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Homomorphisms from the Coxeter graph

Marko Orel [a,b,c,*], Draženka Višnjić [b]

[a] *University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia*
[b] *University of Primorska, IAM, Muzejski trg 2, 6000 Koper, Slovenia*
[c] *IMFM, Jadranska 19, 1000 Ljubljana, Slovenia*

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Let $S_n(\mathbb{F}_2)$ be the set of all $n \times n$ symmetric matrices with coefficients in the binary field $\mathbb{F}_2 = \{0, 1\}$, and let $SGL_n(\mathbb{F}_2)$ be its subset formed by invertible matrices. Let $\widehat{\Gamma}_n$ be the graph with the vertex set $S_n(\mathbb{F}_2)$ where a pair of vertices $\{A, B\}$ form an edge if and only if $\text{rank}(A - B) = 1$. Similarly, let Γ_n be the subgraph in $\widehat{\Gamma}_n$, which is induced by the set $SGL_n(\mathbb{F}_2)$. Graph Γ_n generalizes the well-known Coxeter graph, which is isomorphic to Γ_3. Motivated by research topics in coding theory, matrix theory, and graph theory, this paper represents the first step towards the characterization of all graph homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ where n, m are positive integers. Here, the case $n = 3$ is solved.

* Corresponding author at: University of Primorska, FAMNIT, Glagoljaška 8, 6000 Koper, Slovenia.
E-mail addresses: marko.orel@upr.si (M. Orel), drazenka.visnjic@iam.upr.si (D. Višnjić).

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1. Introduction

The graph $\widehat{\Gamma}_n$ has a long history of study: see for example the books [45, Chapter 5], [9, Subsection 9.5 D] and the references therein for some research related to linear algebra/geometry and combinatorics, respectively. On the other hand, the graph Γ_n was introduced fairly recently by the first author in [33], and studied further by both authors in [35]. There are multiple reasons to investigate Γ_n.

Firstly, it is a natural generalization of the Coxeter graph Γ_3, which is one of the four (currently) known vertex-transitive graphs of order at least three that do not have a Hamilton cycle (cf. [15, Section 3.3]). For odd n, the graph Γ_n is vertex-transitive, and it is still not known if it has a Hamilton cycle for each $n > 3$ [33, Open Problem 16].

The second reason to study the graph Γ_n, which is also the main motivation for the research presented in this paper, is its recently discovered connection with binary self-dual codes [35]. In fact, for odd $n \geq 3$, each linear self-dual code C in $\mathbb{F}_2^{n+1}$ is identified with a certain subset $\mathcal{F}_C$ of matrices A in $SGL_n(\mathbb{F}_2)$ that are characterized by the graph distances $d_{\Gamma_n}(A, I)$ and $d_{\widehat{\Gamma}_n}(A, I)$. Here, I is the identity matrix and $\mathbb{F}_2^{n+1}$ is the vector space formed by all column vectors $\mathbf{x} = (x_1, \ldots, x_{n+1})^\top$ with $x_i \in \mathbb{F}_2$, which has the standard basis $\{\mathbf{e}_1 = (1, 0, \ldots, 0)^\top, \ldots, \mathbf{e}_{n+1} = (0, \ldots, 0, 1)^\top\}$. Self-dual codes are important because many codes with the best parameters are self-dual (cf. [39]). The characterization of permutation inequivalent binary self-dual codes was investigated by Conway, Pless and Sloane [12,13,37,38] who succeeded for lengths ≤ 30. Later, such characterization was improved to $n + 1 \leq 40$ [1,2,4–8,19] (see also the database [20]). The asymptotic behavior of the number of permutation inequivalent binary self-dual codes was discovered by Hou [21]. By considering 'classical' automorphisms of the graph Γ_n that fix the identity matrix, we were able to strengthen a result of Janusz [26] and showed that the group of all $n \times n$ binary orthogonal matrices acts transitively on the set of all self-dual codes in $\mathbb{F}_2^{n+1}$ [35, Theorem 8.10]. We expect that a characterization of all graph homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ will provide us additional information and a deeper understanding of binary self-dual codes.

Lastly, we mention that the characterization of graph homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ is valuable in the research area of *preserver problems* as it generalizes the *fundamental theorem of geometry* of symmetric matrices in the case of a binary field. Namely, the fundamental theorem characterizes all automorphisms of the graph $\widehat{\Gamma}_n$ (for corresponding results on rectangular, alternate, hermitian, and symmetric matrices over general fields see the book [45]). Fundamental theorems of geometry were applied numerous times in the past to solve other preserver problems (see for example [41–43]). To increase the applicability, in the last decades a lot of effort was made in order to generalize fundamental theorems either by characterizing all endomorphisms of the graphs involved [14,22–25,28,31,44] or by replacing matrices with tensors (cf. [10,11,27]). Papers [29,30] contain the characterization of all endomorphisms of the graph with the vertex set formed by all $n \times n$ invertible hermitian matrices over a finite field $\mathbb{F}_{q^2}$ with $q \geq 4$ where edges are defined in the same way as in the graph Γ_n. In comparison with the result in [29,30], the problem of characterizing all graph homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ has an additional difficulty because the codomain graph can be much 'larger' than the domain graph. This obstacle turned out to be nontrivial in some related results [36], despite a much stronger condition, which includes the injectivity of maps, was assumed. Finally, we mention that the usual approach to prove a (generalization of a) fundamental theorem of geometry of some matrices is to characterize maximal cliques first (a.k.a. *maximal adjacent sets* [11] or *maximal coherent sets* [10]), and then benefit from the fact that maximal cliques are mapped inside maximal cliques by graph homomorphisms. This approach does not work for the characterization of the graph homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ because neither Γ_n nor $\widehat{\Gamma}_m$ has triangles (i.e. maximal cliques are just edges). Instead, we focus on the shortest odd cycles in Γ_n and $\widehat{\Gamma}_m$ (see Lemma 6.1, Corollary 6.2 and Corollary 6.3).

The rest of the paper is organized as follows. In Section 2, we recall prerequisites needed to understand the main result of this paper, which is stated in Theorem 2.1. In Section 3, we recall and develop some elementary tools from linear algebra. Section 4 is dedicated to vanishing sums of rank-one matrices in $S_n(\mathbb{F}_2)$, which provide a lot of information on small cycles in graphs Γ_n and $\widehat{\Gamma}_m$. In Section 5, the 7-cycles in the graph Γ_n are investigated. The final part of the proof of Theorem 2.1 is presented in Section 6.

2. Prerequisites and the statement of the main result

All graphs in this paper are simple, and finite (except in Lemma 6.1 and Corollary 6.2). Given a graph $G = (V, E)$ with the vertex set V and the edge set E, a subgraph $G' = (V', E')$ in G is *induced* by a set $U \subseteq V$ if $V' = U$ and $E' = \big\{\{u, v\} \in E : u, v \in U\big\}$. Given graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a *graph homomorphism* $\Phi : G_1 \to G_2$ is a map $\Phi : V_1 \to V_2$ such that $\{\Phi(u), \Phi(v)\} \in E_2$ whenever $\{u, v\} \in E_1$. If it is bijective and $\{\Phi(u), \Phi(v)\} \in E_2 \Leftrightarrow \{u, v\} \in E_1$, then it is a *graph isomorphism*. If $G_1 = G_2$, then a graph homomorphism and a graph isomorphism are a *graph endomorphism* and a *graph automorphism*, respectively. We often name these objects just as homomorphism, isomorphism, endomorphism, and automorphism, without mentioning the word 'graph'.

In the literature, a homomorphism $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ is referred also as an *adjacency preserving* map. It is a map $\Phi : SGL_n(\mathbb{F}_2) \to S_m(\mathbb{F}_2)$ that obeys the implication

$$\operatorname{rank}(A - B) = 1 \Longrightarrow \operatorname{rank}\big(\Phi(A) - \Phi(B)\big) = 1$$

for all $A, B \in SGL_n(\mathbb{F}_2)$. Similarly, an *automorphism* Ψ of the graph $\widehat{\Gamma}_m$ is often referred as a bijection on $S_m(\mathbb{F}_2)$ that *preserves the adjacency in both directions*. It obeys the equivalence

$$\operatorname{rank}(A - B) = 1 \Longleftrightarrow \operatorname{rank}\big(\Psi(A) - \Psi(B)\big) = 1$$

for all $A, B \in S_m(\mathbb{F}_2)$. Automorphisms of $\widehat{\Gamma}_m$ were characterized in [46] for $m \geq 3$ and in [16] for $m = 2$ (see also [45, Theorem 5.4] and [47]). If $m \neq 3$, then each automorphism is of the form

$$\Psi(A) = PAP^\top + B \qquad (A \in S_m(\mathbb{F}_2)) \tag{1}$$

for some fixed $P \in GL_m(\mathbb{F}_2)$ and $B \in S_m(\mathbb{F}_2)$ where $GL_m(\mathbb{F}_2)$ denotes the set of all $m \times m$ invertible matrices over $\mathbb{F}_2$. If $m = 3$, then the automorphism group of $\widehat{\Gamma}_m$ has an additional generator, which can be the map Ψ_τ, defined by

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \longmapsto \begin{pmatrix} a_{11} + a_{23} & a_{12} + a_{23} & a_{13} + a_{23} \\ a_{12} + a_{23} & a_{22} & a_{23} \\ a_{13} + a_{23} & a_{23} & a_{33} \end{pmatrix} \tag{2}$$

on $S_3(\mathbb{F}_2)$. In [31, Theorem 1.1] it was shown that each endomorphism of $\widehat{\Gamma}_m$ is an automorphism if $m \geq 3$. The same claim is true for the graph Γ_n if $n \geq 3$ [33, Theorem 8] despite the characterization of its endomorphisms = automorphisms is not yet known if $n > 3$.[1] In the next paragraph, we briefly present a different characterization of the automorphisms of $\widehat{\Gamma}_3$, which is fully described in [32, pp. 395-396] and implicitly indicated in [48, p. 512].

Observe that the map Ψ_τ is *additive*, i.e. $\Psi_\tau(A+B) = \Psi_\tau(A) + \Psi_\tau(B)$ for all $A, B \in S_3(\mathbb{F}_2)$. If Ψ is any automorphism of $\widehat{\Gamma}_3$, then the automorphism $\Psi' : A \mapsto \Psi(A) - \Psi(0)$ fixes the zero matrix. Since it is a composition of affine maps of the forms (1) and (2), it must be additive. Moreover, the map Ψ' permutes the set

$$\mathcal{R}_1 := \{\mathbf{e}_1^2, \mathbf{e}_2^2, \mathbf{e}_3^2, (\mathbf{e}_1 + \mathbf{e}_2)^2, (\mathbf{e}_1 + \mathbf{e}_3)^2, (\mathbf{e}_2 + \mathbf{e}_3)^2, (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)^2\}, \tag{3}$$

which consists of all seven rank-one matrices in $S_3(\mathbb{F}_2)$. Here, we use the abbreviation $\mathbf{x}^2 := \mathbf{x}\mathbf{x}^\top$ for a column vector $\mathbf{x}$. Observe that the sum of the seven matrices in $\mathcal{R}_1$ is zero and each six of them form a basis of the vector space $S_3(\mathbb{F}_2)$ (in the language of finite geometry, the rank-one matrices form an *arc* of size seven). Hence, if σ is any permutation of the set $\mathcal{R}_1$, then σ is uniquely extendable to an additive (= linear) automorphism of $\widehat{\Gamma}_3$ that we denote by Ψ_σ. Therefore, the automorphisms (= endomorphisms) of $\widehat{\Gamma}_3$ are precisely the maps

$$A \mapsto \Psi_\sigma(A) + B \qquad (A \in S_3(\mathbb{F}_2))$$

where σ ranges over all permutations of the set $\mathcal{R}_1$ and B ranges over all matrices in $S_3(\mathbb{F}_2)$. In this way, the index τ in the map Ψ_τ is the transposition permutation that swaps matrices $(\mathbf{e}_2 + \mathbf{e}_3)^2, (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)^2$ and fixes all other rank-one matrices in $\mathcal{R}_1$. For more information related to the automorphism group of $\widehat{\Gamma}_3$ we refer to [32, pp. 395-396].

[1] It will be included as a research topic in our subsequent paper.

In this and in the subsequent paper, the aim is to characterize all homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$. It is clear that the case $n = 3$ needs a special treatment because such homomorphisms include those that can be factorized as

$$\Gamma_3 \to \widehat{\Gamma}_3 \to \widehat{\Gamma}_m. \tag{4}$$

Hence, the characterization must involve maps Ψ_σ. Another difficulty is the involvement of the inverting map $A \mapsto A^{-1}$. This map is nonlinear and preserves the adjacency relation (and many other rank-related relations) whenever it is defined as already observed in the solutions of many preserver problems [14,25,29,30,33,44]. In fact, the property $\operatorname{rank}(A^{-1}-B^{-1}) = \operatorname{rank}(A-B)$ follows from the equality $A^{-1}-B^{-1} = B^{-1}(B-A)A^{-1}$, which holds for arbitrary invertible matrices A, B that are of the same size and have coefficients in a field. In this paper we treat the case $n = 3$ and leave the case $n > 3$ for a subsequent paper. As it follows from Theorem 2.1, the factorization (4) is always possible (if $m \geq 3$).

We now state the main result of this paper.

Theorem 2.1. *Let $m \geq 3$ be an integer and let $\Phi : \Gamma_3 \to \widehat{\Gamma}_m$ be a graph homomorphism. Then there exist $P \in GL_m(\mathbb{F}_2)$, $B \in S_m(\mathbb{F}_2)$, and a permutation σ of the set $\mathcal{R}_1$ such that either*

$$\Phi(A) = P \begin{pmatrix} \Psi_\sigma(A) & 0 \\ 0 & 0 \end{pmatrix} P^\top + B \qquad \text{for all } A \in SGL_3(\mathbb{F}_2) \tag{5}$$

or

$$\Phi(A) = P \begin{pmatrix} \Psi_\sigma(A^{-1}) & 0 \\ 0 & 0 \end{pmatrix} P^\top + B \qquad \text{for all } A \in SGL_3(\mathbb{F}_2). \tag{6}$$

Remark 2.2. The converse statement is clear from the discussion above, i.e. maps (5) and (6) are always graph homomorphisms $\Gamma_3 \to \widehat{\Gamma}_m$.

Remark 2.3. Regarding the characterization of all homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ where $2 \in \{n, m\}$, note that the graph Γ_2 is just the complete bipartite graph $K_{1,3}$. Hence for each induced subgraph Γ' in $\widehat{\Gamma}_m$, which is either isomorphic to $K_{1,3}$ or to a path on three or two vertices, there exists a homomorphism $\Phi : \Gamma_2 \to \widehat{\Gamma}_m$ with $V(\Gamma')$ as its image. On the other hand, there do not exist homomorphisms $\Phi : \Gamma_n \to \widehat{\Gamma}_2$ if $n \geq 3$. In fact, for $n \geq 3$, Γ_n contains a 7-cycle (see [33]) and therefore its chromatic number is at least three. However, $\widehat{\Gamma}_2$ is the cube graph, which has the chromatic number two. Hence, the existence of such a homomorphism is neglected by [18, Proposition 2.10].

3. Auxiliary results from linear algebra

Recall that a $n \times n$ matrix A over a field $\mathbb{F}$ is *alternate* if $\mathbf{x}^\top A\mathbf{x} = 0$ for all $\mathbf{x} \in \mathbb{F}^n$. Equivalently, A is alternate if and only if it $A^\top = -A$ and the diagonal of A is zero. Alternate matrices have even rank [45, Proposition 1.34]. The next result follows immediately from [45, Corollary 1.36].

Lemma 3.1. *Let $A \in SGL_n(\mathbb{F}_2)$. If A is nonalternate, then there exists $P \in GL_n(\mathbb{F}_2)$ such that $A = PP^\top$.*

The following result is well known (cf. [40, pp. 16-12, 16-16]).

Lemma 3.2. *Let $\mathbb{F}$ be a field and $A \in GL_n(\mathbb{F})$. If $\mathbf{x}, \mathbf{y} \in \mathbb{F}^n$, then $A + \mathbf{x}\mathbf{y}^\top$ is invertible if and only if $\mathbf{y}^\top A^{-1}\mathbf{x} \neq -1$ in which case*

$$(A + \mathbf{x}\mathbf{y}^\top)^{-1} = A^{-1} - \frac{1}{1 + \mathbf{y}^\top A^{-1}\mathbf{x}} A^{-1}\mathbf{x}\mathbf{y}^\top A^{-1}.$$

Corollary 3.3. *Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}_2^n$.*

(i) *The matrix $I + \mathbf{x}^2$ is in $SGL_n(\mathbb{F}_2)$ if and only if $\mathbf{x}^\top\mathbf{x} = 0$ in which case $(I + \mathbf{x}^2)^{-1} = I + \mathbf{x}^2$.*

(ii) *Suppose that $I + \mathbf{x}^2 \in SGL_n(\mathbb{F}_2)$. Then $I + \mathbf{x}^2 + \mathbf{y}^2 \in SGL_n(\mathbb{F}_2)$ if and only if $\mathbf{y}^\top(\mathbf{x} + \mathbf{y}) = 0$ in which case*

$$(I + \mathbf{x}^2 + \mathbf{y}^2)^{-1} = \begin{cases} I + \mathbf{x}^2 + \mathbf{y}^2 & \text{if} \quad \mathbf{y}^\top\mathbf{y} = 0, \\ I + \mathbf{x}^2 + (\mathbf{x} + \mathbf{y})^2 & \text{if} \quad \mathbf{y}^\top\mathbf{y} = 1. \end{cases}$$

(iii) *Suppose that $I + \mathbf{x}^2, I + \mathbf{x}^2 + \mathbf{y}^2 \in SGL_n(\mathbb{F}_2)$. Then $I + \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2 \in SGL_n(\mathbb{F}_2)$ if and only if*

$$\mathbf{z}^\top \left((1 + \mathbf{y}^\top\mathbf{y}) \cdot (\mathbf{x} + \mathbf{y} + \mathbf{z}) + \mathbf{y}^\top\mathbf{y} \cdot (\mathbf{y} + \mathbf{z})\right) = 0.$$

Proof. Part (i) follows directly from Lemma 3.2. If the assumption in (ii) is satisfied, then (i) and the equality $\mathbf{y}^\top\mathbf{x} = \mathbf{x}^\top\mathbf{y}$ in the binary field imply that

$$\mathbf{y}^\top(I + \mathbf{x}^2)^{-1}\mathbf{y} = \mathbf{y}^\top\mathbf{y} + \mathbf{y}^\top\mathbf{x} \cdot \mathbf{x}^\top\mathbf{y} = \mathbf{y}^\top(\mathbf{x} + \mathbf{y}).$$

Since $\mathbb{F}_2 \backslash \{-1\} = \{0\}$, Lemma 3.2 implies that $I + \mathbf{x}^2 + \mathbf{y}^2$ is invertible if and only if $\mathbf{y}^\top(\mathbf{x} + \mathbf{y}) = 0$. Moreover, in this case,

$$\begin{aligned}(I + \mathbf{x}^2 + \mathbf{y}^2)^{-1} &= (I + \mathbf{x}^2)^{-1} + (I + \mathbf{x}^2)^{-1}\mathbf{y}^2(I + \mathbf{x}^2)^{-1} \\ &= I + \mathbf{x}^2 + (\mathbf{y}^2 + \mathbf{x}^\top\mathbf{y} \cdot \mathbf{x}\mathbf{y}^\top)(I + \mathbf{x}^2)\end{aligned}$$

Table 1
Possible selections of vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in Lemma 3.4.

$\mathbf{x}_2^\top \mathbf{x}_2$	$\mathbf{x}_1^\top \mathbf{x}_3$	$\mathbf{x}_3^\top \mathbf{x}_3$	$\mathbf{x}$	$\mathbf{y}$	$\mathbf{z}$
0	0	0	$\mathbf{x}_1+\mathbf{x}_2$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_3$
0	0	1	$\mathbf{x}_1+\mathbf{x}_2$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_2+\mathbf{x}_3$
0	1	0	$\mathbf{x}_1+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2$	$\mathbf{x}_2+\mathbf{x}_3$
0	1	1	$\mathbf{x}_1+\mathbf{x}_2$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_3$
1	0	0	$\mathbf{x}_1+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2$
1	0	1	$\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$
1	1	0	$\mathbf{x}_1+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_2+\mathbf{x}_3$
1	1	1	$\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3$	$\mathbf{x}_1+\mathbf{x}_2$

$$
\begin{aligned}
&= I + \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{x}^\top \mathbf{y} \cdot \mathbf{x}\mathbf{y}^\top + \mathbf{y}^\top \mathbf{x} \cdot \mathbf{y}\mathbf{x}^\top + \mathbf{x}^\top \mathbf{y} \cdot \mathbf{y}^\top \mathbf{x} \cdot \mathbf{x}\mathbf{x}^\top \\
&= I + \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{y}^\top \mathbf{y} \cdot (\mathbf{x}\mathbf{y}^\top + \mathbf{y}\mathbf{x}^\top + \mathbf{x}\mathbf{x}^\top) \\
&= I + \mathbf{x}^2 + (1 + \mathbf{y}^\top \mathbf{y}) \cdot \mathbf{y}^2 + \mathbf{y}^\top \mathbf{y} \cdot (\mathbf{x}+\mathbf{y})^2,
\end{aligned}
$$

which completes the proof (ii).

Suppose the assumptions in (iii) are satisfied. Then, by considering both possibilities $\mathbf{y}^\top \mathbf{y} \in \{0, 1\}$, we deduce that in the binary field

$$
\mathbf{z}^\top (I + \mathbf{x}^2 + \mathbf{y}^2)^{-1} \mathbf{z} = \mathbf{z}^\top \left((1 + \mathbf{y}^\top \mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}+\mathbf{z}) + \mathbf{y}^\top \mathbf{y} \cdot (\mathbf{y}+\mathbf{z}) \right).
$$

The claim now follows from Lemma 3.2. □

Lemma 3.4. *Let $n \geq 3$ and suppose that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{F}_2^n$ are linearly independent vectors such that*

$$
\begin{aligned}
&\mathbf{x}_1^\top \mathbf{x}_1 = 0, \quad \mathbf{x}_2^\top (\mathbf{x}_1 + \mathbf{x}_2) = 0, \\
&\mathbf{x}_3^\top \left((1 + \mathbf{x}_2^\top \mathbf{x}_2) \cdot (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3) + \mathbf{x}_2^\top \mathbf{x}_2 \cdot (\mathbf{x}_2 + \mathbf{x}_3) \right) = 0.
\end{aligned}
\tag{7}
$$

Then there exist pairwise distinct $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{\mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_1+\mathbf{x}_3, \mathbf{x}_2+\mathbf{x}_3, \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3\}$ such that

$$
\mathbf{x}^\top \mathbf{x} = 0, \quad \mathbf{y}^\top (\mathbf{x}+\mathbf{y}) = 0, \quad \mathbf{z}^\top \left((1 + \mathbf{y}^\top \mathbf{y}) \cdot (\mathbf{x}+\mathbf{y}+\mathbf{z}) + \mathbf{y}^\top \mathbf{y} \cdot (\mathbf{y}+\mathbf{z}) \right) = 0. \tag{8}
$$

Moreover, if $\mathbf{x}_2^\top \mathbf{x}_2 = 0$ and $\mathbf{x}_3^\top \mathbf{x}_3 = 1$, then we can select $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$.

Proof. In principle there are 2^6 possibilities for the six values $\mathbf{x}_i^\top \mathbf{x}_j$ $(1 \leq i \leq j \leq 3)$. However, only $2^3 = 8$ of them can satisfy the conditions (7). These are described by the values $\mathbf{x}_2^\top \mathbf{x}_2$, $\mathbf{x}_1^\top \mathbf{x}_3$, $\mathbf{x}_3^\top \mathbf{x}_3$ in the first three columns of Table 1. In fact, together with (7) they determine the other three values $\mathbf{x}_1^\top \mathbf{x}_1$, $\mathbf{x}_1^\top \mathbf{x}_2$, $\mathbf{x}_2^\top \mathbf{x}_3$. We now select $\mathbf{x}, \mathbf{y}, \mathbf{z}$ as described in the last three columns of Table 1. In particular, if $\mathbf{x}_2^\top \mathbf{x}_2 = 0$ and $\mathbf{x}_3^\top \mathbf{x}_3 = 1$, then we can select $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$. □

Lemma 3.5. *Let* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_2^n$ *be linearly independent column-vectors such that* $0 = \mathbf{x}_1^\top \mathbf{x}_1 = \mathbf{x}_2^\top \mathbf{x}_2 = \mathbf{x}_1^\top \mathbf{x}_2$*. Then there exist* $\mathbf{x}_3 \in \mathbb{F}_2^n$ *such that* $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ *are linearly independent, and either*

$$\mathbf{x}_1^\top \mathbf{x}_3 = 0, \quad \mathbf{x}_2^\top \mathbf{x}_3 = 1 = \mathbf{x}_3^\top \mathbf{x}_3 \tag{9}$$

or

$$\mathbf{x}_2^\top \mathbf{x}_3 = 0, \quad \mathbf{x}_1^\top \mathbf{x}_3 = 1 = \mathbf{x}_3^\top \mathbf{x}_3. \tag{10}$$

Proof. Observe that the assumptions imply that $n \geq 3$. Given $\mathbf{x} = (x_1, \ldots, x_n)^\top$ in $\mathbb{F}_2^n$ let $\operatorname{supp}(\mathbf{x}) := \{i \in \{1, \ldots, n\} : x_i = 1\}$ be its support. Since $\mathbf{x}_1, \mathbf{x}_2$ are linearly independent, it follows that $\operatorname{supp}(\mathbf{x}_1) \neq \operatorname{supp}(\mathbf{x}_2)$. In particular,

$$\operatorname{supp}(\mathbf{x}_1) \not\supseteq \operatorname{supp}(\mathbf{x}_2) \tag{11}$$

or

$$\operatorname{supp}(\mathbf{x}_2) \not\supseteq \operatorname{supp}(\mathbf{x}_1). \tag{12}$$

If (11) holds, then we select $i \in \operatorname{supp}(\mathbf{x}_2) \backslash \operatorname{supp}(\mathbf{x}_1)$. Otherwise, we select $i \in \operatorname{supp}(\mathbf{x}_1) \backslash \operatorname{supp}(\mathbf{x}_2)$. The column vector $\mathbf{x}_3 := \mathbf{e}_i$ satisfies either (9) or (10), depending on the two choices for i.

Since $\mathbf{x}_1^\top \mathbf{x}_1 = 0 = \mathbf{x}_2^\top \mathbf{x}_2$, the numbers $|\operatorname{supp}(\mathbf{x}_1)|$, $|\operatorname{supp}(\mathbf{x}_2)|$ are both even. Hence, $\mathbf{x}_3 \neq \mathbf{x}_1, \mathbf{x}_2$. From $\mathbf{x}_1^\top \mathbf{x}_2 = 0$ we further deduce that $|\operatorname{supp}(\mathbf{x}_1) \cap \operatorname{supp}(\mathbf{x}_2)|$ is even. Therefore,

$$\begin{aligned} |\operatorname{supp}(\mathbf{x}_1 + \mathbf{x}_2)| &= |\operatorname{supp}(\mathbf{x}_1) \backslash \operatorname{supp}(\mathbf{x}_2)| + |\operatorname{supp}(\mathbf{x}_2) \backslash \operatorname{supp}(\mathbf{x}_1)| \\ &= \big|\operatorname{supp}(\mathbf{x}_1) \backslash \big(\operatorname{supp}(\mathbf{x}_1) \cap \operatorname{supp}(\mathbf{x}_2)\big)\big| + \big|\operatorname{supp}(\mathbf{x}_2) \backslash \big(\operatorname{supp}(\mathbf{x}_1) \cap \operatorname{supp}(\mathbf{x}_2)\big)\big| \end{aligned}$$

is even. Hence, $\mathbf{x}_3 \neq \mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent. □

4. Vanishing sums of rank-one matrices and small cycles in $\widehat{\boldsymbol{\Gamma}}_n$

In this paper, we denote a k-cycle in a graph $G = (V, E)$ as $[v_1, v_2, \ldots, v_k]$. That is, $v_1, \ldots, v_k \in V$ are distinct vertices such that $\{v_1, v_2\}$, $\{v_2, v_3\}$, …, $\{v_{k-1}, v_k\}$, $\{v_k, v_1\}$ are edges in E. By rotating the vertices or by reversing their order we get the same cycle.

Clearly, if $[X_1, X_2, \ldots, X_k]$ is a k-cycle in $\widehat{\Gamma}_n$ or in Γ_n, then matrices

$$M_1 = X_2 - X_1,\ M_2 = X_3 - X_2,\ \ldots,\ M_{k-1} = X_k - X_{k-1},\ M_k = X_1 - X_k$$

are of rank one and satisfy

$$M_1 + M_2 + \cdots + M_k = 0. \tag{13}$$

The aim of Section 4 is to describe rank-one matrices in $S_n(\mathbb{F}_2)$ that satisfy (13) for small k. In what follows, $\langle \mathbf{x}_1, \ldots, \mathbf{x}_m \rangle$ denotes the vector space that is spanned by vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m$.

Lemma 4.1.

(i) *There do not exist rank-one matrices $M_1, M_2, M_3 \in S_n(\mathbb{F}_2)$ such that $M_1 + M_2 + M_3 = 0$.*

(ii) *There do not exist rank-one matrices $M_1, M_2, M_3, M_4, M_5 \in S_n(\mathbb{F}_2)$ such that $M_1 + M_2 + M_3 + M_4 + M_5 = 0$.*

Proof. Any rank-one matrix can be written as $M_i = \mathbf{x}_i^2$ for some $\mathbf{x}_i \in \mathbb{F}_2^n \backslash \{0\}$.

To prove (ii), choose any rank-one matrices $M_1, \ldots, M_5 \in S_n(\mathbb{F}_2)$. Let D be the dimension of the vector space, which is spanned by vectors $\mathbf{x}_1, \ldots, \mathbf{x}_5$. If $D = 1$, then $M_1 = \cdots = M_5$ and $M_1 + M_2 + M_3 + M_4 + M_5 = M_1 \neq 0$. If $D \geq 3$, then there are distinct $a, b, c \in \{1, \ldots, 5\}$ such that $\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_c$ are linearly independent. Hence, $\operatorname{rank}(M_1 + \cdots + M_5) \geq \operatorname{rank}(M_a + M_b + M_c) - \operatorname{rank}(M_d + M_e) \geq 3 - 2 = 1 > 0$ where $\{d, e\} = \{1, \ldots, 5\} \backslash \{a, b, c\}$. If $D = 2$, then there are $a, b \in \{1, \ldots, 5\}$ such that $\mathbf{x}_a, \mathbf{x}_b$ are linearly independent and $\mathbf{x}_c, \mathbf{x}_d, \mathbf{x}_e \in \langle \mathbf{x}_a, \mathbf{x}_b \rangle \backslash \{0\} = \{\mathbf{x}_a, \mathbf{x}_b, \mathbf{x}_a + \mathbf{x}_b\}$ where $\{c, d, e\} = \{1, \ldots, 5\} \backslash \{a, b\}$. Hence, $M_c + M_d + M_e = \alpha \mathbf{x}_a^2 + \beta \mathbf{x}_b^2 + \gamma (\mathbf{x}_a + \mathbf{x}_b)^2$ for some nonnegative integers α, β, γ with $\alpha + \beta + \gamma = 3$. Consequently,

$$M_a + M_b + M_c + M_d + M_e = (1 + \alpha + \gamma)\mathbf{x}_a^2 + (1 + \beta + \gamma)\mathbf{x}_b^2 + \gamma\big(\mathbf{x}_a \mathbf{x}_b^\top + \mathbf{x}_b \mathbf{x}_a^\top\big). \tag{14}$$

Clearly, (14) is nonzero if $\gamma \neq 0$. For $\gamma = 0$,

$$(1 + \alpha + \gamma) + (1 + \beta + \gamma) = \alpha + \beta = 1 \pmod 2$$

whence $(1 + \alpha + \gamma) \neq 0$ or $(1 + \beta + \gamma) \neq 0$. Therefore, (14) is nonzero.

Statement (i) follows from (ii) by selecting $M_4 = M_5$. □

By Lemma 4.1, there are neither 3-cycles nor 5-cycles in graphs $\widehat{\Gamma}_n$ and Γ_n. It was observed already in [33] that there exist 7-cycles in Γ_n for $n \geq 3$. Hence, the same is true for graph $\widehat{\Gamma}_n$ with $n \geq 3$. To summarize, the *odd girth* (i.e. the length of the shortest odd cycle) of Γ_n and $\widehat{\Gamma}_n$ is 7 whenever $n \geq 3$.

Lemma 4.2. *Suppose that $M_1 + \cdots + M_7 = 0$ for some rank-one matrices $M_1, \ldots, M_7 \in S_3(\mathbb{F}_2)$. Then $\{M_1, \ldots, M_7\} = \mathcal{R}_1$.*

Proof. By Lemma 4.1, $M_1, \ldots, M_7$ are all distinct. Eq. (3) implies the claim. □

Lemma 4.3. *Let $n \geq 3$ and assume that $M_1 + M_2 + \cdots + M_7 = 0$ for some rank-one matrices $M_1, \ldots, M_7 \in S_n(\mathbb{F}_2)$. Then there exist linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{F}_2^n$ such that*

$$\{M_1, \ldots, M_7\} = \{\mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_3^2, (\mathbf{x}_1 + \mathbf{x}_2)^2, (\mathbf{x}_1 + \mathbf{x}_3)^2, (\mathbf{x}_2 + \mathbf{x}_3)^2, (\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3)^2\}. \quad (15)$$

Proof. Write $M_i = \mathbf{y}_i^2$ for each i. Let D be the dimension of the vector space, which is spanned by vectors $\mathbf{y}_1, \ldots, \mathbf{y}_7$. If $D \leq 2$, then there are at most three possibilities for matrices M_i. In particular, there exist distinct i, j such that $M_i = M_j$. Hence, $M_1 + M_2 + \cdots + M_7 = 0$ transforms into a vanishing sum of five rank-one matrices, which is not possible by Lemma 4.1. If $D \geq 4$, then there are distinct $a, b, c, d \in \{1, \ldots, 7\}$ such that $\mathbf{y}_a, \mathbf{y}_b, \mathbf{y}_c, \mathbf{y}_d$ are linearly independent. Hence, $\operatorname{rank}(M_1+\cdots+M_7) \geq \operatorname{rank}(M_a+M_b+M_c+M_d) - \operatorname{rank}(M_e+M_f+M_g) \geq 4-3 = 1 > 0$ where $\{e, f, g\} = \{1, \ldots, 7\} \backslash \{a, b, c, d\}$. Hence, $D = 3$. Let $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ be any linearly independent vectors selected among $\mathbf{y}_1, \ldots, \mathbf{y}_7$. Then $\mathbf{y}_i \in \langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$ for all i. If $P \in GL_n(\mathbb{F}_2)$ is any matrix with first three columns equal to $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$, then for each i we can write

$$M_i = P \begin{pmatrix} \dot{M}_i & 0 \\ 0 & 0 \end{pmatrix} P^\top$$

where $\dot{M}_i \in S_3(\mathbb{F}_2)$ is of rank one, and $\dot{M}_1 + \cdots + \dot{M}_7 = 0$. By Lemma 4.2, we have $\{\dot{M}_1, \ldots, \dot{M}_7\} = \mathcal{R}_1$ and consequently (15). □

Lemma 4.4. *Let $n \geq 3$ and assume that $M_1 + M_2 + \cdots + M_6 = 0$ for some $M_1, \ldots, M_6 \in S_n(\mathbb{F}_2)$ of rank one. Then, there exist nonzero vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{F}_2^n$ such that the multiset $\{M_1, \ldots, M_6\}$ equals $\{\mathbf{x}_1^2, \mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_2^2, \mathbf{x}_3^2, \mathbf{x}_3^2\}$.*

Proof. Write $M_i = \mathbf{y}_i^2$ for each i. Let D be the dimension of the vector space, which is spanned by vectors $\mathbf{y}_1, \ldots, \mathbf{y}_6$. If $D = 1$, then $\mathbf{y}_1 = \cdots = \mathbf{y}_6$, and we can select $\mathbf{x}_1 = \mathbf{x}_2 = \mathbf{x}_3 = \mathbf{y}_1$. If $D = 2$, then $\langle \mathbf{y}_1, \ldots, \mathbf{y}_6 \rangle = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \{0, \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_2\}$ for appropriate linearly independent column vectors $\mathbf{x}_1, \mathbf{x}_2$. Hence, $\mathbf{y}_i^2 \in \{\mathbf{x}_1^2, \mathbf{x}_2^2, (\mathbf{x}_1 + \mathbf{x}_2)^2\}$ for each i. Since, $M_1 + \cdots + M_6 = 0$, each of the three matrices $\mathbf{x}_1^2, \mathbf{x}_2^2, (\mathbf{x}_1 + \mathbf{x}_2)^2$ equals to an even number of matrices $\mathbf{y}_i^2$ $(i = 1, \ldots, 6)$, which ends the proof. If $D \geq 4$, then there are distinct $a, b, c, d \in \{1, \ldots, 6\}$ such that $\mathbf{y}_a, \mathbf{y}_b, \mathbf{y}_c, \mathbf{y}_d$ are linearly independent. Hence,

$$0 = \operatorname{rank}(M_1 + \cdots + M_6) \geq \operatorname{rank}(M_a + M_b + M_c + M_d) - \operatorname{rank}(M_e + M_f) \geq 4 - 2 = 2$$

where $\{e, f\} = \{1, \ldots, 6\} \backslash \{a, b, c, d\}$, a contradiction.

Finally, assume that $D = 3$. Then there are distinct $a, b, c \in \{1, \ldots, 6\}$ such that $\mathbf{y}_a, \mathbf{y}_b, \mathbf{y}_c$ are linearly independent. Let $P \in GL_n(\mathbb{F}_2)$ be such that $P\mathbf{y}_a = \mathbf{e}_1$, $P\mathbf{y}_b = \mathbf{e}_2$, $P\mathbf{y}_c = \mathbf{e}_3$. If $\dot{M}_i$ is the top-left 3×3 block of matrix $PM_iP^\top$, then $\dot{M}_i \in S_3(\mathbb{F}_2)$ is of rank one and $\dot{M}_1 + \cdots + \dot{M}_6 = 0$. Since the rank-one matrices in $S_3(\mathbb{F}_2)$ form an arc of size seven (in the language of finite geometry), it follows that $\{1, \ldots, 6\} = \{i_1, \ldots, i_6\}$ where $\dot{M}_{i_1} = \dot{M}_{i_2}$, $\dot{M}_{i_3} = \dot{M}_{i_4}$, $\dot{M}_{i_5} = \dot{M}_{i_6}$. Consequently, $M_{i_1} = M_{i_2}$, $M_{i_3} = M_{i_4}$, $M_{i_5} = M_{i_6}$ and we can select $\mathbf{x}_1 := \mathbf{y}_{i_1}$, $\mathbf{x}_2 := \mathbf{y}_{i_3}$, $\mathbf{x}_3 := \mathbf{y}_{i_5}$. □

Corollary 4.5. *Let $n \geq 3$ and assume that $M_1 + M_2 + M_3 + M_4 = 0$ for some rank-one matrices $M_1, \ldots, M_4 \in S_n(\mathbb{F}_2)$. Then there exist nonzero vectors $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_2^n$ such that the multiset $\{M_1, M_2, M_3, M_4\}$ equals $\{\mathbf{x}_1^2, \mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_2^2\}$.*

Proof. It follows directly from Lemma 4.4 if we select $M_5 = M_6$ to be an arbitrary rank-one matrix in $S_n(\mathbb{F}_2)$. □

Lemma 4.6. *Let $n \geq 3$ and assume that $M_1 + M_2 + \cdots + M_8 = 0$ for some matrices $M_1, \ldots, M_8 \in S_n(\mathbb{F}_2)$ of rank one. Then, either there exist nonzero vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4 \in \mathbb{F}_2^n$ such that the multiset $\{M_1, \ldots, M_8\}$ equals $\{\mathbf{x}_1^2, \mathbf{x}_1^2, \mathbf{x}_2^2, \mathbf{x}_2^2, \mathbf{x}_3^2, \mathbf{x}_3^2, \mathbf{x}_4^2, \mathbf{x}_4^2\}$, or $n \geq 4$ and there exist linearly independent vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \in \mathbb{F}_2^n$ such that $\{M_1, \ldots, M_8\}$ equals*

$$\{\mathbf{y}_1^2, \mathbf{y}_2^2, \mathbf{y}_3^2, \mathbf{y}_4^2, (\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_3)^2, (\mathbf{y}_1 + \mathbf{y}_2 + \mathbf{y}_4)^2, (\mathbf{y}_1 + \mathbf{y}_3 + \mathbf{y}_4)^2, (\mathbf{y}_2 + \mathbf{y}_3 + \mathbf{y}_4)^2\}. \quad (16)$$

Proof. If $M_i = M_j$ for some distinct i and j, then the result follows from Lemma 4.4. This is always the case if $n = 3$ because $S_3(\mathbb{F}_2)$ contains only seven distinct matrices of rank one. Hence we may assume that $n \geq 4$, all matrices $M_1 =: \mathbf{y}_1^2, \ldots, M_8 =: \mathbf{y}_8^2$ are distinct, and $\dim\langle \mathbf{y}_1, \ldots, \mathbf{y}_8 \rangle \geq 4$. If there exists linearly independent vectors $\mathbf{y}_{i_1}, \ldots, \mathbf{y}_{i_r}$ with $r \geq 5$, then

$$\begin{aligned} 0 &= \operatorname{rank}\left(\sum_{k=1}^{r} \mathbf{y}_{i_k}^2 + \sum_{j \notin \{i_1, \ldots, i_r\}} \mathbf{y}_j^2\right) \\ &\geq \operatorname{rank}\left(\sum_{k=1}^{r} \mathbf{y}_{i_k}^2\right) - \operatorname{rank}\left(\sum_{j \notin \{i_1, \ldots, i_r\}} \mathbf{y}_j^2\right) \geq 5 - 3 = 2, \end{aligned}$$

a contradiction. Hence, $\dim\langle \mathbf{y}_1, \ldots, \mathbf{y}_8 \rangle = 4$. We may assume that $\mathbf{y}_1, \ldots, \mathbf{y}_4$ are linearly independent and $\mathbf{y}_5, \ldots, \mathbf{y}_8 \in \langle \mathbf{y}_1, \ldots, \mathbf{y}_4 \rangle$. Let $P \in GL_n(\mathbb{F}_2)$ be such that $P\mathbf{y}_i = \mathbf{e}_i$ for all $i \leq 4$. Then, $P\mathbf{y}_5, \ldots, P\mathbf{y}_8 \in \langle \mathbf{e}_1, \ldots, \mathbf{e}_4 \rangle$. Since

$$\mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_4^2 + \sum_{i=5}^{8} (P\mathbf{y}_i)^2 = 0,$$

we deduce that matrix

$$\mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2 + \mathbf{e}_4^2 + (P\mathbf{y}_i)^2$$

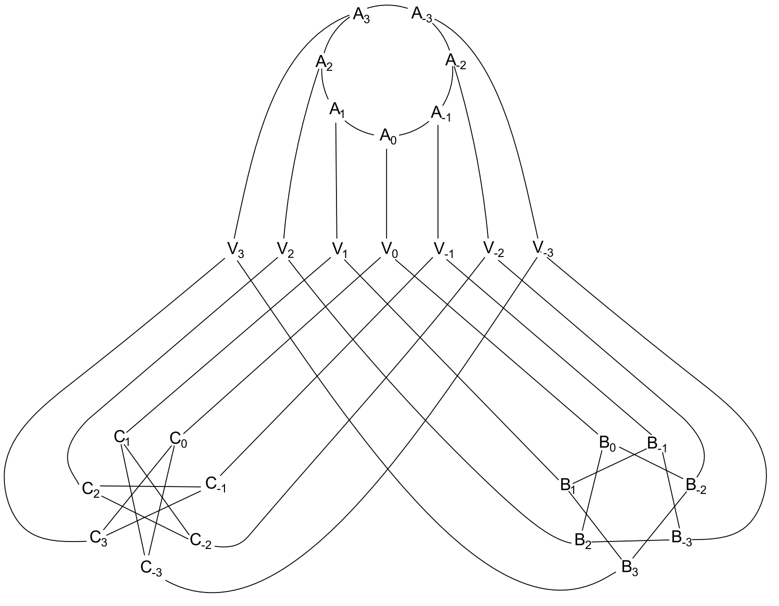

Fig. 1. Graph Γ_3 is the Coxeter graph.

is of rank three for all $i \in \{5,6,7,8\}$. Hence the determinant of its top-left 4-by-4 submatrix vanishes. If $\mathbf{z}_i \in \mathbb{F}_2^4$ denotes the vector obtained from the first four entries of $P\mathbf{y}_i$, then by Lemma 3.2 we deduce that $\mathbf{z}_i^\top I_4 \mathbf{z}_i = 1$ for all $i \in \{5,6,7,8\}$. That is, $\mathbf{z}_i$ has odd number of nonzero entries. Since $P\mathbf{y}_5, \dots, P\mathbf{y}_8 \in \langle \mathbf{e}_1, \dots, \mathbf{e}_4 \rangle$ and matrices $PM_1P^\top, \dots, PM_8P\top$ are all distinct, it follows that

$$\{P\mathbf{y}_5, \dots, P\mathbf{y}_8\} = \{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_4, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_4, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_4\}$$

and consequently $\{M_1, \dots, M_8\}$ equals (16). □

Remark 4.7. Observe that vectors $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4$ in Lemma 4.6 have the property that each five-element subset in $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3, \mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_4, \mathbf{y}_1+\mathbf{y}_3+\mathbf{y}_4, \mathbf{y}_2+\mathbf{y}_3+\mathbf{y}_4\}$ spans the whole space $\langle \mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \mathbf{y}_4 \rangle$. We apply this fact in the proof of Lemma 6.9.

5. The Coxeter graph and the Coxeter-like graphs

It was observed by the first author of this paper that Γ_3 is isomorphic to the well-known Coxeter graph [33]. In Fig. 1, we provide a labeling of the vertices that is similar to the one in [3] but has an additional advantage to incorporate the notion of the inverse of a matrix. Namely, we write $V(\Gamma_3) = SGL_3(\mathbb{F}_2) = \{V_i, A_i, B_i, C_i : i \in \mathbb{F}_7\}$ where $\mathbb{F}_7 = \{-3,-2,-1,0,1,2,3\}$ is the field with seven elements and

$$V_3 = \begin{pmatrix} 0&0&1\\0&1&1\\1&1&0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 1&1&1\\1&0&1\\1&1&0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0&0&1\\0&1&1\\1&1&1 \end{pmatrix}, \quad C_3 = \begin{pmatrix} 1&1&0\\1&0&0\\0&0&1 \end{pmatrix},$$

$$V_2 = \begin{pmatrix} 0&1&0\\ 1&1&1\\ 0&1&1 \end{pmatrix},\quad A_2 = \begin{pmatrix} 0&1&0\\ 1&0&1\\ 0&1&1 \end{pmatrix},\quad B_2 = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&1 \end{pmatrix},\quad C_2 = \begin{pmatrix} 1&1&0\\ 1&1&1\\ 0&1&1 \end{pmatrix},$$

$$V_1 = \begin{pmatrix} 1&0&1\\ 0&0&1\\ 1&1&1 \end{pmatrix},\quad A_1 = \begin{pmatrix} 1&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix},\quad B_1 = \begin{pmatrix} 1&0&1\\ 0&1&1\\ 1&1&1 \end{pmatrix},\quad C_1 = \begin{pmatrix} 1&0&1\\ 0&0&1\\ 1&1&0 \end{pmatrix},$$

$$V_0 = \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&1 \end{pmatrix},\quad A_0 = \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&0 \end{pmatrix},\quad B_0 = \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix},\quad C_0 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix},$$

$$V_{-1} = \begin{pmatrix} 1&1&0\\ 1&0&1\\ 0&1&0 \end{pmatrix}, A_{-1} = \begin{pmatrix} 0&0&1\\ 0&1&0\\ 1&0&1 \end{pmatrix}, B_{-1} = \begin{pmatrix} 0&1&1\\ 1&0&1\\ 1&1&1 \end{pmatrix}, C_{-1} = \begin{pmatrix} 1&1&0\\ 1&1&1\\ 0&1&0 \end{pmatrix},$$

$$V_{-2} = \begin{pmatrix} 0&1&1\\ 1&0&0\\ 1&0&1 \end{pmatrix}, A_{-2} = \begin{pmatrix} 1&1&1\\ 1&0&0\\ 1&0&1 \end{pmatrix}, B_{-2} = \begin{pmatrix} 1&0&0\\ 0&1&1\\ 0&1&0 \end{pmatrix}, C_{-2} = \begin{pmatrix} 0&1&1\\ 1&1&1\\ 1&1&0 \end{pmatrix},$$

$$V_{-3} = \begin{pmatrix} 1&1&1\\ 1&1&0\\ 1&0&0 \end{pmatrix}, A_{-3} = \begin{pmatrix} 1&1&1\\ 1&1&0\\ 1&0&1 \end{pmatrix}, B_{-3} = \begin{pmatrix} 0&1&1\\ 1&1&0\\ 1&0&0 \end{pmatrix}, C_{-3} = \begin{pmatrix} 0&1&0\\ 1&1&0\\ 0&0&1 \end{pmatrix}$$

(compare with [35, Figure 1]; see also figures in [33,34]). Observe that $E(\Gamma_3)$ consists precisely of the edges

$$\{V_i, A_i\}, \{V_i, B_i\}, \{V_i, C_i\}, \{A_i, A_{i+1}\}, \{B_i, B_{i+2}\}, \{C_i, C_{i+3}\} \tag{17}$$

as i ranges over $\mathbb{F}_7$. Moreover,

$$V_i^{-1} = V_{-i},\ A_i^{-1} = A_{-i},\ B_i^{-1} = B_{-i},\ C_i^{-1} = C_{-i}$$

for all $i \in \mathbb{F}_7$. In particular, matrices $I = V_0, A_0, B_0, C_0$ are the fixed points of the inverting map $A \mapsto A^{-1}$ on $SGL_3(\mathbb{F}_2)$.

Recall that an s-arc in a graph Γ is a sequence $(u_0, u_1, \dots, u_s)$ of vertices such that $u_i \sim_\Gamma u_{i+1}$ for all $0 \le i < s$ and $u_{j-1} \ne u_{j+1}$ for all $0 < j < s$ (cf. [17]). A weaker result than Lemma 5.1, which involves 2-arcs in Γ_n, is proved in [33].

Lemma 5.1. *Let $n \ge 3$ and let (U_0, U_1, U_2, U_3) be any 3-arc in Γ_n formed by pairwise distinct matrices. If $n \ge 4$, assume that U_0 and U_3 are not adjacent. Then (U_0, U_1, U_2, U_3) lies on a 7-cycle in Γ_n, i.e. there exist $U_{-3}, U_{-2}, U_{-1} \in SGL_n(\mathbb{F}_2)$ such that $[U_{-3}, U_{-2}, U_{-1}, U_0, U_1, U_2, U_3]$ is a 7-cycle in Γ_n.*

Proof. The Coxeter graph Γ_3 has no 4-cycles (the length of the shortest cycle in Γ_3 is actually seven). Hence, it follows from the assumptions that $\operatorname{rank}(U_0 - U_3) \in \{2, 3\}$ for all $n \ge 3$. Accordingly we separate two cases.

Case 1. Let $\operatorname{rank}(U_0 - U_3) = 3$. Since $\operatorname{rank}(U_0 - U_3)$ is odd, at most one of the matrices U_0 and U_3 is alternate. We may assume that U_0 is nonalternate (otherwise we consider the 3-arc (U_3, U_2, U_1, U_0) instead). By Lemma 3.1, there exists an invertible matrix P such that $U_0 = PP^\top$. Let $U_i' = P^{-1}U_i(P^{-1})^\top$ for $i = 0, 1, 2, 3$. Since the map $X \mapsto PXP^\top$ is an automorphism of Γ_n it suffices to prove that the 3-arc (U_0', U_1', U_2', U_3') lies on a

7-cycle in Γ_n. Since $\operatorname{rank}(U'_{i-1} - U'_i) = 1$ for all i and $\operatorname{rank}(U'_0 - U'_3) = 3$, there exist linearly independent vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{F}_2^n$ such that $U'_i = U'_{i-1} + \mathbf{x}_i^2$ for all $i = 1, 2, 3$. Since matrices

$$U'_0 = I,\ U'_1 = I + \mathbf{x}_1^2,\ U'_2 = I + \mathbf{x}_1^2 + \mathbf{x}_2^2,\ U'_3 = I + \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2$$

are all in $SGL_n(\mathbb{F}_2)$, Corollary 3.3 implies (7). By Lemma 3.4, there exist pairwise distinct $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{\mathbf{x}_1+\mathbf{x}_2, \mathbf{x}_1+\mathbf{x}_3, \mathbf{x}_2+\mathbf{x}_3, \mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3\}$ that satisfy (8). By Corollary 3.3, matrices

$$U'_{-1} := I + \mathbf{x}^2,\ U'_{-2} := I + \mathbf{x}^2 + \mathbf{y}^2,\ U'_{-3} := I + \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2$$

are all in $SGL_n(\mathbb{F}_2)$. Since $U'_{-3} - U'_3 = \mathbf{w}^2$ for $\mathbf{w} \in \{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_3, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\} \backslash \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ it follows that $\operatorname{rank}(U'_{-3} - U_3) = 1$ and (U'_0, U'_1, U'_2, U'_3) lies on the 7-cycle $[U'_{-3}, U'_{-2}, U'_{-1}, U'_0, U'_1, U'_2, U'_3]$.

Case 2. Let $\operatorname{rank}(U_0 - U_3) = 2$. Since $\operatorname{rank}(U_1 - U_2)$ is odd, at most one of the matrices U_1 and U_2 is alternate. We may assume that U_1 is nonalternate (otherwise we consider the 3-arc (U_3, U_2, U_1, U_0) instead). By Lemma 3.1, there exists an invertible matrix P such that $U_1 = PP^\top$. Let $U'_i = P^{-1} U_i (P^{-1})^\top$ for $i = 0, 1, 2, 3$, in particular,

$$U'_1 = I.$$

Since the map $X \mapsto PXP^\top$ is an automorphism of Γ_n it suffices to prove that the 3-arc (U'_0, U'_1, U'_2, U'_3) lies on a 7-cycle in Γ_n. Since $\operatorname{rank}(U'_{i-1} - U'_i) = 1$ for all i, there are nonzero vectors $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{F}_2^n$ such that $U'_i = U'_{i-1} + \mathbf{x}_{i-1}^2$ for all $i = 1, 2, 3$. Since matrices U'_0, U'_1, U'_2, U'_3 are all distinct and $\operatorname{rank}(U_0 - U_3) = 2$, it follows that $\mathbf{x}_1$, $\mathbf{x}_2$ are linearly independent, while $\mathbf{x}_0 = \mathbf{x}_1 + \mathbf{x}_2$. Since matrices

$$U'_2 = I + \mathbf{x}_1^2, \quad U'_3 = I + \mathbf{x}_1^2 + \mathbf{x}_2^2, \quad U'_0 = I + (\mathbf{x}_1 + \mathbf{x}_2)^2$$

are all invertible, we deduce from Corollary 3.3 that

$$\begin{aligned} 0 &= \mathbf{x}_1^\top \mathbf{x}_1, \\ 0 &= \mathbf{x}_2^\top (\mathbf{x}_1 + \mathbf{x}_2), \\ 0 &= (\mathbf{x}_1 + \mathbf{x}_2)^\top (\mathbf{x}_1 + \mathbf{x}_2) = \mathbf{x}_1^\top \mathbf{x}_1 + \mathbf{x}_2^\top \mathbf{x}_2. \end{aligned}$$

Hence, $0 = \mathbf{x}_1^\top \mathbf{x}_1 = \mathbf{x}_2^\top \mathbf{x}_2 = \mathbf{x}_1^\top \mathbf{x}_2$. By Lemma 3.5, there exists $\mathbf{x}_3 \in \mathbb{F}_2^n$ such that $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ are linearly independent and either (9) or (10) is true. Consequently, equalities in (7) are all satisfied. In particular, Corollary 3.3 yields

$$U'_{-3} := I + \mathbf{x}_1^2 + \mathbf{x}_2^2 + \mathbf{x}_3^2 \in SGL_n(\mathbb{F}_2).$$

Moreover, by Lemma 3.4, there exist pairwise distinct $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_3, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\}$, with $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, such that (8) is true. By Corollary 3.3, matrices

$$I + \mathbf{x}^2,\ U'_{-1} := I + \mathbf{x}^2 + \mathbf{y}^2,\ U'_{-2} := I + \mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2$$

are all in $SGL_n(\mathbb{F}_2)$. Since, $U'_0 = I + \mathbf{x}^2$ and $U'_{-2} - U'_{-3} = \mathbf{w}^2$ where $\mathbf{w} \in \{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_1 + \mathbf{x}_3, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3\} \backslash \{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$, we conclude that (U'_0, U'_1, U'_2, U'_3) is contained in the 7-cycle $[U'_{-3}, U'_{-2}, U'_{-1}, U'_0, U'_1, U'_2, U'_3]$. $\square$

Proposition 5.2. *There are 24 7-cycles in the Coxeter graph* Γ_3*, namely:*

$$[A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3], \tag{18}$$

$$[B_1, B_3, B_{-2}, B_0, B_2, B_{-3}, B_{-1}], \tag{19}$$

$$[C_{-2}, C_1, C_{-3}, C_0, C_3, C_{-1}, C_2], \tag{20}$$

$$[A_{i+1}, A_i, A_{i-1}, V_{i-1}, B_{i-1}, B_{i+1}, V_{i+1}] \quad i \in \mathbb{F}_7, \tag{21}$$

$$[B_{i+2}, B_i, B_{i-2}, V_{i-2}, C_{i-2}, C_{i+2}, V_{i+2}] \quad i \in \mathbb{F}_7, \tag{22}$$

$$[C_{i+3}, C_i, C_{i-3}, V_{i-3}, A_{i-3}, A_{i+3}, V_{i+3}] \quad i \in \mathbb{F}_7. \tag{23}$$

Proof. Clearly, the above are distinct 7-cycles in Γ_3 and there are 24 of them. Suppose now that $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$ is an arbitrary 7-cycle in Γ_3. If it contains only vertices of the form A_i, or only vertices of the form B_i, or only vertices of the form C_i, then $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$ must be one of the three 7-cycles in (18)-(20). Assume now that $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$ contains a vertex V_j for some $j \in \mathbb{F}_7$. We may assume that $X_6 = V_j$. Then, $\{X_0, X_5\} \in \{\{A_j, B_j\}, \{B_j, C_j\}, \{C_j, A_j\}\}$. If necessary, we rewrite the 7-cycle in the reversed way as $[X_5, X_4, X_3, X_2, X_1, X_0, X_6]$ to achieve that $(X_0, X_5) \in \{(A_j, B_j), (B_j, C_j), (C_j, A_j)\}$. We separate the three cases accordingly.

Case 1. Let $X_0 = A_j$ and $X_5 = B_j$. By (17),

$$X_1 \in \{A_{j-1}, A_{j+1}\} \quad \text{and} \quad X_4 \in \{B_{j-2}, B_{j+2}\}.$$

Hence, $\{X_2, X_3\} \in \{\{V_k, A_k\}, \{V_k, B_k\}\}$ for some $k \in \mathbb{F}_7$. In the latter case, $X_3 \in \{B_{j+3}, B_{j-3}\}$ and $X_2 \in \{V_{j+3}, V_{j-3}\}$, which is not possible because X_2 and X_1 must be adjacent. Therefore, $X_3 \in \{V_{j-2}, V_{j+2}\}$ and $X_2 \in \{A_{j-2}, A_{j+2}\}$, which results in two possibilities for the 7-cycle $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$:

$$[A_j, A_{j-1}, A_{j-2}, V_{j-2}, B_{j-2}, B_j, V_j] \quad \text{and} \quad [A_j, A_{j+1}, A_{j+2}, V_{j+2}, B_{j+2}, B_j, V_j].$$

The 7-cycle on the left-hand side equals (21) for $j = i + 1$, while the 7-cycle on the right-hand side can be rewritten in the reversed way as (21) for $j = i - 1$.

Case 2. Let $X_0 = B_j$ and $X_5 = C_j$. We proceed as in Case 1, to obtain two possibilities for $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$:

$$[B_j, B_{j-2}, B_{j+3}, V_{j+3}, C_{j+3}, C_j, V_j] \quad \text{and} \quad [B_j, B_{j+2}, B_{j-3}, V_{j-3}, C_{j-3}, C_j, V_j].$$

These are 7-cycles of the form (22) for $j = i + 2$ and $j = i - 2$, respectively.

Case 3. Let $X_0 = C_j$ and $X_5 = A_j$. We proceed as in Case 1, to obtain two possibilities for $[X_0, X_1, X_2, X_3, X_4, X_5, X_6]$:

$$[C_j, C_{j-3}, C_{j+1}, V_{j+1}, A_{j+1}, A_j, V_j] \quad \text{and} \quad [C_j, C_{j+3}, C_{j-1}, V_{j-1}, A_{j-1}, A_j, V_j].$$

These are 7-cycles of the form (23) for $j = i + 3$ and $j = i - 3$, respectively. □

It is well known that the Coxeter graph Γ_3 is *3-arc transitive*, i.e. for each 3-arcs (X_0, X_1, X_2, X_3) and (Y_0, Y_1, Y_2, Y_3) in Γ_3 there exists an automorphism Ψ of Γ_3 such that $\Psi(X_i) = Y_i$ for $i = 0, 1, 2, 3$. Actually, it is even *3-arc regular* as mentioned in [17, Section 4.6], which means that for the 3-arcs (X_0, X_1, X_2, X_3) and (Y_0, Y_1, Y_2, Y_3) the automorphism Ψ with such property is unique. In the proof of Lemma 5.4, we rely on 3-arc transitivity. We provide an additional proof of this property for reader's convenience. For this purpose recall that given a matrix $P \in GL_3(\mathbb{F}_2)$, the maps $X \mapsto PXP^\top$ and $X \mapsto X^{-1}$ are both automorphisms of Γ_3.[2]

Lemma 5.3. *The Coxeter graph Γ_3 is 3-arc transitive.*

Proof. Let (U_0, U_1, U_2, U_3) be a 3-arc in Γ_3. Since U_0 is a 3×3 invertible symmetric matrix, it is not alternate. By Lemma 3.1, there exists $P \in GL_3(\mathbb{F}_2)$ with $PU_0P^\top = I$. Since the map $\Psi_0 : X \mapsto PXP^\top$ is an automorphism of Γ_3, we can replace matrices U_i by $\Psi_0(U_i) = PU_iP^\top$. Hence, we may assume without loss of generality that $U_0 = I = V_0$. From Fig. 1 we deduce that $U_1 = A_0$ or $U_1 = B_0$ or $U_1 = C_0$. Define

$$Q = I \quad \text{or} \quad Q = C_0 \quad \text{or} \quad Q = B_0,$$

respectively. Since the map $\Psi_1 : X \mapsto QXQ^\top$ is an automorphism of Γ_3 such that $\Psi_1(V_0) = V_0$ and $\Psi_1(U_1) = A_0$, we may further assume without loss of generality that $U_1 = A_0$. In this case, $U_2 = A_1$ or $U_2 = A_{-1}$. Let the automorphism Ψ_2 be the identity map or the inverting map $X \mapsto X^{-1}$, respectively. Since $\Psi_2(V_0) = V_0$, $\Psi_2(A_0) = A_0$, and $\Psi_2(U_2) = A_1$ we may further assume that $U_2 = A_1$. Then, $U_3 = A_2$ or $U_3 = V_1$. Let the automorphism Ψ_3 be the identity map or the map $X \mapsto A_0X^{-1}A_0^\top$, respectively. Since $\Psi_3(V_0) = V_0$, $\Psi_3(A_0) = A_0$, $\Psi_3(A_1) = A_1$, $\Psi_3(U_3) = A_2$, we see that each 3-arc can be mapped to the 3-arc (V_0, A_0, A_1, A_2) by an automorphism of Γ_3. Hence, the Coxeter graph is 3-arc transitive. □

Lemma 5.4. *In the Coxeter graph Γ_3,*

(i) *each 3-arc lies on the unique 7-cycle;*

[2] Actually, maps of these forms generate the whole automorphism group of Γ_3 as proved in [33] by applying the counting argument.

(ii) *for each pair of 7-cycles* $\mathcal{C}_1$ *and* $\mathcal{C}_2$ *there exists an automorphism* Ψ *of* Γ_3 *such that* $\Psi(\mathcal{C}_1) = \mathcal{C}_2$*.*

Proof. (i) Let (U_0, U_1, U_2, U_3) be a 3-arc in Γ_3. By Lemma 5.3, there exists an automorphism Ψ of Γ_3 such that $\Psi(A_i) = U_i$ for $i = 0, 1, 2, 3$. By Proposition 5.2, the 3-arc (A_0, A_1, A_2, A_3) lies on the unique 7-cycle, namely (18). Its Ψ-image is the unique 7-cycle that contains (U_0, U_1, U_2, U_3).

(ii) Pick any 3-arcs (W_0, W_1, W_2, W_3) and (U_0, U_1, U_2, U_3) in $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. By Lemma 5.3, there exists an automorphism Ψ of Γ_3 such that $\Psi(W_i) = U_i$ for $i = 0, 1, 2, 3$. By (i), $\Psi(\mathcal{C}_1) = \mathcal{C}_2$. □

Remark 5.5. Observe that the number of 3-arcs in each 7-cycle in the Coxeter graph equals $2 \cdot 7 = 14$ if we consider both directions of the 7-cycle. By Lemma 5.4, each 3-arc lies on the unique 7-cycle. Hence, Proposition 5.2 implies that the number of 3-arcs in Γ_3 equals $24 \cdot 14 = 336$. There are also exactly 336 automorphisms of the Coxeter graph (cf. [33]). Consequently, Lemma 5.3 provides an additional proof of the fact that the Coxeter graph is 3-arc regular.

6. Proof of Theorem 2.1

Lemma 6.1 and Corollary 6.2 represent the first step towards the characterization of all graph homomorphisms $\Phi : \Gamma_3 \to \widehat{\Gamma}_m$. They have a potential of being helpful also in finding solutions to other preserver problems.

Lemma 6.1. *Let* Λ_1 *and* Λ_2 *be two (possibly infinite) simple graphs and* $\Phi : \Lambda_1 \to \Lambda_2$ *a homomorphism. If* $n \geq 1$ *and* $[v_1, \ldots, v_{2n+1}]$ *is an odd cycle in* Λ_1*, then there exist* $1 \leq k \leq n$ *and distinct* $i_1, \ldots, i_{2k+1} \in \{1, \ldots, 2n+1\}$ *such that* $[\Phi(v_{i_1}), \ldots, \Phi(v_{i_{2k+1}})]$ *is an odd cycle in* Λ_2*.*

Proof. Let Λ_1' and Λ_2' be the (finite) subgraphs in Λ_1 and Λ_2, respectively, which are induced by the sets $\{v_1, \ldots, v_{2n+1}\}$ and $\{\Phi(v_1), \ldots, \Phi(v_{2n+1})\}$, respectively. Then the restriction of the map Φ to the set $\{v_1, \ldots, v_{2n+1}\}$ is a homomorphism between Λ_1' and Λ_2'. Since Λ_1' contains an odd cycle, its chromatic number $\chi(\Lambda_1')$ is at least 3. By the well known inequality (cf. [18, Proposition 2.10]), we deduce that $3 \leq \chi(\Lambda_1') \leq \chi(\Lambda_2')$. Hence, Λ_2' is not bipartite, which means that it posses an odd cycle (cf. [18, Proposition 1.4]). □

Corollary 6.2 follows directly from Lemma 6.1.

Corollary 6.2. *Let* Λ_1 *and* Λ_2 *be two (possibly infinite) simple graphs with the same odd girth* g*. If* $\Phi : \Lambda_1 \to \Lambda_2$ *is a homomorphism, then it maps* g*-cycles to* g*-cycles.*

Corollary 6.3. *Let* $n, m \geq 3$*. If* $\Phi : \Gamma_n \to \widehat{\Gamma}_m$ *is a homomorphism, then it maps 7-cycles to 7-cycles.*

Proof. Since the odd girths of Γ_n and $\widehat{\Gamma}_m$ both equal seven, the claim follows from Corollary 6.2. □

For an edge $\{X, Y\}$ in Γ_n or $\widehat{\Gamma}_m$ we will slightly abuse the terminology in the proofs. Namely, we will refer as an 'edge' also the corresponding rank-one matrix $X - Y = Y - X$. By Lemma 4.3, all edges in a 7-cycle are distinct.

Lemma 6.4. *Let $\{N_i : i \in \mathbb{F}_7\} = \mathcal{R}_1$ be any enumeration of rank-one matrices in $S_3(\mathbb{F}_2)$. Then there are exactly two homomorphisms $\Phi : \Gamma_3 \to \widehat{\Gamma}_3$ that satisfy $\Phi(I) = I$ and $\Phi(V_i) - \Phi(A_i) = N_i$ for all $i \in \mathbb{F}_7$. Namely, they are given by*

$$\Phi_1(X) = \Psi_{\sigma_1}(X) - \Psi_{\sigma_1}(I) + I \quad \text{and} \quad \Phi_2(X) = \Psi_{\sigma_2}(X^{-1}) - \Psi_{\sigma_2}(I) + I \tag{24}$$

for all $X \in SGL_3(\mathbb{F}_2)$ where σ_1, σ_2 are the unique suitable permutations of $\mathcal{R}_1$.

Proof. Observe from the definitions of A_i and V_i that $\{V_i - A_i : i \in \mathbb{F}_7\} = \mathcal{R}_1$. If σ_1 is the permutation of $\mathcal{R}_1$ that maps $V_i - A_i$ to N_i for all $i \in \mathbb{F}_7$, then

$$\Phi_1(V_i) - \Phi_1(A_i) = \Psi_{\sigma_1}(V_i) - \Psi_{\sigma_1}(A_i) = \Psi_{\sigma_1}(V_i - A_i) = N_i \qquad (i \in \mathbb{F}_7).$$

Likewise, if σ_2 is the permutation that maps $V_i - A_i$ to N_{-i} for all $i \in \mathbb{F}_7$, then

$$\Phi_2(V_i) - \Phi_2(A_i) = \Psi_{\sigma_2}(V_i^{-1} - A_i^{-1}) = \Psi_{\sigma_2}(V_{-i} - A_{-i}) = N_i \qquad (i \in \mathbb{F}_7).$$

Clearly, both maps Φ_1, Φ_2 are homomorphisms and fix the identity. It remains to prove that there exist at most two homomorphisms Φ from the claim.

Let $\Phi : \Gamma_3 \to \widehat{\Gamma}_3$ be a homomorphism with $\Phi(I) = I$ and $\Phi(V_i) - \Phi(A_i) = N_i$ for all $i \in \mathbb{F}_7$. In particular, the images

$$\Phi(V_0) = V_0 \quad \text{and} \quad \Phi(A_0) = \Phi(V_0) + N_0 \tag{25}$$

are determined. For $i \in \mathbb{F}_7$ observe that each of the 3-arcs (A_{i+1}, A_i, V_i, B_i), $(A_{i+1}, A_i, A_{i-1}, V_{i-1})$, $(B_{i+1}, V_{i+1}, A_{i+1}, A_i)$, $(V_{i+2}, A_{i+2}, A_{i+1}, A_i)$ contain the edge $A_{i+1} - A_i$. By Lemma 5.1, each of them lies on a 7-cycle, which is mapped into another 7-cycle by Corollary 6.3. Hence, all three edges of each 3-arc are mapped into three distinct edges. In particular,

$$\Phi(A_{i+1}) - \Phi(A_i) \neq N_i, N_{i-1}, N_{i+1}, N_{i+2}.$$

Consequently,

$$\Phi(A_{i+1}) - \Phi(A_i) \in \{N_{i-2}, N_{i+3}, N_{i-3}\}. \tag{26}$$

We claim that

$$\Phi(A_{i+1}) - \Phi(A_i) \neq N_{i-3}$$

for all $i \in \mathbb{F}_7$.

Suppose that $\Phi(A_{j+1}) - \Phi(A_j) = N_{j-3}$ for some j. Since the image of $[A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3]$ is a 7-cycle, we deduce that

$$\Phi(A_{j-1}) - \Phi(A_j) \neq N_{j-3},$$

which by (26) implies that

$$\Phi(A_{j-1}) - \Phi(A_j) =: N \in \{N_{j+2}, N_{j+3}\}. \tag{27}$$

Since $(A_{j+3}, A_{j+2}, A_{j+1}, A_j)$ is a 3-arc, it follows as above that matrices $\Phi(A_{j+1}) - \Phi(A_j), \Phi(A_{j+2}) - \Phi(A_{j+1}), \Phi(A_{j+3}) - \Phi(A_{j+2})$ are all distinct. By (26),

$$\begin{aligned}
&\Phi(A_{j+2}) - \Phi(A_{j+1}) \in \{N_{j-1}, N_{j-2}\},\\
&\Phi(A_{j+3}) - \Phi(A_{j+2}) \in \{N_j, N_{j-1}, N_{j-2}\}.
\end{aligned}$$

If $\{\Phi(A_{j+2}) - \Phi(A_{j+1}), \Phi(A_{j+3}) - \Phi(A_{j+2})\} = \{N_{j-1}, N_{j-2}\}$, then from the image of the 8-cycle $[V_{j+3}, A_{j+3}, A_{j+2}, A_{j+1}, A_j, V_j, C_j, C_{j+3}]$ we deduce that

$$\begin{aligned}
&N_{j+3} + N_{j-1} + N_{j-2} + N_{j-3} + N_j + \big(\Phi(V_j) - \Phi(C_j)\big)\\
&+\big(\Phi(C_j) - \Phi(C_{j+3})\big) + \big(\Phi(C_{j+3}) - \Phi(V_{j+3})\big) = 0,
\end{aligned}$$

which contradicts Lemma 4.6 because matrices $N_{j+3}, N_{j-1}, N_{j-2}, N_{j-3}, N_j$ are all distinct. Hence,

$$\Phi(A_{j+3}) - \Phi(A_{j+2}) = N_j.$$

Since $[A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3]$ is mapped into a 7-cycle, we deduce that

$$\Phi(A_{j-3}) - \Phi(A_{j-2}) \neq N_j,$$

which by (26) implies that

$$\Phi(A_{j-3}) - \Phi(A_{j-2}) =: \dot{N} \in \{N_{j+1}, N_{j+2}\}. \tag{28}$$

From the image of the 8-cycle $[V_{j-3}, A_{j-3}, A_{j-2}, A_{j-1}, A_j, V_j, C_j, C_{j-3}]$ we get

$$\begin{aligned}
&N_{j-3} + \dot{N} + \ddot{N} + N + N_j + \big(\Phi(V_j) - \Phi(C_j)\big)\\
&+\big(\Phi(C_j) - \Phi(C_{j-3})\big) + \big(\Phi(C_{j-3}) - \Phi(V_{j-3})\big) = 0
\end{aligned} \tag{29}$$

where

$$\ddot{N} := \Phi(A_{j-2}) - \Phi(A_{j-1}) \in \{N_{j+1}, N_{j+2}, N_{j+3}\} \tag{30}$$

by (26). Since $(A_{j-3}, A_{j-2}, A_{j-1}, A_j)$ is a 3-arc, we deduce as above that rank-one matrices $N, \dot{N}, \ddot{N}$ are all distinct. Hence, it follows from (27), (28), (30) that matrices $N_{j-3}, \dot{N}, \ddot{N}, N, N_j$ are all distinct, and (29) contradicts Lemma 4.6.

We proved that $\Phi(A_{i+1}) - \Phi(A_i) \neq N_{i-3}$ for all $i \in \mathbb{F}_7$. By (26),

$$\Phi(A_{i+1}) - \Phi(A_i) \in \{N_{i-2}, N_{i+3}\} \qquad (i \in \mathbb{F}_7)$$

that is

$$\begin{aligned}
\Phi(A_1) &- \Phi(A_0) \in \{N_{-2},\ N_3\},\\
\Phi(A_3) &- \Phi(A_2) \in \{N_0,\ \ N_{-2}\},\\
\Phi(A_{-2}) &- \Phi(A_{-3}) \in \{N_2,\ \ N_0\},\\
\Phi(A_0) &- \Phi(A_{-1}) \in \{N_{-3},\ N_2\},\\
\Phi(A_2) &- \Phi(A_1) \in \{N_{-1},\ N_{-3}\},\\
\Phi(A_{-3}) &- \Phi(A_3) \in \{N_1,\ \ N_{-1}\},\\
\Phi(A_{-1}) &- \Phi(A_{-2}) \in \{N_3,\ \ N_1\}.
\end{aligned}$$

Since $\big[\Phi(A_{-3}), \Phi(A_{-2}), \Phi(A_{-1}), \Phi(A_0), \Phi(A_1), \Phi(A_2), \Phi(A_3)\big]$ is a 7-cycle, it follows that matrices $\Phi(A_{i+1}) - \Phi(A_i)$ with $i \in \mathbb{F}_7$ are all distinct. Hence, there are only two possibilities for values $\Phi(A_i)$ with $i \neq 0$. Either

$$\Phi(A_i) = \Phi(A_{i-1}) + N_{i-3} \qquad (i \in \mathbb{F}_7) \tag{31}$$

or

$$\Phi(A_i) = \Phi(A_{i-1}) + N_{i+2} \qquad (i \in \mathbb{F}_7). \tag{32}$$

Recall that values $\Phi(A_0)$, $\Phi(V_0)$ are already determined in (25). The values $\Phi(A_i)$ for $i \neq 0$ are determined either by (31) or (32). To complete the proof it suffices to show that values $\Phi(V_i)$ for $i \neq 0$ and $\Phi(B_i), \Phi(C_i)$ for all $i \in \mathbb{F}_7$ are uniquely determined by values $\Phi(V_0), \Phi(A_i)$, $i \in \mathbb{F}_7$.

Firstly recall from the assumption that

$$\Phi(V_i) = \Phi(A_i) + N_i. \tag{33}$$

Choose arbitrary $i \in \mathbb{F}_7$. Since the edges of each 7-cycles

$$\big[\Phi(V_i), \Phi(B_i), \Phi(B_{i+2}), \Phi(V_{i+2}), \Phi(A_{i+2}), \Phi(A_{i+1}), \Phi(A_i)\big]$$

and

$$\big[\Phi(V_i), \Phi(B_i), \Phi(B_{i-2}), \Phi(V_{i-2}), \Phi(A_{i-2}), \Phi(A_{i-1}), \Phi(A_i)\big] \tag{34}$$

are all distinct, it follows that

$$\begin{aligned}\Phi(B_i) - \Phi(V_i) \in \mathcal{R}_1 \backslash \big\{&N_{i+2}, N_{i-2}, N_i, \Phi(A_{i+2}) - \Phi(A_{i+1}),\\ &\Phi(A_{i+1}) - \Phi(A_i), \Phi(A_i) - \Phi(A_{i-1}), \Phi(A_{i-1}) - \Phi(A_{i-2})\big\}.\end{aligned}$$

In the cases (31) and (32) we deduce that

$$\Phi(B_i) = \Phi(V_i) + N_{i+1} \qquad (i \in \mathbb{F}_7) \tag{35}$$

and

$$\Phi(B_i) = \Phi(V_i) + N_{i-1} \qquad (i \in \mathbb{F}_7), \tag{36}$$

respectively. Moreover, in cases (31), (35) and (32), (36) we deduce from the 7-cycle (34) that

$$\Phi(B_i) - \Phi(B_{i-2}) \in \mathcal{R}_1 \backslash \{N_{i+1}, N_{i-1}, N_{i-2}, N_{i+3}, N_{i-3}, N_i\}$$

and

$$\Phi(B_i) - \Phi(B_{i-2}) \in \mathcal{R}_1 \backslash \{N_{i-1}, N_{i-3}, N_{i-2}, N_{i+1}, N_{i+2}, N_i\},$$

respectively. That is,

$$\Phi(B_i) - \Phi(B_{i-2}) = N_{i+2} \qquad (i \in \mathbb{F}_7) \tag{37}$$

and

$$\Phi(B_i) - \Phi(B_{i-2}) = N_{i+3} \qquad (i \in \mathbb{F}_7), \tag{38}$$

respectively. Finally, since edges in each of the 7-cycles

$$\big[\Phi(V_i), \Phi(C_i), \Phi(C_{i+3}), \Phi(V_{i+3}), \Phi(B_{i+3}), \Phi(B_{i-2}), \Phi(B_i)\big]$$

and

$$\big[\Phi(V_i), \Phi(C_i), \Phi(C_{i-3}), \Phi(V_{i-3}), \Phi(B_{i-3}), \Phi(B_{i+2}), \Phi(B_i)\big]$$

are distinct, we deduce that

$$\Phi(C_i) - \Phi(V_i) \in \mathcal{R}_1 \backslash \big\{ \Phi(B_{i+3}) - \Phi(V_{i+3}), \Phi(B_{i-3}) - \Phi(V_{i-3}),$$
$$\Phi(B_i) - \Phi(V_i), \Phi(B_{i-2}) - \Phi(B_{i+3}), \Phi(B_i) - \Phi(B_{i-2}),$$
$$\Phi(B_{i-3}) - \Phi(B_{i+2}), \Phi(B_{i+2}) - \Phi(B_i) \big\}.$$

That is,

$$\Phi(C_i) = \Phi(V_i) + N_{i+3} \qquad (i \in \mathbb{F}_7) \tag{39}$$

if (31), (35), (37), and

$$\Phi(C_i) = \Phi(V_i) + N_{i-3} \qquad (i \in \mathbb{F}_7) \tag{40}$$

if (32), (36), (38). To summarize, values $\Phi(V_0), \Phi(A_0)$ are determined in (25). Values $\Phi(A_i)$ with $i \neq 0$ are determined in (31) or (32). If (31), then values $\Phi(V_i), \Phi(B_i), \Phi(C_i)$ are determined in (33), (35), (39). If (32), then values $\Phi(V_i), \Phi(B_i), \Phi(C_i)$ are determined in (33), (36), (40). □

Corollary 6.5. *A graph homomorphism* $\Phi : \Gamma_3 \to \widehat{\Gamma}_3$ *is either of the form*

$$\Phi(X) = \Psi_\sigma(X) + Y \qquad (X \in SGL_3(\mathbb{F}_2)) \tag{41}$$

or

$$\Phi(X) = \Psi_\sigma(X^{-1}) + Y \qquad (X \in SGL_3(\mathbb{F}_2)) \tag{42}$$

for some $Y \in S_3(\mathbb{F}_3)$ *and some permutation* σ *of the set* $\mathcal{R}_1$*.*

Proof. Obviously, the maps (41) and (42) are both graph homomorphisms. Conversely, let $\Phi : \Gamma_3 \to \widehat{\Gamma}_3$ be any homomorphism. Then the map, defined by $\Phi'(X) := \Phi(X) - \Phi(I) + I$ for all $X \in SGL_3(\mathbb{F}_2)$, is a homomorphism as well. Moreover, $\Phi'(I) = I$. To complete the proof it suffices to show that

$$\{\Phi'(A_i) - \Phi'(V_i) : i \in \mathbb{F}_7\} = \mathcal{R}_1. \tag{43}$$

Namely, then Lemma 6.4 implies that Φ' fits one of the two forms in (24) and therefore Φ is either of the form (41) or (42) where $\sigma \in \{\sigma_1, \sigma_2\}$ and $Y \in \{\Phi(I) - \Psi_{\sigma_1}(I), \Phi(I) - \Psi_{\sigma_2}(I)\}$.

Recall by Lemma 4.3 that the rank-one matrices, which define the edges in a 7-cycle, are all distinct. If $i \in \mathbb{F}_7$ is arbitrary, then

$$\big[\Phi'(V_i), \Phi'(A_i), \Phi'(A_{i+1}), \Phi'(V_{i+1}), \Phi'(C_{i+1}), \Phi'(C_{i-3}), \Phi'(C_i)\big], \tag{44}$$
$$\big[\Phi'(V_i), \Phi'(A_i), \Phi'(A_{i+1}), \Phi'(A_{i+2}), \Phi'(V_{i+2}), \Phi'(B_{i+2}), \Phi'(B_i)\big] \tag{45}$$

are both 7-cycles by Corollary 6.3. In particular,

$$\Phi'(A_i) - \Phi'(V_i) \notin \{\Phi'(A_{i+1}) - \Phi'(V_{i+1}), \Phi'(A_{i+2}) - \Phi'(V_{i+2})\} \qquad (i \in \mathbb{F}_7). \tag{46}$$

Suppose that $\Phi'(A_j) - \Phi'(V_j) = \Phi'(A_{j+3}) - \Phi'(V_{j+3}) =: M_1$ for some $j \in \mathbb{F}_7$. Let $M_2 := \Phi'(A_j) - \Phi'(A_{j+1})$, $M_3 := \Phi'(A_{j+1}) - \Phi'(A_{j+2})$, $M_4 := \Phi'(A_{j+2}) - \Phi'(A_{j+3})$. From the 7-cycle (45) with $i = j$ we deduce that $M_1 \neq M_2, M_3$. From the 7-cycle (44) with $i = j + 2$ we deduce that $M_1 \neq M_4$. Since

$$\big[\Phi'(A_{-3}), \Phi'(A_{-2}), \Phi'(A_{-1}), \Phi'(A_0), \Phi'(A_1), \Phi'(A_2), \Phi'(A_3)\big]$$

is a 7-cycle, we deduce that rank-one matrices M_1, M_2, M_3, M_4 are all distinct. Let $\{M_5, M_6, M_7\} := \mathcal{R}_1 \backslash \{M_1, M_2, M_3, M_4\}$. Consider the image of the 9-cycle

$$[A_j, A_{j+1}, A_{j+2}, A_{j+3}, V_{j+3}, B_{j+3}, B_{j-2}, B_j, V_j].$$

Since $\Phi'(A_j) - \Phi'(V_j) = \Phi'(A_{j+3}) - \Phi'(V_{j+3})$, the edges in the image produce a vanishing sum of seven rank-one matrices. From Lemma 4.2 and the definition of matrices M_2, M_3, M_4 it follows that

$$\begin{aligned}\big\{&\Phi'(B_j) - \Phi'(V_j), \Phi'(B_{j+3}) - \Phi'(V_{j+3}),\\ &\Phi'(B_j) - \Phi'(B_{j-2}), \Phi'(B_{j+3}) - \Phi'(B_{j-2})\big\} = \{M_1, M_5, M_6, M_7\}.\end{aligned} \tag{47}$$

Since (A_j, V_j, B_j) and $(A_{j+3}, V_{j+3}, B_{j+3})$ are both 2-arcs that are contained in some 3-arcs, and therefore in some 7-cycles by Lemma 5.1, it follows that $\Phi'(B_j) \neq \Phi'(A_j)$ and $\Phi'(B_{j+3}) \neq \Phi'(A_{j+3})$. Consequently, (47) and the definition of matrix M_1 imply that

$$M_1 \in \big\{\Phi'(B_j) - \Phi'(B_{j-2}), \Phi'(B_{j+3}) - \Phi'(B_{j-2})\big\}.$$

However, the 7-cycle (45) with $i = j - 2$ and $i = j + 3$ imply that

$$M_1 \notin \big\{\Phi'(B_j) - \Phi'(B_{j-2}), \Phi'(B_{j+3}) - \Phi'(B_{j-2})\big\}$$

a contradiction. Therefore

$$\Phi'(A_i) - \Phi'(V_i) \neq \Phi'(A_{i+3}) - \Phi'(V_{i+3}) \qquad (i \in \mathbb{F}_7). \tag{48}$$

Since $i \in \mathbb{F}_7$ is arbitrary in both (46) and (48), it follows that the seven rank-one matrices $\Phi'(A_i) - \Phi'(V_i); i \in \mathbb{F}_7$ are all distinct, which proves (43). □

Suppose that $\Phi : \Gamma_3 \to \widehat{\Gamma}_m$ is a graph homomorphism where $m \geq 3$. By Corollary 6.3, each 7-cycle in the Coxeter graph Γ_3 is mapped into a seven cycle in $\widehat{\Gamma}_m$. By Proposition 5.2,

$$[\Phi(A_{-3}), \Phi(A_{-2}), \Phi(A_{-1}), \Phi(A_0), \Phi(A_1), \Phi(A_2), \Phi(A_3)], \tag{49}$$
$$[\Phi(B_1), \Phi(B_3), \Phi(B_{-2}), \Phi(B_0), \Phi(B_2), \Phi(B_{-3}), \Phi(B_{-1})], \tag{50}$$
$$[\Phi(C_{-2}), \Phi(C_1), \Phi(C_{-3}), \Phi(C_0), \Phi(C_3), \Phi(C_{-1}), \Phi(C_2)], \tag{51}$$
$$[\Phi(A_{i+1}), \Phi(A_i), \Phi(A_{i-1}), \Phi(V_{i-1}), \Phi(B_{i-1}), \Phi(B_{i+1}), \Phi(V_{i+1})] \quad i \in \mathbb{F}_7, \tag{52}$$
$$[\Phi(B_{i+2}), \Phi(B_i), \Phi(B_{i-2}), \Phi(V_{i-2}), \Phi(C_{i-2}), \Phi(C_{i+2}), \Phi(V_{i+2})] \quad i \in \mathbb{F}_7, \tag{53}$$
$$[\Phi(C_{i+3}), \Phi(C_i), \Phi(C_{i-3}), \Phi(V_{i-3}), \Phi(A_{i-3}), \Phi(A_{i+3}), \Phi(V_{i+3})] \quad i \in \mathbb{F}_7 \tag{54}$$

are 7-cycles in $\widehat{\Gamma}_m$. By Lemma 4.3, the seven edges in each of them (i.e. the differences of consecutive vertices) are of the form (15) where $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \in \mathbb{F}_2^m$ are linearly independent. Hence, we may associate a 3-dimensional vector subspace in $\mathbb{F}_2^m$, which is of the form $\langle \mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3 \rangle$, to each 7-cycle in (49)-(54). We denote these vector space by $W_A(\Phi), W_B(\Phi), W_C(\Phi), W_{AB}^i(\Phi), W_{BC}^i(\Phi), W_{CA}^i(\Phi)$, respectively. If the definition of Φ is clear from the context, we simply write

$$W_A, W_B, W_C, W_{AB}^i, W_{BC}^i, W_{CA}^i.$$

Recall that a 3-dimensional subspace in $\mathbb{F}_2^m$ has seven nonzero elements, one for each edge of a 7-cycle. Moreover, it is spanned by any selection of its 4 distinct nonzero vectors. Given any vector subspace W in $\mathbb{F}_2^m$, we denote by $W^{(2)} = \{\mathbf{w}^2 : \mathbf{w} \in W\}$ the corresponding set of rank-one matrices. Conversely, given a rank one matrix $M = \mathbf{w}^2$ we define $M^{1/2} := \mathbf{w}$.

Lemma 6.6. *Let $m \geq 3$ be an integer and $\Phi : \Gamma_3 \to \widehat{\Gamma}_m$ a graph homomorphism. Then there exists a vector subspace W in $\mathbb{F}_2^m$ of dimension at most 4 such that $\Phi(X) - \Phi(Y) \in W^{(2)}$ whenever $X, Y \in SGL_3(\mathbb{F}_2)$ satisfy* $\operatorname{rank}(X - Y) = 1$.

Proof. Let $W_A = \langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle$ for some linearly independent $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}_2^m$. Further denote $\Phi(A_i) - \Phi(A_{i-1}) = \mathbf{z}_i^2$ for all $i \in \mathbb{F}_7$. Then, $\langle \mathbf{x}, \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{z}_i : i \in \mathbb{F}_7 \rangle$. Since the 7-cycles (49) and (52) with $i = 0$ both contain edges $\Phi(A_1) - \Phi(A_0)$ and $\Phi(A_0) - \Phi(A_{-1})$, it follows that $W_{AB}^0 = \langle \mathbf{z}_0, \mathbf{z}_1, \mathbf{w} \rangle$ for some $\mathbf{w} \in \mathbb{F}_2^m$. We set $W = \langle \mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{w} \rangle$. Clearly, $\dim W \leq 4$.

Observe from Fig. 1 that the Φ-image of each edge $\{X, Y\}$ in Γ_3 is contained in at least one of the 7-cycles that are associated to the vector spaces

$$W_{AB}^i \quad (i \in \mathbb{F}_7), \tag{55}$$
$$W_{BC}^i \quad (i \in \mathbb{F}_7).$$

We need to show that all of them are contained in W. Obviously, $W_{AB}^0 \subseteq W$. Given $i \in \mathbb{F}_7$, consider the subspace W_{AB}^{i+2} and its associated 7-cycle. It shares two edges, $\Phi(A_{i+1}) - \Phi(V_{i+1}) := \mathbf{u}_i^2$ and $\Phi(V_{i+1}) - \Phi(B_{i+1}) := \mathbf{v}_i^2$, with the 7-cycle associated to

W_{AB}^i. By the recursion, we may assume that $\mathbf{u}_i, \mathbf{w}_i \in W$. Since $\mathbf{z}_{i+2}, \mathbf{z}_{i+3} \in W$ and $\mathbf{u}_i, \mathbf{w}_i, \mathbf{z}_{i+2}, \mathbf{z}_{i+3}$ correspond to four distinct edges of the 7-cycle associated to W_{AB}^{i+2}, it follows that $W_{AB}^{i+2} \subseteq W$. Since 7 and 2 are coprime, it follows that all subspaces in (55) are contained in W. In particular, $W_B \subseteq W$. For general $i \in \mathbb{F}_7$, the 7-cycle associated to W_{BC}^i has four edges

$$\Phi(B_{i+2}) - \Phi(B_i), \Phi(B_i) - \Phi(B_{i-2}), \Phi(B_{i-2}) - \Phi(V_{i-2}), \Phi(V_{i+2}) - \Phi(B_{i+2}) \tag{56}$$

contained already in the 7-cycles that are associated to the spaces in (55). Consequently, matrices (56) are in $W^{(2)}$ and therefore $W_{BC}^i \subseteq W$. □

The dimension 4 in Lemma 6.6 actually drops to 3 as we eventually prove in Lemma 6.9. For the proof we require two additional lemmas.

Lemma 6.7. *Let W be a vector space over $\mathbb{F}_2$ of dimension three, and let*

$$W \backslash \{0\} = \{\mathbf{z}_{-3}, \mathbf{z}_{-2}, \mathbf{z}_{-1}, \mathbf{z}_0, \mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3\}$$

be any enumeration of nonzero vectors. Then, there exists $i \in \mathbb{F}_7$ such that

$$\langle \mathbf{z}_i, \mathbf{z}_{i-1} \rangle \cap \langle \mathbf{z}_{i-2}, \mathbf{z}_{i-3} \rangle \cap \langle \mathbf{z}_{i-4}, \mathbf{z}_{i-5} \rangle = \{0\}. \tag{57}$$

Proof. Suppose the claim is false. Consider the vector spaces

$$\begin{aligned} W_1 &= \langle \mathbf{z}_3, \mathbf{z}_2 \rangle, & W_2 &= \langle \mathbf{z}_1, \mathbf{z}_0 \rangle, & W_3 &= \langle \mathbf{z}_{-1}, \mathbf{z}_{-2} \rangle, \\ W_4 &= \langle \mathbf{z}_{-3}, \mathbf{z}_3 \rangle, & W_5 &= \langle \mathbf{z}_2, \mathbf{z}_1 \rangle, & W_6 &= \langle \mathbf{z}_0, \mathbf{z}_{-1} \rangle. \end{aligned}$$

They are all of dimension two, and W_j, W_{j+1}, W_{j+2} are pairwise distinct for $j \in \{1, 2, 3, 4\}$. Since the vector space $W_j \cap W_{j+1} \cap W_{j+2}$ is not trivial by the assumption, it follows that

$$\begin{gathered} \dim(W_j \cap W_{j+1} \cap W_{j+2}) = 1 \\ \text{and} \\ W_j \cap W_{j+1} \cap W_{j+2} = W_j \cap W_{j+1} = W_j \cap W_{j+2} = W_{j+1} \cap W_{j+2} \end{gathered} \tag{58}$$

for all $j \in \{1, 2, 3, 4\}$. Let $\mathbf{w}$ be the nonzero vector in $W_1 \cap W_2 \cap W_3$. Note that W contains exactly three vector subspaces of dimension two, which contain a fixed nonzero vector. In the case of vector $\mathbf{w}$, these are W_1, W_2, W_3. If $j = 2$, then (58) implies that $W_2 \cap W_3 \cap W_4 = \langle \mathbf{w} \rangle$. Hence, $\mathbf{w} \in W_4$ and $W_4 \in \{W_1, W_2, W_3\}$. Since W_2, W_3, W_4 are pairwise distinct it follows that $W_4 = W_1$. If we select $j = 3$ and $j = 4$, we observe in the same way that $W_5 = W_2$ and $W_6 = W_3$, consecutively. The last two equalities imply that

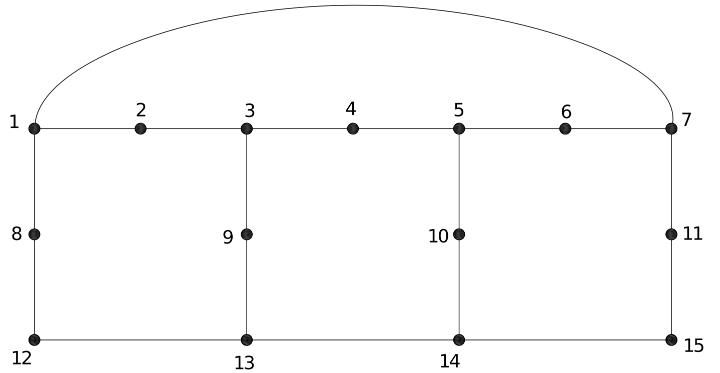

Fig. 2. Graph Γ from Lemma 6.8.

$$\mathbf{z}_2 = \mathbf{z}_0 + \mathbf{z}_1, \tag{59}$$

$$\mathbf{z}_0 = \mathbf{z}_{-1} + \mathbf{z}_{-2}, \tag{60}$$

respectively. Equality (59) implies that

$$\mathbf{z}_2 \in W_1 \cap W_2 = W_1 \cap W_2 \cap W_3 \subseteq W_3 = \{0, \mathbf{z}_{-1}, \mathbf{z}_{-2}, \mathbf{z}_{-1} + \mathbf{z}_{-2}\}.$$

Since $\mathbf{z}_2$ is nonzero and different from $\mathbf{z}_{-1}, \mathbf{z}_{-2}$, it follows that $\mathbf{z}_2 = \mathbf{z}_{-1} + \mathbf{z}_{-2}$. By (60), $\mathbf{z}_2 = \mathbf{z}_0$, a contradiction. $\square$

Lemma 6.8. *Let Γ be the graph in Fig. 2 and let $\Phi : \Gamma \to \widehat{\Gamma}_m$ be a homomorphism where $m \geq 3$. Denote $\mathbf{z}_i = \big(\Phi(i) - \Phi(i+1)\big)^{\frac{1}{2}}$ for $i \in \{1, 2, 3, 4, 5, 6\}$. Suppose that there exists a vector subspace $\widetilde{W} \subseteq \mathbb{F}_2^m$ of dimension three such that $\big(\Phi(i) - \Phi(i+1)\big)^{\frac{1}{2}} \in \widetilde{W}$ for $i \in \{12, 13, 14\}$ and*

$$\big(\Phi(j) - \Phi(k)\big)^{\frac{1}{2}} \in \widetilde{W} \tag{61}$$

for at least four edges $\{j, k\}$ in the 7-cycle $[1, 2, 3, 4, 5, 6, 7]$. Then, (61) holds for all edges $\{j, k\}$ in Γ or

$$\dim\big(\langle \mathbf{z}_1, \mathbf{z}_2\rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4\rangle \cap \langle \mathbf{z}_5, \mathbf{z}_6\rangle\big) = 1.$$

Proof. By Corollary 6.2, Φ maps 7-cycles into 7-cycles. By Lemma 4.3, vectors $\big(\Phi(j) - \Phi(k)\big)^{\frac{1}{2}}$ form a vector space of dimension three (without the zero vector) as $\{j, k\}$ runs over the edges in the cycle $[1, 2, 3, 4, 5, 6, 7]$. By the assumption, four of these vectors are in $\widetilde{W}$. Hence, all seven of them are in $\widetilde{W}$.

Suppose now that there exists an edge $\{j, k\}$ in Γ such that (61) is not true. Then one of its endpoints is in the set $\{8, 9, 10, 11\}$. We may assume that $j \in \{8, 9, 10, 11\}$. We claim that

$$\big(\Phi(12)-\Phi(13)\big)^{\frac{1}{2}} = \mathbf{z}_1+\mathbf{z}_2, \tag{62}$$

$$\big(\Phi(13)-\Phi(14)\big)^{\frac{1}{2}} = \mathbf{z}_3+\mathbf{z}_4, \tag{63}$$

$$\big(\Phi(14)-\Phi(15)\big)^{\frac{1}{2}} = \mathbf{z}_5+\mathbf{z}_6 \tag{64}$$

and vectors

$$\big(\Phi(8)-\Phi(1)\big)^{\frac{1}{2}},\ \big(\Phi(8)-\Phi(12)\big)^{\frac{1}{2}},\ \big(\Phi(9)-\Phi(3)\big)^{\frac{1}{2}},\ \big(\Phi(9)-\Phi(13)\big)^{\frac{1}{2}}, \tag{65}$$

$$\big(\Phi(10)-\Phi(5)\big)^{\frac{1}{2}},\big(\Phi(10)-\Phi(14)\big)^{\frac{1}{2}},\big(\Phi(11)-\Phi(7)\big)^{\frac{1}{2}},\big(\Phi(11)-\Phi(15)\big)^{\frac{1}{2}} \tag{66}$$

are not in $\widetilde{W}$. To prove this claim we separate two cases.

Case 1. Let $j=8$. Consider the image of the 7-cycle $[1,2,3,9,13,12,8]$. By Lemma 4.3, the edges of the image correspond to nonzero elements of a vector space of dimension three, three of them being the vectors $\mathbf{z}_1, \mathbf{z}_2, \big(\Phi(12)-\Phi(13)\big)^{\frac{1}{2}} \in \widetilde{W}$ while either $\big(\Phi(8)-\Phi(1)\big)^{\frac{1}{2}}$ or $\big(\Phi(8)-\Phi(12)\big)^{\frac{1}{2}}$ is not in $\widetilde{W}$. Denote this vector by $\mathbf{v}$. Then, the image of the 7-cycle is associated to vectors

$$\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_1+\mathbf{z}_2, \mathbf{v}, \mathbf{v}+\mathbf{z}_1, \mathbf{v}+\mathbf{z}_2, \mathbf{v}+\mathbf{z}_1+\mathbf{z}_2.$$

Consequently, (62) holds and vectors (65) are not in $\widetilde{W}$. To prove (63), (64), (66), it remains to repeat the procedure above, firstly with the 7-cycle $[3,4,5,10,14,13,9]$ and then with the 7-cycle $[5,6,7,11,15,14,10]$.

Case 2. Let $j\in\{9,10,11\}$. We proceed as in Case 1 where the order of the 7-cycles $[1,2,3,9,13,12,8]$, $[3,4,5,10,14,13,9]$, and $[5,6,7,11,15,14,10]$ is suitably adjusted.

To continue, denote $\mathbf{u} = \big(\Phi(9)-\Phi(3)\big)^{\frac{1}{2}}$. By (65), $\mathbf{u}\notin\widetilde{W}$. Since vectors $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \mathbf{z}_4$ correspond to four edges on a 7-cycle, it follows that

$$\begin{aligned}\dim\big(\langle \mathbf{z}_1,\mathbf{z}_2\rangle \cap \langle \mathbf{z}_3,\mathbf{z}_4\rangle\big) &= \dim\langle \mathbf{z}_1,\mathbf{z}_2\rangle + \dim\langle \mathbf{z}_3,\mathbf{z}_4\rangle - \dim\langle \mathbf{z}_1,\mathbf{z}_2,\mathbf{z}_3,\mathbf{z}_4\rangle \\ &= 2+2-3 = 1. \end{aligned} \tag{67}$$

Let $\mathbf{x}$ be the nonzero vector in $\langle \mathbf{z}_1,\mathbf{z}_2\rangle \cap \langle \mathbf{z}_3,\mathbf{z}_4\rangle$ and fix $\mathbf{y}\in\langle \mathbf{z}_3,\mathbf{z}_4\rangle\backslash\langle\mathbf{x}\rangle$. Then, $\langle\mathbf{x},\mathbf{y}\rangle = \langle \mathbf{z}_3,\mathbf{z}_4\rangle$. By considering the image of the 7-cycle $[3,4,5,10,14,13,9]$, the equality (63), and Lemma 4.3 we can conclude that

$$\{\big(\Phi(9)-\Phi(13)\big)^{\frac{1}{2}}, \big(\Phi(10)-\Phi(5)\big)^{\frac{1}{2}}, \big(\Phi(10)-\Phi(14)\big)^{\frac{1}{2}}\} = \{\mathbf{u}+\mathbf{x}, \mathbf{u}+\mathbf{y}, \mathbf{u}+\mathbf{x}+\mathbf{y}\}. \tag{68}$$

Similarly, by considering the image of the 7-cycle $[1,2,3,9,13,12,8]$, the equality (62), and Lemma 4.3 we can conclude that

$$\{\big(\Phi(9)-\Phi(13)\big)^{\frac{1}{2}}, \big(\Phi(8)-\Phi(1)\big)^{\frac{1}{2}}, \big(\Phi(8)-\Phi(12)\big)^{\frac{1}{2}}\} = \{\mathbf{u}+\mathbf{z}_1, \mathbf{u}+\mathbf{z}_2, \mathbf{u}+\mathbf{z}_1+\mathbf{z}_2\}. \tag{69}$$

Observe that $\mathbf{u}+\mathbf{y}$ is not a member of the set (69) because the opposite would imply that $\mathbf{y} \in \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$ and hence $\mathbf{y} \in \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4 \rangle = \langle \mathbf{x} \rangle$, i.e. $\mathbf{y} = \mathbf{x}$, a contradiction. Similarly, $\mathbf{u}+\mathbf{x}+\mathbf{y}$ is not a member of (69) because the opposite would imply that $\mathbf{x}+\mathbf{y} \in \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$ and thus $\mathbf{y} \in \langle \mathbf{z}_1, \mathbf{z}_2 \rangle$. Hence, the intersection of (68) and (69) consists of a single vector $\big(\Phi(9) - \Phi(13)\big)^{\frac{1}{2}} = \mathbf{u}+\mathbf{x}$. Therefore,

$$\{\big(\Phi(10) - \Phi(5)\big)^{\frac{1}{2}}, \big(\Phi(10) - \Phi(14)\big)^{\frac{1}{2}}\} = \{\mathbf{u}+\mathbf{y}, \mathbf{u}+\mathbf{x}+\mathbf{y}\}.$$

Consequently, by applying (64), we see that five edges in the image of the 7-cycle $[5, 6, 7, 11, 15, 14, 10]$ are associated to vectors

$$\mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_5 + \mathbf{z}_6, \mathbf{u}+\mathbf{y}, \mathbf{u}+\mathbf{x}+\mathbf{y}.$$

Recall from (68), (65), (66) that $\mathbf{u}+\mathbf{y} \notin \widetilde{W} \supseteq \{\mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_5 + \mathbf{z}_6\}$. Therefore, $\mathbf{z}_5, \mathbf{z}_6, \mathbf{u}+\mathbf{y}$ are linearly independent, and by Lemma 4.3, the seven vectors associated to the edges in the image of the 7-cycle $[5, 6, 7, 11, 15, 14, 10]$ are

$$\mathbf{z}_5, \mathbf{z}_6, \mathbf{z}_5 + \mathbf{z}_6, \mathbf{u}+\mathbf{y}, \mathbf{u}+\mathbf{y}+\mathbf{z}_5, \mathbf{u}+\mathbf{y}+\mathbf{z}_6, \mathbf{u}+\mathbf{y}+\mathbf{z}_5+\mathbf{z}_6. \tag{70}$$

Consequently, $\mathbf{u}+\mathbf{x}+\mathbf{y}$ is among the three vectors at the end of (70), i.e. $\mathbf{x} \in \langle \mathbf{z}_5, \mathbf{z}_6 \rangle$. Therefore, $\mathbf{x} \in \langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4 \rangle \cap \langle \mathbf{z}_5, \mathbf{z}_6 \rangle$ and (67) implies that

$$1 \leq \dim\big(\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4 \rangle \cap \langle \mathbf{z}_5, \mathbf{z}_6 \rangle\big) \leq \dim\big(\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4 \rangle\big) = 1.$$

Hence, $\dim\big(\langle \mathbf{z}_1, \mathbf{z}_2 \rangle \cap \langle \mathbf{z}_3, \mathbf{z}_4 \rangle \cap \langle \mathbf{z}_5, \mathbf{z}_6 \rangle\big) = 1$. □

Lemma 6.9. *In Lemma 6.6, we can select W such that* $\dim W = 3$.

Proof. It is clear from (17) and Proposition 5.2 that each edge in Γ_3 lies on some 7-cycle. Moreover, if $\operatorname{rank}(X - Y) = 1$ for some $X, Y \in SGL_3(\mathbb{F}_2)$, then $\big(\Phi(X) - \Phi(Y)\big)^{\frac{1}{2}}$ lies in at least one of the 3-dimensional vectors spaces

$$W_A(\Phi), W_B(\Phi), W_C(\Phi), W^i_{AB}(\Phi), W^i_{BC}(\Phi), W^i_{CA}(\Phi) \ (i \in \mathbb{F}_7) \tag{71}$$

that are associated to the 7-cycles (49)-(54). There are 24 vectors spaces in (71), and by Lemma 6.6 all of them are contained in a vector space W of dimension $\dim W \in \{3, 4\}$. We will prove that we can select W of dimension $\dim W = 3$.

Suppose that this is not true. Then, (at least) two of the vector spaces in (71) are distinct and $\dim W = 4$. Consequently, W contains $\left[\begin{smallmatrix}4\\3\end{smallmatrix}\right]_2 = 15$ distinct 3-dimensional vector subspaces. Since $24 > 15$, two of the vector spaces in (71) must be equal. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be the two 7-cycles in the Coxeter graph Γ_3 whose images correspond to these two vector spaces. Similarly, let $\mathcal{C}_A$ be the 7-cycle $[A_{-3}, A_{-2}, A_{-1}, A_0, A_1, A_2, A_3]$. By Lemma 5.4, there exists an automorphism Ψ of Γ_3 such that $\Psi(\mathcal{C}_A) = \mathcal{C}_1$. Let

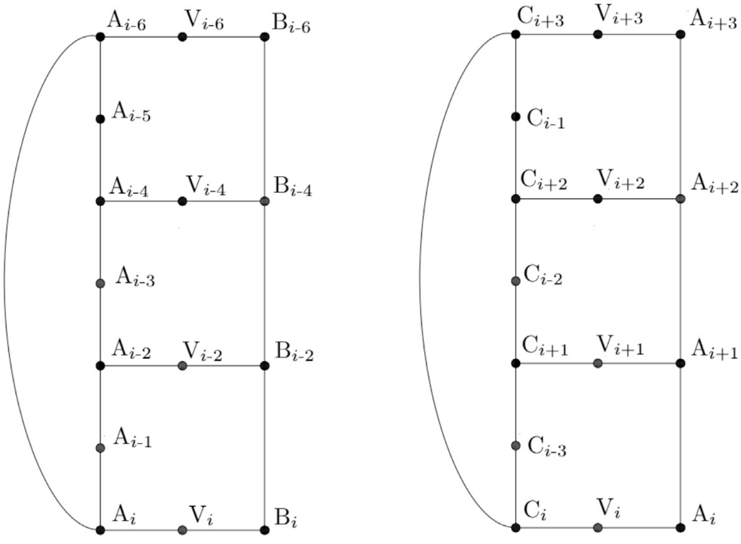

Fig. 3. Two subgraphs of the Coxeter graph.

$$W_A = W_A(\Phi \circ \Psi),\ W_B = W_B(\Phi \circ \Psi),\ W_C = W_C(\Phi \circ \Psi), \tag{72}$$

$$W_{AB}^i = W_{AB}^i(\Phi \circ \Psi),\ W_{BC}^i = W_{BC}^i(\Phi \circ \Psi),\ W_{CA}^i = W_{CA}^i(\Phi \circ \Psi)\ (i \in \mathbb{F}_7). \tag{73}$$

From Proposition 5.2 it follows that W_A is equal to one of the other spaces in (72)-(73). We separate the five cases accordingly, and show that all of them lead to a contradiction. We denote $\mathbf{z}_i := \big((\Phi \circ \Psi)(A_i) - (\Phi \circ \Psi)(A_{i-1})\big)^{1/2}$ for all $i \in \mathbb{F}_7$, that is, $W_A \backslash \{0\} = \{\mathbf{z}_i : i \in \mathbb{F}_7\}$.

Case 1. Let $W_A = W_{AB}^{i_0}$ for some $i_0 \in \mathbb{F}_7$. Firstly suppose that $W_A = W_{AB}^i$ for all i. Then observe from Fig. 1 that sets $(W_{AB}^i)^{(2)}$ cover all edges in the image of $\Phi \circ \Psi$ with the exception of the edges of the form $(\Phi\circ\Psi)(C_{j+3}) - (\Phi\circ\Psi)(C_j)$ and $(\Phi\circ\Psi)(V_{j+3}) - (\Phi\circ\Psi)(C_{j+3})$. However, both these edges are part of the 7-cycle associated to W_{BC}^{j-2}, which has four edges covered by sets $(W_{AB}^i)^{(2)}$. Hence, $W_{BC}^{j-2} = W_{AB}^i = W_A$. This contradicts the assumption that two spaces in (71) are distinct. Therefore there exists $i_1 \in \mathbb{F}_7$ such that $W_A \neq W_{AB}^{i_1}$. In particular, (at least) four edges in $(W_{AB}^{i_1})^{(2)}$ are not in $(W_A)^{(2)}$. Since $(\Phi \circ \Psi)(A_{i_1+1}) - (\Phi \circ \Psi)(A_{i_1})$ and $(\Phi \circ \Psi)(A_{i_1}) - (\Phi \circ \Psi)(A_{i_1-1})$ are both in $(W_A)^{(2)}$, it follows that $(\Phi \circ \Psi)(A_{i_1+1}) - (\Phi \circ \Psi)(V_{i_1+1}) \notin (W_A)^{(2)}$ or $(\Phi \circ \Psi)(V_{i_1+1}) - (\Phi \circ \Psi)(B_{i_1+1}) \notin (W_A)^{(2)}$, that is, $W_A \neq W_{AB}^{i_1+2}$. If we repeat the process we deduce that spaces $W_{AB}^{i_1+4}, W_{AB}^{i_1+6}, \ldots$ are all different from W_A. In particular, $W_A \neq W_{AB}^{i_0}$, a contradiction.

Case 2. Let $W_A = W_B$. By Lemma 6.7, there exists $i \in \mathbb{F}_7$ such that (57) holds. Consider the subgraph Γ' of the Coxeter graph Γ_3 in the left part of Fig. 3. There exist an obvious isomorphism $\Psi' : \Gamma \to \Gamma'$ where Γ is the graph in Fig. 2/Lemma 6.8. By Case 1,

$$W_{AB}^{i-1} \neq W_A. \tag{74}$$

If we apply Lemma 6.8 to the homomorphism $\Phi \circ \Psi \circ \Psi' : \Gamma \to \widehat{\Gamma}_m$ and $\widetilde{W} := W_A = W_B$ we get a contradiction with (74).

Case 3. Let $W_A = W_C$. Then, for each $i \in \mathbb{F}_7$ there exists $j_i \in \mathbb{F}_7$ such that $\mathbf{z}_{j_i} := \big((\Phi \circ \Psi)(C_i) - (\Phi \circ \Psi)(C_{i-3})\big)^{1/2}$ and $\{j_i : i \in \mathbb{F}_7\} = \mathbb{F}_7$. By Lemma 6.7, there exists $i \in \mathbb{F}_7$ such that

$$\langle \mathbf{z}_{j_i}, \mathbf{z}_{j_{i-3}} \rangle \cap \langle \mathbf{z}_{j_{i+1}}, \mathbf{z}_{j_{i-2}} \rangle \cap \langle \mathbf{z}_{j_{i+2}}, \mathbf{z}_{j_{i-1}} \rangle = \{0\}.$$

Consider the subgraph Γ' of the Coxeter graph Γ_3 in the right part of Fig. 3. There exist an obvious isomorphism $\Psi' : \Gamma \to \Gamma'$ where Γ is the graph in Fig. 2/Lemma 6.8. If we apply Lemma 6.8 to the homomorphism $\Phi \circ \Psi \circ \Psi' : \Gamma \to \widehat{\Gamma}_m$ and $\widetilde{W} := W_A = W_C$, we deduce that $\Phi \circ \Psi \circ \Psi'$ maps all edges of Γ into $\widetilde{W}^{(2)}$. Since four edges in the 7-cycle $[A_{i+2}, A_{i+1}, A_i, V_i, B_i, B_{i+2}, V_{i+2}]$, which corresponds to W_{AB}^{i+1}, lie in Γ', we deduce that $W_{AB}^{i+1} = \widetilde{W} = W_A$, a contradiction by Case 1.

Case 4. Let $W_A = W_{BC}^{i_0}$ for some $i_0 \in \mathbb{F}_7$. For each $i \in \mathbb{F}_7$, observe that $\{0, \mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{z}_i + \mathbf{z}_{i+1}\} \subseteq W_{AB}^i \cap W_A$. Since $W_{AB}^i \neq W_A$ by Case 1 for all $i \in \mathbb{F}_7$, Lemma 4.3 implies that

$$W_{AB}^i \backslash \{0\} = \{\mathbf{z}_i, \mathbf{z}_{i+1}, \mathbf{z}_i + \mathbf{z}_{i+1}, \mathbf{w}_i, \mathbf{w}_i + \mathbf{z}_i, \mathbf{w}_i + \mathbf{z}_{i+1}, \mathbf{w}_i + \mathbf{z}_i + \mathbf{z}_{i+1}\}$$

for some $\mathbf{w}_i \notin W_A$. Consequently, $\big((\Phi \circ \Psi)(B_{i+1}) - (\Phi \circ \Psi)(B_{i-1})\big)^{1/2}$ or $\big((\Phi \circ \Psi)(B_{i-1}) - (\Phi \circ \Psi)(V_{i-1})\big)^{1/2}$ is different from $\mathbf{z}_i + \mathbf{z}_{i+1}$, and is therefore not a member of W_A. Hence, $W_{BC}^{i+1} \neq W_A$. Since i is arbitrary, we deduce that $W_{BC}^i \neq W_A$ for all $i \in \mathbb{F}_7$, which contradicts the assumption $W_A = W_{BC}^{i_0}$.

Case 5. Let $W_A = W_{CA}^{i_0}$ for some $i_0 \in \mathbb{F}_7$. By Case 3, $W_C \neq W_A$, which means that the 7-cycle associated to W_C has (at least) four edges not in $W_A^{(2)}$. In particular, $W_A \neq W_{CA}^{i_0+1}$. Consequently, the four edges of the 7-cycles associated to $W_{CA}^{i_0+1}$ that are not covered by the 7-cycles associated by W_A and $W_{CA}^{i_0}$, are not in $W_A^{(2)}$. In particular,

$$(\Phi \circ \Psi)(C_{i_0-3}) - (\Phi \circ \Psi)(C_{i_0+1}) \notin W_A^{(2)}. \tag{75}$$

Consider the 8-cycle $[A_{i_0+1}, A_{i_0+2}, A_{i_0+3}, A_{i_0-3}, V_{i_0-3}, C_{i_0-3}, C_{i_0+1}, V_{i_0+1}]$. Then,

$$\begin{aligned}
&\big((\Phi \circ \Psi)(A_{i_0+1}) - (\Phi \circ \Psi)(A_{i_0+2})\big) + \big((\Phi \circ \Psi)(A_{i_0+2}) - (\Phi \circ \Psi)(A_{i_0+3})\big)\\
&+\big((\Phi \circ \Psi)(A_{i_0+3}) - (\Phi \circ \Psi)(A_{i_0-3})\big) + \big((\Phi \circ \Psi)(A_{i_0-3}) - (\Phi \circ \Psi)(V_{i_0-3})\big)\\
&+\big((\Phi \circ \Psi)(V_{i_0-3}) - (\Phi \circ \Psi)(C_{i_0-3})\big) + \big((\Phi \circ \Psi)(C_{i_0-3}) - (\Phi \circ \Psi)(C_{i_0+1})\big)\\
&+\big((\Phi \circ \Psi)(C_{i_0+1}) - (\Phi \circ \Psi)(V_{i_0+1})\big) + \big((\Phi \circ \Psi)(V_{i_0+1}) - (\Phi \circ \Psi)(A_{i_0+1})\big) = 0
\end{aligned}$$

where the first five rank-one matrices are contained in $W_A^{(2)} = (W_{CA}^{i_0})^{(2)}$. Since $\dim W_A < 4$, it follows from Lemma 4.6, Remark 4.7, and (75) that $(\Phi \circ \Psi)(C_{i_0-3}) - (\Phi \circ \Psi)(C_{i_0+1})$ is equal to a matrix in the last line above, i.e.

$$\big((\Phi\circ\Psi)(C_{i_0+1})-(\Phi\circ\Psi)(V_{i_0+1})\big)\quad\text{or}\quad\big((\Phi\circ\Psi)(V_{i_0+1})-(\Phi\circ\Psi)(A_{i_0+1})\big).$$

Since these three rank-one matrices correspond to three edges in the $(\Phi\circ\Psi)$-image of the 7-cycle $[C_{i_0-3}, C_{i_0+1}, V_{i_0+1}, A_{i_0+1}, A_{i_0}, V_{i_0}, C_{i_0}]$, we get a contradiction by Lemma 4.3/Lemma 4.1 (ii). □

Let $m \le n$. Given $X = [x_{ij}] \in S_m(\mathbb{F}_2)$ and distinct indices $s_1, s_2, \dots, s_m \in \{1,\dots,n\}$ let $\{s_{(1)},\dots,s_{(m)}\} = \{s_1,\dots,s_m\}$ where $s_{(1)} < s_{(2)} < \cdots < s_{(m)}$, and define the matrix $X^{(n)}_{\{s_1,s_2,\dots,s_m\}} =: [\widetilde{x}_{ij}] \in S_n(\mathbb{F}_2)$ by

$$\widetilde{x}_{ij} := \begin{cases} 0 & \text{if } \{i,j\} \nsubseteq \{s_1,s_2,\dots,s_m\},\\ x_{kl} & \text{if } \{i,j\} = \{s_{(k)}, s_{(l)}\}.\end{cases}$$

Example 6.10. If $X \in S_3(\mathbb{F}_2)$, then

$$X^{(4)}_{\{1,2,3\}} = X^{(4)}_{\{3,1,2\}} = \begin{pmatrix} X & 0\\ 0 & 0\end{pmatrix} \quad\text{and}\quad X^{(4)}_{\{2,3,4\}} = \begin{pmatrix} 0 & 0\\ 0 & X\end{pmatrix}$$

where zeros are of appropriate size.

The proof of Theorem 2.1 follows easily from Lemma 6.11. We plan to apply this lemma also in the characterization of graph homomorphisms $\Gamma_n \to \widehat{\Gamma}_m$ for general n in the future paper.

Lemma 6.11. *Let $m \ge 3$ be an integer, $\Phi : \Gamma_3 \to \widehat{\Gamma}_m$ a graph homomorphism, and W the vector space of dimension three from Lemma 6.9. If $Y \in S_m(\mathbb{F}_2)$ is any matrix in the image of Φ, $1 \le s < t < u \le m$ are integers, $\{\mathbf{x}_s, \mathbf{x}_t, \mathbf{x}_u\}$ is any basis of W, and $Q \in GL_m(\mathbb{F}_2)$ is any invertible matrix with i-th column equal to $\mathbf{x}_i$ for all $i \in \{s,t,u\}$, then there exists a graph homomorphism $\widehat{\Phi} : \Gamma_3 \to \widehat{\Gamma}_3$ such that*

$$\Phi(X) = Q\widehat{\Phi}(X)^{(m)}_{\{s,t,u\}}Q^\top + Y \qquad (X \in SGL_3(\mathbb{F}_2)). \tag{76}$$

Proof. By the assumption, there exists $X \in SGL_3(\mathbb{F}_2)$ such that $Y = \Phi(X)$. If $\widetilde{X} \in SGL_3(\mathbb{F}_2)$ is any matrix, then we can find matrices $X_0 = X, X_1, \dots, X_{d-1}, X_d = \widetilde{X}$ in $SGL_3(\mathbb{F}_2)$ such that $\operatorname{rank}(X_{i+1} - X_i) = 1$ for all i. Then $\Phi(X_{i+1}) - \Phi(X_i) =: M_i \in W^{(2)}$ and $\Phi(\widetilde{X}) - \Phi(X) = \sum_{i=0}^{d-1} M_i$. Hence, $\Phi(\widetilde{X}) = QZQ^\top + Y$ where matrix $Z = [z_{ij}]$ satisfies $z_{ij} = 0$ whenever $\{i,j\} \nsubseteq \{s,t,u\}$. If we define $\widehat{\Phi} : SGL_3(\mathbb{F}_2) \to S_3(\mathbb{F}_2)$ by

$$\widehat{\Phi}(\widetilde{X}) =: \begin{pmatrix} z_{ss} & z_{st} & z_{su}\\ z_{st} & z_{tt} & z_{tu}\\ z_{su} & z_{tu} & z_{uu}\end{pmatrix},$$

then we deduce (76). Since Φ is a graph homomorphism, so is $\widehat{\Phi}$. □

Proof of Theorem 2.1. It follows directly from Corollary 6.5 and Lemma 6.11 where we select $(s, t, u) = (1, 2, 3)$. □

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

No data was used for the research described in the article.

References

[1] C. Aguilar-Melchor, P. Gaborit, J.-L. Kim, L. Sok, P. Solé, Classification of extremal and s-extremal binary self-dual codes of length 38, IEEE Trans. Inf. Theory 58 (2012) 2253–2262, https://doi.org/10.1109/TIT.2011.2177809.
[2] K. Betsumiya, M. Harada, A. Munemasa, A complete classification of doubly even self-dual codes of length 40, Electron. J. Comb. 19 (2012) P18, https://doi.org/10.37236/2593.
[3] N. Biggs, Three remarkable graphs, Can. J. Math. 25 (1973) 397–411, https://doi.org/10.4153/CJM-1973-040-1.
[4] R.T. Bilous, Enumeration of the binary self-dual codes of length 34, J. Comb. Math. Comb. Comput. 59 (2006) 173–211.
[5] R.T. Bilous, G.H.J. van Rees, An enumeration of binary self-dual codes of length 32, Des. Codes Cryptogr. 26 (2002) 61–86, https://doi.org/10.1023/A:1016544907275.
[6] I. Bouyukliev, M. Dzhumalieva-Stoeva, V. Monev, Classification of binary self-dual codes of length 40, IEEE Trans. Inf. Theory 61 (2015) 4253–4258, https://doi.org/10.1109/TIT.2015.2440266.
[7] S. Bouyuklieva, I. Bouyukliev, An algorithm for classification of binary self-dual codes, IEEE Trans. Inf. Theory 58 (2012) 3933–3940, https://doi.org/10.1109/TIT.2012.2190134.
[8] S. Bouyuklieva, I. Bouyukliev, M. Harada, Some extremal self-dual codes and unimodular lattices in dimension 40, Finite Fields Appl. 21 (2013) 67–83, https://doi.org/10.1016/j.ffa.2013.01.009.
[9] A.E. Brouwer, A.M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.
[10] W.L. Chooi, K.H. Kwa, M.-H. Lim, Coherence invariant maps on tensor products, Linear Algebra Appl. 516 (2017) 24–46, https://doi.org/10.1016/j.laa.2016.11.023.
[11] W.L. Chooi, J. Lau, Adjacency preserving maps on symmetric tensors, Linear Algebra Appl. 690 (2024) 27–58, https://doi.org/10.1016/j.laa.2024.02.029.

[12] J.H. Conway, V. Pless, On the enumeration of self-dual codes, J. Comb. Theory, Ser. A 28 (1980) 26–53, https://doi.org/10.1016/0097-3165(80)90057-6.

[13] J.H. Conway, V. Pless, N.J.A. Sloane, The binary self-dual codes of length up to 32: a revised enumeration, J. Comb. Theory, Ser. A 60 (1992) 183–195, https://doi.org/10.1016/0097-3165(92)90003-D.

[14] C. de Seguins Pazzis, P. Šemrl, Hua's fundamental theorem of geometry of rectangular matrices over EAS division rings, J. Algebra 439 (2015) 159–187, https://doi.org/10.1016/j.jalgebra.2015.04.040.

[15] T. Dobson, A. Malnič, D. Marušič, Symmetry in Graphs, Cambridge University Press, Cambridge, 2022.

[16] J.-F. Gao, Z.-X. Wan, R.-Q. Feng, D.-J. Wang, Geometry of symmetric matrices and its applications III, Algebra Colloq. 3 (1996) 135–146.

[17] C. Godsil, G. Royle, Algebraic Graph Theory, Springer-Verlag, New York, 2001.

[18] R. Hammack, W. Imrich, S. Klavžar, Handbook of Product Graphs, second ed., CRC Press, Boca Raton, FL, 2011.

[19] M. Harada, A. Munemasa, Classification of self-dual codes of length 36, Adv. Math. Commun. 6 (2012) 229–235, https://doi.org/10.3934/amc.2012.6.229.

[20] M. Harada, A. Munemasa, Database of binary self-dual codes, Available at https://www.math.is.tohoku.ac.jp/~munemasa/research/codes/sd2.htm. (Accessed 25 January 2025).

[21] X. Hou, On the asymptotic number of inequivalent binary self-dual codes, J. Comb. Theory, Ser. A 114 (2007) 522–544, https://doi.org/10.1016/j.jcta.2006.07.003.

[22] L.-P. Huang, Z. Huang, C.-K. Li, N.-S. Sze, Graphs associated with matrices over finite fields and their endomorphisms, Linear Algebra Appl. 447 (2014) 2–25, https://doi.org/10.1016/j.laa.2013.12.030.

[23] L.-P. Huang, J.-Q. Huang, K. Zhao, On endomorphisms of alternating forms graph, Discrete Math. 338 (2015) 110–121, https://doi.org/10.1016/j.disc.2014.10.017.

[24] W.-L. Huang, P. Šemrl, Adjacency preserving maps on Hermitian matrices, Can. J. Math. 60 (2008) 1050–1066, https://doi.org/10.4153/CJM-2008-047-1.

[25] W.-L. Huang, P. Šemrl, The optimal version of Hua's fundamental theorem of geometry of square matrices–the low dimensional case, Linear Algebra Appl. 498 (2016) 21–57.

[26] G.J. Janusz, Parametrization of self-dual codes by orthogonal matrices, Finite Fields Appl. 13 (2007) 450–491, https://doi.org/10.1016/j.ffa.2006.05.001.

[27] K.H. Kwa, Coherence invariant maps on order-3 symmetric tensors, Linear Multilinear Algebra 72 (2024) 597–614, https://doi.org/10.1080/03081087.2022.2160424.

[28] M. Orel, A note on adjacency preservers on Hermitian matrices over finite fields, Finite Fields Appl. 15 (2009) 441–449, https://doi.org/10.1016/j.ffa.2009.02.005.

[29] M. Orel, Adjacency preservers on invertible Hermitian matrices I, Linear Algebra Appl. 499 (2016) 99–128, https://doi.org/10.1016/j.laa.2014.10.034.

[30] M. Orel, Adjacency preservers on invertible Hermitian matrices II, Linear Algebra Appl. 499 (2016) 129–146, https://doi.org/10.1016/j.laa.2014.10.033.

[31] M. Orel, Adjacency preservers, symmetric matrices, and cores, J. Algebr. Comb. 35 (2012) 633–647, https://doi.org/10.1007/s10801-011-0318-0.

[32] M. Orel, Nonstandard rank-one nonincreasing maps on symmetric matrices, Linear Multilinear Algebra 67 (2) (2019) 391–432, https://doi.org/10.1080/03081087.2017.1419456.

[33] M. Orel, On generalizations of the Petersen and the Coxeter graph, Electron. J. Comb. 22 (2015) P4.27, https://doi.org/10.37236/3759.

[34] M. Orel, Preserver problems over finite fields, in: J. Simmons (Ed.), Finite Fields: Theory, Fundamental Properties and Applications, Nova Science Publishers, New York, 2017, pp. 1–54.

[35] M. Orel, Draženka Višnjić, The distance function on Coxeter-like graphs and self-dual codes, Finite Fields Appl. 103 (2025) 102580, https://doi.org/10.1016/j.ffa.2025.102580.

[36] M. Pankov, The graphs of non-degenerate linear codes, J. Comb. Theory, Ser. A 195 (2023) 105720, https://doi.org/10.1016/j.jcta.2022.105720.

[37] V. Pless, A classification of self-orthogonal codes over GF(2), Discrete Math. 3 (1972) 209–246, https://doi.org/10.1016/0012-365X(72)90034-9.

[38] V. Pless, N.J.A. Sloane, On the classification and enumeration of self-dual codes, J. Comb. Theory, Ser. A 18 (3) (1975) 313–335, https://doi.org/10.1016/0097-3165(75)90042-4.

[39] E.M. Rains, N.J.A. Sloane, Self-dual codes, in: V.S. Pless, W.C. Huffman, R.A. Brualdi (Eds.), Handbook of Coding Theory, vols. I, II, North-Holland, Amsterdam, 1998, pp. 177–294.

[40] M. Tsatsomeros, Matrix equalities and inequalities, in: L. Hogben (Ed.), Handbook of Linear Algebra, second ed., CRC Press, Boca Raton (FL), 2014, pp. 16-1–16-18.

[41] P. Šemrl, Hua's fundamental theorems of the geometry of matrices and related results, Linear Algebra Appl. 361 (2003) 161–179, https://doi.org/10.1016/S0024-3795(02)00339-7.
[42] P. Šemrl, Maps on matrix and operator algebras, Jahresber. Dtsch. Math.-Ver. 108 (2006) 91–103.
[43] P. Šemrl, Maps on matrix spaces, Linear Algebra Appl. 413 (2006) 364–393, https://doi.org/10.1016/j.laa.2005.03.011.
[44] P. Šemrl, The optimal version of Hua's fundamental theorem of geometry of rectangular matrices, Mem. Am. Math. Soc. 232 (1089) (2014), https://doi.org/10.1090/memo/1089, vi+74 pp.
[45] Z.-X. Wan, Geometry of matrices, In memory of Professor L.K. Hua (1910–1985), World Scientific Publishing Co., Inc., River Edge, NJ, 1996.
[46] Z.-X. Wan, Geometry of symmetric matrices and its applications II, Algebra Colloq. 1 (1994) 201–224.
[47] Z.-X. Wan, The graph of binary symmetric matrices of order 3, Northeast. Math. J. 11 (1995) 1–2.
[48] W.C. Waterhouse, Linear transformations preserving symmetric rank one matrices, J. Algebra 125 (1989) 502–518, https://doi.org/10.1016/0021-8693(89)90179-8.