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# Asymmetry of copulas with a given opposite diagonal section

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ABSTRACT

In this paper we investigate bivariate copulas with a given opposite diagonal section. We determine the exact lower bound for all such copulas and derive an explicit formula for the maximal asymmetry for copulas with a given opposite diagonal section. As a special case, we consider the situation when the opposite diagonal section is symmetric and unimodal, which is true in many applications. To obtain our results, we also calculate the maximal asymmetry of all distributions with fixed marginals, which is an interesting result by itself.

## 1. Introduction

In the recent years, copulas are a vivid area of research, since they connect multivariate distributions with their marginals, due to Sklar's theorem. It is often the case that, given a random vector, we have additional information about the copula connecting its components. In particular, we may know the values of the copula at some points or on a given set. The first question is the existence of a copula with given values on a specified set. In case that existence is assured, in some situations it is also necessary to determine upper and lower bound for copulas with such values.

Bivariate copulas with prescribed value at a single point and their bounds are studied in [14]. The generalization to the multivariate case was explored only recently in [8]. The case of two points for the trivariate case is solved in [3] and generalized to the multivariate setting in [8]. Bivariate copulas with prescribed values on a horizontal or a vertical section are considered in [5].

Another important situation is when either the diagonal section or the opposite diagonal section of a bivariate copula is prescribed. The paper [2] describes the importance of the diagonal and opposite diagonal sections of copulas in statistics. They are used in the investigations of lower and upper tail dependence of random variables, especially when the data demonstrate either comonotone or countermonotone behavior. Such situations are often encountered in finance and risk management. In particular, positive tail dependence is closely connected to the main diagonal section of a copula, while nonpositive tail dependence is connected to the opposite diagonal section of a copula.

The lower bound for copulas with a given diagonal section has long been known and it is the Bertino copula [6], the problem for the upper bound was open for 15 years and solved recently in [13]. This result enabled the investigation of other properties of such copulas, in particular their asymmetry, also known as non-exchangeability.

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While most standard families of copulas contain only symmetric copulas, recent applications require asymmetric ones, since data often exhibits non-symmetric behavior. The study of asymmetry of copulas was initiated in [10,15] and further explored in [7,11,12], to name just a few. The maximal possible asymmetry for copulas with a given diagonal section was investigated in [13].

As already explained, the diagonal sections of copulas are well studied in the literature, while there are only a few results on opposite diagonal sections, such as [2,9]. Although many of the results on opposite diagonal sections can be obtained from analogous results on diagonal sections by using $y$-reflection, it turns out that this is not the case when studying asymmetry properties of copulas.

In this paper we determine the bounds for bivariate copulas with a given opposite diagonal section and study the maximal asymmetry of such copulas. The lower bound was already given in [9]. The upper bound is a consequence of the lower bound for copulas with a given diagonal section. These results are then used to derive an explicit formula for the maximal asymmetry for copulas with a given opposite diagonal section. This problem turns out to be more involved than in the case of diagonal section. In fact, it requires to determine the maximal asymmetry of all distributions with fixed marginals. If the two marginals of a bivariate distribution are continuous and equal, then the maximal asymmetry of all distributions with these marginals can be derived directly from the asymmetry of copulas. In the case that the marginals are not equal, however, this seems to be a different problem which has not been considered yet.

In order to derive the formula for the maximal asymmetry for copulas with a given opposite diagonal section, we compute the asymmetry function, i.e., the maximal asymmetry that a copula with a given opposite diagonal section can have at a fixed point in the unit square. This essentially means that we simplify originally complicated optimization problem over a non-tangible domain to simpler optimization problems over compact subsets of the unit square. It turns out that calculating the asymmetry function is relatively easy for points $(x, y)$ with $0 \le x \le \frac{1}{2} \le y \le 1$ (see Proposition 7.2), in fact, for such points it is equal to the difference between the upper and lower bound for such copulas. In the case $(x, y) \in [0, \frac{1}{2}]^2$, we need to consider the partial horizontal and vertical sections $C(x, \frac{1}{2})$ and $C(\frac{1}{2}, y)$ of copulas. We choose particular partial horizontal and vertical sections $\varphi_0$ and $\tau_0$, show that there exists a copula with these sections and a given opposite diagonal section (see Proposition 5.6), and later prove that the chosen sections are optimal to obtain the maximal asymmetry (see Propositions 6.7 and 7.1).

Our results can be applied in finance, where testing asset returns for asymmetry and nonpositive tail dependence can improve portfolio management. They can be useful for pricing and risk assessment in insurance, since measuring asymmetry in claims data based on tail dependence may be crucial to detect possible frauds. Furthermore, testing for nonpositive tail dependence in climate variables aids in understanding extreme weather events. These applications can be done using goodness-of-fit-tests for copula models, where a test might evaluate whether a chosen copula family adequately captures the observed dependence structure, using asymmetry measure as a criterion. Additionally, our results can be used for hypothesis testing and in simulation studies, helping validate the robustness and accuracy of chosen models.

The paper is structured as follows. In Section 2 we introduce the notation and provide the exact lower bound for copulas with a given opposite diagonal section. In Section 3 we state our main result, which gives an explicit formula for the maximal asymmetry for copulas with a given opposite diagonal section, and outline its proof. Section 4 is devoted to the investigation of the partial horizontal and vertical sections of such copulas. In Section 5 we introduce partial vertical and horizontal sections $\varphi_0$ and $\tau_0$ and prove the existence of a copula with the prescribed opposite diagonal section $\omega$ and sections $\varphi_0$ and $\tau_0$. We then determine the maximal asymmetry of all distributions with fixed marginals in Section 6. Section 7 contains the proof of the main result from Section 3. Finally, in Section 8 we consider the case when the opposite diagonal section is symmetric and unimodal, which holds for many of the most commonly used families of copulas. Throughout the whole paper we illustrate the results with the example of the opposite diagonal section of the product copula. Some technical proofs are provided in Appendices A and B.

## 2. Exact lower bound for copulas with a given opposite diagonal section

In this section we give explicit formulas of the upper and lower bound for copulas with a given opposite diagonal section. The upper bound was already determined in [9]. The lower bound is an easy consequence of the results from [13].

We start with the notation and some basic definitions. We denote by $\mathbb{I}$ the unit interval $[0, 1]$ and by $\mathcal{C}$ the set of all copulas. If $C$ is a copula we denote by $\delta_C$ its *diagonal section*, i.e., a function $\delta_C(x) = C(x, x)$, and by $\omega_C$ its *opposite diagonal section*, i.e., a function $\omega_C(x) = C(x, 1 - x)$. We denote by $C^t$ its *transpose*, i.e., copula $C^t(x, y) = C(y, x)$ and by $C^\sigma$ its *y-reflection*, i.e., copula $C^\sigma(x, y) = x - C(x, 1 - y)$. For more details see [4,14].

The following proposition gives the characterization of opposite diagonal sections.

**Proposition 2.1** *([9]). For a function $\omega : \mathbb{I} \to \mathbb{I}$ the following conditions are equivalent:*

($i$) *$\omega$ is an opposite diagonal section of some copula,*
($ii$) *$\omega$ satisfies the conditions*

($a$) $0 \le \omega(x) \le \min\{x, 1 - x\}$ *for all* $x \in \mathbb{I}$,
($b$) $|\omega(y) - \omega(x)| \le |y - x|$ *for all* $x, y \in \mathbb{I}$.

We denote the set of all functions $\omega : \mathbb{I} \to \mathbb{I}$ satisfying the equivalent conditions of Proposition 2.1 by $\Omega$. If $C$ is a copula with $\omega$ its opposite diagonal section, we denote by $\omega^t$ the opposite diagonal section of copula $C^t$. Observe that for $x \in \mathbb{I}$ the following holds:

$$\omega^t(x) = C^t(x, 1-x) = C(1-x, x) = \omega(1-x).$$

Note that the diagonal section $\delta_C$ of any copula is increasing and 2-Lipschitz, while the opposite diagonal section $\omega_C$ is 1-Lipschitz and not monotone (except in the trivial case $\omega = 0$). Furthermore, copulas $C$ and $C^t$ have the same diagonal section $\delta_C = \delta_{C^t}$, but this is not necessarily true for the opposite diagonal section.

**Definition 2.2.** We denote the upper and lower bound for copulas with a given opposite diagonal section $\omega \in \Omega$ by

(*i*) $\overline{A}_\omega(x, y) = \sup\{C(x, y) : C \in \mathcal{C}, \omega_C = \omega\}$,
(*ii*) $\underline{A}_\omega(x, y) = \inf\{C(x, y) : C \in \mathcal{C}, \omega_C = \omega\}$.

Since the set of all copulas $\mathcal{C}$ is compact in the uniform topology, the above supremum and infimum are actually maximum and minimum. The explicit formula for the upper bound $\overline{A}_\omega$ is given in the following proposition. We use abbreviations $x \wedge y = \min\{x, y\}$ and $x \vee y = \max\{x, y\}$.

**Proposition 2.3** *([9]). For an opposite diagonal section $\omega$ we have*

$$\overline{A}_\omega(x, y) = W(x, y) + \min_{t\in[x\wedge(1-y), x\vee(1-y)]} \omega(t).$$

*The function $\overline{A}_\omega$ is always a copula.*

In order to give the formula for the lower bound, we recall the definition of the total variation of a function (see e.g. [1, Section 4.4]).

**Definition 2.4.** For a function $f : \mathbb{I} \to \mathbb{R}$ and $0 \le x \le y \le 1$ the *total variation* of $f$ on $[x, y]$ is defined by

$$\mathrm{TV}_x^y(f) = \sup\left\{ \sum_{i=1}^n |f(x_i) - f(x_{i-1})| : x = x_0 < x_1 < \ldots < x_n = y,\ n \in \mathbb{N} \cup \{0\} \right\}.$$

For convenience we extend the definition of $\mathrm{TV}_x^y(f)$ to the case when $y < x$ by letting $\mathrm{TV}_x^y(f) = -\mathrm{TV}_y^x(f)$. If $\mathrm{TV}_0^1(f)$ is finite then $\mathrm{TV}_x^y(f) = \mathrm{TV}_0^y(f) - \mathrm{TV}_0^x(f)$ for any $x, y \in \mathbb{I}$. The formula for the lower bound $\underline{A}_\omega$, given in our next theorem, is a consequence of the results in [13].

**Theorem 2.5.** *For an opposite diagonal section $\omega$ we have*

$$\underline{A}_\omega(x, y) = \max\left\{ W(x, y), \min\{0, x+y-1\} + \tfrac{1}{2}\left(\omega(x) + \omega(1-y) + \mathrm{TV}_{x\wedge(1-y)}^{x\vee(1-y)}(\omega)\right)\right\}.$$

*The function $\underline{A}_\omega$ is 2-increasing on the triangle $\{(x, y) \in \mathbb{I}^2 : x + y \le 1\}$ and on the triangle $\{(x, y) \in \mathbb{I}^2 : x + y \ge 1\}$.*

**Proof.** Given any copula $C$, its $y$-reflection $C^\sigma$ is also a copula, so it respects the upper bound [13, Proposition 3.1]

$$C^\sigma(x, y) \le \min\left\{x, y, \max\{x, y\} - \tfrac{1}{2}\left(\hat{\delta}(x) + \hat{\delta}(y) + \mathrm{TV}_{x\wedge y}^{x\vee y}(\hat{\delta})\right)\right\},$$

where $\delta$ is the diagonal section of $C^\sigma$ and $\hat{\delta}(x) = x - \delta(x)$. Since the opposite diagonal section of $C$ satisfies

$$\omega(x) = C(x, 1-x) = x - C^\sigma(x, x) = x - \delta_{C^\sigma}(x) = \hat{\delta}(x),$$

we have $\omega = \hat{\delta}$. It follows that

$$\begin{aligned} C(x, y) &= x - C^\sigma(x, 1-y) \\ &\ge x - \min\left\{x, 1-y, \max\{x, 1-y\} - \tfrac{1}{2}\left(\hat{\delta}(x) + \hat{\delta}(1-y) + \mathrm{TV}_{x\wedge(1-y)}^{x\vee(1-y)}(\hat{\delta})\right)\right\} \\ &= \max\left\{0, x-(1-y), x - \max\{x, 1-y\} + \tfrac{1}{2}\left(\omega(x) + \omega(1-y) + \mathrm{TV}_{x\wedge(1-y)}^{x\vee(1-y)}(\omega)\right)\right\} \\ &= \max\left\{W(x, y), \min\{0, x+y-1\} + \tfrac{1}{2}\left(\omega(x) + \omega(1-y) + \mathrm{TV}_{x\wedge(1-y)}^{x\vee(1-y)}(\omega)\right)\right\} \\ &= \underline{A}_\omega(x, y). \end{aligned}$$

This lower bound is attained either by the copula $U_\delta^\sigma$ (if $x + y \le 1$) or by the copula $(U_\delta^\sigma)^t$ (otherwise) by [13, Theorem 3.6]. Since $\underline{A}_\omega = U_\delta^\sigma$ on the triangle $\{(x, y) \in \mathbb{I}^2 : x + y \le 1\}$, the function $\underline{A}_\omega$ is 2-increasing there, and similarly for the upper triangle. □

We illustrate the results of this section with the following example.

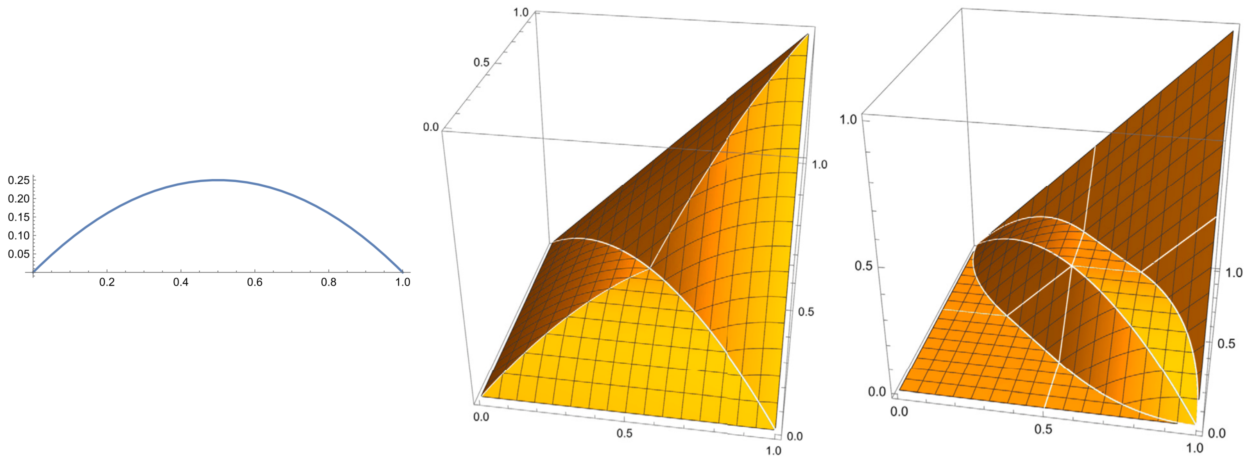



**Fig. 1.** The graphs of $\omega(x)$ (left), $\overline{A}_\omega(x,y)$ (middle) and $\underline{A}_\omega(x,y)$ (right) from Example 2.6.

**Example 2.6.** Take $C=\Pi$ the product copula and $\omega(x)=\omega_\Pi(x)=x(1-x)$ its opposite diagonal section. Then

$$\overline{A}_\omega(x,y)=\begin{cases} x-x^2, & \text{if } 0\le x\le \frac{1}{2} \text{ and } x\le y\le 1-x;\\ y-y^2, & \text{if } 0\le x\le 1 \text{ and } 0\le y\le \min\{x,1-x\};\\ y-(1-x)^2, & \text{if } \frac{1}{2}\le x\le 1 \text{ and } 1-x\le y\le x;\\ x-(1-y)^2, & \text{if } 0\le x\le 1 \text{ and } \max\{x,1-x\}\le y\le 1;\end{cases}$$

and

$$\underline{A}_\omega(x,y)=\begin{cases} y-y^2, & \text{if } \frac{1}{2}\le x\le 1 \text{ and } 1-x\le y\le \min\{\sqrt{1-x},\frac{1}{2}\};\\ x-x^2, & \text{if } 0\le x\le \frac{1}{2} \text{ and } 1-x\le y\le 1-x^2;\\ \frac{1}{4}, & \text{if } \frac{1}{2}\le x\le \frac{3}{4} \text{ and } \frac{1}{2}\le y\le \frac{5}{4}-x;\\ x-(1-y)^2 & \text{if } 0\le x\le \frac{1}{2} \text{ and } \max\{1-\sqrt{x},\frac{1}{2}\}\le y\le 1-x;\\ y-(1-x)^2 & \text{if } \frac{1}{2}\le x\le 1 \text{ and } (x-1)^2\le y\le 1-x;\\ x+y-\frac{3}{4}, & \text{if } \frac{1}{4}\le x\le \frac{1}{2} \text{ and } \frac{3}{4}-x\le y\le \frac{1}{2};\\ W(x,y), & \text{otherwise.}\end{cases}$$

The graphs of functions $\omega$, $\overline{A}_\omega$, and $\underline{A}_\omega$ are depicted in Fig. 1.

## 3. Main result and outline of the proof

The rest of the paper is devoted to the maximal asymmetry of all copulas with a given opposite diagonal section. In this section we state our main result and give a sketch of its proof. The details are provided in the subsequent sections.

Recall the definition of the asymmetry $\mu$ of a copula $C$

$$\mu(C)=\max\{C(x,y)-C(y,x):\ (x,y)\in\mathbb{I}^2\}.$$

The maximum is attained since $C$ is continuous.

**Definition 3.1.** For a given opposite diagonal section $\omega\in\Omega$ we define

$$\gamma_\omega=\sup\{\mu(C):\ C\in\mathcal{C},\omega_C=\omega\},$$

i.e., the *maximal asymmetry* of all copulas with opposite diagonal section $\omega$.

Since the set of copulas $\mathcal{C}$ is compact in supremum norm, the above supremum is attained by some copula $C\in\mathcal{C}$, so it is a maximum. We have

$$\begin{aligned}\gamma_\omega&=\max\{\mu(C):\ C\in\mathcal{C},\omega_C=\omega\}=\max\{\max\{C(x,y)-C(y,x):\ (x,y)\in\mathbb{I}^2\}:\ C\in\mathcal{C},\omega_C=\omega\}\\&=\max\{\max\{C(x,y)-C(y,x):\ C\in\mathcal{C},\omega_C=\omega\}:\ (x,y)\in\mathbb{I}^2\}=\max\{\Gamma_\omega(x,y):\ (x,y)\in\mathbb{I}^2\},\end{aligned}$$

where

$$\Gamma_\omega(x,y) = \max_{C\in\mathcal{C},\,\omega_C=\omega}(C(x,y)-C(y,x)) \tag{1}$$

is the *asymmetry function*.

We split the unit square into eight triangles with the lines $y=x$, $y=1-x$, $y=\frac{1}{2}$, and $x=\frac{1}{2}$. We will consider the function $\Gamma_\omega : \mathbb{I}^2 \to \mathbb{R}$ for each triangle separately. We denote

$$\mathscr{G} = \{(x,y)\in\mathbb{I}^2 : \ x,y\in[0,\tfrac{1}{2}], x\le y\}, \text{ and}$$
$$\mathscr{H} = \{(x,y)\in\mathbb{I}^2 : \ x\in[0,\tfrac{1}{2}], y\in[\tfrac{1}{2},1], x+y\le 1\}.$$

In order to state our main result, we define an auxiliary function. For any opposite diagonal section $\omega$ let $h_\omega : \mathscr{G}\to\mathbb{I}$ be defined by

$$h_\omega(x,y) = \min_{t\in[y,\frac{1}{2}]}(\omega(1-t) - \underline{A}_\omega(1-t,x)). \tag{2}$$

A closed form of $h_\omega(x,y)$ is given in Lemma 4.2.

**Theorem 3.2.** *Given an opposite diagonal section $\omega\in\Omega$, maximal asymmetry $\gamma_\omega$ of all copulas with opposite diagonal section $\omega$ is*

$$\gamma_\omega = \max\left\{\max_{(x,y)\in\mathscr{G}}\Gamma_\omega(x,y), \max_{(x,y)\in\mathscr{H}}\Gamma_\omega(x,y), \max_{(x,y)\in\mathscr{G}}\Gamma_{\omega^t}(x,y), \max_{(x,y)\in\mathscr{H}}\Gamma_{\omega^t}(x,y)\right\},$$

*where for any $(x,y)\in\mathscr{G}$*

$$\Gamma_\omega(x,y) = \min\{\overline{A}_\omega(x,\tfrac{1}{2}), \omega(\tfrac{1}{2}) - \underline{A}_\omega(\tfrac{1}{2},x) + \overline{A}_\omega(x,\tfrac{1}{2}) - \underline{A}_\omega(y,\tfrac{1}{2}), h_\omega(x,y)\}, \tag{3}$$

*and for any $(x,y)\in\mathscr{H}$*

$$\Gamma_\omega(x,y) = \overline{A}_\omega(x,y) - \underline{A}_\omega(y,x). \tag{4}$$

In the paper [13] the maximal asymmetry of copulas with a given diagonal section was determined. It is the maximum of the asymmetry function over the unit square $\mathbb{I}^2$, where, for a given diagonal section $\delta$, the asymmetry function is expressed as the difference between the upper and lower bound of all copulas with diagonal section $\delta$. The copula that attains the maximal asymmetry is obtained by splicing the lower and upper bound of all copulas with diagonal section $\delta$, i.e., gluing them along the main diagonal.

One could expect that the maximal asymmetry of copulas with a given opposite diagonal section $\omega$ could be obtained by applying a $y$-reflection to the set of copulas with a given diagonal section and using the result from [13]. However, it turns out that this is not the case, because the so obtained measure would calculate what could be called the asymmetry with respect to the opposite diagonal, i.e., $C(x,y)-\widehat{C}(y,x)$, where $\widehat{C}$ is the survival copula of copula $C$. For a concrete example, where the reflection method does not work, see Example 8.2. The example shows that the maximal asymmetry of all copulas with the opposite diagonal section of the product copula $\Pi$ differs from the maximal asymmetry of all copulas with the diagonal section of $\Pi$.

Furthermore, even the idea of splicing used in [13] does not work in the present case, since the asymmetry function $\Gamma_\omega$ is not equal to the difference $\overline{A}_\omega(x,y) - \underline{A}_\omega(y,x)$ in general. As we will prove in Theorem 3.2, $\Gamma_\omega$ is equal to this difference for points in the triangle $\mathscr{H}$ but not in the triangle $\mathscr{G}$. The intuition behind this property is that if a point $(x,y)$ is close to the main diagonal then the value $\Gamma_\omega(x,y)$ is relatively small while the difference $\overline{A}_\omega(x,y) - \underline{A}_\omega(y,x)$ might be large. In fact, $\overline{A}_\omega(x,y)$ and $\underline{A}_\omega(y,x)$ need not match along the main diagonal.

As it turns out, the asymmetry function $\Gamma_\omega$ in the triangle $\mathscr{G}$ is connected to the maximal asymmetry of all distributions with fixed marginals. After a thorough review of the literature, we could not find the asymmetry of distributions being studied anywhere. Therefore, we consider it in Section 6. The obtained results may be of independent interest to the reader.

Since the proof of the formula (3) in Theorem 3.2 is quite technical, we give here an outline of the proof. Fix an opposite diagonal section $\omega$. For a copula $C$ with opposite diagonal section $\omega$, we first consider its partial horizontal and vertical sections $C(x,\frac{1}{2})$ and $C(\frac{1}{2},y)$ for $x,y\in[0,\frac{1}{2}]$. We study their properties in Section 4. When the patch $C|_{[0,\frac{1}{2}]^2}$ of copula $C$ is rescaled to have value 1 at $(\frac{1}{2},\frac{1}{2})$ and appropriately extended to the entire plane $\mathbb{R}^2$, the so obtained function is a continuous cumulative distribution function with marginals which are essentially rescaled partial sections of $C$. Using the result on the asymmetry function for all (continuous) distribution functions with fixed marginals (see Theorem 6.4), we calculate the asymmetry function on $\mathscr{G}$ for copulas with given partial horizontal and vertical sections. In Section 5 we define for a given point $(x_0,y_0)\in\mathscr{G}$ specific partial horizontal and vertical sections $\varphi_0$ and $\tau_0$. In Proposition 6.7 we show that the value of the asymmetry function at $(x_0,y_0)$ over all possible partial sections is maximized precisely at $\varphi_0$ and $\tau_0$. In Proposition 5.6 we prove that partial sections $\varphi_0$ and $\tau_0$ are indeed realizable by a copula with opposite diagonal section $\omega$. Finally, we prove formula (3) in Proposition 7.1.

We prove formula (4) in Proposition 7.2 and, using reflections, we prove Theorem 3.2 in Section 7.

## 4. On partial vertical and horizontal sections of copulas

Next, we consider partial vertical and horizontal sections $C(x, \frac{1}{2})$ and $C(\frac{1}{2}, y)$ for $x, y \in [0, \frac{1}{2}]$ of any copula $C$ with a given opposite diagonal section $\omega \in \Omega$. These will be needed in the construction of copulas with prescribed opposite diagonal section.

First we define the partial sections of the bounds $\overline{A}_\omega$ and $\underline{A}_\omega$.

**Definition 4.1.** For an arbitrary opposite diagonal section $\omega \in \Omega$ define $\underline{\varphi}, \overline{\varphi}, \underline{\tau}, \overline{\tau} : [0, \frac{1}{2}] \to \mathbb{I}$ with

$$(i)\quad \underline{\varphi}(x) = \underline{A}_\omega(x, \tfrac{1}{2}) = \max\left\{0, x - \tfrac{1}{2} + \tfrac{1}{2}\left(\omega(x) + \omega(\tfrac{1}{2}) + \mathrm{TV}_x^{\frac{1}{2}}(\omega)\right)\right\};$$
$$(ii)\quad \overline{\varphi}(x) = \overline{A}_\omega(x, \tfrac{1}{2}) = \min_{t\in[x,\frac{1}{2}]} \omega(t);$$
$$(iii)\quad \underline{\tau}(y) = \underline{A}_\omega(\tfrac{1}{2}, y) = \max\left\{0, y - \tfrac{1}{2} + \tfrac{1}{2}\left(\omega(\tfrac{1}{2}) + \omega(1-y) + \mathrm{TV}_{\frac{1}{2}}^{1-y}(\omega)\right)\right\};$$
$$(iv)\quad \overline{\tau}(y) = \overline{A}_\omega(\tfrac{1}{2}, y) = \min_{t\in[\frac{1}{2},1-y]} \omega(t).$$

For any opposite diagonal section $\omega$ recall the function $h_\omega : \mathscr{G} \to \mathbb{I}$ defined in (2):

$$h_\omega(x, y) = \min_{t\in[y,\frac{1}{2}]} (\omega(1-t) - \underline{A}_\omega(1-t, x)).$$

Later on, the function $h_\omega$ will serve as the bound for the variation of partial vertical sections of copulas with opposite diagonal section $\omega$.

We define an additional auxiliary function $q$. Let $q : [0, \frac{1}{2}] \to [\frac{1}{2}, 1]$ be defined by

$$q(x) = \begin{cases} \frac{1}{2}, & \text{if } \underline{A}_\omega(\frac{1}{2}, x) > 0; \\ \max\{t \in [\frac{1}{2}, 1-x]; \underline{A}_\omega(t, x) = 0\}, & \text{if } \underline{A}_\omega(\frac{1}{2}, x) = 0. \end{cases} \tag{5}$$

Note that $q(x)$ is well defined, since the set in the definition is nonempty if $\underline{A}_\omega(\frac{1}{2}, x) = 0$.

Using function $q$, the next lemma gives a simplified formula for the function $h_\omega$. The proof can be found in Appendix A.

**Lemma 4.2.** *Let the functions $h_\omega$ and $q$ be defined by (2) and (5). If $q(x) > \frac{1}{2}$, then*

$$h_\omega(x, y) = \begin{cases} \min\{\overline{A}_\omega(\frac{1}{2}, 1-q(x)), \underline{A}_\omega(q(x), y)\}, & \text{if } x \le y \le 1-q(x); \\ \overline{A}_\omega(\frac{1}{2}, y), & \text{if } 1-q(x) \le y \le \frac{1}{2}; \end{cases}$$

*if $q(x) = \frac{1}{2}$, then*

$$h_\omega(x, y) = \omega(1-y) - \underline{A}_\omega(1-y, x).$$

Next example, a continuation of Example 2.6, illustrates the above definitions.

**Example 4.3.** Take $\omega(x) = \omega_\Pi(x) = x(1-x)$. Then

$$\underline{\varphi}(x) = \underline{\tau}(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{4}; \\ x - \frac{1}{4}, & \text{if } \frac{1}{4} \le x \le \frac{1}{2}; \end{cases}$$

and $\overline{\varphi}(x) = \overline{\tau}(x) = x(1-x)$.

Next, we have

$$q(x) = \begin{cases} 1 - \sqrt{x}, & \text{if } 0 \le x \le \frac{1}{4}; \\ \frac{1}{2}, & \text{if } \frac{1}{4} \le x \le \frac{1}{2}; \end{cases}$$

and

$$h_\omega(x, y) = \begin{cases} y - x, & \text{if } 0 \le x \le \frac{1}{2} \text{ and } x \le y \le \sqrt{x}; \\ y - y^2, & \text{if } 0 \le x \le \frac{1}{2} \text{ and } \sqrt{x} \le y \le \frac{1}{2}. \end{cases}$$

The graphs of functions $\overline{\varphi} = \overline{\tau}$, $\underline{\varphi} = \underline{\tau}$, $q$ and $h_\omega$ are shown in Fig. 2.

We collect some properties of function $h_\omega$ in the following lemma. The proof can be found in Appendix A.

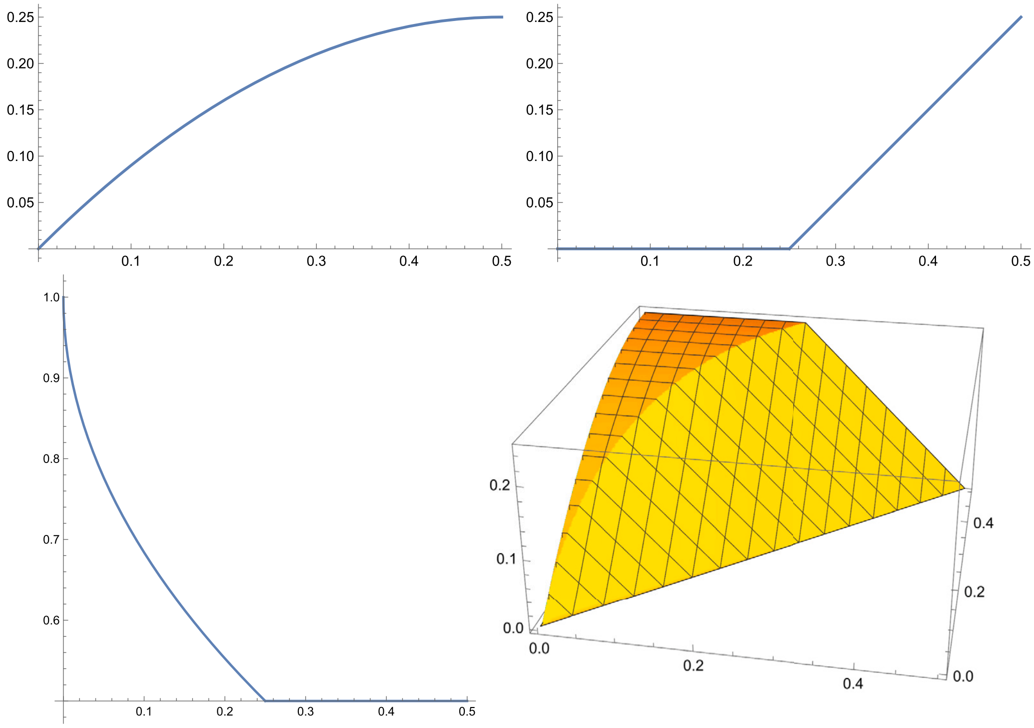



**Fig. 2.** The graphs of $\overline{\varphi}(x) = \overline{\tau}(x)$ (upper left), $\underline{\varphi}(x) = \underline{\tau}(x)$ (upper right), $q(x)$ (lower left) and $h_\omega(x, y)$ (lower right) from Example 4.3.

**Lemma 4.4.** *For an arbitrary opposite diagonal section $\omega$ the function $h_\omega$ is nonnegative, increasing and 1-Lipschitz in the second variable, and $h_\omega(x,x) = 0$ for all $x \in [0, \frac{1}{2}]$.*

The main result of this section describes the fundamental properties of partial vertical and horizontal sections of copulas with given opposite diagonal section $\omega$.

**Proposition 4.5.** *Take a copula $C$ with an opposite diagonal section $\omega$. Define $\varphi : [0, \frac{1}{2}] \to \mathbb{I}$ with $\varphi(x) = C(x, \frac{1}{2})$. Then it holds that $\underline{\varphi}(x) \le \varphi(x) \le \overline{\varphi}(x)$ for every $x \in [0, \frac{1}{2}]$, $\varphi$ is increasing and*

$$\varphi(x_2) - \varphi(x_1) \le h_{\omega^t}(x_1, x_2)$$

*for every $x_1, x_2$ with $0 \le x_1 \le x_2 \le \frac{1}{2}$.*

*Similarly, define $\tau : [0, \frac{1}{2}] \to \mathbb{I}$ with $\tau(y) = C(\frac{1}{2}, y)$. Then it holds that $\underline{\tau}(y) \le \tau(y) \le \overline{\tau}(y)$ for every $y \in [0, \frac{1}{2}]$, $\tau$ is increasing and*

$$\tau(y_2) - \tau(y_1) \le h_\omega(y_1, y_2)$$

*for every $y_1, y_2$ with $0 \le y_1 \le y_2 \le \frac{1}{2}$.*

**Proof.** The bounds for $\varphi$ (respectively $\tau$) follow from the definitions of $\underline{\varphi}$ and $\overline{\varphi}$ (respectively $\underline{\tau}$ and $\overline{\tau}$). The increasingness of $\varphi$ (respectively $\tau$) follows from the fact that for $x, y \in [0, \frac{1}{2}]$, $\varphi$ (respectively $\tau$) is defined as a vertical (respectively horizontal) section of copula $C$ at $y = \frac{1}{2}$ (respectively $x = \frac{1}{2}$).

Suppose $0 \le y_1 \le y_2 \le \frac{1}{2}$. Take $t_1 \in [y_2, \frac{1}{2}]$ such that

$$h_\omega(y_1, y_2) = \omega(1 - t_1) - \underline{A}_\omega(1 - t_1, y_1).$$

Since $C \in \mathcal{C}$, it holds that

$$V_C([\tfrac{1}{2}, 1 - t_1] \times [y_1, t_1]) = C(1 - t_1, t_1) + C(\tfrac{1}{2}, y_1) - C(1 - t_1, y_1) - C(\tfrac{1}{2}, t_1) \ge 0.$$

Then

$$\tau(t_1) - \tau(y_1) = C(\tfrac{1}{2}, t_1) - C(\tfrac{1}{2}, y_1) \le \omega(1 - t_1) - C(1 - t_1, y_1) \le \omega(1 - t_1) - \underline{A}_\omega(1 - t_1, y_1) = h_\omega(y_1, y_2).$$

But $\tau$ is increasing and $t_1 \geq y_2$, hence

$$\tau(y_2) - \tau(y_1) \leq \tau(t_1) - \tau(y_1) \leq h_\omega(y_1, y_2),$$

and the assertion is proved.

Lastly, to prove that $\varphi(x_2) - \varphi(x_1) \leq h_{\omega^t}(x_1, x_2)$ for every $x_1, x_2$ with $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$, take $t_2 \in [x_2, \frac{1}{2}]$ such that $h_{\omega^t}(x_1, x_2) = \omega^t(1 - t_2) - \underline{A}_{\omega^t}(1 - t_2, x_1)$. Recall that $C \in \mathcal{C}$, hence

$$V_C([x_1, t_2] \times [\tfrac{1}{2}, 1 - t_2]) = C(t_2, 1 - t_2) + C(x_1, \tfrac{1}{2}) - C(x_1, 1 - t_2) - C(t_2, \tfrac{1}{2}) \geq 0.$$

Then

$$\begin{aligned}\varphi(t_2) - \varphi(x_1) &= C(t_2, \tfrac{1}{2}) - C(x_1, \tfrac{1}{2}) \leq \omega(t_2) - C(x_1, 1 - t_2) \\ &\leq \omega(t_2) - \underline{A}_\omega(x_1, 1 - t_2) = \omega^t(1 - t_2) - \underline{A}^t_\omega(1 - t_2, x_1) = \omega^t(1 - t_2) - \underline{A}_{\omega^t}(1 - t_2, x_1) = h_{\omega^t}(x_1, x_2).\end{aligned}$$

But $\varphi$ is increasing and $t_2 \geq x_2$, hence

$$\varphi(x_2) - \varphi(x_1) \leq \varphi(t_2) - \varphi(x_1) \leq h_{\omega^t}(x_1, x_2),$$

and the final assertion is proved. □

## 5. Copulas with prescribed opposite diagonal and partial vertical and horizontal sections

In this section we prove the existence of a copula with prescribed opposite diagonal and specific partial vertical and horizontal sections. Its existence will be needed in Section 7.

We first construct specific partial sections. Fix a function $\omega \in \Omega$ and a point $x_0 \in [0, \frac{1}{2}]$. We define

$$\hat{x} = \min\{x \in [x_0, \tfrac{1}{2}]; \underline{\varphi}(x) = \overline{\varphi}(x_0)\}, \tag{6}$$

where $\underline{\varphi}$ and $\overline{\varphi}$ are functions defined in Section 2. Note that $\hat{x}$ is well defined since $\underline{\varphi}(x_0) \leq \overline{\varphi}(x_0)$, $\underline{\varphi}(\frac{1}{2}) = \overline{\varphi}(\frac{1}{2}) \geq \overline{\varphi}(x_0)$ and $\underline{\varphi}$ and $\overline{\varphi}$ are continuous, so that the set is nonempty and the minimum is attained.

Now, we define functions $\varphi_0, \tau_0 : [0, \frac{1}{2}] \to \mathbb{I}$ by

$$\varphi_0(x) = \begin{cases} \overline{\varphi}(x), & \text{if } 0 \leq x \leq x_0; \\ \overline{\varphi}(x_0), & \text{if } x_0 \leq x \leq \hat{x}; \\ \underline{\varphi}(x), & \text{if } \hat{x} \leq x \leq \frac{1}{2}; \end{cases} \qquad \text{and} \tag{7}$$

$$\tau_0(y) = \begin{cases} \underline{\tau}(y), & \text{if } 0 \leq y \leq x_0; \\ \underline{\tau}(x_0) + h_\omega(x_0, y), & \text{if } x_0 \leq y \leq \frac{1}{2}. \end{cases} \tag{8}$$

The function $\varphi_0$ is well defined by the definition of $\hat{x}$ and the function $\tau_0$ is well defined since $h_\omega(x_0, x_0) = 0$ by Lemma 4.4. Furthermore, the functions have the following properties, so they are possible partial sections of a copula with opposite diagonal $\omega$ at $y = \frac{1}{2}$ and $x = \frac{1}{2}$.

**Proposition 5.1.** *The functions $\varphi_0$ and $\tau_0$ are increasing, we have*

$$\underline{\varphi}(x) \leq \varphi_0(x) \leq \overline{\varphi}(x), \qquad \underline{\tau}(y) \leq \tau_0(y) \leq \overline{\tau}(y)$$

*for every $x, y \in [0, \frac{1}{2}]$. Furthermore,*

$$\varphi_0(x_2) - \varphi_0(x_1) \leq h_{\omega^t}(x_1, x_2), \qquad \text{and} \qquad \tau_0(y_2) - \tau_0(y_1) \leq h_\omega(y_1, y_2)$$

*for any $x_1, x_2, y_1, y_2$ with $0 \leq x_1 \leq x_2 \leq \frac{1}{2}$ and $0 \leq y_1 \leq y_2 \leq \frac{1}{2}$.*

**Proof.** The function $\varphi_0$ is obviously increasing and satisfies $\underline{\varphi}(x) \leq \varphi_0(x) \leq \overline{\varphi}(x)$. The property $\varphi_0(x_2) - \varphi_0(x_1) \leq h_{\omega^t}(x_1, x_2)$ is also obvious, since the functions $\underline{\varphi}$ and $\overline{\varphi}$ have this property by Proposition 4.5.

The function $h_\omega(x_0, y)$ is increasing in $y$, so $\tau_0$ is also increasing. We have for any $y \in [x_0, \frac{1}{2}]$

$$\begin{aligned}\underline{\tau}(x_0) + h_\omega(x_0, y) &= \underline{A}_\omega(\tfrac{1}{2}, x_0) + \min_{t \in [y, \frac{1}{2}]} (\omega(1 - t) - \underline{A}_\omega(1 - t, x_0)) = \min_{t \in [y, \frac{1}{2}]} (\omega(1 - t) + \underline{A}_\omega(\tfrac{1}{2}, x_0) - \underline{A}_\omega(1 - t, x_0)) \\ &\leq \min_{t \in [y, \frac{1}{2}]} \omega(1 - t) = \min_{t \in [\frac{1}{2}, 1 - y]} \omega(t) = \overline{\tau}(y)\end{aligned}$$

and

$$\begin{aligned}
\underline{\tau}(x_0) + h_\omega(x_0, y) &= \underline{A}_\omega(\tfrac{1}{2}, x_0) + \min_{t\in[y,\frac{1}{2}]} \left(\omega(1-t) - \underline{A}_\omega(1-t, x_0)\right) \\
&= \min_{t\in[y,\frac{1}{2}]} \left(\underline{A}_\omega(\tfrac{1}{2}, x_0) + \underline{A}_\omega(1-t, t) - \underline{A}_\omega(1-t, x_0)\right) \\
&= \min_{t\in[y,\frac{1}{2}]} \left(\underline{A}_\omega(\tfrac{1}{2}, x_0) + \underline{A}_\omega(1-t, t) - \underline{A}_\omega(1-t, x_0) - \underline{A}_\omega(\tfrac{1}{2}, t) + \underline{\tau}(t)\right) \\
&= \min_{t\in[y,\frac{1}{2}]} \left(V_{\underline{A}_\omega}([\tfrac{1}{2}, 1-t] \times [x_0, t]) + \underline{\tau}(t)\right) \\
&\geq \min_{t\in[y,\frac{1}{2}]} \underline{\tau}(t) = \underline{\tau}(y),
\end{aligned}$$

since $\underline{A}_\omega$ is 2-increasing on the lower triangle, so $\underline{\tau}(y) \leq \tau_0(y) \leq \overline{\tau}(y)$.

It remains to prove that $\tau_0(y_2) - \tau_0(y_1) \leq h_\omega(y_1, y_2)$ if $0 \leq y_1 \leq y_2 \leq \frac{1}{2}$. We consider three cases. If $y_2 \leq x_0$, the claim is obvious, since the function $\underline{\tau}$ has this property by Proposition 4.5.

Suppose that $y_1 \leq x_0 \leq y_2$. Since we have $h_\omega(y_1, y_2) = \min_{t\in[y_2,\frac{1}{2}]} (\omega(1-t) - \underline{A}_\omega(1-t, y_1))$, there exist $t_1 \in [y_2, \frac{1}{2}]$ such that $h_\omega(y_1, y_2) = \omega(1-t_1) - \underline{A}_\omega(1-t_1, y_1)$. Then

$$\begin{aligned}
\tau_0(y_2) - \tau_0(y_1) &= \underline{\tau}(x_0) + h_\omega(x_0, y_2) - \underline{\tau}(y_1) \\
&= \underline{A}_\omega(\tfrac{1}{2}, x_0) + \min_{t\in[y_2,\frac{1}{2}]} (\omega(1-t) - \underline{A}_\omega(1-t, x_0)) - \underline{A}_\omega(\tfrac{1}{2}, y_1) \\
&\leq \underline{A}_\omega(\tfrac{1}{2}, x_0) + \omega(1-t_1) - \underline{A}_\omega(1-t_1, x_0) - \underline{A}_\omega(\tfrac{1}{2}, y_1) \\
&= \omega(1-t_1) - \underline{A}_\omega(1-t_1, y_1) - \left(\underline{A}_\omega(1-t_1, x_0) - \underline{A}_\omega(1-t_1, y_1) - \underline{A}_\omega(\tfrac{1}{2}, x_0) + \underline{A}_\omega(\tfrac{1}{2}, y_1)\right) \\
&= h_\omega(y_1, y_2) - V_{\underline{A}_\omega}([\tfrac{1}{2}, 1-t_1] \times [y_1, x_0]) \\
&\leq h_\omega(y_1, y_2),
\end{aligned}$$

since $\underline{A}_\omega$ is 2-increasing on the lower triangle.

Suppose finally that $x_0 \leq y_1$. Since $h_\omega(x_0, y_1) = \min_{t\in[y_1,\frac{1}{2}]} (\omega(1-t) - \underline{A}_\omega(1-t, x_0))$ there exists $t_1 \in [y_1, \frac{1}{2}]$ such that $h_\omega(x_0, y_1) = \omega(1-t_1) - \underline{A}_\omega(1-t_1, x_0)$. Similarly, there exist $t_2 \in [y_2, \frac{1}{2}]$ such that $h_\omega(x_0, y_2) = \omega(1-t_2) - \underline{A}_\omega(1-t_2, x_0)$, and $t_3 \in [y_2, \frac{1}{2}]$ such that $h_\omega(y_1, y_2) = \omega(1-t_3) - \underline{A}_\omega(1-t_3, y_1)$. If $t_1 \geq y_2$, then $h_\omega(x_0, y_1) = h_\omega(x_0, y_2)$, so that

$$\tau_0(y_2) - \tau_0(y_1) = \underline{\tau}(x_0) + h_\omega(x_0, y_2) - \underline{\tau}(x_0) - h_\omega(x_0, y_1) = 0 \leq h_\omega(y_1, y_2)$$

by Lemma 4.4. If $t_1 < y_2$ then $\omega(1-t_2) - \underline{A}_\omega(1-t_2, x_0) \leq \omega(1-t_3) - \underline{A}_\omega(1-t_3, x_0)$ since $t_3 \in [y_2, \frac{1}{2}]$, so that

$$\begin{aligned}
\tau_0(y_2) - \tau_0(y_1) &= \underline{\tau}(x_0) + h_\omega(x_0, y_2) - \underline{\tau}(x_0) - h_\omega(x_0, y_1) \\
&= \omega(1-t_2) - \underline{A}_\omega(1-t_2, x_0) - \omega(1-t_1) + \underline{A}_\omega(1-t_1, x_0) \\
&\leq \omega(1-t_3) - \underline{A}_\omega(1-t_3, x_0) - \omega(1-t_1) + \underline{A}_\omega(1-t_1, x_0) \\
&= \omega(1-t_3) - \underline{A}_\omega(1-t_3, y_1) - \omega(1-t_1) + \underline{A}_\omega(1-t_1, y_1) \\
&\quad - \left(\underline{A}_\omega(1-t_3, x_0) - \underline{A}_\omega(1-t_3, y_1) - \underline{A}_\omega(1-t_1, x_0) + \underline{A}_\omega(1-t_1, y_1)\right) \\
&= h_\omega(y_1, y_2) - \left(\omega(1-t_1) - \underline{A}_\omega(1-t_1, y_1)\right) - V_{\underline{A}_\omega}([1-t_3, 1-t_1] \times [x_0, y_1]) \\
&\leq h_\omega(y_1, y_2) - \left(\underline{A}_\omega(1-t_1, t_1) - \underline{A}_\omega(1-t_1, y_1)\right) \\
&\leq h_\omega(y_1, y_2),
\end{aligned}$$

which finishes the proof. □

For the opposite diagonal section $\omega$ from Example 4.3 and $x_0 = \frac{1}{6}$ the specific partial sections constructed above are given in the following example.

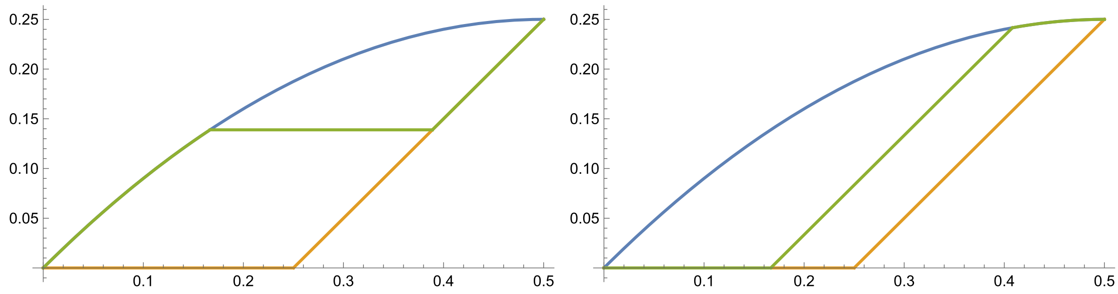



**Fig. 3.** The graphs of functions $\overline{\varphi}(x) \geq \varphi_0(x) \geq \underline{\varphi}(x)$ (left) and functions $\overline{\tau}(x) \geq \tau_0(x) \geq \underline{\tau}(x)$ (right) from Example 5.2.

**Example 5.2.** Take $\omega(x) = \omega_\Pi(x) = x(1-x)$ and $x_0 = \frac{1}{6}$. Then

$$\varphi_0(x) = \begin{cases} x - x^2, & \text{if } 0 \leq x \leq \frac{1}{6}; \\ \frac{5}{36}, & \text{if } \frac{1}{6} \leq x \leq \frac{7}{18}; \\ x - \frac{1}{4}, & \text{if } \frac{7}{18} \leq x \leq \frac{1}{2}; \end{cases}$$

and

$$\tau_0(y) = \begin{cases} 0, & 0 \leq y \leq \frac{1}{6}; \\ y - \frac{1}{6}, & \frac{1}{6} \leq y \leq \frac{1}{\sqrt{6}}; \\ y - y^2; & \frac{1}{\sqrt{6}} \leq y \leq \frac{1}{2}. \end{cases}$$

The graphs of functions $\varphi_0$ and $\tau_0$, together with $\overline{\varphi}, \underline{\varphi}, \overline{\tau}$, and $\underline{\tau}$, are depicted in Fig. 3. We remark that the middle part of the function $\varphi_0$ is always horizontal, while the middle part of the function $\tau_0$ is not necessarily parallel to a section of $\underline{\tau}$ in general.

The desired copula $C$ will later be constructed as a piece-wise function. The following two lemmas provide the function rules for two of the pieces and show that they are 2-increasing. The proofs are postponed to Appendix A.

**Lemma 5.3.** *Let*

$$\Theta = \omega(\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x}) \tag{9}$$

*and let* $s : [x_0, \hat{x}] \times [\frac{1}{2}, 1-\hat{x}] \to \mathbb{I}$ *be a function, defined by* $s(x,y) = \overline{A}_\omega(x_0, y)$ *if* $\Theta = 0$*, and*

$$s(x,y) = \overline{A}_\omega(x_0, y) + \frac{1}{\Theta}\left(\overline{A}_\omega(x, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x})\right)\left(\underline{A}_\omega(\hat{x}, y) - \overline{A}_\omega(x_0, y)\right), \tag{10}$$

*if* $\Theta > 0$*. Then the function* $s$ *is 2-increasing.*

We define also

$$\tilde{x} = q(x_0) = \begin{cases} \frac{1}{2}, & \text{if } \tau_0(x_0) > 0; \\ \max\{t \in [\frac{1}{2}, 1-x_0]; \underline{A}_\omega(t, x_0) = 0\}, & \text{if } \tau_0(x_0) = 0. \end{cases} \tag{11}$$

**Lemma 5.4.** *Let*

$$\Theta' = \omega(\tilde{x}) - \tau_0(1-\tilde{x}) \tag{12}$$

*and let* $r : [\frac{1}{2}, \tilde{x}] \times [x_0, 1-\tilde{x}] \to \mathbb{I}$ *be a function, defined by* $r(x,y) = \tau_0(y)$ *if* $\Theta' = 0$*, and*

$$r(x,y) = \tau_0(y) + \frac{1}{\Theta'}\left(\overline{A}_\omega(x, 1-\tilde{x}) - \tau_0(1-\tilde{x})\right)\left(\underline{A}_\omega(\tilde{x}, y) - \tau_0(y)\right), \tag{13}$$

*if* $\Theta' > 0$*. Then the function* $r$ *is 2-increasing.*

Building upon Example 5.2, we calculate functions $s$ and $r$ (see Lemmas 5.3 and 5.4) for the opposite diagonal of the product copula.

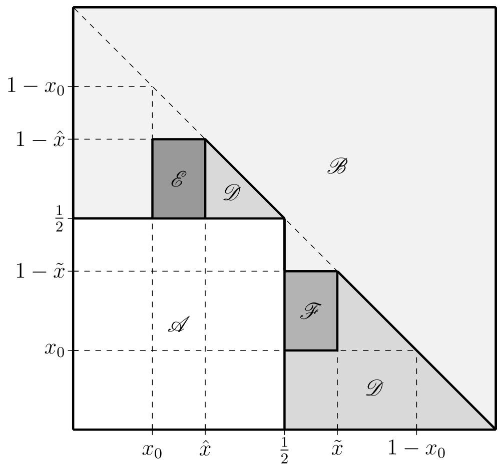



**Fig. 4.** The regions defined in the proof of Proposition 5.6.

**Example 5.5.** Take $\omega(x)=\omega_\Pi(x)=x(1-x)$ and $x_0=\frac{1}{6}$. Since $\Theta=\frac{8}{81}\neq 0$, by (10) we obtain

$$s(x,y)=\frac{5}{36}+\frac{81}{8}\left(-\frac{5}{36}+(1-x)x\right)\left(-\frac{3}{4}+y+(1-y)y\right).$$

On the other hand, $\Theta'=0$, hence $r(x,y)=\tau_0(y)=y-\frac{1}{6}$.

Finally, in the next proposition we construct the sought after copula $C$ with given opposite diagonal section $\omega$ and partial horizontal and vertical sections $\varphi_0$ and $\tau_0$.

**Proposition 5.6.** *For any function $\omega\in\Omega$ and any $x_0\in[0,\frac{1}{2}]$ let $\hat{x}$ be defined by (6) and functions $\varphi_0,\tau_0:[0,\frac{1}{2}]\to\mathbb{I}$ by (7) and (8). Let $A:[0,\frac{1}{2}]^2\to\mathbb{I}$ be any grounded 2-increasing function with marginals $\varphi_0$ and $\tau_0$, i.e. $A(x,\frac{1}{2})=\varphi_0(x)$ and $A(\frac{1}{2},y)=\tau_0(y)$. Then there exists a copula $C$ such that $\omega_C=\omega$ and $C(x,y)=A(x,y)$ for any $(x,y)\in[0,\frac{1}{2}]^2$.*

**Proof.** We will consider two cases, $\varphi_0(x_0)=\overline{A}_\omega(x_0,\frac{1}{2})>0$ and $\varphi_0(x_0)=0$. The definition of the copula $C$ will slightly differ in these cases. If $\varphi_0(x_0)>0$, we define the regions of the unit square $\mathbb{I}^2$ as follows:

$$\mathscr{A}=[0,\tfrac{1}{2}]^2;$$

$$\mathscr{B}=\{(x,y)\in\mathbb{I}^2:\ x+y\geq 1\}\cup[0,x_0]\times[\tfrac{1}{2},1]\cup[0,1]\times[1-\hat{x},1]\cup[\tfrac{1}{2},1]\times[1-\tilde{x},1];$$

$$\mathscr{D}=\{(x,y)\in\mathbb{I}^2:\ x+y\leq 1\}\cap\left([\hat{x},1]\times[\tfrac{1}{2},1]\cup[\tfrac{1}{2},1]\times[0,x_0]\cup[\tilde{x},1]\times[0,1]\right);$$

$$\mathscr{E}=[x_0,\hat{x}]\times[\tfrac{1}{2},1-\hat{x}];$$

$$\mathscr{F}=[\tfrac{1}{2},\tilde{x}]\times[x_0,1-\tilde{x}],$$

where $\hat{x}$ and $\tilde{x}$ are defined by equations (6) and (11). If $\varphi_0(x_0)=0$, the region $\mathscr{E}$ is empty, the regions $\mathscr{A}$ and $\mathscr{F}$ are the same as in the first case and the regions $\mathscr{B}$ and $\mathscr{D}$ are defined as follows:

$$\mathscr{B}=\{(x,y)\in\mathbb{I}^2:\ x+y\geq 1\}\cup[\tfrac{1}{2},1]\times[1-\tilde{x},1];$$

$$\mathscr{D}=\{(x,y)\in\mathbb{I}^2:\ x+y\leq 1\}\cap\left([0,1]\times[\tfrac{1}{2},1]\cup[\tfrac{1}{2},1]\times[0,x_0]\cup[\tilde{x},1]\times[0,1]\right).$$

Fig. 4 shows these regions in the case $\varphi_0(x_0)>0$. In the case $\varphi_0(x_0)=0$ the whole lower triangle of the square $[0,\frac{1}{2}]\times[\frac{1}{2},1]$ belongs to the region $\mathscr{D}$. Note that the regions cover the whole unit square and that they intersect only at their boundaries. Note also, that the region $\mathscr{E}$ has empty interior in the case that $\hat{x}=x_0$ or $\hat{x}=\frac{1}{2}$ and the region $\mathscr{F}$ has empty interior in the case that $\tilde{x}=\frac{1}{2}$ or $\tilde{x}=1-x_0$.

Next we define copula $C$ by

$$C(x,y)=\begin{cases} A(x,y), & \text{if } (x,y)\in\mathscr{A};\\ \overline{A}_\omega(x,y), & \text{if } (x,y)\in\mathscr{B};\\ \underline{A}_\omega(x,y), & \text{if } (x,y)\in\mathscr{D};\\ s(x,y), & \text{if } (x,y)\in\mathscr{E};\\ r(x,y), & \text{if } (x,y)\in\mathscr{F}; \end{cases} \tag{14}$$

where the function $s$ is defined by (10) and the function $r$ by (13). We need to prove that $C$ is well defined, i.e., the function rules coincide on the intersections of the regions, that it is really a copula, i.e., it is grounded, 2-increasing and has uniform marginals, and that $\omega_C=\omega$ and $C(x,y)=A(x,y)$ for any $(x,y)\in[0,\frac{1}{2}]^2$. The last two properties are obvious by definition of $C$ and since opposite diagonal is contained in region $\mathscr{B}$. Groundedness and uniform marginals are also obvious.

We need to prove that $C$ is well defined on the intersections of the regions. This technical part of the proof can be found in the Appendix B.

To finish the proof, we now need to show that $C$ is 2-increasing. Let $R$ be any rectangle in $\mathbb{I}^2$. Then $R$ can be written as a union of finitely many rectangles $R_1,R_2,\dots,R_k$ with pairwise empty intersections of interiors, such that each of them is contained entirely in one of the regions, or it has two vertices on the opposite diagonal and its lower-left triangle lies entirely in region $\mathscr{D}$. The volume $V_C(R)$ of $R$ equals the sum of the volumes $V_C(R_i)$, so it is non-negative as soon as $V_C(R_i)\ge 0$ for all $i$. If $R_i$ is contained entirely in one of the regions, then $V_C(R_i)\ge 0$, because function $A$ is 2-increasing by assumption, functions $s$ and $r$ are 2-increasing by Lemmas 5.3 and 5.4, function $\overline{A}_\omega$ is 2-increasing since it is a copula, and function $\underline{A}_\omega$ is 2-increasing in the triangle $\{(x,y)\in\mathbb{I}^2: x+y\le 1\}$ by Theorem 2.5. So suppose that $R_i=[x,1-y]\times[y,1-x]$, where $(x,y)\in\mathscr{D}$. We have

$$\begin{aligned} V_C(R_i)&=C(x,y)+C(1-y,1-x)-C(x,1-x)-C(1-y,y)\\ &=\underline{A}_\omega(x,y)+\overline{A}_\omega(1-y,1-x)-\omega(x)-\omega(1-y)\\ &=U_\delta^\sigma(x,y)+\overline{A}_\omega(1-y,1-x)-U_\delta^\sigma(x,1-x)-U_\delta^\sigma(1-y,y)\\ &\ge U_\delta^\sigma(x,y)+U_\delta^\sigma(1-y,1-x)-U_\delta^\sigma(x,1-x)-U_\delta^\sigma(1-y,y)\\ &=V_{U_\delta^\sigma}(R_i)\ge 0, \end{aligned}$$

where $U_\delta^\sigma$ is $y$-reflection of the copula $U_\delta$ from [13, Theorem 3.3] and $\delta(x)=x-\omega(x)$, which finishes the proof. □

To conclude this section, we choose an explicit example of a grounded, 2-increasing function $A:[0,\frac{1}{2}]^2\to\mathbb{I}$ with marginals $\varphi_0$ and $\tau_0$ for the case $\omega=\omega_\Pi$, and depict the graphs of both, function $A$ and copula $C$, using the formulas from preceding examples.

**Example 5.7.** Take $\omega(x)=\omega_\Pi(x)=x(1-x)$, $x_0=\frac{1}{6}$ and $A(x,y)=\frac{1}{\omega(\frac{1}{2})}\varphi_0(x)\tau_0(y)$. Then

$$A(x,y)=\begin{cases} 0, & \text{if } 0\le x\le\frac{1}{2} \text{ and } 0\le y\le\frac{1}{6};\\ (x-x^2)(y-\frac{1}{6}), & \text{if } 0\le x\le\frac{1}{6} \text{ and } \frac{1}{6}\le y\le\frac{1}{\sqrt{6}};\\ \frac{5}{36}(y-\frac{1}{6}), & \text{if } \frac{1}{6}\le x\le\frac{7}{18} \text{ and } \frac{1}{6}\le y\le\frac{1}{\sqrt{6}};\\ (x-\frac{1}{4})(y-\frac{1}{6}), & \text{if } \frac{7}{18}\le x\le\frac{1}{2} \text{ and } \frac{1}{6}\le y\le\frac{1}{\sqrt{6}};\\ (x-x^2)(y-y^2), & \text{if } 0\le x\le\frac{1}{6} \text{ and } \frac{1}{\sqrt{6}}\le y\le\frac{1}{2};\\ \frac{5}{36}(y-y^2), & \text{if } \frac{1}{6}\le x\le\frac{7}{18} \text{ and } \frac{1}{\sqrt{6}}\le y\le\frac{1}{2};\\ (x-\frac{1}{4})(y-y^2), & \text{if } \frac{7}{18}\le x\le\frac{1}{2} \text{ and } \frac{1}{\sqrt{6}}\le y\le\frac{1}{2}. \end{cases}$$

Furthermore, let copula $C$ be defined by (14). The graphs of $A$ and $C$ are shown in Fig. 5. The graph of $C$ is colored according to the regions $\mathscr{A},\mathscr{B},\mathscr{D},\mathscr{E},\mathscr{F}$.

## 6. Maximal asymmetry of bivariate distributions

We now turn our attention to the asymmetry of distributions with given marginal distribution functions $F$ and $G$.

**Definition 6.1.** For a distribution function $H$ we define its *asymmetry* by

$$\mu(H)=\sup_{x,y\in\overline{\mathbb{R}}}|H(x,y)-H(y,x)|.$$

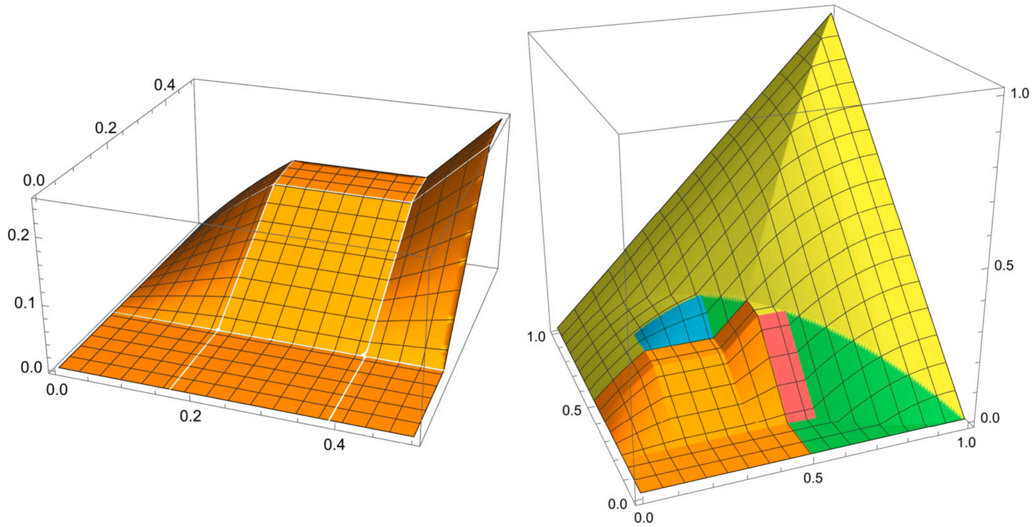



**Fig. 5.** The graphs of $A(x,y)$ (left) and $C(x,y)$ (right) from Example 5.7.

We will investigate maximal asymmetry of distributions with fixed marginals. Given two univariate distributions $F$ and $G$, we denote

$$\mathcal{D}_{F,G} = \left\{ H \text{ bivariate distribution} \mid H(x,\infty) = F(x),\ H(\infty,y) = G(y) \right\}.$$

By Sklar's theorem we have

$$\mathcal{D}_{F,G} = \left\{ H(x,y) = C(F(x),G(y)) \mid C \in \mathcal{C} \right\}. \tag{15}$$

**Definition 6.2.** We define *maximal asymmetry* of bivariate distributions with fixed marginals $F$ and $G$ by

$$\mu_{F,G} = \sup \left\{ \mu(H) \mid H \in \mathcal{D}_{F,G} \right\}.$$

Furthermore, we define an *asymmetry function* $\alpha_{F,G} : \overline{\mathbb{R}}^2 \to [-1,1]$ by

$$\alpha_{F,G}(x,y) = \sup_{H \in \mathcal{D}_{F,G}} \big( H(x,y) - H(y,x) \big).$$

Note that in the case that $F$ and $G$ are CDFs of random variables uniformly distributed on $\mathbb{I}$, the set $\mathcal{D}_{F,G}$ is equal to the set of all copulas $\mathcal{C}$. In this case it is well known that $\mu_{F,G} = \max_{C\in\mathcal{C}} \mu(C) = \frac{1}{3}$, see [10,15]. We extend this result to general marginals $F$ and $G$. In the case that $F$ and $G$ are continuous, we also show that the maximal asymmetry is attained.

As it turns out the maximal asymmetry $\mu_{F,G}$ can be expressed with the difference function $N$ introduced in [16]. We first recall the definition and some properties of $N$. The *difference function* $N : \mathbb{I}^2 \times \mathbb{I}^2 \to \mathbb{I}$ is defined by

$$N(x_1,y_1,x_2,y_2) = \sup_{Q\in\mathcal{Q}} \big( Q(x_2,y_2) - Q(x_1,y_1) \big) \tag{16}$$

for all $(x_1,y_1),(x_2,y_2) \in \mathbb{I}^2$. An explicit formula for $N$ can be found in [16, Equation (7)], however, below we derive a simplified version of the formula (see Proposition 6.3). According to [16, Corollary 11], the supremum in equation (17) is actually attained by some copula, hence

$$N(x_1,y_1,x_2,y_2) = \sup_{C\in\mathcal{C}} \big( C(x_2,y_2) - C(x_1,y_1) \big) = \max_{C\in\mathcal{C}} \big( C(x_2,y_2) - C(x_1,y_1) \big). \tag{17}$$

In fact, by [16, Theorem 10] (see also the comment after the theorem), for a fixed $(x_1,y_1) \in \mathbb{I}^2$ the function $K_{x_1,y_1} : \mathbb{I}^2 \to \mathbb{I}$ defined by

$$K_{x_1,y_1}(x,y) = N(x_1,y_1,x,y) + W(x_1,y_1) \tag{18}$$

for all $(x,y) \in \mathbb{I}^2$ is a copula that satisfies

$$N(x_1,y_1,x_2,y_2) = K_{x_1,y_1}(x_2,y_2) - K_{x_1,y_1}(x_1,y_1).$$

Furthermore, taking $(x_0, y_0) = (x_1, y_1)$ and $\alpha = W(x_1, y_1)$ in [16, Theorem 10], we infer that the copula $K_{x_1,y_1}$ is an exact upper bound for copulas $C$ with fixed value $C(x_1, y_1) = W(x_1, y_1)$. By the proof of [14, Theorem 3.2.3], this exact upper bound can be expressed as

$$K_{x_1,y_1}(x, y) = \min\{x, y, W(x_1, y_1) + (x - x_1)^+ + (y - y_1)^+\}, \tag{19}$$

where $r^+ = \max\{r, 0\}$ for all $r \in \mathbb{R}$. Combining the above we obtain the following.

**Proposition 6.3.** *For all* $(x_1, y_1), (x_2, y_2) \in \mathbb{I}^2$ *we have*

$$N(x_1, y_1, x_2, y_2) = \min\{M(x_2, y_2) - W(x_1, y_1), (x_2 - x_1)^+ + (y_2 - y_1)^+\}.$$

**Proof.** From equations (18) and (19) we obtain

$$\begin{aligned} N(x_1, y_1, x_2, y_2) &= K_{x_1,y_1}(x_2, y_2) - W(x_1, y_1) \\ &= \min\{x_2, y_2, W(x_1, y_1) + (x_2 - x_1)^+ + (y_2 - y_1)^+\} - W(x_1, y_1) \\ &= \min\{M(x_2, y_2) - W(x_1, y_1), (x_2 - x_1)^+ + (y_2 - y_1)^+\}, \end{aligned}$$

and we are done. □

The main theorem of this section gives an explicit formula for $\mu_{F,G}$. To this end, for arbitrary univariate distribution functions $F$ and $G$ introduce an auxiliary function $\Lambda_{F,G} : \overline{\mathbb{R}}^2 \to \mathbb{R}$ with

$$\Lambda_{F,G}(x, y) = \min\left\{F(x), 1 - G(x) + F(x) - F(y), G(y) - G(x)\right\}. \tag{20}$$

**Theorem 6.4.** *For any two univariate distributions $F$ and $G$ we have*

$$\alpha_{F,G}(x, y) = \begin{cases} \Lambda_{F,G}(x, y), & \text{if } x \le y; \\ \Lambda_{G,F}(y, x), & \text{if } x \ge y; \end{cases}$$

*for all* $(x, y) \in \overline{\mathbb{R}}^2$ *and*

$$\mu_{F,G} = \sup_{(x,y)\in\overline{\mathbb{R}}^2} \alpha_{F,G}(x, y).$$

*If $F$ and $G$ are continuous, then there exists a bivariate distribution $H$ with marginals $F$ and $G$ such that $\mu_{F,G} = \mu(H)$.*

**Proof.** Fix univariate distribution functions $F$ and $G$. Using equations (15) and (17) we obtain

$$\alpha_{F,G}(x, y) = \sup_{C\in\mathcal{C}} \big(C(F(x), G(y)) - C(F(y), G(x))\big) = N(F(y), G(x), F(x), G(y)). \tag{21}$$

Proposition 6.3 implies

$$\alpha_{F,G}(x, y) = \min\{M(F(x), G(y)) - W(F(y), G(x)), (F(x) - F(y))^+ + (G(y) - G(x))^+\}.$$

Note that

$$\begin{aligned} M(F(x), G(y)) - W(F(y), G(x)) &= \min\{F(x), G(y)\} - \max\{0, F(y) + G(x) - 1\} \\ &= \min\{F(x), G(y), 1 - G(x) + F(x) - F(y), 1 - F(y) + G(y) - G(x)\}. \end{aligned}$$

Furthermore, since $F$ and $G$ are increasing, one of $F(x) - F(y)$ and $G(y) - G(x)$ is non-negative and the other one is non-positive, so

$$(F(x) - F(y))^+ + (G(y) - G(x))^+ = \max\{F(x) - F(y), G(y) - G(x)\}.$$

Hence,

$$\alpha_{F,G}(x, y) = \min\left\{F(x), G(y), 1 - G(x) + F(x) - F(y), 1 - F(y) + G(y) - G(x), \max\{F(x) - F(y), G(y) - G(x)\}\right\}. \tag{22}$$

Considering the cases $x \le y$ and $x \ge y$ in the last maximum, we obtain

$$\alpha_{F,G}(x, y) = \begin{cases} \min\left\{F(x), 1 - G(x) + F(x) - F(y), G(y) - G(x)\right\}, & \text{if } x \le y; \\ \min\left\{G(y), 1 - F(y) + G(y) - G(x), F(x) - F(y)\right\}, & \text{if } x \ge y; \end{cases} = \begin{cases} \Lambda_{F,G}(x, y), & \text{if } x \le y; \\ \Lambda_{G,F}(y, x), & \text{if } x \ge y. \end{cases}$$

Since $|H(x, y) - H(y, x)| = \max\{H(x, y) - H(y, x), H(y, x) - H(x, y)\}$, we infer

$$\mu_{F,G} = \sup_{H \in \mathcal{D}_{F,G}} \sup_{x,y \in \overline{\mathbb{R}}} \max\{H(x,y) - H(y,x), H(y,x) - H(x,y)\}$$
$$= \max\{ \sup_{x,y \in \overline{\mathbb{R}}} \alpha_{F,G}(x,y), \sup_{x,y \in \overline{\mathbb{R}}} \alpha_{F,G}(y,x)\} = \sup_{x,y \in \overline{\mathbb{R}}} \alpha_{F,G}(x,y),$$

which proves the second claim.

Now assume that $F$ and $G$ are continuous. Since they are distribution functions, they are continuous also as functions $\overline{\mathbb{R}} \to \mathbb{I}$. By equation (22) function $\alpha_{F,G}(x,y)$ is also continuous. Since $\overline{\mathbb{R}}^2$ is a compact set, the supremum $\sup_{x,y \in \overline{\mathbb{R}}} \alpha_{F,G}(x,y)$ is attained at some point $(x_0, y_0) \in \overline{\mathbb{R}}^2$, so that

$$\mu_{F,G} = \alpha_{F,G}(x_0,y_0) = N(F(y), G(x), F(x), G(y))$$

by the equation above and by equation (21). Hence, equation (17) implies that there exists a copula $C$ such that

$$\mu_{F,G} = C(F(x_0), G(y_0)) - C(F(y_0), G(x_0)) = H(x_0,y_0) - H(y_0,x_0),$$

where $H(x,y) = C(F(x), G(y))$. It follows that $\mu_{F,G} = \mu(H)$, which finishes the proof. □

As already mentioned in the introduction, if the marginal distribution functions $F$ and $G$ are equal and continuous, then the maximal asymmetry of distributions with marginals $F$ and $G$ is equal to the maximal asymmetry of copulas, i.e., $\mu_{F,F} = \frac{1}{3}$. Indeed,

$$\mu_{F,F} = \sup\{\mu(H) \mid H \in \mathcal{D}_{F,F}\} = \sup\{ \sup_{x,y \in \overline{\mathbb{R}}} (H(x,y) - H(y,x)) \mid H \in \mathcal{D}_{F,F}\}$$
$$= \sup\{ \sup_{x,y \in \overline{\mathbb{R}}} (C(F(x), F(y)) - C(F(y), F(x))) \mid C \in \mathcal{C}\}$$
$$= \sup\{ \sup_{u,v \in \mathbb{I}} (C(u,v) - C(v,u)) \mid C \in \mathcal{C}\} = \frac{1}{3},$$

by [10,15].

On the other hand, if for marginal distribution functions $F$ and $G$ there exists a point $t \in \mathbb{R}$ such that $F(t) = 0$ and $G(t) = 1$, then

$$H(\infty, t) - H(t, \infty) = G(t) - F(t) = 1 \le \mu_{F,G} \le 1,$$

hence $\mu_{F,G} = 1$.

To illustrate Theorem 6.4, we give an example.

**Example 6.5.** Let

$$F(x) = \begin{cases} 0, & \text{if } x < 0; \\ x, & \text{if } 0 \le x < \frac{1}{3}; \\ \frac{3}{2}x - \frac{1}{6}, & \text{if } \frac{1}{3} \le x < \frac{2}{3}; \\ \frac{1}{2}x + \frac{1}{2}, & \text{if } \frac{2}{3} \le x < 1; \\ 1, & \text{if } 1 \le x; \end{cases} \qquad G(y) = \begin{cases} 0, & \text{if } y < 0; \\ y, & \text{if } 0 \le y < \frac{1}{3}; \\ \frac{1}{2}y + \frac{1}{6}, & \text{if } \frac{1}{3} \le y < \frac{2}{3}; \\ \frac{3}{2}y - \frac{1}{2}, & \text{if } \frac{2}{3} \le y < 1; \\ 1, & \text{if } 1 \le y. \end{cases}$$

Fig. 6 shows the graphs of functions $F$, $G$, and $\alpha_{F,G}$. The function $\alpha_{F,G}$ has three local maxima in the interior of the unit square, namely,

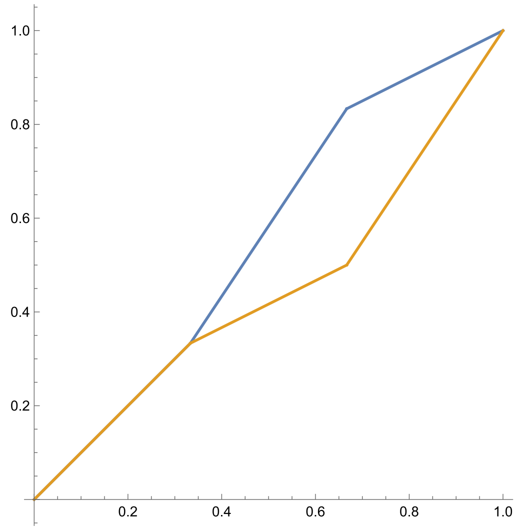
$$\alpha_{F,G}(\tfrac{5}{21}, \tfrac{13}{21}) = \tfrac{5}{21} \approx 0.2381, \ \alpha_{F,G}(\tfrac{2}{3}, \tfrac{11}{12}) = \tfrac{3}{8} \approx 0.3750, \text{ and } \alpha_{F,G}(\tfrac{31}{45}, \tfrac{19}{45}) = \tfrac{17}{45} \approx 0.3778.$$

It follows that $\mu_{F,G} = \frac{17}{45}$. Note that $F \ge G$ but the maximal asymmetry is attained in the lower triangle $y \le x$ of $\mathbb{I}^2$.

Theorem 6.4 gives the maximal asymmetry for all marginal distribution functions, not necessarily continuous. If $F$ and $G$ are cumulative distribution functions of discrete random variables $X$ and $Y$ with finite range, then the function $\alpha_{F,G}$ is piecewise constant with finite range, hence $\mu_{F,G}$ is attained in this case as well. In the next example we illustrate Theorem 6.4 for discrete random variables $X$ and $Y$.

**Example 6.6.** Let $X$ and $Y$ be discrete random variables given by

$$X : \begin{pmatrix} 1 & 3 & 5 \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} \quad \text{and} \quad Y : \begin{pmatrix} 2 & 4 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



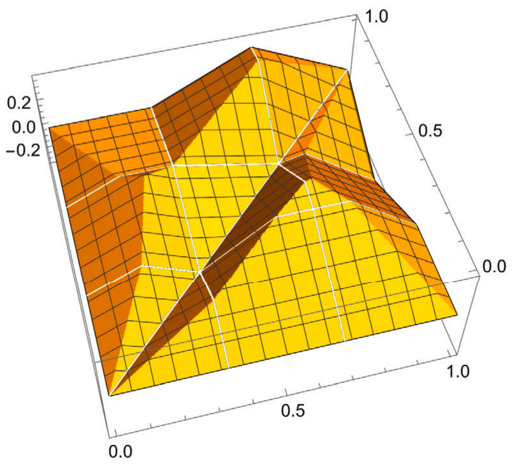




**Fig. 6.** The graphs of $F(x)$ (left, top curve), $G(y)$ (left, bottom curve) and $\alpha_{F,G}(x,y)$ (right) from Example 6.5.

and $F,G$ their respective cumulative distribution functions. Then the function $\alpha_{F,G}(x,y)$ equals zero as soon as $x<1$ or $y<1$. On each of the rectangles $[i,i+1)\times[j,j+1)$ for $i,j=1,\dots,5$, $\alpha_{F,G}$ is constant. The matrix $A=[a_{ij}]_{i,j=1}^{5}$, where $a_{ij}=\alpha_{F,G}(i,j)$, is equal to

$$A=\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & -\frac{1}{6} & \frac{1}{3} & 0 \end{bmatrix}$$

by formula (20). It follows that $\mu_{F,G}=\frac{1}{2}$. Suppose the distribution of a random vector $(X,Y)$ is given by a probability mass function

| $X\backslash Y$ | 2 | 4 |
|---|---|---|
| 1 | $\frac{1}{6}$ | $\frac{1}{6}$ |
| 3 | 0 | $\frac{1}{3}$ |
| 5 | $\frac{1}{3}$ | 0 |

and $H$ is its cumulative distribution function. Note that $H(4,3)=\frac{1}{6}$ and $H(3,4)=\frac{2}{3}$. Hence, this $H$ attains the maximal asymmetry value $\frac{1}{2}$.

Now, we fix $\omega\in\Omega$ with $\omega(\frac{1}{2})>0$ and points $x_0,y_0\in\mathbb{I}$ such that $0\le x_0\le y_0\le\frac{1}{2}$. We transform the function $\varphi_0$ defined in equation (7) to a univariate distribution function

$$F_0(x)=\begin{cases}0, & \text{if } x<0;\\ \frac{\varphi_0(x)}{\omega(\frac{1}{2})}, & \text{if } 0\le x\le\frac{1}{2};\\ 1, & \text{if } \frac{1}{2}<x.\end{cases}$$

Similarly as $F_0$, we extend functions $F,G_0,G:\ [0,\frac{1}{2}]\to\mathbb{R}$ defined with $F(x)=\frac{\varphi(x)}{\omega(\frac{1}{2})}$, $G_0(y)=\frac{\tau_0(y)}{\omega(\frac{1}{2})}$ and $G(y)=\frac{\tau(y)}{\omega(\frac{1}{2})}$ to $\overline{\mathbb{R}}$. Here recall that $\tau_0$ is defined in equation (8) and $\varphi,\tau$ are defined in Proposition 4.5.

In next proposition we show that the expression $\Lambda_{F,G}(x_0,y_0)$ attains its maximal value when $F=F_0$ and $G=G_0$. This result will be used in next section to determine the asymmetry function for copulas with prescribed opposite diagonal section.

**Proposition 6.7.** *For extensions of partial sections $\varphi,\tau$ of copulas with a given opposite diagonal section $\omega$ where $\omega(\frac{1}{2})>0$ it holds that*

$$\Lambda_{F_0,G_0}(x_0,y_0)\ge\Lambda_{F,G}(x_0,y_0).$$

**Proof.** By definition,

$$\Lambda_{F_0,G_0}(x_0,y_0)=\frac{1}{\omega(\frac{1}{2})}\min\{\varphi_0(x_0),\omega(\tfrac{1}{2})-\tau_0(x_0)+\varphi_0(x_0)-\varphi_0(y_0),\tau_0(y_0)-\tau_0(x_0)\}$$

and

$$\Lambda_{F,G}(x_0,y_0)=\frac{1}{\omega(\frac{1}{2})}\min\{\varphi(x_0),\omega(\tfrac{1}{2})-\tau(x_0)+\varphi(x_0)-\varphi(y_0),\tau(y_0)-\tau(x_0)\}.$$

The comparison of first terms is straightforward:

$$\varphi_0(x_0)=\overline{\varphi}(x_0)\geq\varphi(x_0)\geq\omega(\tfrac{1}{2})\Lambda_{F,G}(x_0,y_0).$$

Next, estimate second terms:

$$\begin{aligned}\omega(\tfrac{1}{2})-\tau_0(x_0)+\varphi_0(x_0)-\varphi_0(y_0)&=\omega(\tfrac{1}{2})-\underline{\tau}(x_0)+\overline{\varphi}(x_0)-\max\{\overline{\varphi}(x_0),\underline{\varphi}(y_0)\}\\&=\omega(\tfrac{1}{2})-\underline{\tau}(x_0)+\min\{0,\overline{\varphi}(x_0)-\underline{\varphi}(y_0)\}\\&\geq\omega(\tfrac{1}{2})-\tau(x_0)+\min\{0,\varphi(x_0)-\varphi(y_0)\}\\&=\omega(\tfrac{1}{2})-\tau(x_0)+\varphi(x_0)-\varphi(y_0)\\&\geq\omega(\tfrac{1}{2})\Lambda_{F,G}(x_0,y_0).\end{aligned}$$

Only the last term of min in the definition of $\Lambda$ is left for us to estimate:

$$\begin{aligned}\tau_0(y_0)-\tau_0(x_0)&=\underline{\tau}(x_0)+h_\omega(x_0,y_0)-\underline{\tau}(x_0)\\&=h_\omega(x_0,y_0)\geq\tau(y_0)-\tau(x_0)\\&\geq\omega(\tfrac{1}{2})\Lambda_{F,G}(x_0,y_0).\end{aligned}$$

Hence, $\Lambda_{F_0,G_0}(x_0,y_0)\geq\Lambda_{F,G}(x_0,y_0)$, which concludes the proof. □

## 7. Asymmetry of copulas with given opposite diagonal section

In this section we prove our main result, Theorem 3.2. In the next two propositions we determine values of the asymmetry function $\Gamma_\omega$ defined in (1) for the points $(x,y)$ lying in the triangles $\mathscr{G}$ and $\mathscr{H}$.

**Proposition 7.1.** *Given $(x_0,y_0)\in\mathscr{G}$ and an opposite diagonal section $\omega\in\Omega$, it holds that*

$$\Gamma_\omega(x_0,y_0)=\min\{\overline{A}_\omega(x_0,\tfrac{1}{2}),\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0)+\overline{A}_\omega(x_0,\tfrac{1}{2})-\underline{A}_\omega(y_0,\tfrac{1}{2}),h_\omega(x_0,y_0)\}.$$

**Proof.** If $\omega(\frac{1}{2})=0$, then both sides of the equality are 0, so suppose that $\omega(\frac{1}{2})>0$. Let $C$ be any copula with $\omega_C=\omega$ and let $\varphi$ and $\tau$ be its partial horizontal and vertical sections as defined in Section 4. Let distribution functions $F$ and $G$ be defined as in Section 6. Then $C(x,y)=\omega(\frac{1}{2})H(x,y)$ for some distribution function $H$ with marginals $F$ and $G$. By Theorem 6.4 we have

$$C(x_0,y_0)-C(y_0,x_0)=\omega(\tfrac{1}{2})(H(x_0,y_0)-H(y_0,x_0))\leq\omega(\tfrac{1}{2})\Lambda_{F,G}(x_0,y_0).$$

Let $\varphi_0$ and $\tau_0$ be defined as in Section 5 and distribution functions $F_0$ and $G_0$ be defined as in Section 6. Then by Proposition 6.7 we have that $\Lambda_{F,G}(x_0,y_0)\leq\Lambda_{F_0,G_0}(x_0,y_0)$. This implies

$$C(x_0,y_0)-C(y_0,x_0)\leq\omega(\tfrac{1}{2})\Lambda_{F_0,G_0}(x_0,y_0)=\min\{\varphi_0(x_0),\omega(\tfrac{1}{2})-\tau_0(x_0)+\varphi_0(x_0)-\varphi_0(y_0),\tau_0(y_0)-\tau_0(x_0)\}.$$

Since $y_0\geq x_0$, we have by equations (7) and (8) that

$$\begin{aligned}\varphi_0(x_0)&=\overline{A}_\omega(x_0,\tfrac{1}{2}),\\\tau_0(x_0)&=\underline{A}_\omega(\tfrac{1}{2},x_0),\\\varphi_0(y_0)&=\max\{\overline{A}_\omega(x_0,\tfrac{1}{2}),\underline{A}_\omega(y_0,\tfrac{1}{2})\},\ \text{and}\\\tau_0(y_0)-\tau_0(x_0)&=h_\omega(x_0,y_0),\end{aligned}$$

so

$$C(x_0,y_0)-C(y_0,x_0)\leq\min\{\overline{A}_\omega(x_0,\tfrac{1}{2}),\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0),\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0)+\overline{A}_\omega(x_0,\tfrac{1}{2})-\underline{A}_\omega(y_0,\tfrac{1}{2}),h_\omega(x_0,y_0)\}.$$

Since

$$h_\omega(x_0,y_0)=\min_{t\in[y_0,\frac{1}{2}]}(\omega(1-t)-\underline{A}_\omega(1-t,x_0))\le\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0),$$

the second term can be omitted and we obtain

$$C(x_0,y_0)-C(y_0,x_0)\le\min\{\overline{A}_\omega(x_0,\tfrac{1}{2}),\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0)+\overline{A}_\omega(x_0,\tfrac{1}{2})-\underline{A}_\omega(y_0,\tfrac{1}{2}),h_\omega(x_0,y_0)\}.$$

Since $F_0$ and $G_0$ are continuous, there exists a distribution function $H_0$ with marginals $F_0$ and $G_0$ that attains the maximal asymmetry at the point $(x_0,y_0)$, see Theorem 6.4. The same theorem implies that

$$H_0(x_0,y_0)-H_0(y_0,x_0)=\Lambda_{F_0,G_0}(x_0,y_0).$$

Now, let $A:[0,\frac{1}{2}]^2\to\mathbb{I}$ be defined by $A(x,y)=\omega(\frac{1}{2})H_0(x,y)$. Then $A$ is grounded 2-increasing function with marginals $\varphi_0$ and $\tau_0$. By Proposition 5.6 there exists a copula with opposite diagonal section $\omega$, that matches with $A$ on the square $[0,\frac{1}{2}]^2$. It follows that

$$C(x_0,y_0)-C(y_0,x_0)=\omega(\tfrac{1}{2})\Lambda_{F_0,G_0}(x_0,y_0),$$

so the bound

$$\min\{\overline{A}_\omega(x_0,\tfrac{1}{2}),\omega(\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},x_0)+\overline{A}_\omega(x_0,\tfrac{1}{2})-\underline{A}_\omega(y_0,\tfrac{1}{2}),h_\omega(x_0,y_0)\}$$

is attained. □

**Proposition 7.2.** *Given $(x_0,y_0)\in\mathscr{H}$ and an opposite diagonal section $\omega\in\Omega$, it holds that*

$$\Gamma_\omega(x_0,y_0)=\overline{A}_\omega(x_0,y_0)-\underline{A}_\omega(y_0,x_0).$$

**Proof.** For any copula $C\in\mathcal{C}$ with $\omega_C=\omega$ we have

$$C(x_0,y_0)-C(y_0,x_0)\le\overline{A}_\omega(x_0,y_0)-\underline{A}_\omega(y_0,x_0),$$

so that

$$\Gamma_\omega(x_0,y_0)\le\overline{A}_\omega(x_0,y_0)-\underline{A}_\omega(y_0,x_0).$$

In order to prove the equality we need to find a copula where the bound is attained. Let

$$C(x,y)=\begin{cases}\frac{1}{\omega(\frac{1}{2})}\overline{A}_\omega(x,\frac{1}{2})\underline{A}_\omega(\frac{1}{2},y), & \text{if } (x,y)\in[0,\frac{1}{2}]^2;\\ \overline{A}_\omega(x,y), & \text{if } (x,y)\in[0,\frac{1}{2}]\times[\frac{1}{2},1]\cap\{(x,y)\in\mathbb{I}^2:\ x+y\le1\};\\ \underline{A}_\omega(x,y), & \text{if } (x,y)\in[\frac{1}{2},1]\times[0,1]\cup\{(x,y)\in\mathbb{I}^2:\ x+y\ge1\}.\end{cases}$$

In the case $\omega(\frac{1}{2})=0$ we understand the first expression to be 0. The function $C$ is well defined, since for every $x\in[0,\frac{1}{2}]$ and $y=\frac{1}{2}$ the first and the second expression are equal to $\overline{A}_\omega(x,\frac{1}{2})$, for every $y\in[0,\frac{1}{2}]$ and $x=\frac{1}{2}$ the first and the third expression are equal to $\underline{A}_\omega(\frac{1}{2},y)$, and for every $x\in[0,\frac{1}{2}]$ and $y=1-x$ the second and the third expression are equal to $\omega(x)$. It is obviously grounded and has uniform marginals, and we prove that it is 2-increasing in the same way as in Proposition 5.6. Since $(x_0,y_0)\in\mathscr{H}$, we have

$$C(x_0,y_0)-C(y_0,x_0)=\overline{A}_\omega(x_0,y_0)-\underline{A}_\omega(y_0,x_0),$$

so the bound is attained. □

We are now ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** Propositions 7.1 and 7.2 determine the function $\Gamma_\omega$ on triangles $\mathscr{G}$ and $\mathscr{H}$. Using reflections, the value of function $\Gamma_\omega$ on the remaining six triangles can be expressed in terms of the value of $\Gamma_\omega$ and $\Gamma_{\omega^t}$ on $\mathscr{G}$ and $\mathscr{H}$.

If $x,y\in\mathbb{I}, y\le x$, then

$$\begin{aligned}\Gamma_\omega(x,y)&=\max_{C\in\mathcal{C},\omega_C=\omega}(C(x,y)-C(y,x))=\max_{C\in\mathcal{C},\omega_C=\omega}(C^t(y,x)-C^t(x,y))\\&=\max_{C^t\in\mathcal{C},\omega^t_C=\omega}(C(y,x)-C(x,y))=\max_{C\in\mathcal{C},\omega_C=\omega^t}(C(y,x)-C(x,y))\\&=\Gamma_{\omega^t}(y,x).\end{aligned}$$

Furthermore, if $x,y\in\mathbb{I},x+y\ge1$, then $(1-y)+(1-x)\le1$ and

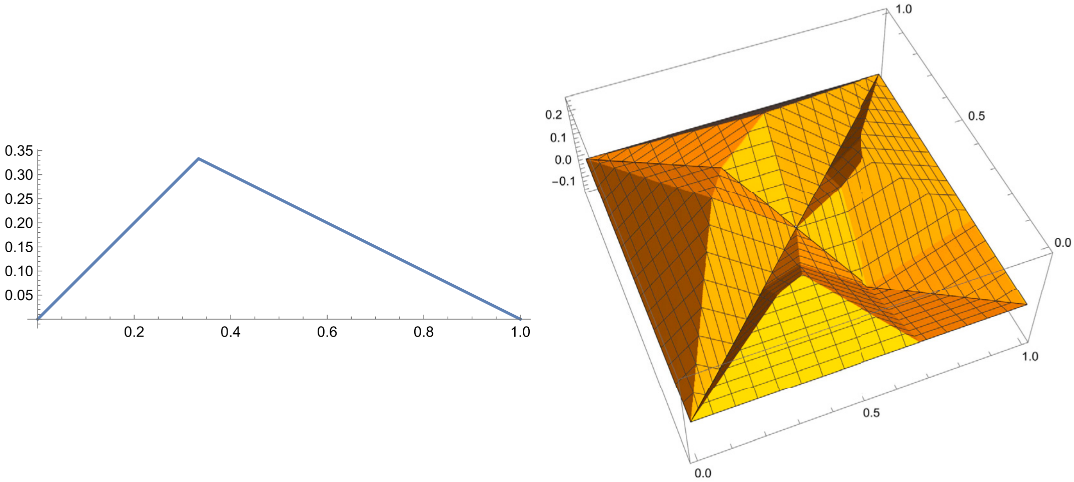



**Fig. 7.** The graphs of $\omega(x)$ (left) and $\Gamma_\omega(x,y)$ (right) from Example 7.3.

$$
\begin{aligned}
\Gamma_\omega(x,y) &= \max_{C\in\mathcal{C},\,\omega_C=\omega}(C(x,y)-C(y,x))\\
&= \max_{C\in\mathcal{C},\,\omega_C=\omega}((x+y-1+\widehat{C}(1-x,1-y))-(y+x-1+\widehat{C}(1-y,1-x)))\\
&= \max_{C\in\mathcal{C},\,\omega_C=\omega}(\widehat{C}(1-x,1-y)-\widehat{C}(1-y,1-x))\\
&= \max_{\widehat{C}\in\mathcal{C},\,\omega_{\widehat{C}}=\omega}(C(1-x,1-y)-C(1-y,1-x))\\
&= \max_{C\in\mathcal{C},\,\omega_C=\omega^t}(C(1-x,1-y)-C(1-y,1-x))\\
&= \Gamma_{\omega^t}(1-x,1-y)=\Gamma_\omega(1-y,1-x).
\end{aligned}
$$

This means that, when looking for maximal asymmetry $\gamma_\omega$ of all copulas with opposite diagonal section $\omega$, we need to maximize the function $\Gamma_\omega$ only over the lower triangle $x+y\le 1$ of the unit square, and if we consider also the function $\Gamma_{\omega^t}$, the triangles $\mathscr{G}$ and $\mathscr{H}$ are enough. It follows that

$$
\gamma_\omega=\max\left\{\max_{(x,y)\in\mathscr{G}}\Gamma_\omega(x,y),\ \max_{(x,y)\in\mathscr{H}}\Gamma_\omega(x,y),\ \max_{(x,y)\in\mathscr{G}}\Gamma_{\omega^t}(x,y),\ \max_{(x,y)\in\mathscr{H}}\Gamma_{\omega^t}(x,y)\right\},
$$

which finishes the proof. □

In the following example we apply the theorem to a specific opposite diagonal section.

**Example 7.3.** Let $\omega(x)=\min\{x,\frac{1}{2}(1-x)\}$. Fig. 7 shows the graphs of $\omega$ and $\Gamma_\omega$. The maximum of $\Gamma_\omega$ is attained at the point $x=\frac{1}{4}$ and $y=\frac{1}{2}$, where

$$
\Gamma_\omega(\tfrac{1}{4},\tfrac{1}{2})=\overline{A}_\omega(\tfrac{1}{4},\tfrac{1}{2})-\underline{A}_\omega(\tfrac{1}{2},\tfrac{1}{4})=\tfrac{1}{4}-0=\tfrac{1}{4}.
$$

It follows that $\gamma_\omega=\frac{1}{4}$.

The next example demonstrates how the maximal asymmetry varies when the opposite diagonal section is changing. We consider a one-parametric family of piecewise linear opposite diagonal sections and present a graph of the relationship between the parameter and the maximal asymmetry.

**Example 7.4.** For $t\in[0,\frac{1}{3}]$ let $\omega_t$ be a piecewise linear function with its graph connecting the points $(0,0)$, $(\frac{1}{3},\frac{1}{3})$, $(\frac{2}{3},t)$ and $(1,0)$, i.e.,

$$
\omega_t(x)=\begin{cases} x, & \text{if } 0\le x\le \frac{1}{3};\\ (3t-1)x+\frac{2}{3}-t, & \text{if } \frac{1}{3}\le x\le\frac{2}{3};\\ 3t(1-x), & \text{if } \frac{2}{3}\le x\le 1.\end{cases}
$$

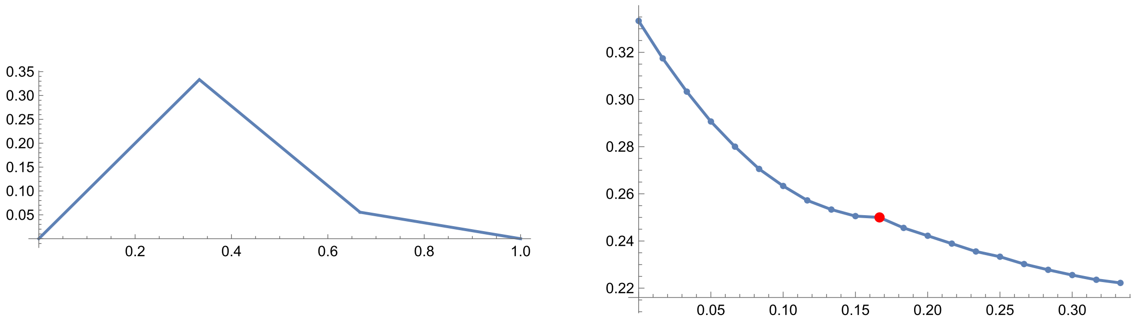



**Fig. 8.** The graphs of $\omega_t$ for $t = \frac{1}{18}$ (left) and $\gamma_{\omega_t}$ as a function of $t$ (right) from Example 7.4. The red point corresponds to the value of $\gamma_\omega$ from Example 7.3.

Fig. 8 shows the graph of $\omega_t$ for $t = \frac{1}{18}$ and the approximate graph of the value of $\gamma_{\omega_t}$ as a function of $t$. Using Mathematica software [17], we computed the approximate values of $\gamma_{\omega_t}$ for $t = \frac{i}{60}$, where $i = 0, 1, \dots, 20$. Note that at $t = 0$ we get $\gamma_{\omega_0} = \frac{1}{3}$, the maximal asymmetry over the set of all copulas. For $t = \frac{1}{6}$ we obtain $\omega$ from Example 7.3 which has $\gamma_\omega = \frac{1}{4}$ (the point in the graph is depicted in red). For $t = \frac{1}{3}$ we get $\gamma_{\omega_t} = \frac{2}{9}$, which can be exactly determined using the results from Section 8.

Note that the shape of the graph $\gamma_{\omega_t}$ as a function of $t$ changes at $t = \frac{1}{6}$, which is precisely when the function $\omega_t$ turns from convex to concave on the interval $[\frac{1}{3}, 1]$. This example demonstrates that the relationship between the parameter $t$ and the maximal asymmetry $\gamma_{\omega_t}$ depending on $t$ may be difficult to predict.

## 8. Simple symmetric opposite diagonal section

In this section we derive a formula for the maximal asymmetry of copulas with given opposite diagonal section $\omega$ in the case when $\omega$ is a *simple symmetric* opposite diagonal section, i.e., $\omega$ is increasing on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$, and $\omega(1-x) = \omega(x)$ for all $x \in \mathbb{I}$. In this case the formula is simple enough to be applied in practice.

Note that for a simple symmetric opposite diagonal section $\omega$ we have

$$\gamma_\omega = \max \left\{ \max_{(x,y)\in\mathscr{G}} \Gamma_\omega(x,y), \max_{(x,y)\in\mathscr{H}} \Gamma_\omega(x,y) \right\}$$

by Theorem 3.2. For every $(x, y)$ with $x + y \le 1$ we have

$$\overline{A}_\omega(x,y) = \min_{t\in[x,1-y]} \omega(t) = \min\{\omega(x), \omega(1-y)\}, \tag{23}$$

$$\underline{A}_\omega(x,y) = \max\left\{0, x+y-1+\tfrac{1}{2}\left(\omega(x)+\omega(1-y)+\mathrm{TV}_x^{1-y}(\omega)\right)\right\}. \tag{24}$$

Let $(x, y) \in \mathscr{H}$. Then

$$\begin{aligned} \overline{A}_\omega(x,y) &= \omega(x), \\ \underline{A}_\omega(y,x) &= \max\left\{0, x+y-1+\tfrac{1}{2}\left(\omega(y)+\omega(1-x)+\mathrm{TV}_y^{1-x}(\omega)\right)\right\} \\ &= \max\left\{0, x+y-1+\tfrac{1}{2}\left(\omega(y)+\omega(1-x)+\omega(y)-\omega(1-x)\right)\right\} \\ &= \max\left\{0, x+y-1+\omega(y)\right\}, \end{aligned} \tag{25}$$

since $\omega$ is increasing on $[0, \frac{1}{2}]$ and decreasing on $[\frac{1}{2}, 1]$, so that, by Theorem 3.2,

$$\begin{aligned} \Gamma_\omega(x,y) = \overline{A}_\omega(x,y) - \underline{A}_\omega(y,x) &= \omega(x) - \max\left\{0, x+y-1+\omega(y)\right\} \\ &= \min\left\{\omega(x), 1-x-y+\omega(x)-\omega(y)\right\} \\ &= \min\left\{\omega(x), \omega(x)-x+(1-y)-\omega(y)\right\}. \end{aligned}$$

Note that function $(1-y) - \omega(y)$ is decreasing in $y$ since $\omega$ is 1-Lipschitz, hence, $\Gamma_\omega(x, y)$ is also decreasing in $y$. This implies

$$\max_{(x,y)\in\mathscr{H}} \Gamma_\omega(x,y) = \max_{x\in[0,\frac{1}{2}]} \max_{y\in[\frac{1}{2},1-x]} \Gamma_\omega(x,y) = \max_{x\in[0,\frac{1}{2}]} \Gamma_\omega(x,\tfrac{1}{2}) = \max_{x\in[0,\frac{1}{2}]} \min\left\{\omega(x), \omega(x)-x+\tfrac{1}{2}-\omega(\tfrac{1}{2})\right\}.$$

Function $\omega(x)$ is increasing on $[0,\frac{1}{2}]$ with value 0 at $x=0$ while function $\omega(x)-x+\frac{1}{2}-\omega(\frac{1}{2})$ is decreasing on $[0,\frac{1}{2}]$ with value 0 at $x=\frac{1}{2}$. Hence, the above maximum is attained at the point where $\omega(x)=\omega(x)-x+\frac{1}{2}-\omega(\frac{1}{2})$, i.e. at $x=\frac{1}{2}-\omega(\frac{1}{2})$, and is equal to

$$\max_{(x,y)\in\mathscr{H}}\Gamma_\omega(x,y)=\omega\big(\tfrac{1}{2}-\omega(\tfrac{1}{2})\big).$$

For any $(x,y)\in[0,\frac{1}{2}]^2$ equation (24) implies

$$\underline{A}_\omega(x,y)=\max\left\{0,x+y-1+\tfrac{1}{2}\left(\omega(x)+\omega(1-y)+2\omega(\tfrac{1}{2})-\omega(x)-\omega(1-y)\right)\right\}=\max\left\{0,x+y-1+\omega(\tfrac{1}{2})\right\} \tag{26}$$

since $\omega$ is symmetric and simple and $x\le\frac{1}{2}\le 1-y$. Now let $(x,y)\in\mathscr{G}$. For any $t\in[y,\frac{1}{2}]$ we have $(x,t)\in\mathscr{G}$ and $(x,1-t)\in\mathscr{H}$, so by equation (25) we have

$$\underline{A}_\omega(1-t,x)=\max\{0,x-t+\omega(1-t)\}.$$

Thus, using equation (2), we obtain

$$\begin{aligned}h_\omega(x,y)&=\min_{t\in[y,\frac{1}{2}]}(\omega(1-t)-\underline{A}_\omega(1-t,x))=\min_{t\in[y,\frac{1}{2}]}(\omega(1-t)-\max\{0,x-t+\omega(1-t)\})\\&=\min_{t\in[y,\frac{1}{2}]}\min\{\omega(1-t),t-x\}=\min\{\min_{t\in[y,\frac{1}{2}]}\omega(1-t),\min_{t\in[y,\frac{1}{2}]}(t-x)\}.\end{aligned}$$

Function $\omega(1-t)$ is increasing on $[0,\frac{1}{2}]$, and so is $t-x$, hence

$$h_\omega(x,y)=\min\{\omega(1-y),y-x\}.$$

Furthermore, using equations (23) and (26), we have

$$\begin{aligned}&\overline{A}_\omega(x,\tfrac{1}{2})=\min\{\omega(x),\omega(\tfrac{1}{2})\}=\omega(x),\\&\underline{A}_\omega(\tfrac{1}{2},x)=\max\{0,x-\tfrac{1}{2}+\omega(\tfrac{1}{2})\},\\&\underline{A}_\omega(y,\tfrac{1}{2})=\max\{0,y-\tfrac{1}{2}+\omega(\tfrac{1}{2})\},\end{aligned}$$

so by Theorem 3.2 we get

$$\begin{aligned}\Gamma_\omega(x,y)&=\min\left\{\omega(x),\omega(\tfrac{1}{2})-\max\{0,x-\tfrac{1}{2}+\omega(\tfrac{1}{2})\}+\omega(x)-\max\{0,y-\tfrac{1}{2}+\omega(\tfrac{1}{2})\},\min\{\omega(1-y),y-x\}\right\}\\&=\min\left\{\omega(x),\min\{\omega(\tfrac{1}{2}),\tfrac{1}{2}-x\}+\min\{\omega(x),\omega(x)-y+\tfrac{1}{2}-\omega(\tfrac{1}{2})\},\min\{\omega(1-y),y-x\}\right\}\\&=\min\left\{\omega(x),\ \omega(\tfrac{1}{2})+\omega(x),\ \tfrac{1}{2}-x+\omega(x),\ \omega(x)-y+\tfrac{1}{2},1-x-y+\omega(x)-\omega(\tfrac{1}{2}),\ \omega(1-y),\ y-x\right\}.\end{aligned}$$

Note that $\omega(x)$ is smaller than any of the expressions $\omega(x)+\omega(\frac{1}{2})$, $\frac{1}{2}-x+\omega(x)$, and $\omega(x)-y+\frac{1}{2}$. Furthermore, $\omega(1-y)=\omega(y)\ge\omega(x)$ since $\omega$ is symmetric, increasing on $[0,\frac{1}{2}]$, and $0\le x\le y\le\frac{1}{2}$. Therefore,

$$\Gamma_\omega(x,y)=\min\left\{\omega(x),1-x-y+\omega(x)-\omega(\tfrac{1}{2}),y-x\right\},$$

and

$$\max_{(x,y)\in\mathscr{G}}\Gamma_\omega(x,y)=\max_{x\in[0,\frac{1}{2}]}\max_{y\in[x,\frac{1}{2}]}\min\left\{\omega(x),1-x-y+\omega(x)-\omega(\tfrac{1}{2}),y-x\right\}.$$

For a fixed $x\in[0,\frac{1}{2}]$, function $y-x$ is increasing in $y$ while function $1-x-y+\omega(x)-\omega(\frac{1}{2})$ is decreasing in $y$. Their graphs intersect at $y=\frac{1}{2}(1+\omega(x)-\omega(\frac{1}{2}))\in[x,\frac{1}{2}]$, since $\omega(x)\le\omega(\frac{1}{2})$ and $\omega(\frac{1}{2})-\omega(x)\le\frac{1}{2}-x$, so that $y\ge\frac{1}{2}(\frac{1}{2}+x)\ge\frac{1}{2}(x+x)=x$. Their value at the intersection is $\frac{1}{2}+\frac{1}{2}\omega(x)-\frac{1}{2}\omega(\frac{1}{2})-x$, hence,

$$\max_{y\in[x,\frac{1}{2}]}\min\left\{\omega(x),1-x-y+\omega(x)-\omega(\tfrac{1}{2}),y-x\right\}=\min\{\omega(x),\tfrac{1}{2}+\tfrac{1}{2}\omega(x)-\tfrac{1}{2}\omega(\tfrac{1}{2})-x\}.$$

Function $\omega(x)$ is increasing on $[0,\frac{1}{2}]$ with value 0 at $x=0$ while function $\frac{1}{2}+\frac{1}{2}\omega(x)-\frac{1}{2}\omega(\frac{1}{2})-x$ is decreasing on $[0,\frac{1}{2}]$ with value 0 at $x=\frac{1}{2}$. Being continuous functions, their graphs intersect at some point $x_1\in[0,\frac{1}{2}]$ that satisfies the equation

$$\omega(x_1)=1-\omega(\tfrac{1}{2})-2x_1.$$

Note that $x_1$ is unique since the left hand side of this equation is increasing in $x_1$ and the right hand side is strictly decreasing. This implies

$$\max_{(x,y)\in\mathscr{G}} \Gamma_\omega(x,y) = \max_{x\in[0,\frac{1}{2}]} \min\{\omega(x), \tfrac{1}{2} + \tfrac{1}{2}\omega(x) - \tfrac{1}{2}\omega(\tfrac{1}{2}) - x\} = \omega(x_1).$$

Note that $x_1 = \frac{1}{2}(1 - \omega(\frac{1}{2}) - \omega(x_1)) \geq \frac{1}{2}(1 - \omega(\frac{1}{2}) - \omega(\frac{1}{2})) = \frac{1}{2} - \omega(\frac{1}{2})$, hence,

$$\gamma_\omega = \max\{\omega(x_1), \omega(\tfrac{1}{2} - \omega(\tfrac{1}{2}))\} = \omega(x_1).$$

This proves the following.

**Theorem 8.1.** *Let $\omega \in \Omega$ be a simple symmetric opposite diagonal section. Then the maximal asymmetry of copulas with opposite diagonal section $\omega$ is equal to*

$$\gamma_\omega = \omega(x_1),$$

*where $x_1 \in [0, \frac{1}{2}]$ is the unique solution of the equation*

$$\omega(x) = 1 - \omega(\tfrac{1}{2}) - 2x.$$

We finish the paper by calculating the maximal asymmetry of all copulas with opposite diagonal section equal to the opposite diagonal section of the product copula.

**Example 8.2.** Let $\omega(x) = \omega_\Pi(x) = x(1-x)$, which is a simple symmetric opposite diagonal section. The equation $\omega(x) = 1 - \omega(\frac{1}{2}) - 2x$ is equivalent to $x^2 - 3x + \frac{3}{4} = 0$ and has a solution $x_1 = \frac{3-\sqrt{6}}{2} \in [0, \frac{1}{2}]$, so that

$$\gamma_{\omega_\Pi} = \omega(x_1) = \sqrt{6} - \tfrac{9}{4} \approx 0.1995.$$

Note, however, that the $y$-reflection transforms $\omega$ into $\delta(x) = \delta_\Pi(x) = x^2$. In this case the maximal asymmetry of copulas with diagonal section $\delta$ equals $\mu_{\delta_\Pi} = \frac{15}{64} \approx 0.2344$, as shown in [13, Example 4.11].

## 9. Concluding remarks

We have considered a class of bivariate copulas with a prescribed opposite diagonal section. For such copulas, we have determined the exact lower bound. Using this result, we have derived a formula for their maximal asymmetry in Theorem 3.2. In practical applications, copulas often have a symmetric and unimodal opposite diagonal section. In this case our formula simplifies significantly, see Theorem 8.1.

In Section 6 we have found the maximal asymmetry of all distributions with fixed marginals. The maximal value is attained whenever marginal distributions are continuous or discrete with finite range. In the discrete case, our results might have applications to contingency tables.

Future work on the subject could include the study of maximal asymmetry of all copulas with prescribed values on a given set. For example, instead of a diagonal or an opposite diagonal section one could consider copulas with a fixed value at a single point or copulas with prescribed vertical or horizontal sections. One could also consider a generalization of the problem studied in this paper. In particular, one could replace transposition $C \mapsto C^t$ with an involution $\kappa$ on the set of bivariate copulas and investigate the difference between the copula $C$ from some special class and its image $\kappa(C)$.

## CRediT authorship contribution statement

**Damjana Kokol Bukovšek:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Investigation, Conceptualization. **Blaž Mojškerc:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Investigation, Conceptualization. **Nik Stopar:** Writing – review & editing, Writing – original draft, Visualization, Methodology, Investigation, Conceptualization.

## Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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## Appendix A. Proofs of selected results

In the first appendix we give the proofs of Lemmas 4.2, 4.4, 5.3, and 5.4.

### A.1. Proof of Lemma 4.2

Suppose first that $q(x) > \frac{1}{2}$. It follows that $\underline{A}_\omega(\frac{1}{2}, x) = 0$. By the definition of $q(x)$ we have $\underline{A}_\omega(t, x) > 0$ for any $t \in (q(x), 1 - x]$. Since, for any $t \le 1 - x$,

$$\underline{A}_\omega(t, x) = \max\left\{0, t + x - 1 + \tfrac{1}{2}\left(\omega(t) + \omega(1-x) + \mathrm{TV}_t^{1-x}(\omega)\right)\right\},$$

we have

$$\underline{A}_\omega(q(x), x) = 0 = q(x) + x - 1 + \tfrac{1}{2}\left(\omega(q(x)) + \omega(1-x) + \mathrm{TV}_{q(x)}^{1-x}(\omega)\right).$$

For any $t \in [q(x), 1 - x]$ we have

$$\begin{aligned}
\omega(t) - \underline{A}_\omega(t, x) &= \omega(t) - \left(t + x - 1 + \tfrac{1}{2}\left(\omega(t) + \omega(1-x) + \mathrm{TV}_t^{1-x}(\omega)\right)\right) \\
&= 1 - x - t + \tfrac{1}{2}\left(\omega(t) - \omega(1-x) - \mathrm{TV}_t^{1-x}(\omega)\right) \\
&= 1 - x - t + \tfrac{1}{2}\left(\omega(t) - \omega(1-x) - \mathrm{TV}_t^{1-x}(\omega)\right) + q(x) + x - 1 + \tfrac{1}{2}\left(\omega(q(x)) + \omega(1-x) + \mathrm{TV}_{q(x)}^{1-x}(\omega)\right) \\
&= q(x) - t + \tfrac{1}{2}\left(\omega(q(x)) + \omega(t) + \mathrm{TV}_{q(x)}^{t}(\omega)\right) \\
&= \underline{A}_\omega(q(x), 1 - t).
\end{aligned}$$

This implies that we have for any $y \in [x, 1 - q(x)]$

$$\begin{aligned}
h_\omega(x, y) &= \min_{t \in [y, \frac{1}{2}]} \left(\omega(1-t) - \underline{A}_\omega(1-t, x)\right) \\
&= \min_{t \in [\frac{1}{2}, 1-y]} \left(\omega(t) - \underline{A}_\omega(t, x)\right) \\
&= \min\{\min_{t \in [\frac{1}{2}, q(x)]} \left(\omega(t) - \underline{A}_\omega(t, x)\right), \min_{t \in [q(x), 1-y]} \left(\omega(t) - \underline{A}_\omega(t, x)\right)\} \\
&= \min\{\min_{t \in [\frac{1}{2}, q(x)]} \omega(t), \min_{t \in [q(x), 1-y]} \underline{A}_\omega(q(x), 1-t)\} \\
&= \min\{\overline{A}_\omega(\tfrac{1}{2}, 1 - q(x)), \underline{A}_\omega(q(x), y)\}.
\end{aligned}$$

Furthermore, we have for any $y \in [1 - q(x), \frac{1}{2}]$

$$\begin{aligned}
h_\omega(x, y) &= \min_{t \in [y, \frac{1}{2}]} \left(\omega(1-t) - \underline{A}_\omega(1-t, x)\right) = \min_{t \in [\frac{1}{2}, 1-y]} \left(\omega(t) - \underline{A}_\omega(t, x)\right) \\
&= \min_{t \in [\frac{1}{2}, 1-y]} \omega(t) = \overline{A}_\omega(\tfrac{1}{2}, y).
\end{aligned}$$

Note that $\underline{A}_\omega(q(x), 1 - q(x)) = \omega(q(x)) - \underline{A}_\omega(q(x), x)$ by above, thus

$$\underline{A}_\omega(q(x), 1 - q(x)) = \omega(q(x)) = \overline{A}_\omega(q(x), 1 - q(x)) \ge \overline{A}_\omega(\tfrac{1}{2}, 1 - q(x)),$$

so that for $y = 1 - q(x)$ both expressions match.

Suppose now that $q(x) = \frac{1}{2}$. Then $\underline{A}_\omega(t, x) > 0$ for every $t \in (\frac{1}{2}, 1 - x]$. It follows that for any such $t$ and thus also for $t = \frac{1}{2}$ the value of $\underline{A}_\omega(t, x)$ equals the second expression in the maximum, i.e.,

$$\underline{A}_\omega(\tfrac{1}{2}, x) = x - \tfrac{1}{2} + \tfrac{1}{2}\left(\omega(\tfrac{1}{2}) + \omega(1-x) + \mathrm{TV}_{1/2}^{1-x}(\omega)\right),$$

and

$$\underline{A}_\omega(t, x) = t + x - 1 + \tfrac{1}{2}\left(\omega(t) + \omega(1-x) + \mathrm{TV}_t^{1-x}(\omega)\right)$$

for any $t \in [\frac{1}{2}, 1 - x]$. Now, for any $y \in [x, \frac{1}{2}]$ and $t \in [y, \frac{1}{2}]$ we have

$$\begin{aligned}\omega(1-t)-\underline{A}_\omega(1-t,x) &= \omega(1-t)-\left(x-t+\tfrac{1}{2}\left(\omega(1-t)+\omega(1-x)+\mathrm{TV}_{1-t}^{1-x}(\omega)\right)\right)\\ &= \omega(1-y)-\left(x-y+\tfrac{1}{2}\left(\omega(1-y)+\omega(1-x)+\mathrm{TV}_{1-y}^{1-x}(\omega)\right)\right)\\ &\quad + t-y+\tfrac{1}{2}\left(\omega(1-t)-\omega(1-y)-\mathrm{TV}_{1-t}^{1-y}(\omega)\right)\\ &= \omega(1-y)-\underline{A}_\omega(1-y,x)+\tfrac{1}{2}\left(t-y-(\omega(1-y)-\omega(1-t))\right)+\tfrac{1}{2}\left(t-y-\mathrm{TV}_{1-t}^{1-y}(\omega)\right)\\ &\geq \omega(1-y)-\underline{A}_\omega(1-y,x),\end{aligned}$$

since the function $\omega$ is 1-Lipschitz and

$$\mathrm{TV}_{1-t}^{1-y}(\omega) \leq (1-y)-(1-t)=t-y.$$

This implies

$$h_\omega(x,y)=\min_{t\in[y,\frac{1}{2}]}\left(\omega(1-t)-\underline{A}_\omega(1-t,x)\right)=\omega(1-y)-\underline{A}_\omega(1-y,x),$$

which finishes the proof. □

### A.2. *Proof of Lemma 4.4*

Denote with $\mathscr{G}=\{(x,y)\in[0,\frac{1}{2}]^2:\ x\leq y\}$ the domain of the function $h_\omega$. First, we show that $h_\omega(x,y)\geq 0$ for all $(x,y)\in\mathscr{G}$. Indeed, take some $t\in[y,\frac{1}{2}]$. Then

$$\omega(1-t)-\underline{A}_\omega(1-t,x)=\underline{A}_\omega(1-t,t)-\underline{A}_\omega(1-t,x)\geq 0$$

since $x\leq t$ and $\underline{A}_\omega$ is a copula, hence increasing in the second variable. Taking the minimum over all $t\in[y,\frac{1}{2}]$, gives nonnegativity of $h_\omega$.

Next, we prove that $h_\omega(x,x)=0$ for all $x\in[0,\frac{1}{2}]$. Indeed, take some $x\in[0,\frac{1}{2}]$. Then

$$0\leq h_\omega(x,x)=\min_{t\in[x,\frac{1}{2}]}\left(\omega(1-t)-\underline{A}_\omega(1-t,x)\right)\leq \omega(1-x)-\underline{A}_\omega(1-x,x)=0,$$

and we are done.

To show the increasingness of $h_\omega$ in the second variable, take some $0\leq x\leq y_1\leq y_2\leq\frac{1}{2}$. Then

$$h_\omega(x,y_1)=\min_{t\in[y_1,\frac{1}{2}]}\left(\omega(1-t)-\underline{A}_\omega(1-t,x)\right)\leq \min_{t\in[y_2,\frac{1}{2}]}\left(\omega(1-t)-\underline{A}_\omega(1-t,x)\right)=h_\omega(x,y_2),$$

since the second minimum is taken over a subinterval of the interval, over which the first minimum is taken.

Lastly, we show that the function $h_\omega$ is 1-Lipschitz in the second variable. To this end, take some $0\leq x\leq y_1\leq y_2\leq\frac{1}{2}$. Consider first the case $q(x)=\frac{1}{2}$ from Lemma 4.2. Then the function $h_\omega$ can be written as

$$\begin{aligned}h_\omega(x,y_2) &= \omega(1-y_2)-\underline{A}_\omega(1-y_2,x)\\ &= \omega(1-y_2)-(1-y_2+x-1)-\tfrac{1}{2}\left(\omega(1-y_2)+\omega(1-x)+\mathrm{TV}_{(1-x)\wedge(1-y_2)}^{(1-x)\vee(1-y_2)}(\omega)\right)\\ &= \omega(1-y_2)-(x-y_2)-\tfrac{1}{2}\left(\omega(1-y_2)+\omega(1-x)+\mathrm{TV}_{1-y_2}^{1-x}(\omega)\right).\end{aligned}$$

Similarly, we can write $h_\omega(x,y_1)$. Then

$$\begin{aligned}h_\omega(x,y_2)-h_\omega(x,y_1) &= \omega(1-y_2)-(x-y_2)-\tfrac{1}{2}\left(\omega(1-x)+\omega(1-y_2)+\mathrm{TV}_{1-y_2}^{1-x}(\omega)\right)\\ &\quad -\omega(1-y_1)+(x-y_1)+\tfrac{1}{2}\left(\omega(1-x)+\omega(1-y_1)+\mathrm{TV}_{1-y_1}^{1-x}(\omega)\right)\\ &= \tfrac{1}{2}(\omega(1-y_2)-\omega(1-y_1))+(y_2-y_1)+\tfrac{1}{2}\left(\mathrm{TV}_{1-y_1}^{1-x}(\omega)-\mathrm{TV}_{1-y_2}^{1-x}(\omega)\right)\\ &= \tfrac{1}{2}\left(-\mathrm{TV}_{1-y_2}^{1-y_1}(\omega)-(\omega(1-y_1)-\omega(1-y_2))\right)+(y_2-y_1).\end{aligned}$$

Recall that $|\omega(1-y_1)-\omega(1-y_2)|\leq\mathrm{TV}_{1-y_2}^{1-y_1}(\omega)$, hence

$$\tfrac{1}{2}\left(-\mathrm{TV}_{1-y_2}^{1-y_1}(\omega)-(\omega(1-y_1)-\omega(1-y_2))\right)\leq 0.$$

Then $h_\omega(x,y_2)-h_\omega(x,y_1)\leq y_2-y_1$, which implies 1-Lipschitz property of $h_\omega$ in the second variable.

In the case of $q(x) > \frac{1}{2}$ from Lemma 4.2, we have

$$h_\omega(x,y) = \begin{cases} \min\{\overline{A}_\omega(\frac{1}{2}, 1-q(x)), \underline{A}_\omega(q(x), y)\}, & \text{if } x \le y \le 1-q(x); \\ \overline{A}_\omega(\frac{1}{2}, y), & \text{if } 1-q(x) \le y \le \frac{1}{2}. \end{cases}$$

The first expression in the minimum is constant with respect to $y$, while the second is 1-Lipschitz, hence the minimum over both is 1-Lipschitz. The function $\overline{A}_\omega(\frac{1}{2}, y)$ is also 1-Lipschitz in $y$, which concludes the proof. □

### A.3. Proof of Lemma 5.3

We have $\Theta = \overline{A}_\omega(\hat{x}, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x}) \ge 0$. The claim is obvious if $\Theta = 0$, so suppose that $\Theta > 0$. The function $s$ is the sum of two terms, the first one is clearly 2-increasing. The second term is a product of two functions, the first one depending only on $x$ and the second only on $y$, which we will show are increasing. The 2-increasingness of $s$ will follow. The first factor, $\overline{A}_\omega(x, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x})$, is obviously increasing in $x$. To show increasingness of the second factor $s_2(y) = \underline{A}_\omega(\hat{x}, y) - \overline{A}_\omega(x_0, y)$, we choose $y_1, y_2$, such that $\frac{1}{2} \le y_1 \le y_2 \le 1-\hat{x}$. If $\underline{A}_\omega(\hat{x}, \frac{1}{2}) = \underline{\varphi}(\hat{x}) = \overline{\varphi}(x_0) = 0$, then $\hat{x} = x_0$, so $\Theta = 0$, a contradiction. Hence, $\underline{A}_\omega(\hat{x}, \frac{1}{2}) > 0$. Then $\underline{A}_\omega(\hat{x}, y_1) \ge \underline{A}_\omega(\hat{x}, \frac{1}{2}) > 0$, so that

$$\begin{aligned} \underline{A}_\omega(\hat{x}, y_1) &= \max\left\{ W(\hat{x}, y_1), \min\{0, \hat{x} + y_1 - 1\} + \tfrac{1}{2}\left(\omega(\hat{x}) + \omega(1-y_1) + \mathrm{TV}_{\hat{x}\wedge(1-y_1)}^{\hat{x}\vee(1-y_1)}(\omega)\right)\right\} \\ &= \hat{x} + y_1 - 1 + \tfrac{1}{2}\left(\omega(\hat{x}) + \omega(1-y_1) + \mathrm{TV}_{\hat{x}}^{1-y_1}(\omega)\right) \text{ and} \\ \overline{A}_\omega(x_0, y_1) &= W(x_0, y_1) + \min_{t\in[x_0\wedge(1-y_1), x_0\vee(1-y_1)]} \omega(t) = \min_{t\in[x_0, 1-y_1]} \omega(t). \end{aligned}$$

Similarly,

$$\begin{aligned} \underline{A}_\omega(\hat{x}, y_2) &= \hat{x} + y_2 - 1 + \tfrac{1}{2}\left(\omega(\hat{x}) + \omega(1-y_2) + \mathrm{TV}_{\hat{x}}^{1-y_2}(\omega)\right) \text{ and} \\ \overline{A}_\omega(x_0, y_2) &= \min_{t\in[x_0, 1-y_2]} \omega(t). \end{aligned}$$

Now we calculate

$$\begin{aligned} s_2(y_2) - s_2(y_1) &= \underline{A}_\omega(\hat{x}, y_2) - \overline{A}_\omega(x_0, y_2) - \underline{A}_\omega(\hat{x}, y_1) + \overline{A}_\omega(x_0, y_1) \\ &= y_2 - y_1 + \tfrac{1}{2}\left(\omega(1-y_2) - \omega(1-y_1) - \mathrm{TV}_{1-y_2}^{1-y_1}(\omega)\right) - \min_{t\in[x_0, 1-y_2]} \omega(t) + \min_{t\in[x_0, 1-y_1]} \omega(t). \end{aligned}$$

Let $t_1 \in [x_0, 1-y_1]$ and $t_2 \in [x_0, 1-y_2]$ be such that

$$\min_{t\in[x_0,1-y_1]} \omega(t) = \omega(t_1) \quad \text{and} \quad \min_{t\in[x_0,1-y_2]} \omega(t) = \omega(t_2).$$

Obviously, $t_2 \le t_1$. If $t_1 = t_2$, then

$$\begin{aligned} s_2(y_2) - s_2(y_1) &= y_2 - y_1 + \tfrac{1}{2}\left(\omega(1-y_2) - \omega(1-y_1) - \mathrm{TV}_{1-y_2}^{1-y_1}(\omega)\right) \\ &= \tfrac{1}{2}\left(y_2 - y_1 - (\omega(1-y_1) - \omega(1-y_2))\right) + \tfrac{1}{2}\left(y_2 - y_1 - \mathrm{TV}_{1-y_2}^{1-y_1}(\omega)\right) \ge 0, \end{aligned}$$

since the function $\omega$ is 1-Lipschitz and

$$\mathrm{TV}_{1-y_2}^{1-y_1}(\omega) \le (1-y_1) - (1-y_2) = y_2 - y_1,$$

hence, we are done. If $t_2 < t_1$, it holds that $t_2 \le 1-y_2 < t_1 \le 1-y_1$. Now,

$$\begin{aligned} s_2(y_2) - s_2(y_1) &= y_2 - y_1 + \tfrac{1}{2}\left(\omega(1-y_2) - \omega(1-y_1) - \mathrm{TV}_{1-y_2}^{1-y_1}(\omega)\right) - \omega(t_2) + \omega(t_1) \\ &\ge y_2 - y_1 + \tfrac{1}{2}\left(\omega(1-y_2) - \omega(1-y_1) - y_2 + y_1\right) - \omega(t_2) + \omega(t_1) \\ &= \tfrac{1}{2}\left(y_2 - y_1 + \omega(1-y_2) - \omega(1-y_1)\right) - \omega(t_2) + \omega(t_1) \\ &\ge \tfrac{1}{2}\left(y_2 - y_1 + \omega(1-y_2) - \omega(1-y_1)\right) - \omega(1-y_2) + \omega(t_1) \\ &= \tfrac{1}{2}\left(y_2 - y_1 + (\omega(t_1) - \omega(1-y_2)) - (\omega(1-y_1) - \omega(t_1))\right) \\ &\ge \tfrac{1}{2}\left(y_2 - y_1 - (t_1 - (1-y_2)) - (1-y_1-t_1)\right) = 0, \end{aligned}$$

which finishes the proof. □

### A.4. Proof of Lemma 5.4

As in the proof of Lemma 5.3 we have $\Theta' \geq 0$ and we assume without loss of generality that $\Theta' > 0$. It follows that $\tilde{x} > \frac{1}{2}$ and $\tau_0(x_0) = 0$. The function $\overline{A}_\omega(x, 1-\tilde{x}) - \tau_0(1-\tilde{x})$ is increasing in $x$, we need to prove that $r_2(y) = \underline{A}_\omega(\tilde{x}, y) - \tau_0(y)$ is increasing in $y$. Since $y \leq 1-\tilde{x}$, Lemma 4.2 implies that

$$\begin{aligned} r_2(y) &= \underline{A}_\omega(\tilde{x}, y) - \tau_0(y) = \underline{A}_\omega(\tilde{x}, y) - \underline{\tau}(x_0) - h_\omega(x_0, y) = \underline{A}_\omega(\tilde{x}, y) - \min\{\overline{A}_\omega(\tfrac{1}{2}, 1-\tilde{x}), \underline{A}_\omega(\tilde{x}, y)\} \\ &= \max\{\underline{A}_\omega(\tilde{x}, y) - \overline{A}_\omega(\tfrac{1}{2}, 1-\tilde{x}), 0\}, \end{aligned}$$

is increasing, which finishes the proof. □

## Appendix B. Proof of Proposition 5.6

In this appendix we finalize the proof of Proposition 5.6 by showing that the copula $C$ given in (14) is well defined.

### B.1. Proof that copula C is well defined

We consider all line segments where the regions $\mathscr{A}, \mathscr{B}, \mathscr{D}, \mathscr{E}, \mathscr{F}$ intersect, see Fig. 4. In the case $\varphi_0(x_0) > 0$ these line segments are:

(i) $\{(x, y) \in \mathbb{I}^2; x \in [0, x_0], y = \frac{1}{2}\} \subseteq \mathscr{A} \cap \mathscr{B}$: We have

$$A(x, \tfrac{1}{2}) = \varphi_0(x) = \overline{\varphi}(x) = \overline{A}_\omega(x, \tfrac{1}{2}).$$

(ii) $\{(x, y) \in \mathbb{I}^2; x \in [x_0, \hat{x}], y = \frac{1}{2}\} \subseteq \mathscr{A} \cap \mathscr{E}$: If $\Theta > 0$, the second term in (10) equals 0 because of

$$\underline{A}_\omega(\hat{x}, \tfrac{1}{2}) - \overline{A}_\omega(x_0, \tfrac{1}{2}) = \underline{\varphi}(\hat{x}) - \overline{\varphi}(x_0) = 0.$$

It follows that we have for any $\Theta \geq 0$

$$A(x, \tfrac{1}{2}) = \varphi_0(x) = \overline{\varphi}(x_0) = \overline{A}_\omega(x_0, \tfrac{1}{2}) = s(x, \tfrac{1}{2}).$$

(iii) $\{(x, y) \in \mathbb{I}^2; x \in [\hat{x}, \frac{1}{2}], y = \frac{1}{2}\} \subseteq \mathscr{A} \cap \mathscr{D}$: We have

$$A(x, \tfrac{1}{2}) = \varphi_0(x) = \underline{\varphi}(x) = \underline{A}_\omega(x, \tfrac{1}{2}).$$

(iv) $\{(x, y) \in \mathbb{I}^2; x = \frac{1}{2}, y \in [1-\tilde{x}, \frac{1}{2}]\} \subseteq \mathscr{A} \cap \mathscr{B}$: If $\tilde{x} = \frac{1}{2}$, then $y = \frac{1}{2}$ and

$$A(\tfrac{1}{2}, \tfrac{1}{2}) = \omega(\tfrac{1}{2}) = \underline{A}_\omega(\tfrac{1}{2}, \tfrac{1}{2}).$$

If $\tilde{x} > \frac{1}{2}$, then $\tau_0(x_0) = 0$, thus

$$A(\tfrac{1}{2}, y) = \tau_0(y) = h_\omega(x_0, y) = \overline{A}_\omega(\tfrac{1}{2}, y)$$

by Lemma 4.2.

(v) $\{(x, y) \in \mathbb{I}^2; x = \frac{1}{2}, y \in [x_0, 1-\tilde{x}]\} \subseteq \mathscr{A} \cap \mathscr{F}$: If $\Theta' > 0$, the second term in (13) equals 0, because $\tau_0(1-\tilde{x}) = \overline{A}_\omega(\frac{1}{2}, 1-\tilde{x})$ as proved above. It follows that we have for any $\Theta' \geq 0$

$$A(\tfrac{1}{2}, y) = \tau_0(y) = r(\tfrac{1}{2}, y).$$

(vi) $\{(x, y) \in \mathbb{I}^2; x = \frac{1}{2}, y \in [0, x_0]\} \subseteq \mathscr{A} \cap \mathscr{D}$: We have

$$A(\tfrac{1}{2}, y) = \tau_0(y) = \underline{\tau}(y) = \underline{A}_\omega(\tfrac{1}{2}, y).$$

(vii) $\{(x, y) \in \mathbb{I}^2; x = x_0, y \in [\frac{1}{2}, 1-\hat{x}]\} \subseteq \mathscr{E} \cap \mathscr{B}$: If $\Theta > 0$, the second term in (10) equals 0, so we have

$$s(x_0, y) = \overline{A}_\omega(x_0, y)$$

for any $\Theta \geq 0$.

(viii) $\{(x, y) \in \mathbb{I}^2; x \in [x_0, \hat{x}], y = 1-\hat{x}\} \subseteq \mathscr{E} \cap \mathscr{B}$: If $\Theta > 0$, we have

$$\begin{aligned} s(x, 1-\hat{x}) &= \overline{A}_\omega(x_0, 1-\hat{x}) + \frac{1}{\Theta}\left(\overline{A}_\omega(x, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x})\right)\left(\underline{A}_\omega(\hat{x}, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x})\right) \\ &= \overline{A}_\omega(x_0, 1-\hat{x}) + \overline{A}_\omega(x, 1-\hat{x}) - \overline{A}_\omega(x_0, 1-\hat{x}) = \overline{A}_\omega(x, 1-\hat{x}), \end{aligned}$$

since $\underline{A}_\omega(\hat{x},1-\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})=\omega(\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})=\Theta$. If $\Theta=0$, we have

$$s(x,1-\hat{x})=\overline{A}_\omega(x_0,1-\hat{x})=\omega(\hat{x})=\overline{A}_\omega(x,1-\hat{x}),$$

since $\Theta=0$ implies that the function $\overline{A}_\omega(x,1-\hat{x})$ is constant for $x\in[x_0,\hat{x}]$.

(ix) $\{(x,y)\in\mathbb{I}^2; x=\hat{x}, y\in[\frac{1}{2},1-\hat{x}]\}\subseteq\mathscr{E}\cap\mathscr{D}$: If $\Theta>0$, we have

$$\begin{aligned}s(\hat{x},y)&=\overline{A}_\omega(x_0,y)+\frac{1}{\Theta}\left(\overline{A}_\omega(\hat{x},1-\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})\right)\left(\underline{A}_\omega(\hat{x},y)-\overline{A}_\omega(x_0,y)\right)\\&=\overline{A}_\omega(x_0,y)+\underline{A}_\omega(\hat{x},y)-\overline{A}_\omega(x_0,y)=\underline{A}_\omega(\hat{x},y),\end{aligned}$$

since $\overline{A}_\omega(\hat{x},1-\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})=\omega(\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})=\Theta$. If $\Theta=0$, then $\omega(\hat{x})=\underline{A}_\omega(\hat{x},1-\hat{x})=\overline{A}_\omega(x_0,1-\hat{x})$. It was proved in the proof of Lemma 5.3 that under the condition that $\overline{A}_\omega(x_0,\frac{1}{2})=\underline{A}_\omega(\hat{x},\frac{1}{2})>0$ the function $s_2(y)=\underline{A}_\omega(\hat{x},y)-\overline{A}_\omega(x_0,y)$ is increasing in $y$ on the interval $[\frac{1}{2},1-\hat{x}]$. Since $s_2(\frac{1}{2})=\underline{A}_\omega(\hat{x},\frac{1}{2})-\overline{A}_\omega(x_0,\frac{1}{2})=0$ and $s_2(1-\hat{x})=\underline{A}_\omega(\hat{x},1-\hat{x})-\overline{A}_\omega(x_0,1-\hat{x})=0$, the function $s_2$ is zero on that interval, so that

$$s(\hat{x},y)=\overline{A}_\omega(x_0,y)=\underline{A}_\omega(\hat{x},y).$$

(x) $\{(x,y)\in\mathbb{I}^2; x\in[\hat{x},\frac{1}{2}], y=1-x\}\subseteq\mathscr{B}\cap\mathscr{D}$: We have

$$\overline{A}_\omega(x,1-x)=\omega(x)=\underline{A}_\omega(x,1-x).$$

(xi) $\{(x,y)\in\mathbb{I}^2; x\in[\tilde{x},1], y=1-x\}\subseteq\mathscr{B}\cap\mathscr{D}$: We have

$$\overline{A}_\omega(x,1-x)=\omega(x)=\underline{A}_\omega(x,1-x).$$

(xii) $\{(x,y)\in\mathbb{I}^2; x\in[\frac{1}{2},\tilde{x}], y=x_0\}\subseteq\mathscr{F}\cap\mathscr{D}$: If $\Theta'=0$ then

$$r(x,x_0)=\tau_0(x_0)=0=\underline{A}_\omega(x,x_0),$$

as soon as $\tilde{x}>\frac{1}{2}$, since $\underline{A}_\omega(x,x_0)=0$ for all $x\le\tilde{x}$ by definition of $\tilde{x}$. If $\tilde{x}=\frac{1}{2}$ then $x=\frac{1}{2}$ and $r(\frac{1}{2},x_0)=\tau_0(x_0)=\underline{A}_\omega(\frac{1}{2},x_0)$ again. If $\Theta'>0$, then $\tilde{x}>\frac{1}{2}$, so $\tau_0(x_0)=0$, which implies

$$r(x,x_0)=\tau_0(x_0)=0=\underline{A}_\omega(x,x_0)$$

since the second term in (13) equals 0 and $\underline{A}_\omega(x,x_0)=0$ for all $x\le\tilde{x}$.

(xiii) $\{(x,y)\in\mathbb{I}^2; x=\tilde{x}, y\in[x_0,1-\tilde{x}]\}\subseteq\mathscr{F}\cap\mathscr{D}$: If $\Theta'>0$, then

$$\begin{aligned}r(\tilde{x},y)&=\tau_0(y)+\frac{1}{\Theta'}\left(\overline{A}_\omega(\tilde{x},1-\tilde{x})-\tau_0(1-\tilde{x})\right)\left(\underline{A}_\omega(\tilde{x},y)-\tau_0(y)\right)\\&=\tau_0(y)+\underline{A}_\omega(\tilde{x},y)-\tau_0(y)=\underline{A}_\omega(\tilde{x},y),\end{aligned}$$

since $\overline{A}_\omega(\tilde{x},1-\tilde{x})-\tau_0(1-\tilde{x})=\omega(\tilde{x})-\tau_0(1-\tilde{x})=\Theta'$. If $\Theta'=0$ and $\tilde{x}=\frac{1}{2}$, then by Lemma 4.2

$$\begin{aligned}r(\tilde{x},y)&=\tau_0(y)=\tau_0(x_0)+h_\omega(x_0,y)=\underline{A}_\omega(\tfrac{1}{2},x_0)+\omega(1-y)-\underline{A}_\omega(1-y,x_0)\\&=x_0-\tfrac{1}{2}+\tfrac{1}{2}\left(\omega(\tfrac{1}{2})+\omega(1-x_0)+\mathrm{TV}_{1/2}^{1-x_0}(\omega)\right)+\omega(1-y)\\&\quad-\left(1-y+x_0-1+\tfrac{1}{2}\left(\omega(1-y)+\omega(1-x_0)+\mathrm{TV}_{1-y}^{1-x_0}(\omega)\right)\right)\\&=y-\tfrac{1}{2}+\tfrac{1}{2}\left(\omega(\tfrac{1}{2})+\omega(1-y)+\mathrm{TV}_{1/2}^{1-y}(\omega)\right)\\&=\underline{A}_\omega(\tfrac{1}{2},y).\end{aligned}$$

If $\Theta'=0$ and $\tilde{x}>\frac{1}{2}$, then $\tau_0(1-\tilde{x})\le\overline{A}_\omega(\frac{1}{2},1-\tilde{x})\le\omega(\tilde{x})=\tau_0(1-\tilde{x})$, so that $\overline{A}_\omega(\frac{1}{2},1-\tilde{x})=\omega(\tilde{x})$, and by Lemma 4.2

$$r(\tilde{x},y)=\tau_0(y)=\min\{\overline{A}_\omega(\tfrac{1}{2},1-\tilde{x}),\underline{A}_\omega(\tilde{x},y)\}=\min\{\omega(\tilde{x}),\underline{A}_\omega(\tilde{x},y)\}=\underline{A}_\omega(\tilde{x},y).$$

(xiv) $\{(x,y)\in\mathbb{I}^2; x\in[\frac{1}{2},\tilde{x}], y=1-\tilde{x}\}\subseteq\mathscr{F}\cap\mathscr{B}$: If $\Theta'>0$, then

$$\begin{aligned}r(x,1-\tilde{x})&=\tau_0(1-\tilde{x})+\frac{1}{\Theta'}\left(\overline{A}_\omega(x,1-\tilde{x})-\tau_0(1-\tilde{x})\right)\left(\underline{A}_\omega(\tilde{x},1-\tilde{x})-\tau_0(1-\tilde{x})\right)\\&=\tau_0(1-\tilde{x})+\overline{A}_\omega(x,1-\tilde{x})-\tau_0(1-\tilde{x})=\overline{A}_\omega(x,1-\tilde{x}),\end{aligned}$$

since $\underline{A}_\omega(\tilde{x}, 1-\tilde{x}) - \tau_0(1-\tilde{x}) = \omega(\tilde{x}) - \tau_0(1-\tilde{x}) = \Theta'$. If $\Theta' = 0$ and $\tilde{x} = \frac{1}{2}$, then $x = \frac{1}{2}$ and $r(\frac{1}{2}, \frac{1}{2}) = \tau_0(\frac{1}{2}) = \overline{A}_\omega(\frac{1}{2}, \frac{1}{2})$. If $\Theta' = 0$ and $\tilde{x} > \frac{1}{2}$, then $\tau_0(x_0) = 0$ and $\tau_0(1-\tilde{x}) = \overline{A}_\omega(\frac{1}{2}, 1-\tilde{x}) = \omega(\tilde{x}) = \overline{A}_\omega(\tilde{x}, 1-\tilde{x})$ as above, which implies that $\overline{A}_\omega(x, 1-\tilde{x})$ is constant for $x \in [\frac{1}{2}, \tilde{x}]$, so that

$$r(x, 1-\tilde{x}) = \tau_0(1-\tilde{x}) = \overline{A}_\omega(x, 1-\tilde{x}).$$

In the case $\varphi_0(x_0) = 0$ the line segments (iv)–(vi) and (xi)–(xiv) are the same as in the previous case, and instead of the line segments (i)–(iii) and (vii)–(x) we have

(xv) $\{(x, y) \in \mathbb{I}^2; x \in [0, \frac{1}{2}], y = \frac{1}{2}\} \subseteq \mathscr{A} \cap \mathscr{D}$: Since $\varphi_0(x_0) = \overline{A}_\omega(x_0, \frac{1}{2}) = 0$, also $\underline{A}_\omega(x_0, \frac{1}{2}) = 0$, so $\hat{x} = x_0$ and thus $\varphi_0 = \underline{\varphi}$. It follows that

$$A(x, \tfrac{1}{2}) = \varphi_0(x) = \underline{\varphi}(x) = \underline{A}_\omega(x, \tfrac{1}{2}).$$

(xvi) $\{(x, y) \in \mathbb{I}^2; x \in [0, \frac{1}{2}], y = 1 - x\} \subseteq \mathscr{B} \cap \mathscr{D}$: We have

$$\overline{A}_\omega(x, 1-x) = \omega(x) = \underline{A}_\omega(x, 1-x).$$

This proves that the copula $C$ is well defined. □

## Data availability

No data was used for the research described in the article.

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