equiv 0 \bmod 3, \text{ where } n \text{ is odd} \end{cases}.$$

2. The second theorem is a summary of lower and upper bounds for bundles with reflections. More precisely, we consider $C_n \Box^{\varphi} C_m$ where $\varphi_{n-1,0}$ is ρ_0, ρ_1, or ρ_2 and $\varphi_{i,i+1} = id$ for all other i.

Theorem 2 *Let $m \geq 3$ and $n \geq 6$, let $\varphi_{n-1,0}$ be a reflection and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\left\lfloor \frac{m}{3} \right\rfloor n < \gamma_{r2}(C_n \Box^{\varphi} C_m) \leq \left\lceil \frac{m}{3} \right\rceil (n + 1).$$

2 Preliminaries

In 2-rainbow domination, the goal is to assign a subset of the color set $\{1, 2\}$ to every vertex of G such that every vertex with the empty set assigned has both colors in its neighborhood. Such an assignment is called a *2-rainbow dominating function* (2RD function) of the graph G. The weight of assignment f is the value

$w(f) = \sum_{v\in V(G)} |f(v)|$, where $|f(v)|$ is the number of colors assigned to vertex v. We also say that G is *2RD-colored* by f. A vertex is *2RD-dominated* if either it is assigned a nonempty set of colors, or it has both colors in its neighborhood. A vertex is said to be *colored* if $f(v) \neq \emptyset$ and is *not colored* or *uncolored* otherwise. The *2-rainbow domination number* $\gamma_{r2}(G)$ equals the minimum weight over all 2RD functions of G.

Two graphs G and H are called *isomorphic*, in symbols $G \simeq H$, if there exists a bijection φ from $V(G)$ onto $V(H)$ that preserves adjacency and nonadjacency. In other words, a bijection $\varphi : V(G) \to V(H)$ is an *isomorphism* when: $\varphi(i)\varphi(j) \in E(H)$ if and only if $ij \in E(G)$. An isomorphism of a graph G onto itself is called an *automorphism*. The identity automorphism on G will be denoted by id_G or shortly id. The *cycle* C_n on n vertices is defined by $V(C_n) = \{0, 1, \dots, n-1\}$ and $ij \in E(C_n)$ if $i \equiv j \pm 1 \bmod n$.

In the previous section we have defined a Cartesian graph bundle with base B and fibre G, $B\square^{\varphi}G$. It may be interesting to note that while it is well-known that a graph can have only one representation as a product (up to isomorphism and up to the order of factors) (Hammack et al. 2011), there may be many different graph bundle representations of the same graph (Zmazek and Žerovnik 2002). If all $\varphi_{u,v}$ are identity automorphisms, then the graph bundle is isomorphic to the product $X = B\square^{\varphi}G = B\square G$. Furthermore, it is well-known that if the base graph is a tree, then the graph bundle is always isomorphic to a product, i.e. $X = T\square^{\varphi}G \simeq T\square G$ for any graph G, any tree T and any assignment of automorphisms φ (Pisanski and Vrabec 1982; Pisanski et al. 1983).

For our considerations, it is important to note that a graph bundle over a cycle can always be relabeled in a way that all but at most one automorphism are identities (Klavžar and Mohar 1995). Fixing $V(C_n) = \{0, 1, 2, \dots, n-1\}$, we denote $\varphi_{n-1,0} = \alpha$, $\varphi_{i-1,i} = id$ for $i = 1, 2, \dots, n-1$, and write $C_n\square^{\varphi}G = C_n\square^{\alpha}G$. Some more examples of Cartesian graph bundles can be found in Erveš and Žerovnik (2013).

For later reference, recall from (Brezovnik et al. 2024) that a 2RDF of the Cartesian product of cycles $C_m\square C_n$ may be based on the pattern

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \cdots \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{1}$$

Moreover, it is easy to write explicit formula for the values of a 2RDF, namely

$$f_1(i,j) = \begin{cases} 0, & i \not\equiv j \bmod 3 \\ 2{-}i \bmod 2, & i \equiv j \bmod 3 \end{cases}.$$

Alternative is to define a 2RDF as

$$f_2(i,j) = \begin{cases} 0, & i \not\equiv j \bmod 3 \\ 2-j \bmod 2, & i \equiv j \bmod 3 \end{cases},$$

which results in the pattern

$$\begin{bmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \dots \\ \dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \quad (2)$$

Below we recall some known results that will be needed in the continuation as a starting point of generalization from products to bundles.

The characterization of $C_m \square C_n$ that attain the general lower bound for 2-rainbow domination number is given in Stępień et al. (2015).

Theorem 3 (Stępień et al. 2015) *If either* $m \equiv 0 \bmod 3$ *and* $n \equiv 0 \bmod 6$, *or* $m \equiv 0 \bmod 6$ *and* $n \equiv 0 \bmod 3$, *then*

$$\gamma_{r2}(C_m \square C_n) = \frac{1}{3}mn.$$

The next propositions from (Brezovnik et al. 2024) provide bounds for 2-rainbow domination number of Cartesian product of two cycles of different sizes.

Proposition 4 (Brezovnik et al. 2024) *Let* $m \geq 3$ *and* $n \geq 3$. *Write* $m = 3k + \ell$, *where* $\ell \equiv m \bmod 3$. *Then*

$$\gamma_{r2}(C_m \square C_n) \geq kn + \ell\frac{n}{2} = \frac{mn}{3} + \ell\frac{n}{6}.$$

Theorem 5 (*Brezovnik et al.* 2024) *Let* $m \geq 3$ *and* $n \geq 6$, $n = 6a + b$, $n \equiv b \pmod 6$. *Then we have*

a) *if* $b \neq 4$ *then* $\gamma_{r2}(C_m \square C_n) \leq \lceil \frac{m}{3} \rceil (n+1)$,
b) *if* $b = 4$ *then* $\gamma_{r2}(C_m \square C_n) \leq \lceil \frac{m}{3} \rceil (n+2)$.

3 Bundles of cycles over cycles

Automorphisms of a cycle C_m are of two types. A cyclic shift σ_ℓ of the cycle by ℓ elements or briefly *cyclic* ℓ*-shift*, $0 \leq \ell < m$, maps u_i to $u_{i+\ell}$ (index modulo m). As a special case we have the identity ($\ell = 0$, $\sigma_0 = id$). Other automorphisms of cycles are *reflections*. If C_m is a cycle on odd number of vertices, then all the reflections have exactly one fixed point. If the number m is even, then we have reflections without fixed points and reflections with two fixed points.

More formally, we define:

- **cyclic ℓ-shift** σ_ℓ, defined as $\sigma_\ell(i) = i + \ell$ for $i = 0, 1, \dots, m-1$. (Recall the convention $i + \ell \equiv (i + \ell) \mod m$.)
- **reflection with no fixed points** ρ_0, defined as $\rho_0(i) = m - i - 1$ for $i = 0, 1, \dots, m-1$. (For m even there is no fixed points.)
- **reflection with one fixed point** ρ_1, defined as $\rho_1(i) = m - i - 1$ for $i = 0, 1, \dots, m-1$. (For m odd, there is exactly one fixed point, $\rho_1(\frac{m-1}{2}) = m - \frac{m-1}{2} - 1 = \frac{m-1}{2}$.)
- **reflection with two fixed points** ρ_2, defined as $\rho_2(0) = 0$ and $\rho_2(i) = m - i$ for $i = 1, 2, \dots, m-1$. (For m even, there is the second fixed point $\rho_2(\frac{m}{2}) = m - \frac{m}{2} = \frac{m}{2}$.)

In Fig. 1, the graph bundle $C_4 \Box^\varphi C_4$ is depicted, where $\varphi = \sigma_1(i)$ represents a cyclic shift with $\ell = 1$. Meanwhile, Fig. 2 illustrates the graph bundle $C_5 \Box^\varphi C_3$, where $\varphi = \rho_1(i)$ represents a reflection with exactly one fixed point (namely, v_1).

3.1 Lower bounds on $\gamma_{r2}(C_n \Box^\varphi C_m)$

In this subsection, we will establish a lower bound on the 2-rainbow domination number of all kinds of bundles of cycle over cycle. We will do so by using the discharging argument, following the ideas of Shao et al. (2018) and (Shao et al. 2019). In case when the nontrivial automorphism is a reflection, we can show a slightly stronger inequality. Let us first recall the next definitions.

For a fixed m the set of vertices $C^i = \{(i, 1), (i, 2), \cdots, (i, m)\}$, $i \in [n]$ is called *the i-th column* of $C_n \Box^\varphi C_m$. Moreover, we define $s_i = \sum_{x \in C^i} |f(x)|$, where $f(x)$ is a 2RDF of $C_n \Box^\varphi C_m$.

The next lemma, analogue to Lemma 3.3 in Brezovnik et al. (2024), is the main tool for achieving the lower bound.

Lemma 6 *Let f be a $\gamma_{r2}(C_n \Box^\varphi C_m)$-function, $m \geq 3$. Write $m = 3k + \ell$, where $\ell \equiv m \pmod 3$. Then*

a) $s_{i-1} + s_{i+1} \geq 2m - 4s_i = 6k + 2\ell - 4s_i$,
b) *if, in addition,* $k \geq s_{min} = \min\{s_{i-1}, s_{i+1}\}$, *then*

$$s_{max} \geq 2m - 4s_i - s_{min} \geq 5k + 2\ell - 4s_i,$$

where $s_{max} = \max\{s_{i-1}, s_{i+1}\}$.

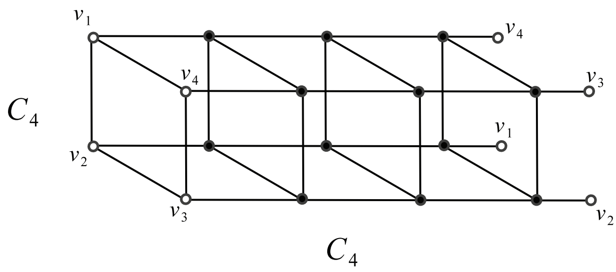

Fig. 1 Graph bundle $C_4\square^{\varphi}C_4$, where $\varphi = \sigma_1(i)$

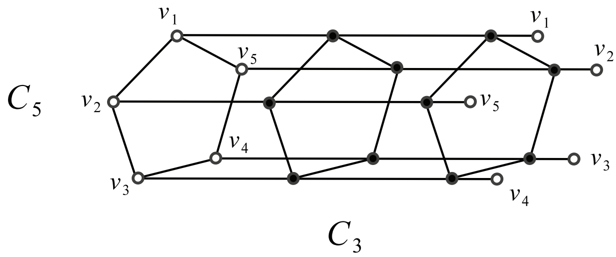

Fig. 2 Graph bundle $C_5\square^{\varphi}C_3$, where $\varphi = \rho_1(i)$

Proof Assume $m \geq 3$. Observe that any three consecutive columns, say $\mathcal{C}^{i-1}, \mathcal{C}^{i}, \mathcal{C}^{i+1}$, together with edges incident to $\mathcal{C}^{i-1}$ form a subgraph of the bundle that is isomorphic to the product $C_n\square C_3$. The rest of the proof repeats the arguments used in the proof of Lemma 3.3 in Brezovnik et al. (2024). □

Proposition 7 *Let $m \geq 3$ and $n \geq 6$, let $\varphi_{n-1,0}$ be any automorphism and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\gamma_{r2}(C_n\square^{\varphi}C_m) \geq \left(\left\lfloor \frac{m}{3} \right\rfloor + \alpha\right)n,$$

where $\alpha = \begin{cases} 0, & m \equiv 0 \bmod 3 \\ \frac{1}{2}, & m \equiv 1 \bmod 3 \\ 1, & m \equiv 2 \bmod 3 \end{cases}.$

Proof Write $m \equiv \ell \bmod 3$. Let us first consider the case when $\ell = 0$. Our goal is to prove that we need at least $n\frac{m}{3}$ colors to properly 2RD-dominate all the vertices of $C_n\square^{\varphi}C_m$. Let f be a 2RDF with the minimum weight. Write $m = 3k$. Our goal is to redistribute all the colors in such a way that there are at least k colors present in each column. By doing this, we will demonstrate that there are, in total, at least nk of them. In that manner, we introduce a redistribution rule for the weights s_i, transforming them into new weights labeled as s_i' for all $i \in [n]$. We will now do the following. If in the $\mathcal{C}^i$-th column we have at least k colors, then we set $s_i' := k$ and redistribute $\frac{1}{2}\Delta s_i$ to every of the neighbouring columns, where $\Delta s_i := (s_i - k)$. On the other hand, if $s_i < k$, then we accept the colors from columns $\mathcal{C}^{i-1}$ and $\mathcal{C}^{i+1}$ at the column $\mathcal{C}^i$. We then have $s_i' = s_i + \frac{1}{2}\Delta s_{i-1} + \frac{1}{2}\Delta s_{i+1}$ and Lemma 6a assures that $s_i' \geq k$. Note that by Lemma 6a this holds also in the case when only one neighbor has weight larger than k. The defined redistribution of weights thus assures that if s_{i-1} or s_{i+1} was greater than k, then s_{i-1}' or s_{i+1}' again is greater or equal to k, respectively. Applying this redistribution of weights to all $i \in [n]$, we achieve that for every i, $s_i' \geq k$. Hence $\gamma_{r2}(C_n \square^{\varphi} C_m) = \sum_i s_i \geq \sum_i s_i' \geq nk$, as needed.

Suppose now that $\ell = 1$ (i.e $m = 3k + 1$). Then we have to prove, that we need at least $\frac{m}{3} + \frac{n}{2}$ colors to properly 2RD-dominate all the vertices of $C_n \square^{\varphi} C_m$. By the same arguments as in the first part of this proof (using Lemma 6) and adjusting the threshold value to $k + \frac{1}{2}$, it can be shown that we can properly redistribute all the colors in such a way that every column has weight at least $k + \frac{1}{2}$.

The case when $\ell = 2$ (i.e $m = 3k + 2$) is analogous, we omit the details. □

Later it will be shown that this lower bound is tight as we will construct such 2RD assignments. The lower bound can however not be met when the nontrivial automorphism of the bundle is a reflection.

Proposition 8 *Let $m \geq 3$ and $n \geq 6$, let $\varphi_{n-1,0}$ be a reflection and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\gamma_{r2}(C_n \square^{\varphi} C_m) > \left\lfloor \frac{m}{3} \right\rfloor n.$$

Proof First, assume that $m = 3k$. Recall that after starting the patterns (1) or (2), their continuation is uniquely determined. As $\varphi_{n-1,0}$ is a reflection, there are only a few possible combinations of 2RD function on the first and the last columns, for one of them see the sketch below. In particular, observe the sequence of arrows between columns $n-1$ and 0 in the rightmost matrix.

$$\left[\begin{array}{ccc||ccc} .. & .. & .. & .. & .. & .. \\ .. & 0 & 1 & 0 & 1 & .. \\ .. & 0 & 0 & 2 & 0 & .. \\ .. & 1 & 0 & 0 & 0 & .. \\ .. & 0 & 2 & 0 & 1 & .. \\ .. & 0 & 0 & 2 & 0 & .. \\ .. & 2 & 0 & 0 & 0 & .. \\ .. & .. & .. & .. & .. & .. \\ \hline .. & n\text{-}2 & n\text{-}1 & 0 & 1 & .. \end{array}\right] \mapsto \left[\begin{array}{ccc||cc} .. & .. & .. & .. & .. \\ .. & 0 & 1 \rightarrow & - & .. \\ .. & 0 & 0 \leftarrow & \leftarrow & .. \\ .. & 1 & 0 - & \rightarrow & .. \\ .. & 0 & 2 \rightarrow & - & .. \\ .. & 0 & 0 \leftarrow & \leftarrow & .. \\ .. & 2 & 0 - & \rightarrow & .. \\ .. & .. & .. & .. & .. \\ \hline .. & n\text{-}2 & n\text{-}1 & 0 & .. \end{array}\right] \left[\begin{array}{c} \cdots \\ 1 \leftrightarrow 6 \\ 2 \leftrightarrow 5 \\ 3 \leftrightarrow 4 \\ 4 \leftrightarrow 3 \\ 5 \leftrightarrow 2 \\ 6 \leftrightarrow 1 \\ \cdots \\ \hline \varphi_{n-1,0} \end{array}\right]. \quad (3)$$

In column $n-1$, direction upside down, we have the sequence $(\dots, \rightarrow, \leftarrow, -, \dots)$ while in column 0 we have $(\dots, -, \leftarrow, \rightarrow, \dots)$. Clearly, the arrows of the two sequences can not fit, even if shifted vertically. Hence, additional colors are needed to obtain a 2RDF. Moreover, if we delete column $n-1$, the pattern $(\dots, \rightarrow, \leftarrow, -, \dots)$ repeats at column $n-2$. We conclude that for any n, $\gamma_{r2}(C_n\Box^{\varphi}C_{3k}) > nk$. It is straightforward to see that similar reasoning leads to conclusion $\gamma_{r2}(C_n\Box^{\varphi}C_m) > n\left\lfloor \frac{m}{3} \right\rfloor$ for arbitrary m. □

In the following we consider separately the upper bounds for shifts and reflections.

3.2 Upper bounds for bundles with cyclic shifts

Consider $X = C_n\Box^{\varphi}C_m$ where all $\varphi_{n-1,0} = \sigma_{\ell}$ and $\varphi_{i,i+1} = id$ for all other i.

Proposition 9 *Let $m \geq 3$ and $n \geq 6$, and let $\varphi_{n-1,0} = \sigma_{\ell}$ and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\gamma_{r2}(C_n\Box^{\varphi}C_m) \leq \left\lceil \frac{m}{3} \right\rceil (n+\beta),$$

where

$$\beta = \begin{cases} 0, & n+\ell \equiv 0 \bmod 3, \text{ where } n \text{ is even} \\ 1, & n+\ell \equiv 1,2,5 \bmod 6 \\ 2, & n+\ell \equiv 4 \bmod 6 \\ & \text{or } n+\ell \equiv 0 \bmod 3, \text{ where } n \text{ is odd} \end{cases}.$$

Proof We need to analyze optimal constructions that combine the ideas from Theorems 3 and 5. Here we only sketch the main idea. Observe that along the base cycle, the pattern is gradually shifted and a shift corresponding to shift automorphism just changes the shifting angle. For each case, consider the first and last columns and make a specific correction so that each zero-valued element should have both colors in its neighborhood. Technical details are left to the reader. (Are analog to the arguments proving Corollary 4.7 in Brezovnik et al. (2024).) □

3.3 Upper bounds for bundles with reflections

Theorem 2 follows from the two propositions that will be proved in this section. First we will introduce four transformations of the graph $C_n\Box^{\varphi}C_m$ that increase the length of the fibre by one or two.

3.3.1 Transformations

Let $G = C_n\Box^{\varphi}C_m$, where $\varphi_{n-1,0}$ is a reflection. First, we define two transformations assuming that $\varphi_{n-1,0}$ has a fixed vertex v with neighbors u and w on the cycle. Applies to reflections with one or two fixed points.

T1 Delete vertex v and replace it with two adjacent vertices v_a and v_b. Add edges uv_a, $v_a v_b$ and $v_b w$. Define reflection $\tilde{\varphi}_{n-1,0}$ with $\tilde{\varphi}_{n-1,0}(v_a) = v_b$, $\tilde{\varphi}_{n-1,0}(v_b) = v_a$ and $\tilde{\varphi}_{n-1,0}(x) = \varphi_{n-1,0}(x)$ otherwise.

T3 Add two vertices v_a and v_b. Add edges uv_a, $v_a v$, vv_b and $v_b w$. Define reflection $\tilde{\varphi}_{n-1,0}$ with $\tilde{\varphi}_{n-1,0}(v_a) = v_b$, $\tilde{\varphi}_{n-1,0}(v_b) = v_a$ and $\tilde{\varphi}_{n-1,0}(x) = \varphi_{n-1,0}(x)$ otherwise.

Now assume that $\varphi_{n-1,0}$ reflects two adjacent vertices, say u and w, on the cycle. Applies to reflections with one or no fixed points. Formally, $\tilde{\varphi}_{n-1,0}(u) = w$ and $\tilde{\varphi}_{n-1,0}(w) = u$. Two more transformations are defined as follows.

T2 Delete edge wu, add a vertex v, and edges uv and vw. Define reflection $\tilde{\varphi}_{n-1,0}$ with $\tilde{\varphi}_{n-1,0}(v) = v$, and $\tilde{\varphi}_{n-1,0}(x) = \varphi_{n-1,0}(x)$ otherwise.

T4 Delete edge wu, add vertices v_a and v_b, and edges uv_a, $v_a v_b$, and $v_b w$. Define reflection $\tilde{\varphi}_{n-1,0}$ with $\tilde{\varphi}_{n-1,0}(v_a) = v_b$, $\tilde{\varphi}_{n-1,0}(v_b) = v_a$ and $\tilde{\varphi}_{n-1,0}(x) = \varphi_{n-1,0}(x)$ otherwise.

A simplified representation of all four transformations are depicted on Fig. 3.

In the tables that we use for presenting the assignments, the vertices correspond to rows. Below we show how the transformations act on these tables.

T1 Delete the $\frac{m-1}{2}$-th row (together with the incident edges) of G and replace it with two additional rows (together with the edges connecting neighboring rows) connecting them into the cycle (see the upper left diagram of Fig. 4, where the vertex v is in the deleted row, and the vertices v_a and v_b are part of the new rows).

T2 Add an additional row between the 0-th and the $(n-1)$-th row (fixed edge) of G (along with the vertical edges connecting the vertices of these rows to the new rows) and connect it to the cycle (see the upper right diagram of Fig. 4, where the vertex v is part of the new row, and the vertices w and u are in the 0-th and the $(n-1)$-th row of G, respectively).

T3 Add two more rows around the $\frac{m-1}{2}$-th row of G (together with the vertical edges connecting the vertices of these rows to the $\frac{m-1}{2}$-th row) and connect them to the cycle (see the lower left diagram of Fig. 4, where the vertex v is in the $\frac{m-1}{2}$-th row, and the vertices v_a and v_b are part of the new rows).

T4 Add two additional rows between the 0-th and the $(n-1)$-th rows (fixed edge) of G (together with the vertical edges connecting the vertices of these rows to the new rows) and connect them to the cycle (see the lower right diagram of

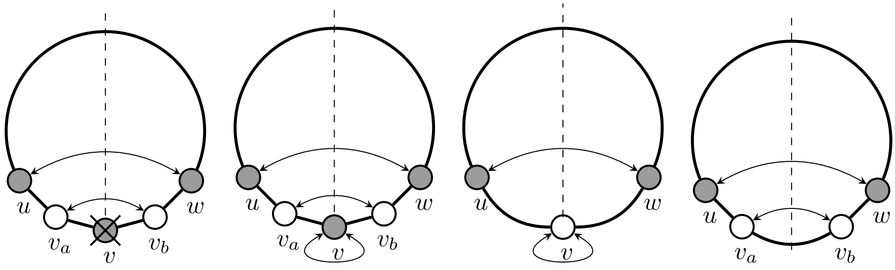

Fig. 3 Four transformations, $T1$, $T3$, $T2$, $T4$. Vertices added by transformations are white

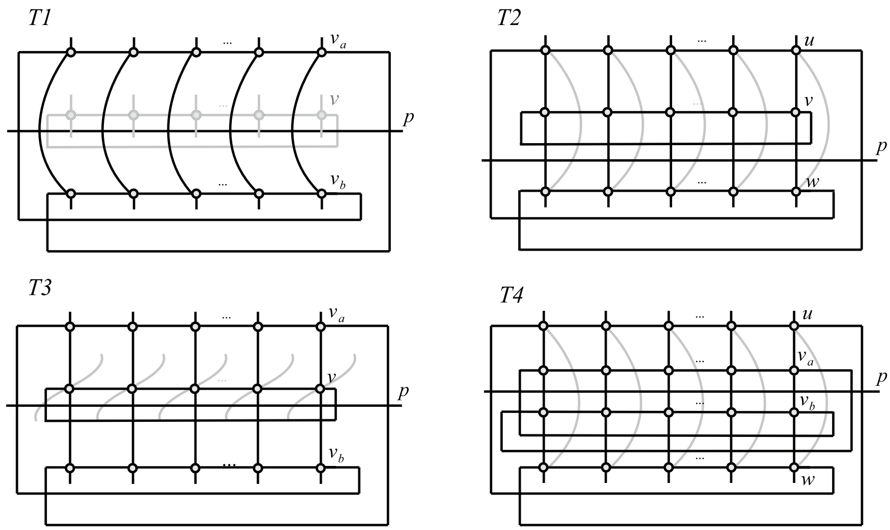

Fig. 4 Transformations $T1 - T4$ of $C_n \square^{\varphi} C_m$. The line p represents the axis of the reflection. Deleted edges are colored grey

Fig. 4, where the vertices v_a and v_b are each in one of those new rows, and the vertices w and u are part of the 0-th and the $(n-1)$-th row of G, respectively).

The following lemmas will ensure that after performing the graph transformations, the 2-rainbow domination number can still be appropriately bounded from above.

Lemma 10 *Let* $G = C_n \square^{\varphi} C_m$, *where* $m \equiv 0 \bmod 3$, *all* $\varphi_{n-1,0} = \rho_1$, *and* $\varphi_{i,i+1} = id$ *for all other* i. *Furthermore, let* f *be a* 2*RDF of* G, *obtained by using Pattern* 1 *or* 2 *and modifying (if needed) the last column and last two rows of* G. *Suppose that* $w(f) \leq \lceil \frac{m}{3} \rceil (n+1)$. *Then for* $G' = C_n \square^{\varphi} C_{m'}$ *obtained from* G *by one of Transformations* Ti, $i = 1, 2, 3, 4$, *we have*

$$\gamma_{r2}(G') \le \left\lceil \frac{m}{3} \right\rceil (n+1).$$

Proof By the assumption of the lemma, we can properly 2RD-dominate vertices of G by using at most $\lceil \frac{m}{3} \rceil (n+1)$ colors. Let $G' = C_n \square^{\varphi} C_{m'}$ be graph, obtained from G by Transformation $Ti, i = 1, 2, 3, 4$. When we apply Transformation $T1$, $\frac{m-1}{2}$-th row of G has been deleted and two additional rows has been added. After applying Transformation $T2$, we add one additional row. After applying Transformations $T3$ or $T4$ two rows were added. We now color the newly created vertices with the colors of corresponding vertices in the initial graph, as shown in Figs. 5, 6, 7 and8. Since f was defined by using Pattern 1 or 2, the colors are evenly distributed across the rows (with a slight deviation through the last column and the last two rows). Therefore, we increase $w(f)$ by at most $3 \cdot \frac{n+1}{3}$ in each case. Then for $m' = m + j, j = 1, 2,$

$$\begin{aligned}\gamma_{r2}(G') \le & \left\lceil \frac{m}{3} \right\rceil (n+1) + 3 \cdot \frac{n+1}{3} \\ = & \left(\frac{m}{3} + 1 \right)(n+1) \\ = & \left(\left\lceil \frac{m+j}{3} \right\rceil \right)(n+1) \\ = & \left\lceil \frac{m'}{3} \right\rceil (n+1),\end{aligned}$$

which ends the proof. □

3.3.2 Upper bound for bundles with reflection

In the following, we state the upper bound for 2-rainbow domination number of $C_n \square^{\varphi} C_m$ that we will demonstrate using the graph transformations mentioned above.

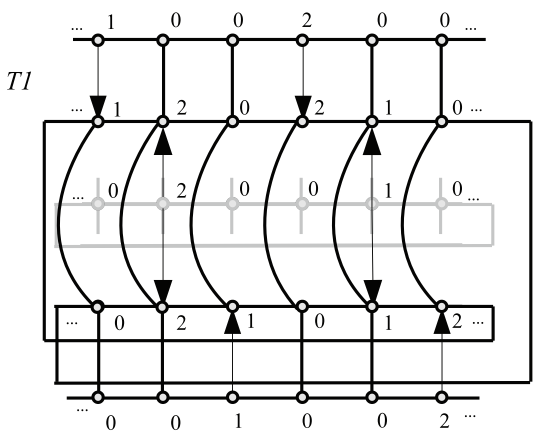

Fig. 5 Coloring of newly formed vertices of graph G' after using Transformation $T1$

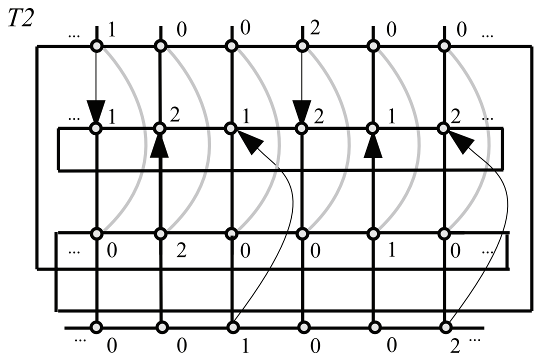

Fig. 6 Coloring of newly formed vertices of graph G' after using Transformation $T2$

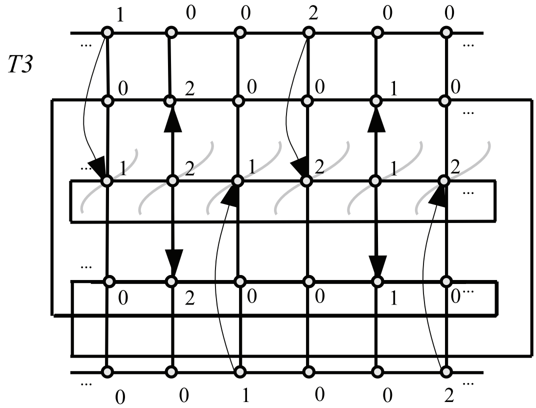

Fig. 7 Coloring of newly formed vertices of graph G' after using Transformation $T3$

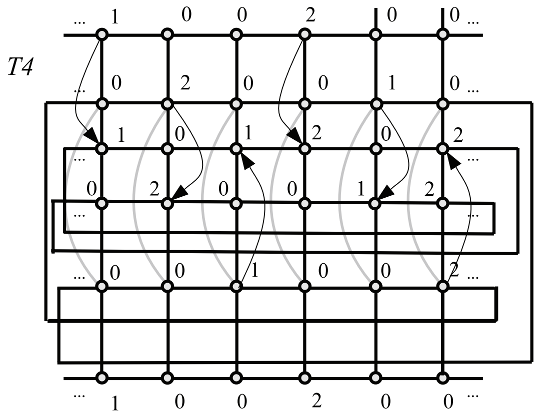

Fig. 8 Coloring of newly formed vertices of graph G' after using Transformation $T4$

Proposition 11 *Let $m \geq 3$ and $n \geq 6$, let $\varphi_{n-1,0}$ be a reflection and $\varphi_{i,i+1} = id$ for all other i. Then*

$$\gamma_{r2}(C_n \Box^{\varphi} C_m) \leq \left\lceil \frac{m}{3} \right\rceil (n+1).$$

Proof of Proposition 11 provides upper bounds by constructing several 2RDF's. Simple inductive arguments then provide existence of 2RDF's for all possible cases.

Proof Let us first consider the case where $m = 3k, k \in \mathbb{Z}^+$. More precisely, we first show that the theorem holds for $m = 3$.

Consider Pattern (1), and let the reflection fix row 2. The pattern starts at column 0 (see the indices in the last row). Note the reflected pattern on the left side, high indices. In the figure below, $n \equiv 0 \bmod 6$.

$$\left[\begin{array}{cccccccc|cccccccc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\ \hline \dots & .. & .. & .. & .. & n\text{-}2 & n\text{-}1 & & 0 & 1 & 2 & .. & .. & .. & \dots \end{array}\right] \left[\begin{array}{c} 1 \leftrightarrow 3 \\ 2 \\ 3 \leftrightarrow 1 \\ \hline \varphi_{n-1,0} \end{array}\right]. \tag{4}$$

It is straightforward to see that one vertex needs to be assigned a color to obtain a 2RDF, and that there is no 2RDF of weight n. A 2RDF is, for example, given by

$$\left[\begin{array}{ccccccc|ccccccc} \dots & 1 & 0 & 0 & 2 & 0 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\ \hline \dots & .. & .. & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots \end{array}\right]. \tag{5}$$

For $n \not\equiv 0 \bmod 6$, in all cases, the 2RDF are obtained similarly, by assigning one color to one vertex. Explicitly, we have, for $n = 6i + 5$

$$\left[\begin{array}{cccccc|ccccccc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & \mathbf{2} & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\ \hline \dots & .. & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots \end{array}\right]. \tag{6}$$

Furthermore, for $n = 6i + 4$

$$\left[\begin{array}{ccccc|ccccccc} \dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\ \dots & 0 & 2 & 0 & \mathbf{1} & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\ \hline \dots & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots \end{array}\right]. \tag{7}$$

Similarly, for $n = 6i + 3$

$$\left[\begin{array}{cccc|ccccccc}
\dots & 1 & 0 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots
\end{array}\right], \tag{8}$$

and for $n = 6i + 2$

$$\left[\begin{array}{ccccccccc|ccccccc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & .. & .. & .. & .. & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots
\end{array}\right]. \tag{9}$$

And finally, for $n = 6i + 1$

$$\left[\begin{array}{cccccccc|ccccccc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \mathbf{2} & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & .. & .. & .. & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots
\end{array}\right]. \tag{10}$$

If we successively apply patterns for $m = 3$, we get patterns for each $m = 3k$ and $n \geq 3$ (note that the patterns for reflections with no fixed point and with two fixed points coincides). As an example, the 2RDF of $C_n \square^{\varphi} C_m$, $m = 12$ and $n = 6i + 2$, with a desired weight is shown below. Note that the pattern can be interpreted as a shift with two fixed points (e.g. row 3 and row 9 are fixed) or as a shift without fixed points (e.g. row i is mapped to row $m - i + 1$ modulo 12). For example, in (11) we have a 2RDF for cases $m = 12$, $n = 6i + 2$ for bundles with reflections having two fixed points.

$$\left[\begin{array}{cccccc|ccccccc}
\dots & 2 & 0 & 0 & 1 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & 2 & 0 & 0 & 1 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & 2 & 0 & 0 & 1 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & 2 & 0 & 0 & 1 & \mathbf{2} & 0 & 0 & 1 & 0 & 0 & 2 & \dots \\
\dots & 0 & 1 & 0 & 0 & 2 & 0 & 2 & 0 & 0 & 1 & 0 & \dots \\
\dots & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & \dots \\
\hline
\dots & .. & .. & .. & n\text{-}2 & n\text{-}1 & 0 & 1 & 2 & .. & .. & .. & \dots
\end{array}\right]
\left[\begin{array}{c}
1 \leftrightarrow 5 \\
2 \leftrightarrow 4 \\
3 \\
4 \leftrightarrow 2 \\
5 \leftrightarrow 1 \\
6 \leftrightarrow 12 \\
7 \leftrightarrow 11 \\
8 \leftrightarrow 10 \\
9 \\
10 \leftrightarrow 8 \\
11 \leftrightarrow 7 \\
12 \leftrightarrow 6 \\
\hline
\varphi_{n-1,0}
\end{array}\right]. \tag{11}$$

As mentioned above, for reflections with no fixed points, the pattern is identical to the one mentioned above. As an example, we provide the pattern for the case when $m = 6$.

$$\left[\begin{array}{cccccc|c} \dots & 2\,0\,0 & 1 & \mathbf{2} & | & 0\,0\,1\,0\,0\,2 & \dots \\ \dots & 0\,1\,0 & 0 & 2 & | & 0\,2\,0\,0\,1\,0 & \dots \\ \dots & 0\,0\,2 & 0 & 0 & | & 1\,0\,0\,2\,0\,0 & \dots \\ \hline \dots & 2\,0\,0 & 1 & \mathbf{2} & | & 0\,0\,1\,0\,0\,2 & \dots \\ \dots & 0\,1\,0 & 0 & 2 & | & 0\,2\,0\,0\,1\,0 & \dots \\ \dots & 0\,0\,2 & 0 & 0 & | & 1\,0\,0\,2\,0\,0 & \dots \\ \hline \dots & ..\,..\,.. & n\text{-}2 & n\text{-}1 & | & 0\,1\,2\,..\,..\,.. & \dots \end{array}\right] \left[\begin{array}{c} 1 \leftrightarrow 6 \\ 2 \leftrightarrow 5 \\ 3 \leftrightarrow 4 \\ 4 \leftrightarrow 3 \\ 5 \leftrightarrow 2 \\ 6 \leftrightarrow 1 \\ \hline \varphi_{n-1,0} \end{array}\right]. \tag{12}$$

Therefore, for each $m = 3k$ it holds

$$\gamma_{r2}(C_n \Box^{\varphi} C_m) \leq \frac{m}{3}(n+1) = \left\lceil \frac{m}{3} \right\rceil (n+1).$$

In the sequel we shall use this results to prove the remaining cases. Lemma 10 ensures that by starting with the graph with $m = 3k$ and using one of Transformations Ti, $i = 1, 2, 3, 4$, we always can properly 2RD-color the extended graph with the appropriate number of colors. Below are Tables 1 and 2 which illustrate how we can transform the initial graph to obtain all possible graphs for the remaining values of m and n. The abbreviation “N. F.” stands for “No Fixed Points”,“ i.e. a reflection without fixed points, while “1 F.” and “2 F.” stands for “1 Fixed Point” and “2 Fixed Points”,” i.e. a reflection with one or two fixed points.

Since the cases for all values of m can be obtained from $C_n \Box^{\varphi} C_m, m = 3k$ with use of the transformations (and since Lemma 10 assures that the weights of 2RDF for all possible values of n do not exceed the upper boundary), the proof is now complete. □

3.4 Conclusion

We have provided lower and upper bounds for the 2-rainbow domination number of Cartesian graph bundles of cycles over cycles with a gap of at most $\frac{1}{3}(2m + 2n + 4)$, the same as known for the Cartesian products of cycles, $C_m \Box C_n$ (Brezovnik et al. 2024). In Brezovnik et al. (2024), it is conjectured that the lower bounds on the products differ from the exact values by at most one constant that depends on m and is independent of n. We believe that by analogy elaborated in this paper, the same would hold for the graph bundles as well.

Table 1 Illustration of how we can obtain graphs $G' = C_n \Box^{\varphi} C_{m'}$ for all remaining values of m' and n from the graph $C_n \Box^{\varphi} C_m, m = 3k$, m odd, by using transformations Ti, $i \in [4]$

Transformation\\m mod 3	1	2
$T1$	N. F	
$T2$	2 F	
$T3$		1 F
$T4$		1 F

Table 2 Illustration of how we can obtain graphs $G' = C_n \square^{\varphi} C_{m'}$ for all remaining values of m' and n from the graph $C_n \square^{\varphi} C_m, m = 3k, m$ even, by using transformations $Ti, i \in [4]$

Transformation\\m mod 3	1	2
$T1$ (2 F.)	1 F	
$T2$ (N. F.)	1 F	
$T3$ (2 F.)		2 F
$T4$ (N. F.)		N. F

At present, the exact values are known only for lengths m and n, so it is natural to ask whether and how it is possible to close the gap between the lower and upper bounds. At least for small m, we claim that it is possible to avoid the tedious analysis by applying an algebraic method that can be used for various graph invariants including the domination type problems (Klavžar and Žerovnik 1996; Pavlic and Žerovnik 2013; Repolusk and Žerovnik 2018; Gabrovšek et al. 2023; Garzón et al. 2022; Martínez et al. 2022). Also, one might make use of the dynamic programming technique that recently led to improved lower bounds for domination number of Cartesian product of cycles and paths (Guichard 2024). Without going into details, the technique would seemingly naturally extend to 2-rainbow domination, and then provide possibly better lower bounds that may, in turn, give rise to lower bounds for Cartesian graph bundles of cycles over cycles. Recall that the graph bundles over paths are isomorphic to the products.

Such a research task may be a challenge for future work. Another avenue for future research may be study of 2-rainbow domination for some other graph products, i.e. strong product, direct product, etc. We refer to the book (Imrich and Klavžar 2000) for further reading on graph products.

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Declarations

Conflict of interest The author declares that they have no Conflict of interest.

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