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Positive self-commutators of positive operators

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Abstract

We consider a positive operator A on a Hilbert lattice such that its self-commutator $C = A^*A - AA^*$ is positive. If A is also idempotent, then it is an orthogonal projection, and so $C = 0$. Similarly, if A is power compact, then $C = 0$ as well. We prove that every positive compact central operator on a separable infinite-dimensional Hilbert lattice $\mathcal{H}$ is a self-commutator of a positive operator. We also show that every positive central operator on $\mathcal{H}$ is a sum of two positive self-commutators of positive operators.

Keywords Banach lattices · Positive operators · Commutators

Mathematics Subject Classification Primary: 46B42 · 47B65 · 47B47

1 Introduction and Preliminaries

Positive commutators of positive operators on Banach lattices have been the subject of extensive research across various contexts. A systematic investigation into their properties began with [5], where the authors examined the spectral properties of the positive commutator $[A, B] := AB - BA$ formed by positive compact operators A and B. They established that such a commutator is always quasinilpotent, and it is contained in the radical of the Banach algebra generated by A and B.

Further developments were seen in [10], where positive commutators of either positive nilpotent or positive power compact operators were explored. Independently, Drnovšek [8] and Gao [14] proved that a positive commutator of positive operators, provided one of them is compact, is necessarily quasinilpotent. Kandić and Šivic proved in [16] that such commutator necessarily belongs to the radical of the Banach algebra generated by A and B. In [17] they were investigating the dimension of

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the algebra generated by two positive $n \times n$ matrices. The most interesting result yields that, under some technical conditions, the unital algebra generated by two positive idempotent operators with a positive commutator is at most nine dimensional. Drnovšek obtained the same result in [9] under different assumptions. In [11] the authors proved that every positive compact operator on a separable Banach lattice $L^p(\mu)$ $(1 \le p < \infty)$ is a commutator of positive operators. Lastly, in [12] authors studied commutators greater than a perturbation of the identity in ordered normed algebras.

In this paper, we explore positive self-commutators of positive operators on Hilbert lattices. Our results are motivated by Radjavi's work [18], which essentially proves that a self-adjoint operator A on a separable Hilbert space is a self-commutator if and only if zero lies within the convex hull of the essential spectrum of A. Notably, Sourour [24] provided a concise proof of this result. According to [7, Theorem 11.5], for every normal operator N on a Hilbert space, there exists a measure space (Ω, Σ, μ) and a function $\phi \in L^\infty(\mu)$ such that N is unitarily equivalent to the multiplication operator M_ϕ acting on $L^2(\mu)$. Furthermore, if μ is σ-finite, then by [2, Example 2.67], multiplication operators of the form M_ϕ, where $\phi \in L^\infty(\mu)$, are precisely orthomorphisms, or equivalently, central operators on the Banach lattice $L^p(\mu)$ for $1 \le p < \infty$. Therefore, the main goal of the paper is to study which positive central operators are self-commutators of positive operators.

The paper is organized as follows. In Section 2, we establish that a positive idempotent operator on a Hilbert lattice with a positive self-commutator is an orthogonal projection. Additionally, we show that the zero operator is the only positive operator on a Hilbert lattice that is a self-commutator of a positive power compact operator. In Section 3, we prove that every positive bounded diagonal operator on a Hilbert lattice $\mathcal{H} \in \{\ell^2, \ell^2_n\}$ factors through $L^2[0,1]$ as a self-product of the form X^*X for some positive linear operator $X\colon \mathcal{H} \to L^2[0,1]$. Furthermore, there exists a positive self-adjoint operator $Y\colon L^2[0,1] \to L^2[0,1]$ such that $Y^2 = XX^*$. This result proves to be particularly useful in Section 4, where we prove the order analog of [18, Theorem 5] stating that every positive compact central operator on a separable infinite-dimensional Hilbert space is a self-commutator of a positive operator. Finally, in Section 5, we establish that every positive central operator on a separable Hilbert lattice can be expressed as a sum of two positive self-commutators of positive operators.

Before we proceed to the results, we briefly recall some facts about Banach and Hilbert lattices and operators acting on them. Let L be a vector lattice with the positive cone L^+. The band

$$S^d := \{x \in L : \ |x| \wedge |y| = 0 \text{ for all } y \in S\}$$

is called the *disjoint complement* of a set S of L. A band B of L is said to be a *projection band* if $L = B \oplus B^d$. An operator A on L is called *positive* if it maps the cone L^+ into itself. An operator A on L is called *negative* if the operator $-A$ is positive.

Let A be a positive operator on a vector lattice L. The *null ideal* $\mathcal{N}(A)$ is the ideal in L defined by

$$\mathcal{N}(A) = \{x \in L : A|x| = 0\}.$$

When $\mathcal{N}(A) = \{0\}$, we say that the operator A is *strictly positive*. The *range ideal* $\mathcal{R}(A)$ of A is the ideal generated by the range of A, that is,

$$\mathcal{R}(A) = \{y \in L : \exists x \in L^+ \text{ such that } |y| \leq Ax\}.$$

A positive operator A on a vector lattice is an *orthomorphism* if every band of L is invariant under A. If there exists a positive real number λ such that $|Ax| \leq \lambda|x|$ for every $x \in L$, then A is called *central*. By [1, Theorem 3.29], an operator on a Banach lattice is central if and only if it is an orthomorphism.

Let $L^p(\mu)$ $(1 \leq p < \infty)$ be a separable Banach lattice. Then by [6, Theorem 7.1] the Banach lattice $L^p(\mu)$ is isometric and order isomorphic to one of the following Banach lattices ℓ_n^p, ℓ^p, $L^p[0,1]$, $\ell^p \oplus L^p[0,1]$ or $\ell_n^p \oplus L^p[0,1]$. If $\mathcal{H}$ is a Hilbert lattice, then by [22, Theorem IV.6.7] it is isometric and order isomorphic to a Hilbert lattice of the form $L^2(\Omega, \Sigma, \mu)$, where Ω is a locally compact Hausdorff space and μ is a strictly positive Radon measure. Therefore, every band B in a Hilbert lattice satisfies $B^d = B^\perp$. If A is a bounded operator on a Hilbert lattice, then by [20, Proposition 1.1], the operator A is positive if and only if the Hilbert space adjoint operator A^* is positive.

For the terminology and details not explained here we refer the reader to [1] or [2].

2 First observations on positive self-commutators

Let A be a positive operator on a Hilbert lattice $\mathcal{H}$ such that its self-commutator $C = A^*A - AA^*$ is also positive. Since $A^*A \geq AA^*$, a simple induction gives that $(A^*A)^n \leq (A^*)^n A^n$ for each positive integer n. It follows that $\|(A^*A)^n\| \leq \|(A^*)^n\|\|A^n\|$, and so $r(A^*A) \leq r(A^*)r(A) = r(A)^2$, where r denotes the spectral radius. In particular, if $r(A) = 0$ then $r(A^*A) = 0$. As A^*A is self-adjoint, we have $A^*A = 0$, and so $A = 0$ and $C = 0$. Therefore, we can assume that $r(A) > 0$.

Let us first consider the following question: which positive operators on a Hilbert lattice $\mathcal{H}$ are self-commutators of positive idempotent operators?

Proposition 2.1 *Let A be a positive idempotent operator on a Hilbert lattice $\mathcal{H}$ such that its self-commutator $C = A^*A - AA^*$ is also positive. Then A is an orthogonal projection, and so $C = 0$.*

Proof Assume first that $\mathcal{N}(A) = \{0\}$ and $\mathcal{R}(A)^d = \{0\}$. Since closed ideals are bands in $\mathcal{H}$, the range ideal $\mathcal{R}(A)$ is dense in $\mathcal{H}$. Therefore, it follows from $ACA = AA^*A - AA^*A = 0$ that $C = 0$ and the operator A is normal. Furthermore,

$$(A - A^*A)^*(A - A^*A) = A^*A - A^*A - A^*A + A^*AA^*A = 0,$$

so that $A - A^*A = 0$. It follows that $A^* = A^*A = A$ completing the proof in this case.

Consider now the general case. Let us define the bands H_1, H_2, H_3 and H_4 by $H_1 = \mathcal{N}(A) \cap \mathcal{R}(A)^d$, $H_2 = \mathcal{N}(A) \cap \mathcal{R}(A)^{dd}$, $H_3 = \mathcal{N}(A)^d \cap \mathcal{R}(A)^{dd}$ and $H_4 = \mathcal{N}(A)^d \cap \mathcal{R}(A)^d$. With respect to the band decomposition $\mathcal{H} = H_1 \oplus H_2 \oplus H_3 \oplus H_4$, the idempotent A has the form

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & X & Z \\ 0 & 0 & E & Y \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where E, X, Y and Z are positive operators on the appropriate bands. The choice of decomposition enables localizing positivity and idempotency conditions. Namely, it follows from $A^2 = A$ that $E^2 = E$, $XE = X$, $EY = Y$ and $XY = Z$. We also have $\mathcal{N}(E) = \{0\}$, $\mathcal{N}(Y) = \{0\}$, $\mathcal{R}(E)^d = \{0\}$ and $\mathcal{R}(X)^d = \{0\}$. We now compute

$$A^*A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & X^*X + E^*E & (X^*X + E^*E)Y \\ 0 & 0 & Y^*(X^*X + E^*E) & Y^*(X^*X + E^*E)Y \end{pmatrix}$$

and

$$AA^* = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & XX^* + XYY^*X^* & X(I + YY^*)E^* & 0 \\ 0 & E(I + YY^*)X^* & EE^* + YY^* & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Comparing the (2, 2)-block in $C = A^*A - AA^* \geq 0$, we obtain that $XX^* = 0$, and so $X = 0$. Comparing the (3, 3)-block, we conclude that $E^*E \geq EE^* + YY^* \geq EE^*$. By the special case, E is an orthogonal projection, and therefore $YY^* = 0$, so that $Y = 0$. It follows that A is an orthogonal projection as well. This completes the proof. □

Suppose that a positive matrix C is a self-commutator of a positive matrix. Then, [5, Theorem 2.1] yields that C is nilpotent. Since C is also hermitian, it needs to be zero. This result can be extended to positive power compact operators on Hilbert lattices. Recall that a bounded operator on a Banach space is said to be *power compact* if some power is a compact operator.

Proposition 2.2 *The zero operator is the only positive operator on a Hilbert lattice that is a self-commutator of a positive power compact operator.*

Proof Suppose that a positive operator C on a Hilbert lattice E is a self-commutator of a positive power compact operator A. Since the operator $C = A^*A - AA^*$ is positive and A is a positive power compact operator, by [10, Lemma 2.3] the operator C is also power compact.

We claim that C is a quasinilpotent operator. First, consider the complexifications $A_{\mathbb{C}}$ and $A_{\mathbb{C}}^*$ of A and A^*, respectively. Since $A_{\mathbb{C}}$ and $A_{\mathbb{C}}^*$ are positive power compact operators that satisfy $A_{\mathbb{C}}^*A_{\mathbb{C}} - A_{\mathbb{C}}A_{\mathbb{C}}^* \geq 0$, by [16, Proposition 4.7], the operators

$A_{\mathbb{C}}^*$ and $A_{\mathbb{C}}$ are simultaneously triangularizable. Suppose that C^n is compact for some $n \in \mathbb{N}$. Then the operator

$$C_{\mathbb{C}}^n = (A_{\mathbb{C}}^* A_{\mathbb{C}} - A_{\mathbb{C}} A_{\mathbb{C}}^*)^n$$

is compact on $E_{\mathbb{C}}$. Since $A_{\mathbb{C}}$ and $A_{\mathbb{C}}^*$ are simultaneusly triangularizable, the diagonal coefficients of the operators $C_{\mathbb{C}}^n$ are all zero. By Ringrose's theorem (see [21, Theorem 1] and [19, Theorem 7.2.3]) the operator $C_{\mathbb{C}}^n$ is quasinilpotent. Hence, $C_{\mathbb{C}}$ and C are quasinilpotent. Since C is self-adjoint, it is the zero operator. □

The compactness assumption cannot be omitted in the last proposition, as the example of the unilateral shift operator on ℓ^2 shows.

3 Positive linear isometries between separable L^p-spaces

In Section 4 we will consider the question which positive operators on Hilbert lattices are self-commutators of positive operators. As a first result, we will prove that every positive compact central operator on a separable infinite-dimensional Hilbert lattice is a self-commutator of a positive operator. In our proof we will use the fact that for every positive diagonal operator D on ℓ_n^2 there exists a positive operator $X\colon \ell_n^2 \to L^2[0,1]$ such that $D = X^*X$ (see Corollary 3.2). A more general result holds also when one replaces ℓ_n^2 by ℓ^2. As a special case, when $D = I$, we obtain that there always exist a positive isometry from $\ell^2 \to L^2[0,1]$.

Proposition 3.1 *For every positive bounded diagonal operator D on ℓ^2 there exists a positive linear operator $X\colon \ell^2 \to L^2[0,1]$ such that $X^*X = D$. Furthermore, there exists a positive self-adjoint operator $Y\colon L^2[0,1] \to L^2[0,1]$ such that $Y^2 = XX^*$.*

Proof Let $(d_n)_{n\in\mathbb{N}}$ be the sequence of diagonal entries of the operator D and let $(e_n)_{n\in\mathbb{N}}$ be the standard basis of ℓ^2. We claim that $X\colon \ell^2 \to L^2[0,1]$ defined as

$$X\colon x \mapsto \sum_{n=1}^{\infty} \sqrt{2^n d_n} \langle x, e_n \rangle \chi_n,$$

where χ_n is the characteristic function of the interval $[\frac{1}{2^n}, \frac{1}{2^{n-1}}]$ is a bounded operator which satisfies $X^*X = D$. To see this, first define $M := \sup_{n\in\mathbb{N}} d_n$ and note that

$$\begin{aligned}\|Xx\|^2 &= \sum_{n=1}^{\infty} 2^n d_n |\langle x, e_n\rangle|^2 \int_{2^{-n}}^{2^{-n+1}} dt = \sum_{n=1}^{\infty} 2^n d_n |\langle x, e_n\rangle|^2 2^{-n} \\ &\le M \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 = M\|x\|^2\end{aligned}$$

shows that X is bounded with $\|X\| \le \sqrt{M}$. From $\langle X^* f, e_k\rangle = \langle f, Xe_k\rangle = \sqrt{2^k d_k} \langle f, \chi_k\rangle$ it follows

$$X^* f = \sum_{k=1}^{\infty} \sqrt{2^k d_k} \langle f, \chi_k \rangle e_k$$

and so

$$\begin{aligned} X^* X x &= \sum_{n=1}^{\infty} \sqrt{2^n d_n} \langle x, e_n \rangle \, X^* \chi_n = \sum_{n=1}^{\infty} \sqrt{2^n d_n} \langle x, e_n \rangle \sum_{k=1}^{\infty} \sqrt{2^k d_k} \langle \chi_n, \chi_k \rangle e_k \\ &= \sum_{n=1}^{\infty} 2^n d_n \langle x, e_n \rangle 2^{-n} e_n = \sum_{n=1}^{\infty} d_n \langle x, e_n \rangle e_n \\ &= Dx. \end{aligned}$$

Similarly as above, a direct calculation shows that for every $f \in L^2[0, 1]$ we have

$$X X^* f = \sum_{k=1}^{\infty} 2^k d_k \langle f, \chi_k \rangle \chi_k.$$

Hence, the positive operator $Y : L^2[0, 1] \to L^2[0, 1]$ defined as

$$Y f = \sum_{k=1}^{\infty} 2^k \sqrt{d_k} \langle f, \chi_k \rangle \chi_k$$

satisfies $Y^2 = X X^*$ and $Y^* = Y$. □

If one replaces the Hilbert lattice ℓ^2 with the finite-dimensional Hilbert lattice ℓ^2_n, then the operator $X : x \mapsto \sum_{k=1}^{n} \langle x, e_k \rangle \sqrt{2^k d_k} \, \chi_k$ satisfies $X^* X = D$. Moreover, as above, a direct calculation shows that for every $f \in L^2[0, 1]$ we have

$$X X^* f = \sum_{k=1}^{n} 2^k d_k \langle f, \chi_k \rangle \chi_k.$$

The positive operator $Y : L^2[0, 1] \to L^2[0, 1]$ defined as

$$Y f = \sum_{k=1}^{n} 2^k \sqrt{d_k} \langle f, \chi_k \rangle \chi_k$$

satisfies $Y^2 = X X^*$ and $Y^* = Y$. This finite-dimensional adaptation of the proof of Proposition 3.1 gives the following result.

Corollary 3.2 *For every positive $n \times n$ diagonal matrix D there exists a positive linear operator $X : \ell^2_n \to L^2[0, 1]$ such that $X^* X = D$. Furthermore, there exists a positive self-adjoint operator $Y : L^2[0, 1] \to L^2[0, 1]$ such that $Y^2 = X X^*$.*

Corollary 3.3 *There exist positive linear isometries* $U : \ell^2 \to L^2[0, 1]$ *and* $U_n : \ell_n^2 \to L^2[0, 1]$ *for each* $n \in \mathbb{N}$.

Proof By Proposition 3.1 there exists a positive linear operator $X : \ell^2 \to L^2[0, 1]$ such that X^*X is the identity operator on ℓ^2. Therefore, $U := X : \ell^2 \to L^2[0, 1]$ is the desired positive linear isometry. To find a positive isometry $U_n : \ell_n^2 \to L^2[0, 1]$ one can apply Corollary 3.2 instead of Proposition 3.1. □

Since ℓ^2 and $L^2[0, 1]$ are isometrically isomorphic as Hilbert spaces, a natural question which arises here is whether there exists a positive surjective linear isometry, i.e., a positive unitary operator $U : \ell^2 \to L^2[0, 1]$. Flores proved in [13] that there even does not exist a regular isomorphism between ℓ^2 and $L^2[0, 1]$. The following proposition shows that there is no positive linear isometry from $L^2[0, 1]$ to ℓ^2. The obstruction comes from lattice structures of both Banach lattices.

Proposition 3.4 *There is no positive linear isometry* $V : L^p[0, 1] \to \ell^p$ *for any* $1 < p < \infty$.

Proof Suppose there exists a positive linear isometry $V : L^p[0, 1] \to \ell^p$. We claim that V is a lattice homomorphism. To see this, it suffices to prove that $f \wedge g = 0$ in $L^p[0, 1]$ implies $Vf \wedge Vg = 0$ in ℓ^p. Therefore, assume that $f \wedge g = 0$ in $L^p[0, 1]$. Then

$$\|V(f + g)\|_p^p = \|f + g\|_p^p = \|f\|_p^p + \|g\|_p^p = \|Vf\|_p^p + \|Vf\|_p^p.$$

Positivity of Vf and Vg implies

$$\sum_{n=1}^{\infty} ((Vf)_n + (Vg)_n)^p = \sum_{n=1}^{\infty} (Vf)_n^p + \sum_{n=1}^{\infty} (Vg)_n^p.$$

We claim that, for each n, one of $(Vf)_n$ and $(Vg)_n$ is nonzero only if the other one is zero. To prove this, it suffices to prove that for positive real numbers a, b and $1 < p < \infty$ we always have $(a+b)^p > a^p + b^p$. Let us introduce the function $f : [0, \infty) \to \mathbb{R}$ defined as $f(x) := (x + b)^p - x^p - b^p$. Then $f'(x) = p\left((x + b)^{p-1} - x^{p-1}\right) > 0$. Hence, f is strictly increasing, and so $(a + b)^p - a^p - b^p = f(a) > f(0) = 0$. This shows that, for each n, $(Vf)_n$ and $(Vg)_n$ cannot be non-zero simultaneously. This proves that $Vf \wedge Vg = 0$, and so, V is a lattice homomorphism.

Since V is an isometric lattice homomorphism, the Banach lattice ℓ^p contains a closed sublattice E which is lattice isometric to the Banach lattice $L^p[0, 1]$. As such, E needs to be without atoms. On the other hand, [4, Theorem 3.2] implies that E is the closed span of an infinite disjoint positive sequence, and so E is atomic. This contradiction shows that V cannot exist. □

It should be noted that the proof of Proposition 3.4 shows that there is no positive operator $T : L^p[0, 1] \to \ell^p$ such that the restriction of T to the positive cone $L^p[0, 1]_+$ is an isometry. For $p = 1$ such operators exist.

Example 3.5 Let us define the linear operator $V\colon L^1[0,1] \to \ell^1$ by

$$(Vf)_n = \int_{2^{-n}}^{2^{-n+1}} f(t)\,dt.$$

Since

$$\sum_{n=1}^{\infty} |(Vf)|_n \le \sum_{n=1}^{\infty} \int_{2^{-n}}^{2^{-n+1}} |f(t)|\,dt = \int_0^1 |f(t)|\,dt = \|f\|_1,$$

it follows that $\|Vf\|_1 \le \|f\|_1$, and so, V is a contraction. Moreover, if f is a non-negative function, then the calculation above yields $\|Vf\|_1 = \|f\|_1$. Positivity of V should be clear.

Although Proposition 3.4 shows that there does not exist a positive linear isometry $V\colon L^p[0,1] \to \ell^p$ for $1 < p < \infty$, spaces ℓ^2 and $L^2[0,1]$ are isometrically isomorphic as Hilbert spaces. For $p = 1$ the situation is more severe as there is no linear isometry from $L^1[0,1]$ to ℓ^1. This follows from the fact that by [3, Example 3.1.2] the Banach space ℓ^1 has an unconditional basis, so that by [3, Theorem 6.3.3] the Banach space $L^1[0,1]$ cannot be embedded into a Banach space with an unconditional basis.

4 Positive central self-commutators

In this section we consider the question which positive central operators on separable infinite-dimensional Hilbert lattices are positive self-commutators of positive operators. We start by proving an order analog of [18, Theorem 5].

Theorem 4.1 *Every positive compact central operator on a separable infinite-dimensional Hilbert lattice is a self-commutator of a positive operator.*

Proof Let C be a positive compact central operator on a separable infinite-dimensional Hilbert lattice $\mathcal{H}$. As mentioned in Section 1, $\mathcal{H}$ is lattice isometric to one of the following Hilbert lattices: ℓ^2, $L^2[0,1]$, $\ell^2 \oplus L^2[0,1]$ and $\ell^2_n \oplus L^2[0,1]$. We will consider each of these cases separately.

Case $\mathcal{H} \cong \ell^2$: Since C is a central operator on ℓ^2, it is a diagonal operator with positive diagonal coefficients $(d_n)_{n\in\mathbb{N}}$. From compactness of C it follows that $d_n \to 0$ as n goes to infinity, so that there exists the largest diagonal coefficient. Without loss of generality we may assume that $d_1 \ge d_k$ for every $k \in \mathbb{N}$. Since for every $\epsilon > 0$ there are only finitely many $n \in \mathbb{N}$ such that $d_n \ge \epsilon$ there exist countably infinitely many sequences $s^{(n)}$ such that

(i) the terms of all sequences $s^{(n)}$ equal the terms of the sequence $(d_n)_{n\in\mathbb{N}}$ with respect to all of their multiplicities;
(ii) the first term $s_1^{(n)}$ of the sequence $s^{(n)}$ is its largest term for each $n \in \mathbb{N}$;
(iii) $s_1^{(n)} \le \frac{d_1}{2^{n-1}}$ for every $n \in \mathbb{N}$.

Let $e_k^{(n)}$ be the standard basis eigenvector that corresponds to the eigenvalue $s_k^{(n)}$ for $n, k \in \mathbb{N}$ and let $\mathcal{H}_n$ be the closed linear span of $\{e_k^{(n)} : k \in \mathbb{N}\}$. Then every $\mathcal{H}_n$ is lattice isometric to ℓ^2. Let us define the isometric lattice isomorphism $U: \ell^2 \to \left(\bigoplus_{n=1}^{\infty} \mathcal{H}_n\right)_2$ as follows. For each $m \in \mathbb{N}$ there exist unique positive integers $k, n \in \mathbb{N}$ such that $e_m = e_k^{(n)}$. We define $Ue_m = e_k^{(n)}$ and extend U by linearity on the linear span of the set of all standard basis vectors. Clearly, the extended mapping is a positive bounded linear operator which can be uniquely extended by continuity to a positive operator on ℓ^2. Since by construction U maps the set $\{e_n : n \in \mathbb{N}\}$ injectively onto itself, we have that U is unitary. Moreover, an easy calculation shows that

$$UCU^{-1} = \begin{pmatrix} D_1 & 0 & 0 & \dots \\ 0 & D_2 & 0 & \dots \\ 0 & 0 & D_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

where D_n is the diagonal operator $\operatorname{Diag}\left(\{s_k^{(n)} : k \in \mathbb{N}\}\right)$ on $\mathcal{H}_n$.

Let us solve an infinite system of operator equations given by $X_1^2 = D_1$ and $X_{n+1}^2 - X_n^2 = D_{n+1}$ for $n \in \mathbb{N}$. If $n = 1$, the positive diagonal operator $X_1 := \sqrt{D_1}$ satisfies $X_1^2 = D_1$. If $n = 2$, then the positive diagonal operator $X_2 := \sqrt{D_1 + D_2}$ solves the operator equation $X_2^2 = X_1^2 + D_2$. Inductively, one can show that for each $n \in \mathbb{N}$ the positive diagonal operator $X_n := \sqrt{D_1 + \cdots + D_n}$ formally solves the operator equation $X_n^2 - X_{n-1}^2 = D_n$. Since the norm of the operator X_n satisfies

$$\|X_n\| = \sqrt{s_1^{(1)} + s_1^{(2)} + \cdots + s_1^{(n)}} \le \sqrt{d_1 + \frac{d_1}{2} + \cdots + \frac{d_1}{2^{n-1}}} \le \sqrt{2d_1},$$

the operator X_n is bounded.

Let us define the operator X on $\ell^2 \cong \left(\bigoplus_{n=1}^{\infty} \ell^2\right)_2$ as the infinite operator matrix

$$X = \begin{pmatrix} 0 & 0 & 0 & \dots \\ X_1 & 0 & 0 & \dots \\ 0 & X_2 & 0 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $\|X_n\| \le \sqrt{2d_1}$ for each $n \in \mathbb{N}$, the operator X is bounded with $\|X\| \le \sqrt{2d_1}$. Self-adjointness of X_n now yields

$$[X^*, X] = \begin{pmatrix} X_1^2 & 0 & 0 & \dots \\ 0 & X_2^2 - X_1^2 & 0 & \dots \\ 0 & 0 & X_3^2 - X_2^2 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} D_1 & 0 & 0 & \dots \\ 0 & D_2 & 0 & \dots \\ 0 & 0 & D_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and so $UCU^{-1} = [X^*, X]$. Since U satisfies $U^* = U^{-1}$, we finally have $C = U^{-1}[X^*, X]U = [U^{-1}X^*U, U^{-1}XU] = [(U^{-1}XU)^*, U^{-1}XU]$.

Case $\mathcal{H} \cong L^2[0,1]$: Since the Banach lattice $L^2[0,1]$ is atomless, by [23, Corollary 1.7] it follows that $C = 0$ on $L^2[0,1]$, so that it is a self-commutator of itself.

Case $\mathcal{H} \cong \ell^2 \oplus L^2[0,1]$: Since C is a positive central operator on $\ell^2 \oplus L^2[0,1]$, it can be written as $C = C_1 \oplus C_2$ where C_1 and C_2 are positive central operators on ℓ^2 and $L^2[0,1]$, respectively. Compactness of C yields compactness of C_1 and C_2. The previous two cases imply that $C_2 = 0$ and $C_1 = [X^*, X]$ for some positive operator X on ℓ^2. Therefore, the operator $C = [X^*, X] \oplus 0 = [X^* \oplus 0, X \oplus 0]$ is a self-commutator.

Case $\mathcal{H} \cong \ell_n^2 \oplus L^2[0,1]$: Since C is a positive central operator on $\ell_n^2 \oplus L^2[0,1]$, it can be written as $C = C_1 \oplus C_2$ where C_1 and C_2 are positive central operators on ℓ_n^2 and $L^2[0,1]$, respectively. Similarly, as in the previous case, it follows that $C_2 = 0$, and so $C = C_1 \oplus 0$ for some positive diagonal $n \times n$ matrix C_1.

By Corollary 3.2 there exist a positive operator $X\colon \ell_n^2 \to L^2[0,1]$ such that $C_1 = X^*X$ and a positive self-adjoint operator $Y\colon L^2[0,1] \to L^2[0,1]$ such $XX^* = Y^2$. With respect to the decomposition $\mathcal{H} \cong \ell_n^2 \oplus \bigoplus_{n=1}^\infty L^2[0,1]$ we define the positive operator Z as an infinite block operator matrix

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ X & 0 & 0 & 0 & \dots \\ 0 & Y & 0 & 0 & \dots \\ 0 & 0 & Y & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

A direct calculation shows

$$Z^*Z = \begin{pmatrix} X^*X & 0 & 0 & 0 & \dots \\ 0 & Y^2 & 0 & 0 & \dots \\ 0 & 0 & Y^2 & 0 & \dots \\ 0 & 0 & 0 & Y^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad \text{and} \quad ZZ^* = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & XX^* & 0 & 0 & \dots \\ 0 & 0 & Y^2 & 0 & \dots \\ 0 & 0 & 0 & Y^2 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $X^*X = C_1$ and $Y^2 = XX^*$, we have

$$[Z^*, Z] = Z^*Z - ZZ^* = \begin{pmatrix} C_1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = C$$

which finishes the proof. □

Corollary 4.2 *Every positive compact central operator of infinite rank on a Hilbert lattice is a self-commutator of a positive operator.*

Proof Let C be a central operator on a Hilbert lattice $\mathcal{H}$. The range ran C of a central operator C is an ideal in $\mathcal{H}$, so that it is equal to the range ideal $\mathcal{R}(C)$. Since the closure $\overline{\mathcal{R}(C)}$ is a closed ideal in an order continuous Banach lattice $\mathcal{H}$, we have that $\overline{\mathcal{R}(C)}$ is a band in $\mathcal{H}$. The band generated by $\mathcal{R}(C)$ is $\mathcal{R}(C)^{dd}$, so that $\mathcal{R}(C)^{dd} \subseteq \overline{\mathcal{R}(C)}$. On the other hand, every band in a normed lattice is a closed ideal from where it follows $\overline{\mathcal{R}(C)} \subseteq \mathcal{R}(C)^{dd}$. Since every band in $\mathcal{H}$ is a projection band, we have

$$\mathcal{H} = \mathcal{R}(C)^{dd} \oplus \mathcal{R}(C)^{d} = \overline{\mathcal{R}(C)} \oplus \mathcal{R}(C)^{d}.$$

With respect to this decomposition, C can be written as $C = C_1 \oplus 0$ where $C_1 = C|_{\overline{\mathcal{R}(C)}}$. Since C_1 is a compact operator, its range is separable, so that $\overline{\mathcal{R}(C)}$ is infinite-dimensional and separable as well. Hence, by Theorem 4.1 the operator C_1 is a self-commutator of a positive operator. To finish the proof we observe that the zero operator is self-commutator of itself. □

Our next goal is to prove that positive central operators with infinite-dimensional kernels on a Hilbert lattice which is either ℓ^2 or $L^2[0,1]$ are always self-commutators of positive operators. Before we prove this result, we need the following lemma.

Lemma 4.3 *Let T be a positive operator on a Hilbert lattice $\mathcal{H}$ which is either ℓ^2 or $L^2[0,1]$. If T is positive semi-definite operator whose square root is a positive operator, then the operators*

$$\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}$$

on the Hilbert lattice $\mathcal{H} \oplus \mathcal{H}$ are self-commutators of positive operators.

Proof We only consider the case $\mathcal{H} = \ell^2$ as the case of $\mathcal{H} = L^2[0,1]$ can be treated similarly. Let $\sqrt{T}$ be the square-root of T. We define the operator A on the Hilbert lattice $\ell^2 \oplus \ell^2 \cong \ell^2 \oplus \bigoplus_{n=1}^{\infty} \ell^2$ as

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ \sqrt{T} & 0 & 0 & 0 & \dots \\ 0 & \sqrt{T} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{T} & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since $\sqrt{T}$ is self-adjoint, a direct calculation shows that $[A^*, A] = T \oplus 0$. Positivity of $\sqrt{T}$ yields that A is a positive operator on $\ell^2 \oplus \ell^2$.

To prove that the operator $0 \oplus T$ is a self-commutator of a positive operator, we first introduce the switch operator $U : \mathcal{H} \oplus \mathcal{H} \to \mathcal{H} \oplus \mathcal{H}$ given by the 2×2 block-operator matrix

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Clearly, U is an isometric self-adjoint isomorphism of the Hilbert lattice $\mathcal{H}\oplus\mathcal{H}$ which satisfies $U = U^{-1} = U^*$. Since

$$U^* \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix} U = \begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix},$$

by the case above, there exists a positive operator A on $\mathcal{H} \oplus \mathcal{H}$ such that $T \oplus 0 = [A^*, A]$, and so $0 \oplus T = U(T \oplus 0)U^* = U[A^*, A]U^* = [UA^*U^*, UAU^*]$. □

We now prove an order analog of [18, Theorem 3].

Theorem 4.4 *Let C be a positive central operator on a separable infinite-dimensional Hilbert lattice ℓ^2 or $L^2[0, 1]$ with infinite dimensional kernel. Then C is a self-commutator of a positive operator.*

Proof Obviously, we may assume that $C \neq 0$ as the zero operator is a self-commutator of itself. Since C is a central operator, it is order continuous and its kernel $\ker C$ equals its absolute kernel which is a projection band. Its orthogonal complement $(\ker C)^\perp = (\ker C)^d$ is the band generated by the range of C. Since $\ell^2 = (\ker C)^\perp \oplus \ker C$, the restriction of C to $(\ker C)^\perp$ is a strictly positive central operator.

Suppose that the underlying Hilbert lattice is ℓ^2. If $\dim(\ker C)^\perp$ is finite, then C is a compact operator, so that it is a self-commutator of a positive operator by Theorem 4.1. If $\dim(\ker C)^\perp$ is infinite, then we can decompose $\ell^2 = (\ker C)^\perp \oplus \ker C \cong \ell^2 \oplus \ell^2$ and the block-operator matrix corresponding to C is of the form

$$\begin{pmatrix} C|_{(\ker C)^\perp} & 0 \\ 0 & 0 \end{pmatrix}.$$

Here the decomposition $\ell^2 = (\ker C)^\perp \oplus \ker C \cong \ell^2 \oplus \ell^2$ is simultaneously a Hilbert space and a Banach lattice decomposition. This implies that $C|_{(\ker C)^\perp}$ is a diagonal operator with positive diagonal entries on the Hilbert lattice $(\ker C)^\perp \cong \ell^2$. As such, it is positive semi-definite operator which admits a positive square root. Lemma 4.3 yields that C is a self-commutator of a positive operator.

When the underlying Hilbert lattice is $L^2[0, 1]$ we argue similarly. Since $C \neq 0$, we can decompose $L^2[0, 1] = (\ker C)^\perp \oplus \ker C \cong L^2[0, 1] \oplus L^2[0, 1]$. Then, with respect to this decomposition, the operator C can be represented as a 2×2 block-operator matrix

$$\begin{pmatrix} C|_{(\ker C)^\perp} & 0 \\ 0 & 0 \end{pmatrix}.$$

Since the operator $C|_{(\ker C)^\perp}$ is a positive central operator on $L^2[0, 1]$, it is a multiplication operator M_φ for some non-negative function $\varphi \in L^\infty[0, 1]$. Since M_φ is positive semi-definite with the positive square root $M_{\sqrt{\varphi}}$, another application of Lemma 4.3 yields that C is a self-commutator of a positive operator. □

Corollary 4.5 *A band projection on ℓ^2 or $L^2[0, 1]$ is a self-commutator of a positive operator if and only if its kernel is infinite-dimensional.*

Proof If P is a band projection with a finite-dimensional kernel, then $I - P$ is a finite-rank projection. Hence, by an application of Wintner-Wielandt's result for the Calkin algebra (see [15]), the operator $P = I - (I - P)$ is not a commutator. The other implication follows from Theorem 4.4. □

We conclude this section with a result stating that invertible positive central operators on Banach lattices are not commutators of positive operators. In the proof of this result we will need the following slight improvement of [12, Theorem 2.1] which for the sake of simplicity we state only for the case of operators on Banach lattices.

Proposition 4.6 *Let A and B be operators on a Banach lattice L with one of them positive or negative. Then there does not exist a positive scalar $c > 0$ such that $AB - BA \geq cI$.*

Theorem 4.7 *An invertible positive central operator on a Banach lattice is not a commutator of two operators with one of them positive or negative.*

Proof Since the center $\mathscr{Z}(L)$ of a Banach lattice L is isomorphic as an f-algebra to an f-algebra $C(K)$ for some compact Hausdorff space K, a positive central operator C is invertible if and only if there exists a positive scalar $\delta > 0$ such that $C \geq \delta I$. Hence, Proposition 4.6 yields that there do not exist bounded operators A and B on L with one of them positive or negative such that $C = AB - BA$. □

5 Positive central operators are sums of two positive self-commutators

It was proved in Theorem 4.4 that every positive central operator with infinite-dimensional kernel on a Hilbert lattice ℓ^2 or $L^2[0,1]$ is a self-commutator of a positive operator. It turns out that the situation becomes very difficult when we consider the separable Hilbert lattice $\ell^2 \oplus L^2[0,1]$. To explain where the difficulty lies, let us consider the following example.

Example 5.1 The operator C on a Hilbert lattice $\mathcal{H} = \ell^2 \oplus L^2[0,1]$ represented with the 2×2 operator matrix

$$\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$$

is a self-commutator of a positive operator on $\mathcal{H}$. To see this, observe first that by Proposition 3.1 there exists a positive isometry $X\colon \ell^2 \to L^2[0,1]$ and a positive self-adjoint operator $Y\colon L^2[0,1] \to L^2[0,1]$ such that $Y^2 = XX^*$. With respect to the Hilbert lattice decomposition $\mathcal{H} \cong \ell^2 \oplus \bigoplus_{n=1}^{\infty} L^2[0,1]$, as in Theorem 4.1 we define the positive operator Z as the infinite block operator matrix

$$Z = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ X & 0 & 0 & 0 & \dots \\ 0 & Y & 0 & 0 & \dots \\ 0 & 0 & Y & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

An easy calculation shows that $[Z^*, Z] = C$ proving that C is a self-commutator of a positive operator.

At first glance it may seem that a similar idea works to prove that the operator C on $\mathcal{H} = \ell^2 \oplus L^2[0, 1]$ represented with a 2×2 block operator matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

is a self-commutator of a positive operator. However, there is a big issue. The proof that the operator from Example 5.1 is a self-commutator of a positive operator heavily relies on the fact that there exists a positive isometry $X : \ell^2 \to L^2[0, 1]$. However, by Proposition 3.4 there are no positive isometries from $L^2[0, 1]$ to ℓ^2.

In this section we prove that every positive central operator on a separable infinite-dimensional Hilbert lattice is a sum of two positive self-commutators of positive operators.

Theorem 5.2 *Every positive central operator C on an infinite-dimensional separable Hilbert lattice is a sum of two positive self-commutators of positive operators.*

Proof Suppose first that $\mathcal{H}$ is either ℓ^2 or $L^2[0, 1]$. Then, $\mathcal{H} \cong \mathcal{H} \oplus \mathcal{H}$ as a Hilbert lattice. Since C is a central operator, with respect to the decomposition $\mathcal{H} \oplus \mathcal{H}$ the operator C can be represented as a 2×2 operator matrix

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & C_2 \end{pmatrix}$$

where C_1 and C_2 are positive central operators on $\mathcal{H}$. If $\mathcal{H} = \ell^2$, then C_1 and C_2 are diagonal operators with positive diagonal entries. If $\mathcal{H} = L^2[0, 1]$, then C_1 and C_2 are multiplication operators with almost everywhere non-negative multiplying functions. Since positive diagonal operators on ℓ^2 and positive multiplication operators on $L^2[0, 1]$ are positive and positive semi-definite whose roots are positive operators, by Lemma 4.3 there exist positive operators A and B on $\mathcal{H} \oplus \mathcal{H}$ such that

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} = [A^*, A] + [B^*, B].$$

Now we consider the case when $\mathcal{H} = \ell^2 \oplus L^2[0, 1]$. Since C is a positive central operator on $\mathcal{H}$, with respect to the decomposition $\ell^2 \oplus L^2[0, 1]$ we can write

$$C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}$$

where C_1 and C_2 are positive central operators on ℓ^2 and $L^2[0, 1]$, respectively. By the cases above, there exist positive operators A_1, B_1 on ℓ^2 and A_2, B_2 on $L^2[0, 1]$ such that $C_1 = [A_1^*, A_1] + [B_1^*, B_1]$ and $C_2 = [A_2^*, A_2] + [B_2^*, B_2]$. Hence,

$$\begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} = \begin{pmatrix} [A_1^*, A_1] + [B_1^*, B_1] & 0 \\ 0 & [A_2^*, A_2] + [B_2^*, B_2] \end{pmatrix}$$
$$= \begin{pmatrix} [A_1^*, A_1] & 0 \\ 0 & [A_2^*, A_2] \end{pmatrix} + \begin{pmatrix} [B_1^*, B_1] & 0 \\ 0 & [B_2^*, B_2] \end{pmatrix}.$$

The last expression can be written as the sum of two self-commutators of positive operators $A_1 \oplus A_2$ and $B_1 \oplus B_2$.

Finally, we consider the last case when the underlying Hilbert lattice $\mathcal{H}$ is of the form $\ell_n^2 \oplus L^2[0, 1]$ for some $n \in \mathbb{N}$. As above, due to the Hilbert lattice isomorphism $L^2[0, 1] \cong L^2[0, 1] \oplus L^2[0, 1]$ we can furthermore decompose $\ell_n^2 \oplus L^2[0, 1] \cong \ell_n^2 \oplus L^2[0, 1] \oplus L^2[0, 1]$ and with respect to this decomposition we can write

$$C = \begin{pmatrix} D & 0 & 0 \\ 0 & M_\varphi & 0 \\ 0 & 0 & M_\psi \end{pmatrix} = \begin{pmatrix} D & 0 & 0 \\ 0 & M_\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_\psi \end{pmatrix}$$

for some non-negative functions φ and $\psi \in L^\infty[0, 1]$ and some non-negative diagonal $n \times n$ matrix D. Since an application of Lemma 4.3 yields that the last operator above is a positive self-commutator of a positive operator, due to the Hilbert lattice isomorphism $\ell_n^2 \oplus L^2[0, 1] \oplus L^2[0, 1] \cong \ell_n^2 \oplus L^2[0, 1] \oplus L^2[0, 1] \oplus L^2[0, 1]$ it suffices to prove that the operator given by the block-operator matrix

$$\begin{pmatrix} D & 0 & 0 \\ 0 & M_\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} \cong \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & M_\varphi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a positive self-commutator. Furthermore, by applying the switch operator we need to prove that the operator

$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_\varphi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a positive self-commutator of a positive operator. By Theorem 4.1 and Theorem 4.4, respectively, there exist positive operators A and B such that

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = [A^*, A] \quad \text{and} \quad \begin{pmatrix} M_\varphi & 0 \\ 0 & 0 \end{pmatrix} = [B^*, B].$$

To conclude the proof, observe that the operator

$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & M_\varphi & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a self-commutator of the positive operator $A \oplus B$. □

We conclude this paper with the following question.

Question 5.3 Is the operator C on the Hilbert lattice $\ell^2 \oplus L^2[0, 1]$ represented with a 2×2 block operator matrix

$$\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$

a self-commutator of a positive operator?

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Declarations

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