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The 2-rainbow domination number of Cartesian product of cycles

Simon Brezovnik, Darja Rupnik Poklukar
FME, University of Ljubljana, Aškerčeva 6, Ljubljana 1000, Slovenia

Janez Žerovnik *
FME, University of Ljubljana, Aškerčeva 6, Ljubljana 1000, Slovenia and *Rudolfovo - Science and Technology Centre Novo Mesto, Podbreznik 15, 8000 Novo mesto, Slovenia*

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Abstract

A *k-rainbow dominating function* (kRDF) of G is a function that assigns subsets of $\{1, 2, ..., k\}$ to the vertices of G such that for vertices v with $f(v) = \emptyset$ we have $\bigcup_{u \in N(v)} f(u) = \{1, 2, ..., k\}$. The *weight* $w(f)$ of a kRDF f is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. The minimum weight of a kRDF of G is called the *k-rainbow domination number* of G, which is denoted by $\gamma_{rk}(G)$. In this paper, we study the 2-rainbow domination number of the Cartesian product of two cycles. Exact values are given for a number of infinite families and we prove lower and upper bounds for all other cases.

Keywords: 2-rainbow domination, domination number, Cartesian product.

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1 Introduction

The Cartesian product is one of the standard graph products [13]. For example, meshes, tori, hypercubes and some of their generalizations are Cartesian products.

Graph domination is one of the most popular topics in graph theory [15, 16, 17]. There are many variants motivated by interesting applications. The k-rainbow domination problem was first studied in [2] and has attracted a lot of attention. For example, in [3], the authors proved that the concept of 2-rainbow domination is equivalent to ordinary domination

*Corresponding author.
E-mail addresses: simon.brezovnik@fs.uni-lj.si (Simon Brezovnik), darja.rupnik@fs.uni-lj.si (Darja Rupnik Poklukar), janez.zerovnik@fs.uni-lj.si (Janez Žerovnik)

in the prism $G\Box K_2$ and established the NP-completeness of determining whether a graph has a 2-rainbow dominating function with a certain weight. Furthermore, in [4] the authors characterize the pairs of graphs G and H for which $\gamma(G\Box H) = \min\{V(G), V(H)\}$. There are also many papers that observe 2-rainbow domination on generalized Petersen graphs, for example [5, 9, 32, 33]. In recent years, research on the 2-rainbow domination and its variants has expanded even further. For example, in [22] the k-rainbow domination on regular graphs was investigated. Meybodi et al. [23] investigated k-rainbow domination in graphs with bounded tree-width. In [18] Kim investigated k-rainbow domination in middle graphs in the context of operations research. In [6] an independent variant of k-rainbow domination on the lexicographic products of graphs was investigated. Recently, Kosari and Asgharsharghi [21] studied the l-distance k-rainbow domination numbers of graphs. For further references, see [1].

In this paper we study 2-rainbow domination numbers of the Cartesian product of two cycles. We provide exact values for a number of infinite families and prove lower and upper bounds for all other cases.

Our main results are summarized in the following two theorems.

For $n \equiv 0 \pmod 6$ the first theorem gives exact values of $\gamma_{r2}(C_m\Box C_n)$ for $m \equiv 0, 2 \pmod 3$ and bounds with gap at most $\frac{1}{2}n$ for the case $m \equiv 1 \pmod 3$.

Theorem 1.1. *Let $m \geq 3$ and $n \geq 6$, $n \equiv 0 \pmod 6$. Then we have*

(a) *if $m \equiv 0 \pmod 3$ then* $\gamma_{r2}(C_m\Box C_n) = \dfrac{m}{3}n\,.$

(b) *if $m \equiv 1 \pmod 3$ then*

$$\left(\frac{m-1}{3} + \frac{1}{2}\right) n \leq \gamma_{r2}(C_m\Box C_n) \leq \frac{m+2}{3}n\,.$$

(c) *if $m \equiv 2 \pmod 3$ then* $\gamma_{r2}(C_m\Box C_n) = \dfrac{m+1}{3}n\,.$

The second theorem is a summary of the lower and upper bounds of the products of cycles, covering all cases. Note that the gap is at most $\frac{1}{2}n + 2\left\lceil\frac{m}{3}\right\rceil$.

Theorem 1.2. *Let $m \geq 3$ and $n \geq 6$. Then*

$$\left(\left\lfloor\frac{m}{3}\right\rfloor + \alpha\right) n \leq \gamma_{r2}(C_m\Box C_n) \leq \min\left\{\left\lceil\frac{m}{3}\right\rceil(n+\beta), \left\lceil\frac{n}{3}\right\rceil(m+\gamma)\right\}\,,$$

where

$$\alpha = \begin{cases} 0, & m \equiv 0 \pmod 3 \\ \frac{1}{2} & m \equiv 1 \pmod 3 \\ 1, & m \equiv 2 \pmod 3 \end{cases}, \qquad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6 \end{cases},$$

and $$\gamma = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 5 \pmod 6 \\ 2, & n \equiv 1,2,3,4 \pmod 6. \end{cases}$$

The upper bounds are given in alternative form as Corollary 4.9.

The rest of the paper is organized as follows. In the next section we recall some basic definitions and some useful previously known results. In Section 3 we prove lower bounds. In Section 4, we study two patterns that allow constructions that yield upper bounds. The final section contains a number of ideas for future research.

2 Preliminaries

A finite, simple and undirected graph $G = (V(G), E(G))$ is given by a set of vertices $V(G)$ and a set of edges $E(G)$. As usual, the edges $\{i, j\} \in E(G)$ are shortly denoted by ij.

A set S is a dominating set if every vertex in the complement $V(G) \setminus S$ is adjacent to a vertex in S. The minimum cardinality of a dominating set of G is called the domination number $\gamma(G)$.

The Cartesian product of two graphs, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$, in which two vertices are adjacent if and only if they are equal in one coordinate and adjacent in the other. The Cartesian product of graphs is one of the standard graph products [13]. The Cartesian product is commutative. In other words: $C_m \Box C_n$ is isomorphic to $C_n \Box C_m$. So if we consider the product of the cycles $C_m \Box C_n$, we can assume $m \leq n$.

For a given vertex $v \in V(G)$, the open neighborhood $N(v)$ consists of the vertices adjacent to v. The degree of vertex v equals $\deg_G(v) = |N(v)|$. The minimum and the maximum degree of a graph G are denoted by $\delta(G)$ and $\Delta(G)$.

Let f be a function that assigns to each vertex a set of colors chosen from the set $\{1, 2, \ldots, k\} = [k]$, with the property that for each $v \in V(G)$ with $f(v) = \emptyset$ we have

$$\bigcup_{u \in N(v)} f(u) = [k].$$

Such a function f is called a k-rainbow dominating function (kRDF) of G. The weight of f, denoted by $w(f)$, is defined as

$$w(f) = \sum_{v \in V(G)} |f(v)|.$$

Recall that $f(v)$ is a set of colors and $|f(v)|$ denotes the number of elements in $f(v)$. The minimum weight of a kRDF on G is called the $k-$rainbow domination number of G, $\gamma_{rk}(G)$ and in this case the function is called $\gamma_{rk}(G)$*-function*. It is clear that for $k = 1$ this definition corresponds to the usual domination.

The following theorems, which connect rainbow domination with (ordinary) domination, will be of interest here.

Theorem 2.1 ([2])**.** *For any graph G we have $\gamma_{rk}(G) = \gamma(G \Box K_k)$.*

Theorem 2.2 ([14])**.** *For any graph G we have $\gamma_{rk}(G) \leq k\gamma(G)$.*

In [19], it was shown that $\gamma(C_3 \Box C_n) = n - \lfloor \frac{n}{4} \rfloor$, $\gamma(C_4 \Box C_n) = n$ for $n \geq 4$ and

$$\gamma(C_5 \Box C_n) = \begin{cases} n, & n \equiv 0 \pmod 5 \\ n + 1, & n \equiv 1, 2, 4 \pmod 5). \end{cases}$$

This result was supplemented in [8], where it was shown that $\gamma(C_5\Box C_n) = n + 2$ for $n \equiv 3 \pmod 5$ and also exact values for $\gamma(C_6\Box C_n)$ and $\gamma(C_7\Box C_n)$ were given. In [7] it was proved that $\lceil \frac{9n}{5} \rceil \leq \gamma(C_8\Box C_n) \leq \lceil \frac{9n}{5} \rceil + 1$ for $n \geq 8$ and exact value for $\gamma(C_9\Box C_n)$ was given.

Considering 2-rainbow domination number of the Cartesian product of two cycles, the well-known inequality is (see [29])

$$\frac{mn}{3} \leq \gamma_{r2}(C_m\Box C_n) \leq 2\gamma(C_m\Box C_n).$$

The 2-rainbow domination number of the products $C_3\Box C_n$ and $C_5\Box C_n$ were studied in [29, 30]. In [31] a complete characterization of graphs $C_m\Box C_n$ was given, for which the 2-rainbow domination number is equal to $\frac{mn}{3}$. A summary of the then known results on the k-rainbow domination of the Cartesian product of cycles appears in [12]. In Table 1 we recall the previously known formulas for 2-rainbow domination numbers for $C_m\Box C_n$.

Result	Ref.
$\gamma_{r2}(C_3\Box C_n) = \begin{cases} n, & n \equiv 0 \pmod 6 \\ n+1, & n \equiv 1,2,3,5 \pmod 6 \\ n+2, & n \equiv 4 \pmod 6 \end{cases}$	[29]
$\gamma_{r2}(C_4\Box C_n) = \begin{cases} \lfloor \frac{3n}{2} \rfloor, & n \equiv 0 \pmod 8 \\ \lfloor \frac{3n}{2} \rfloor + 1, & n \equiv 2,4,5 \pmod 8 \\ \lfloor \frac{3n}{2} \rfloor + 2, & n \equiv 1,3,6,7 \pmod 8 \end{cases}$	[27]
$\gamma_{r2}(C_5\Box C_n) = 2n$	[30]
$\gamma_{r2}(C_8\Box C_n) = 3n$	[27]
$\gamma_{r2}(C_m\Box C_n) = \frac{mn}{3}$, if and only if either $m \equiv 0 \pmod 3, n \equiv 0 \pmod 6$ or $m \equiv 0 \pmod 6, n \equiv 0 \pmod 3$	[31]

Table 1: A summary of previously known results.

3 Lower bounds for 2-rainbow domination of $C_m\Box C_n$

For simplicity, we introduce some more notations. The vertices of $V(C_m\Box C_n)$ are denoted by (i, j) for $i \in [m]$ and $j \in [n]$. The coordinates i and j are taken modulo m and n respectively, so that we identify m and 0, for example. For a fixed (small) m, the set of vertices is $\mathcal{C}^i = \{(i,1),(i,2),\cdots,(i,n)\}$, $i \in [m]$ is called the i-th column of $C_m\Box C_n$.

Let f be a 2RDF of $C_m\Box C_n$ and $s_i = \sum_{x\in\mathcal{C}^i} |f(x)|$. The sequence $(s_1, s_2, \cdots, s_m)$, is called the 2RDF sequence that corresponds to f. We also use $f(i,j) = f(v)$ to denote the value of f at vertex $v = (i,j)$ for $i \in [m]$ and $j \in [n]$.

First, we recall a general bound for regular graphs. We believe that it is well known, although we have not found a reference with a proof. Therefore, for the sake of completeness, we provide a short proof.

Lemma 3.1. *Let G be an r-regular graph. Then* $\gamma_{rk}(G) \geq \dfrac{k}{r+k}|V(G)|$.

Proof. Assume that f is a kRDF and that n^* vertices are colored. Then double count to obtain $rw(f) \geq (|V(G)| - n^*)k$. Apply $n^* \leq w(f)$ and the conclusion follows. □

Cartesian products of cycles are 4-regular graphs, and we consider 2-rainbow domination, so we need a special case of Lemma 3.1, namely $k = 2$ and $r = 4$.

Corollary 3.2. *Let G be a 4-regular graph. Then* $\gamma_{r2}(G) \geq \dfrac{1}{3}|V(G)|$.

Note that the statement also follows from [22, Lemma 2.2, Case (6)].

The next lemma will be useful to obtain better lower bounds for Cartesian products of cycles. In particular, for bounds of $\gamma_{r2}(C_m \Box C_n)$. Recall that $s_i = \sum_{x \in \mathcal{C}^i} |f(x)|$.

Lemma 3.3. *Let f be a $\gamma_{r2}(C_m \Box C_n)$-function. Write $m = 3k + \ell$, where $\ell \equiv m \pmod 3$. Then*

(a) $s_{i-1} + s_{i+1} \geq 2m - 4s_i = 6k + 2\ell - 4s_i$,

(b) *if $k \geq s_{min} = \min\{s_{i-1}, s_{i+1}\}$, then*

$$s_{max} \geq 2m - 4s_i - s_{min} \geq 5k + 2\ell - 4s_i\,,$$

where $s_{max} = \max\{s_{i-1}, s_{i+1}\}$.

Proof. Note that at most s_i vertices of the column $\mathcal{C}^i$ are colored (this holds in the case when all $|f(v)| = 1$). Other (uncolored) vertices in $\mathcal{C}^i$, at least $m - s_i$ of them, have a total demand at least $2(m - s_i)$. Since at most $2s_i$ of this demand can be fulfilled by the colored vertices of $\mathcal{C}^i$, we must have at least $2m - 4s_i$ colors in the neighborhood of $\mathcal{C}^i$. Equivalent to this is $s_{i-1} + s_{i+1} \geq 2m - 4s_i$. So if we use $m = 3k + \ell$, we have

$$s_{i-1} + s_{i+1} \geq 2m - 4s_i = 6k + 2\ell - 4s_i\,,$$

as required. Finally, if $k \geq s_{min} = \min\{s_{i-1}, s_{i+1}\}$, then

$$s_{max} \geq 2m - 4s_i - s_{min} = 6k + 2\ell - 4s_i - s_{min} \geq 5k + 2\ell - 4s_i\,,$$

and the proof is complete. □

The next observation provides lower bounds. The proof is based on the discharging argument and follows the ideas of [27] and [28].

Proposition 3.4. *Let $m \geq 3$ and $n \geq 3$. Write $m = 3k + \ell$, where $\ell \equiv m \pmod 3$. Then*

$$\gamma_{r2}(C_m \Box C_n) \geq kn + \ell\frac{n}{2} = \frac{mn}{3} + \ell\frac{n}{6}\,.$$

Proof. Note that when $m = 3k$, the proof follows directly from Lemma 3.1 In the following, we write the proof for the case when $m = 3k + 1$, since the proof for the case when $m = 3k + 2$ is similar and can therefore be omitted.

Let f be a γ_{r2}-function on the vertex set of $C_m \Box C_n$ and let $(s_1, s_2, \ldots, s_m)$, be the 2RDF sequence corresponding to f. We define a discharging rule in which the columns with sufficiently large s_i give half of their overweight to one or both of the neighboring columns. For this purpose, let f' be a function on the vertex set of $C_m \Box C_n$ that assigns a positive real number to each vertex. Denote by $s'_i = \sum_{x \in \mathcal{C}^i} f'(x)$ and let $(s'_1, s'_2, \ldots, s'_m)$ be the sequence corresponding to f'. Moreover, we define f' such that the following holds:

If $s_i > k + \frac{1}{2}$ then set $s'_i = k + \frac{1}{2}$. If $s_i \le k + \frac{1}{2}$, then

- if $s_{i-1} > k+\frac{1}{2}$ and $s_{i+1} > k+\frac{1}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - (k+\frac{1}{2})) + \frac{1}{2}(s_{i+1} - (k+\frac{1}{2}))$,
- if $s_{i-1} > k + \frac{1}{2}$ and $s_{i+1} < k + \frac{1}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i-1} - (k + \frac{1}{2}))$,
- if $s_{i-1} < k + \frac{1}{2}$ and $s_{i+1} > k + \frac{1}{2}$, then $s'_i = s_i + \frac{1}{2}(s_{i+1} - (k + \frac{1}{2}))$.

We claim that $s'_i \ge k + \frac{1}{2}$ for all i. Assume $s_i \le k + \frac{1}{2}$. Note that, since s_i is an integer, $s_i \le k + \frac{1}{2}$ implies $s_i \le k$. Again, if $s_{i-1} > k$ and $s_{i+1} > k$ then, by Lemma 3.3,

$$\begin{aligned} s'_i &= s_i + \frac{1}{2}(s_{i-1} - (k + \frac{1}{2})) + \frac{1}{2}(s_{i+1} - (k + \frac{1}{2})) \\ &= s_i + \frac{1}{2}(s_{i-1} + s_{i+1}) - (k + \frac{1}{2}) \\ &\ge s_i + 3k + 1 - 2s_i - k - \frac{1}{2} = 2k + \frac{1}{2} - s_i \\ &= k + \frac{1}{2} + (k - s_i) \ge k + \frac{1}{2}. \end{aligned}$$

or, when $s_{min} = \min\{s_{i-1}, s_{i+1}\} \le k$,

$$\begin{aligned} s'_i &= s_i + \frac{1}{2}(s_{max} - (k + \frac{1}{2})) \\ &\ge s_i + 2k + 1 - 2s_i - \frac{1}{4} = 2k + \frac{3}{4} - s_i . \end{aligned}$$

Recall that s_i is an integer, so $s_i \le k + \frac{1}{2}$ is equivalent to $s_i \le k$, and hence

$$s'_i = 2k + \frac{3}{4} - s_i = k + \frac{3}{4} + (k - s_i) > k + \frac{1}{2},$$

which implies $\gamma_{r2}(C_m \Box C_n) = \sum_i s_i \ge \sum_i s'_i \ge n(k + \frac{1}{2})$.

Summarizing, we get

(a) $\gamma_{r2}(C_m \Box C_n) \ge kn$ when $\ell = 0$,

(b) $\gamma_{r2}(C_m \Box C_n) \ge kn + \frac{n}{2}$ when $\ell = 1$, and

(c) $\gamma_{r2}(C_m \Box C_n) \ge kn + n$ when $\ell = 2$.

which in turn implies

$$\gamma_{r2}(C_m \Box C_n) \ge kn + \ell\frac{n}{2} = \frac{mn}{3} + \ell\frac{n}{6}$$

as claimed. □

4 Upper bounds

Recall the characterization of the products where the general lower bound is attained [31]. More precisely, the result is given in the next theorem.

Theorem 4.1 ([31]). *If either $m \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 6$, or $m \equiv 0 \pmod 6$ and $n \equiv 0 \pmod 3$, then*

$$\gamma_{r2}(C_m \Box C_n) = \frac{1}{3} mn \,.$$

For later reference, observe such 2RDF may be based on the pattern

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \cdots \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} . \tag{4.1}$$

Moreover, it is easy to write explicit formula for the values, namely

$$f_1(i,j) = \begin{cases} 0, & i \not\equiv j \pmod 3 \\ 2{-}i \bmod 2, & i \equiv j \pmod 3). \end{cases}$$

The alternative is to define a 2RDF as

$$f_2(i,j) = \begin{cases} 0, & i \not\equiv j \pmod 3 \\ 2{-}j \bmod 2, & i \equiv j \pmod 3) \end{cases}$$

which results in the pattern

$$\begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \cdots \\ \cdots & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & \cdots \\ \cdots & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & \cdots \\ \cdots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \cdots \\ \cdots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \cdots \\ \cdots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} . \tag{4.2}$$

It is easy to see that the first pattern results in 2RDF's with $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$, if $m \equiv 0 \pmod 3$, $n \equiv 0 \pmod 6$. The second pattern provides 2RDF's with $\gamma_{r2}(C_m \Box C_n) = \frac{mn}{3}$ if $m \equiv 0 \pmod 6$, $n \equiv 0 \pmod 3$. Note that $m \geq 6$ is required for the second pattern, while the first pattern can be applied if $m \geq 3$.

Remark. It is worth noting that in both cases we have $s_i = \frac{m}{3}$.

Now we outline constructions that directly imply some upper bounds.

Proposition 4.2. *Let* $m \equiv 2 \pmod 3$ *and* $n \equiv 0 \pmod 6$*. Write* $m = 3k + 2$*. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq kn + n.$$

Proof. First, we provide a 2RDF proving that $\gamma_{r2}(C_5 \Box C_n) \leq 2n$. Start with the pattern (4.2), use the first six rows and replace the 2nd and 3rd row with the union of them.

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \dots \\ \dots & 2 & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & \dots \\ \dots & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \end{bmatrix}$$

Is it obvious that the same construction gives 2RDF's proving that

$$\gamma_{r2}(C_{3k+2} \Box C_n) \leq kn + n,$$

as claimed. □

Proposition 4.3. *Let* $m \equiv 1 \pmod 3$ *and* $n \equiv 0 \pmod 6$*. Write* $m = 3k + 1$*. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq kn + n.$$

Proof. First, we provide a 2RDF proving that $\gamma_{r2}(C_4 \Box C_n) \leq 2n$. Start with the pattern (4.2), use the first six rows, replace the 2nd and 3rd row with the union of them, and replace the 4th and 5th row with the union of them.

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ \dots & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \dots \\ \dots & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \mathbf{2} & \mathbf{1} & \mathbf{0} & \dots \\ \dots & 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 & \dots \end{bmatrix}$$

Is it obvious that the same construction gives 2RDF's proving that

$$\gamma_{r2}(C_{3k+1} \Box C_n) \leq kn + n,$$

as claimed. □

To summarize, we can combine the Propositions 4.2 and 4.3 with Theorem 4.1 to obtain

Proposition 4.4. *Let* $n \equiv 0 \pmod 6$ *and* $m \geq 3$*. Then* $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil n$.

The next propositions provide general upper bounds for the cases when $n \not\equiv 0 \pmod 6$. Below we provide constructions based on the previously studied 2RDF for each possible reminder $b = 0, 1, 2, 3, 4, 5$ where $n \equiv b \pmod 6$. We start with the case $m \equiv 0 \pmod 3$.

Proposition 4.5. *Let* $m \geq 3$*,* $m \equiv 0 \pmod 3$*, and* $n \geq 6$*,* $n \equiv b \pmod 6$*. Hence* $n = 6a + b$ *for some integer* $a \geq 0$*. Then*

(a) *if* $b = 5$ *then* $\gamma_{r2}(C_m \Box C_n) \leq \gamma_{r2}(C_m \Box C_{6a}) + 2m = \frac{m}{3}(n + 1)$,

(b) *if* $b = 4$ *then* $\gamma_{r2}(C_m \Box C_n) \leq \gamma_{r2}(C_m \Box C_{6a}) + 2m = \frac{m}{3}(n + 2)$,

(c) *if* $b = 3$ *then* $\gamma_{r2}(C_m \Box C_n) \le \gamma_{r2}(C_m \Box C_{6a}) + 4\frac{m}{3} = \frac{m}{3}(n+1)$,

(d) *if* $b = 2$ *then* $\gamma_{r2}(C_m \Box C_n) \le \gamma_{r2}(C_m \Box C_{6a}) + m = \frac{m}{3}(n+1)$,

(e) *if* $b = 1$ *then* $\gamma_{r2}(C_m \Box C_n) \le \gamma_{r2}(C_m \Box C_{6a}) + 2\frac{m}{3} = \frac{m}{3}(n+1)$.

Proof. In the following we give explicit constructions for the case $m = 6 = 2 \times 3$ and various n. It is obvious that in general we can simply repeat the pattern of three consecutive rows. The weight of a column is $\frac{m}{3}$, hence the bounds given in proposition.

(a) if $b = 5$, then replace two columns of the 2RDF $(C_m \Box C_{6a+6})$ by their union and observe that the table gives a 2RDF of $(C_m \Box C_{6a+5})$.

$$\left[\begin{array}{ccccccc|ccc|ccc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2
\end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccccc|ccc|cc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \mathbf{2} & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \mathbf{1} & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2 \\
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & \mathbf{2} & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & \mathbf{1} & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 2
\end{array}\right].$$

So if we look at the last 6 columns, which have shrunk to 5 columns, we see that the number of colors used does not change. If instead of $m = 6$ we consider $m = 3k$, three rows, e.g. rows $4 - 6$, are repeated $(k-2)$ times and the same construction is applied. The last 6 columns therefore contain $6 \times k = 2m$ colors.

In the remaining cases, we only give the tables containing the constructions that alter the rightmost columns (in the tables $m = 6$ is chosen).

(b) if $b = 4$, then take (for example) the last four columns and replace them with two columns, each of which is the union of two columns.

$$\left[\begin{array}{ccccccc|cc|cccc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2
\end{array}\right] \rightarrow$$

$$\left[\begin{array}{ccccccc|cc|cc}
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{1} & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \mathbf{2} \\
\dots & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 \\
\dots & 0 & 2 & 0 & 0 & 1 & 0 & 0 & 2 & \mathbf{1} & 0 \\
\dots & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & \mathbf{2}
\end{array}\right].$$

(c) if $b = 3$, then take (for example) the last six columns and replace them with three columns, as follows

$$\begin{bmatrix} \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \{\mathbf{1},\mathbf{2}\} & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 2 \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \{\mathbf{1},\mathbf{2}\} & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 2 \end{bmatrix}.$$

Note that the 2RDF in this case is not a singleton 2RDF. A singleton 2RDF either assigns a singleton to the empty set [10]. We do not know whether there is a singleton 2RDF with the same weight.

(d) if $b = 2$, then take (for example) the last six columns and replace them with two columns, as follows

$$\begin{bmatrix} \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \mathbf{2} \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 2 \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & \mathbf{2} \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 2 \end{bmatrix}.$$

(e) if $b = 1$, then replace the last six columns with an altered column.

$$\begin{bmatrix} \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \\ \cdots & 1 & 0 & 0 & 2 & 0 & 0 & | & 1 & 0 & 0 & 2 & 0 & 0 \\ \cdots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 & 2 & 0 & 0 & 1 & 0 \\ \cdots & 0 & 0 & 1 & 0 & 0 & 2 & | & 0 & 0 & 1 & 0 & 0 & 2 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & \mathbf{1} \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & \mathbf{2} \\ \dots & 1 & 0 & 0 & 2 & 0 & 0 & | & \mathbf{1} \\ \dots & 0 & 2 & 0 & 0 & 1 & 0 & | & 0 \\ \dots & 0 & 0 & 1 & 0 & 0 & 2 & | & \mathbf{2} \end{bmatrix}.$$

□

Now we generalize Proposition 4.5 to arbitrary m.

Proposition 4.6. *Let $m \geq 3$ and $n \geq 6$. If $n \equiv 4 \pmod 6$, then $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil (n+2)$. Otherwise, $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{m}{3} \rceil (n+1)$.*

Proof. (sketch) The bounds are obtained by constructions that combine the ideas from Propositions 4.2, 4.3 and 4.5. The main idea is the following. Start with $C_{\tilde{m}} \Box C_n$ where $\tilde{m} = 3\lceil \frac{m}{3} \rceil$. Note that there is at most $\lceil \frac{m}{3} \rceil$ colors in each column. Apply the constructions as in the proofs of Propositions 4.2, 4.3 and 4.5. Recall that in each of these constructions some columns are deleted and we replace one or two rows by unions of two rows. The total weight is preserved in this way, so the proposition holds. □

The upper bounds provided in Propositions 4.5 and 4.6 have a similar form, and can be written in a condensed way as follows.

Corollary 4.7. *Let $m \geq 3$ and $n \geq 6$. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq \left\lceil \frac{m}{3} \right\rceil (n+\beta), \quad \textit{where} \quad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6. \end{cases}$$

The construction used in Propositions 4.5, 4.6, and Corollary 4.7 are based on the basic assignment (4.1). Constructions based on (4.2) can be used in a similar way and result in slightly different upper bounds.

Proposition 4.8. *Let $m \geq 6$ and $n \geq 3$. Then*

$$\gamma_{r2}(C_m \Box C_n) \leq \left\lceil \frac{n}{3} \right\rceil (m+\gamma), \quad \textit{where} \quad \gamma = \begin{cases} 0, & m \equiv 0 \pmod 6 \\ 1, & m \equiv 5 \pmod 6 \\ 2, & m \equiv 1,2,3,4 \pmod 6. \end{cases}$$

Proof. We give only a brief outline of the proof and omit the detailed arguments, because the ideas are analogous to those previously elaborated in the proofs of Propositions 4.5, 4.6 and Corollary 4.7,

Recall first that for $m \equiv 0 \bmod 6$ and $n \equiv 0 \bmod 3$ Pattern (4.2) returns a 2RDF with weight $\frac{mn}{3}$.

Let us now assume that $n \equiv 0 \bmod 3$ and let $m \equiv d \bmod 6$. We claim that if $d = 5$ then $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{n}{3} \rceil (m+1)$, otherwise, $\gamma_{r2}(C_m \Box C_n) \leq \lceil \frac{n}{3} \rceil (m+2)$. If $d = 5$ then one row is deleted, and the colors of the deleted row are given to one neighboring rows. Formally, row m is defined as a union of the rows m and $m+1$ of the pattern. In any other case, a 2RDF is obtained by deleting some rows and replacing rows 1 and m with unions.

We have thus seen that the cases $n \not\equiv 0 \bmod 3$ can be handled by deleting one or two columns in the pattern. The colors of the deleted column(s) are then used to complete the assignment of columns 1 and n. And we have the upper bound as claimed. □

It seems obvious that the two upper bounds are not equivalent. Now we compare them more closely. To this end we write

$$\begin{aligned} \mathrm{B1}(m,n) &= \left\lceil \frac{m}{3} \right\rceil (n+\beta) \\ &= \frac{1}{3}(m+a)(n+\beta) = \frac{1}{3}mn + \frac{1}{3}an + \frac{1}{3}\beta m + \frac{1}{3}a\beta \qquad (4.3) \\ \mathrm{B2}(m,n) &= \left\lceil \frac{n}{3} \right\rceil (m+\gamma) \\ &= \frac{1}{3}(n+c)(m+\gamma) = \frac{1}{3}mn + \frac{1}{3}\gamma n + \frac{1}{3}cm + \frac{1}{3}\gamma c \qquad (4.4) \end{aligned}$$

where

$$a = \begin{cases} 0, & m \equiv 0 \pmod 3 \\ 1, & m \equiv 2 \pmod 3 \\ 2, & m \equiv 1 \pmod 3 \end{cases}, \quad \beta = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 1,2,3,5 \pmod 6 \\ 2, & n \equiv 4 \pmod 6 \end{cases},$$

$$\gamma = \begin{cases} 0, & n \equiv 0 \pmod 6 \\ 1, & n \equiv 5 \pmod 6 \\ 2, & n \equiv 1,2,3,4 \pmod 6 \end{cases}, \quad \text{and} \quad c = \begin{cases} 0, & n \equiv 0 \pmod 3 \\ 1, & n \equiv 2 \pmod 3 \\ 2, & n \equiv 1 \pmod 3. \end{cases}$$

Note that both B1 and B2 are of the form $\frac{1}{3}mn + \frac{1}{3}(x,y,z)(n,m,1)$, and let us write the values of (x,y,z) in two tables for easier comparison. (See Table 2 and Table 3.)

$\mathrm{B1}(m,n)$	n mod 6	0	1	2	3	4	5
m mod 6	$a\backslash\beta$	0	1	1	1	2	1
0	0	(0,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,2,0)	(0,1,0)
1	2	(2,0,0)	(2,1,2)	(2,1,2)	(2,1,2)	(2,2,4)	(2,1,2)
2	1	(1,0,0)	(1,1,1)	(1,1,1)	(1,1,1)	(1,2,2)	(1,1,1)
3	0	(0,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,2,0)	(0,1,0)
4	2	(2,0,0)	(2,1,2)	(2,1,2)	(2,1,2)	(2,2,4)	(2,1,2)
5	1	(1,0,0)	(1,1,1)	(1,1,1)	(1,1,1)	(1,2,2)	(1,1,1)

Table 2: B1 as a function of m and n.

$\mathrm{B2}(m,n)$	n mod 6	0	1	2	3	4	5
m mod 6	$\gamma\backslash c$	0	2	1	0	2	1
0	0	(0,0,0)	(0,2,0)	(0,1,0)	(0,0,0)	(0,2,0)	(0,1,0)
1	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
2	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
3	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
4	2	(2,0,0)	(2,2,4)	(2,1,2)	(2,0,0)	(2,2,4)	(2,1,2)
5	1	(1,0,0)	(1,2,2)	(1,1,1)	(1.0,0)	(1,2,2)	(1,1,1)

Table 3: B2 as a function of m and n.

In fourteen cases $\mathrm{B1} < \mathrm{B2}$, in other words the first pattern gives rise a better 2RDF. In four cases, $\mathrm{B2} < \mathrm{B1}$. Note that in two cases, the triples are no comparable. In particular, when $m \equiv 2 \pmod 6$ and $n \equiv 3 \pmod 6$ we have

$$\mathrm{B1} = \frac{1}{3}mn + \frac{1}{3}(n+m+1) <> \mathrm{B2} = \frac{1}{3}mn + \frac{1}{3}2m$$

and hence

$$\mathrm{B1} >=< \mathrm{B2} \quad \Longleftrightarrow \quad n+1 >=< m\,.$$

Similarly, when $m \equiv 3 \pmod 6$ and $n \equiv 3 \pmod 6$,

$$\mathrm{B1} = \frac{1}{3}mn + \frac{1}{3}n <> \mathrm{B2} = \frac{1}{3}mn + \frac{1}{3}2m$$

and hence

$$\mathrm{B1} >=< \mathrm{B2} \quad \Longleftrightarrow \quad n >=< 2m\,.$$

We summarize the observations in Table 4.

$m \backslash n$ mod 6	0	1	2	3	4	5
0	=	$\mathrm{B1}(m,n)$	=	$\mathrm{B2}(m,n)$	=	=
1	=	$\mathrm{B1}(m,n)$	=	$\mathrm{B2}(m,n)$	=	=
2	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$	$>=<$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$
3	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$	$>=<$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$
4	=	$\mathrm{B1}(m,n)$	=	$\mathrm{B2}(m,n)$	=	=
5	=	$\mathrm{B1}(m,n)$	=	$\mathrm{B2}(m,n)$	=	=

Table 4: Comparison of B1 and B2.

Finally, we recall that the Cartesian product is commutative, $C_m \Box C_n \simeq C_n \Box C_m$. Therefore, the best upper bound for $\gamma_{r2}(C_m \Box C_n)$ is based on the constructions considered here and is the minimum of the bounds $\mathrm{B1}(m,n), \mathrm{B2}(m,n), \mathrm{B1}(n,m)$, and $\mathrm{B2}(n,m)$. The results are written in Table 5.

$m \backslash n$ mod 6	0	1	2	3	4	5
0	=	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$	$\mathrm{B2}(m,n)$	=	=
1		$*$	$\mathrm{B1}(n,m)$	$\mathrm{B1}(n,m)$	$\mathrm{B1}(n,m)$	$\mathrm{B1}(n,m)$
2			$\mathrm{B1}(m,n)$	$\mathrm{B1}(n,m)$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$
3				$*$	$\mathrm{B1}(m,n)$	$\mathrm{B1}(m,n)$
4					=	=
5						=

where $* = \min\{\mathrm{B1}(m,n), \mathrm{B1}(n,m)\}$.

Table 5: Upper bounds using the commutativity of the Cartesian product.

Explicitly, the best upper bounds for $\gamma_{r2}(C_m \Box C_n)$ are of the form

$$\frac{1}{3}mn + \frac{1}{3}(n,m,1)(x,y,z)$$

with values of (x,y,z) from Table 6.

The bounds can be summarized as follows.

$m \backslash n$ mod 6	0	1	2	3	4	5
0	(0,0,0)	(0,1,0)	(0,1,0)	(0,0,0)	(0,2,0)	(0,1,0)
1		(2,1,2) or (1,2,2)	(1,1,1)	(1,0,0)	(1,2,2)	(0,1,0)
2			(1,1,1)	(1,0,0)	(1,2,2)	(1,1,1)
3				(0,1,0) or (1,0,0)	(0,2,0)	(0,1,0)
4					(2,2,4)	(2,1,2)
5						(1,1,1)

Table 6: Upper bounds in terms of (x, y, z).

Corollary 4.9. *Let $m \geq 6$ and $n \geq 6$. As the Cartesian product is commutative, we can assume $m \geq n$. Then*

$$\gamma_{r2}(C_m \square C_n) \leq \frac{1}{3}mn + \frac{1}{3}\delta\,,$$

where δ can be read from Table 7.

$m \backslash n$ mod6	0	1	2	3	4	5
0	0	m	m	0	$2m$	m
1		$\min\{n+2m+2, 2n+m+2\}$	$n+m+1$	n	$n+2m+2$	m
2			$n+m+1$	n	$n+2m+2$	$n+m+1$
3				$\min\{m,n\}$	$2m$	m
4					$2n+2m+4$	$2n+m+2$
5						$n+m+1$

Table 7: The values of δ as a function of m and n in Corollary 4.9.

5 Conclusions and future work

We have provided lower and upper bounds for the 2-rainbow domination number of $C_m \square C_n$ with a gap of at most $\frac{1}{3}(2m+2n+4)$. The proof of the lower bound is based on a discharging argument and seems to be close to the best possible in most cases. The upper bound, on the other hand, is based on two constructions that are quite rough in some cases, and we believe that it can be improved by carefully analyzing special cases. We conjecture that the lower bounds differ from the exact values by at most one constant, which depends on m and is independent of n

At least for examples with small m, we claim that it is possible to avoid the tedious analysis by applying an algebraic method that can be used for various graph invariants including the domination type problems [11, 20, 24, 26]. Such a research task remains a challenge for future work.

Another interesting line of research, which is a natural extension of this study, is a generalization of the results presented here to graph bundles, a natural generalization of graph products [25].

ORCID iDs

Simon Brezovnik https://orcid.org/0000-0001-6584-6435
Darja Rupnik Poklukar https://orcid.org/0000-0003-3498-0097
Janez Žerovnik https://orcid.org/0000-0002-6041-1106

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