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An infinite family of simple graphs underlying chiral, orientable reflexible and non-orientable rotary maps

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A B S T R A C T

In this paper, we provide the first known infinite family of simple graphs, each of which is the skeleton of a chiral map, a skeleton of a reflexible map on an orientable surface, as well as a skeleton of a reflexible map on a non-orientable surface. This family consists of all lexicographic products $C_n[mK_1]$, where $m \geq 3$, $n = sm$, with s an integer not divisible by 4. This answers a question posed by Wilson in 2002.

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1. Introduction

A *map* is a connected finite graph (called the *skeleton of the map*), viewed as a 1-dimensional CW complex, embedded on a closed surface such that when it is removed from the surface, the connected components (called *faces*) are all homeomorphic to an open disk. The edges and vertices of the skeleton are referred to also as edges and vertices of the map. The map is called *polytopal* provided that the boundary of each face is a cycle in the graph and that every edge lies on the boundary of two distinct faces. The term "polytopal" here comes from the fact that such a map can be considered as an abstract polytope of rank 3; see, for example, [7]. One should also note that polytopal maps are examples of closed 2-cellular embeddings of graphs, in both of the two standard meanings of this notion (see, for example, the discussion in [4, Section 1]). All maps appearing in this paper will be polytopal and all graphs will be simple.

An automorphism of the skeleton of a map that extends to a homeomorhism of the underlying surface is called an *automorphism of the map* and the set of all such automorphisms forms a group, called the *automorphism group of the map*. If the automorphism group is rich enough so that the stabiliser of each vertex contains a cyclic group acting transitively on the neighbouring edges (and faces) and, in addition, so that the stabiliser of each face contains a cyclic group acting transitively on the vertices (and edges) of the face, then the map is called *rotary*. Note that the automorphism group of a rotary map acts transitively on the vertices, on the faces, on the oriented edges (arcs), as well as on incident vertex-face pairs of the map. In particular, all faces in a rotary map have equal co-valence (where by the *co-valence of a face* we mean the number of edges lying on the boundary of the face). If a rotary map has co-valence p and the skeleton has valence q, then we say that the map has *type* $\{p, q\}$.

If, in addition, the face-stabiliser in a rotary map also contains an automorphism that acts as a reflection on the face (which is equivalent to requiring that the stabiliser of an edge in the face stabiliser is non-trivial), then the map is called *reflexible*. Note that every rotary map on a non-orientable surface is reflexible. On the other hand, if the underlying surface is orientable, then rotary maps that are not reflexible exist and are called *chiral*. Reflexible maps can thus be *orientable* (if the underlying surface is orientable) or *non-orientable* (if the surface is non-orientable).

There is a vast literature on different aspects of rotary maps, but typically, the topic is approached from one of the following three points of view:

- Given a fixed (orientable or non-orientable) surface $\mathcal{S}$, classify all rotary maps on $\mathcal{S}$;
- Given a finite group G (or a family thereof), find all rotary maps whose automorphism group is isomorphic to G;
- Given a finite connected graph Γ (or a family thereof), determine all rotary maps whose skeleton is Γ.

This paper falls into the third of the above categories. In particular, we shall address the following question, which was originally posed by Steve Wilson in [10].

Question 1.1. Does there exist a graph which is the skeleton of representatives from all three classes of rotary maps: non-orientable reflexible, orientable reflexible, and chiral?

As was observed already in [10], the answer to this question is affirmative since the complete multipartite graph $K_{3,3,3}$ is an example of such a graph. However, the existence of an infinite family of graphs, each of which underlies maps from all three classes, has not been known until now. The main purpose of this paper is to exhibit such an infinite family. In particular, we shall prove the following theorem:

Theorem 1.2. *Let s be an arbitrary positive integer not divisible by 4, let $m \geq 3$ be an odd integer, let $n = sm$, and let Γ be the lexicographic product $C_n[mK_1]$. Then Γ is the skeleton of*

- *a chiral polytopal map of type $\{mn, 2m\}$ and genus $1 + m(n\frac{m-1}{2} - 1)$, and*
- *an orientable reflexible polytopal map of type $\{n, 2m\}$ and genus $1 + m(n\frac{m-1}{2} - m)$, and*
- *a non-orientable reflexible polytopal map of type $\{2n, 2m\}$ and genus $2 + m(n(m - 1) - m)$.*

2. Rotary maps arising from graphs

In this section, we review some definitions pertaining to rotary maps and their automorphisms.

Let Γ be the skeleton of a polytopal map $\mathcal{M}$ and assume that Γ is a simple graph. A cycle C that constitutes the boundary of a face f is called a *facial cycle* of $\mathcal{M}$. Since the face of a polytopal map of type $\{p, q\}$, with $q \geq 3$, is uniquely determined by its facial cycles, we shall often think of faces both as parts of the underlying surface as well as the cycles in the skeleton. By definition of polytopality of maps, the set $\mathcal{C}$ of all facial cycles has the property that every edge of Γ belongs to precisely two cycles in $\mathcal{C}$. Such a set of cycles in a graph is called a *cycle double cover* (see [6], for example). In short, the set of facial cycles in a polytopal map is a cycle double cover of the skeleton.

Let $\mathcal{C}$ be a cycle double cover of a simple graph Γ. For a vertex $v \in \Gamma$, its *vertex figure with respect to* $\mathcal{C}$ is defined as the graph whose vertices are the edges of Γ incident to v, and two such edges are adjacent in the vertex figure whenever they are consecutive edges of a cycle in $\mathcal{C}$. Note that the valence of a vertex in a vertex figure is at most 2 (since every edge of Γ belongs to exactly two cycles in $\mathcal{C}$). Moreover, if $\mathcal{C}$ is the set of facial cycles of a polytopal map, then the vertex figure at a vertex v of valence q is in fact isomorphic to the cycle C_q (if $q \geq 3$) or to K_2 (if $q = 2$).

The following lemma, providing the converse of the above observations, is a straightforward folklore result and has appeared in similar forms in several publications (see, for example, [8, Lemma 3.9]). Here we provide just a brief sketch of the proof.

Lemma 2.1. *Let* Γ *be a connected simple graph of minimal valence at least* 3 *and let* $\mathcal{C}$ *be a cycle double cover of* Γ *such that the vertex figure of each vertex of* Γ *with respect to* $\mathcal{C}$ *is connected (and thus isomorphic to a cycle). Let* $\mathcal{S}$ *be the topological space obtained from* Γ *(viewed as a 1-dimensional CW complex) by gluing a copy of a closed disk to each cycle of* $\mathcal{C}$ *homeomorphically along the boundary of the disk. Then* $\mathcal{S}$ *is a closed surface and the embedding of* Γ *onto* $\mathcal{S}$ *is a polytopal map* $\mathcal{M}(\Gamma, \mathcal{C})$ *whose set of facial cycles is* $\mathcal{C}$*. Moreover, the automorphism group* $\mathrm{Aut}(\mathcal{M}(\Gamma, \mathcal{C}))$ *consists of all automorphisms of* $\mathrm{Aut}(\Gamma)$ *that induce a permutation of the cycles of* $\mathcal{C}$*.*

The proof of the above lemma can be sketched as follows. First, observe that every point of $\mathcal{S}$ which is an internal point of one of the closed disks glued to Γ has a regular neighbourhood (that is, homeomorphic to an open disk). Similarly, a regular neighbourhood of an internal point of an edge exists due to the fact that every edge of Γ belongs to exactly two cycles in $\mathcal{C}$ (and is thus identified with a point in the boundary of exactly two disks that were glued to Γ). Finally, since the minimal valence of Γ is at least 3 and since every edge belongs to two cycles in $\mathcal{C}$, the vertex figure at each vertex (being connected) is a cycle. This then enables one to find a regular neighbourhood of each point of $\mathcal{S}$ that corresponds to a vertex of Γ. In short, $\mathcal{S}$ is a closed surface with Γ embedded into it. Since the boundary of each face is a cycle of Γ and since every edge lies on the boundaries of two distinct faces, the corresponding map is polytopal. Finally, a simple fact that every homeomorphism between the boundaries of two closed disks can be extended to a homeomorphism between the disks implies that every automorphism of Γ that preserves the set of facial cycles $\mathcal{C}$ extends to a homeomorphism of $\mathcal{S}$. With this we finish the sketch of the proof of Lemma 2.1.

A *flag* of a polytopal map is an incident vertex-edge-face triple. Observe that the automorphism group of a polytopal map acts naturally on the set of flags of the map. In particular, the connectedness implies that the automorphism group is semi-regular (or free) on the flags. Thus, the order of the automorphism group divides the number of flags and is at most twice the number of arcs and at most four times the number of edges.

Given a polytopal rotary map $\mathcal{M}$ and a flag $\Phi = (v, e, f)$ of $\mathcal{M}$, there exist automorphisms σ_1 and σ_2, acting as a 1-step rotation of the face f and around the vertex v, respectively. These automorphisms can be chosen in such a way that $\sigma_1\sigma_2$ is an involution reversing the edge e. The group $\langle\sigma_1, \sigma_2\rangle \leq \mathrm{Aut}(\mathcal{M})$ is denoted by $\mathrm{Aut}^+(\mathcal{M})$, and is often called the *rotational group* of $\mathcal{M}$; the automorphisms σ_1 and σ_2 are then called the *distinguished generators of* $\mathrm{Aut}^+(\mathcal{M})$ *with respect to the base flag* Φ. The rotational group of a rotary map $\mathcal{M}$ has index at most 2 in $\mathrm{Aut}(\mathcal{M})$; in fact, the index of $\mathrm{Aut}^+(\mathcal{M})$ in $\mathrm{Aut}(\mathcal{M})$ is 2 whenever $\mathcal{M}$ is reflexible and orientable, otherwise $\mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$.

It is not difficult to see that whenever $\mathcal{M}$ is reflexible there exists an involutory automorphism ρ of the map fixing both v and f, and interchanging the two edges of f incident to v; in this case, $\mathrm{Aut}(\mathcal{M})$ acts transitively on the flags of $\mathcal{M}$. Moreover, $\langle\rho, \sigma_1\rangle$ (resp. $\langle\rho, \sigma_2\rangle$) is a dihedral group: the full automorphism group of the face f

(resp. of the vertex figure of v). Thus, conjugation by ρ sends σ_i to σ_i^{-1}, for $i = 1, 2$; we shall say in such a case that ρ *inverts* σ_i. Recall that if $\mathcal{M}$ is non-orientable reflexible, then $\mathrm{Aut}^+(\mathcal{M}) = \mathrm{Aut}(\mathcal{M})$ and consequently, the involutory automorphism ρ belongs to $\mathrm{Aut}^+(\mathcal{M})$. In contrast, if $\mathcal{M}$ is orientable reflexible, then $\rho \notin \mathrm{Aut}^+(\mathcal{M})$, and $\mathrm{Aut}(\mathcal{M}) = \langle \sigma_1, \sigma_2, \rho \rangle$.

With the discussion above, the following lemma is straightforward.

Lemma 2.2. *Let $\mathcal{M}$ be a rotary polytopal map of type $\{p, q\}$ with the skeleton Γ. Let σ_1 and σ_2 be the distinguished generators of $G = \mathrm{Aut}^+(\mathcal{M})$ with respect to some base flag Φ. Let v be the vertex of Φ. Then $(\sigma_1\sigma_2)^2 = 1$, σ_1 has order p and maps v to a neighbour w of v, while σ_2 has order q, fixes v and acts transitively on its neighbours. Moreover,*

- *$\mathcal{M}$ is non-orientable (and thus reflexible) if and only if there exists an involution $\rho \in G$ fixing v and inverting σ_1 and σ_2. In this case $|\mathrm{Stab}_G(vw)| = 2$.*
- *$\mathcal{M}$ is reflexible and orientable if and only if there exists an involution $\rho \in \mathrm{Aut}(\mathcal{M})\backslash G$ fixing v and inverting σ_1 and σ_2. In this case $|\mathrm{Stab}_G(vw)| = 1$.*
- *$\mathcal{M}$ is chiral if and only if there exists no involution $\rho \in \mathrm{Aut}(\mathcal{M})$ fixing v and inverting σ_1 and σ_2. In this case, $|\mathrm{Stab}_G(vw)| = 1$.*

The following result, which can be thought of as a converse of the previous lemma, characterises graphs that can be embedded as skeletons of rotary maps. Versions of this result can be found in [5] and [9]. However, additional information that we provide here is somewhat difficult to deduce from the previous results. For this reason, we decided to provide an independent proof.

Theorem 2.3. *Let $q \geq 3$ be an integer, let Γ be a q-valent connected simple graph, let v be a vertex of Γ and let σ_1 and σ_2 be automorphisms of Γ satisfying the following conditions:*

(i) *σ_2 fixes v and the group $\langle \sigma_2 \rangle$ transitively permutes the q neighbours of v;*
(ii) *σ_1 maps v to a neighbour w of v;*
(iii) *$(\sigma_1\sigma_2)^2 = 1$.*

Let $G = \langle \sigma_1, \sigma_2 \rangle$ and let p be the order of σ_1. Then $p \geq 3$, the group G acts transitively on the arcs of Γ and the following holds:

(a) *If $|\mathrm{Stab}_G(vw)| = 1$, then Γ is the skeleton of a polytopal rotary map $\mathcal{M}$ of type $\{p, q\}$ on an orientable surface with $\mathrm{Aut}^+(\mathcal{M}) = G$. In this case, $\mathcal{M}$ is reflexible if and only if there exists an involutory automorphism ρ of Γ normalising G, inverting σ_1 and fixing v.*

(b) *If* $|\mathrm{Stab}_G(vw)| = 2$ *and there exists* $\rho \in G$ *such that* $v\rho = v$ *and* $\sigma_1^\rho = \sigma_1^{-1}$, *then* Γ *is the skeleton of a polytopal non-orientable reflexible map* $\mathcal{M}$ *of type* $\{p, q\}$ *with* $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) = G$.

In both cases, the set of the facial cycles of $\mathcal{M}$ *is the* G*-orbit of the cycle* f *given by the sequence of vertices* $(v, v\sigma_1, v\sigma_1^2, \ldots, v\sigma_1^p)$, *and* σ_1, σ_2 *are the distinguished generators of* $\mathrm{Aut}^+(\mathcal{M})$ *with respect to the flag* (v, e, f), *where* e *is the edge with vertices* v *and* $v\sigma_1^{-1}$. *Moreover, whenever* $\mathcal{M}$ *is reflexible, conjugation by* ρ *(as given above) also inverts* σ_2.

Proof. For convenience, set $u = v\sigma_1^{-1}$. Let us now deduce some consequences of the conditions given in the theorem. First, notice that since $\sigma_1\sigma_2$ is an involution, it follows that $\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}$, and thus

$$w\sigma_2 = v\sigma_1\sigma_2 = v\sigma_2^{-1}\sigma_1^{-1} = v\sigma_1^{-1} = u.$$

Since $\langle\sigma_2\rangle$ transitively permutes the q neighbours of v and $q \geq 3$, this implies that $u \neq w$. Consequently, the orbit of v under $\langle\sigma_1\rangle$ consists of at least three distinct vertices, implying that $p \geq 3$. Moreover, since $\sigma_2\sigma_1\sigma_2 = \sigma_1^{-1}$, the fact that σ_2 is not an involution implies that σ_2 does not invert σ_1.

Next, since $\sigma_2\sigma_1$ is an involution and $v(\sigma_2\sigma_1) = v\sigma_1 = w$, we see that $\sigma_2\sigma_1$ inverts the edge $\{v, w\}$. This shows that the arc-stabiliser $\mathrm{Stab}_G(vw)$ has index 2 in the edge-stabiliser $\mathrm{Stab}_G(e')$ where $e' = \{v, w\}$. Hence, if $|\mathrm{Stab}_G(vw)| = 1$, then $\mathrm{Stab}_G(e') = \langle\sigma_2\sigma_1\rangle$. Observe also that the connectedness of Γ and the fact that $\mathrm{Stab}_G(v)$ is transitive on the neighbourhood of v, together with the existence of an edge-reversing automorphism $\sigma_2\sigma_1$, implies that G is transitive on the arcs of Γ.

Now suppose that there exists $\rho \in \mathrm{Aut}(\Gamma)$ such that $v\rho = v$ and $\sigma_1^\rho = \sigma_1^{-1}$. Let $H = \langle\sigma_1, \sigma_2, \rho\rangle$ (where possibly $H = G$). Note that $v(\sigma_2\rho) = v = v\rho^2$. Moreover, since ρ inverts σ_1, it follows that $\sigma_1^{\rho^2} = \sigma_1$ and $\sigma_1^{\rho^{-1}} = \sigma_1^{-1}$, implying that

$$w(\sigma_2\rho) = (w\sigma_2)\rho = u\rho = v\sigma_1^{-1}\rho = v\rho\sigma_1 = v\sigma_1 = w \quad \text{and}$$
$$w\rho^2 = v\sigma_1\rho^2 = v\rho^2\sigma_1 = v\sigma_1 = w.$$

Therefore, $\sigma_2\rho, \rho^2 \in \mathrm{Stab}_H(vw)$. Since both ρ and ρ^{-1} invert σ_1 but σ_2 does not, we see that $\sigma_2\rho \neq \rho^2$ and $\sigma_2\rho \neq 1$.

If the hypothesis of (b) holds, then we may assume that $\rho \in G$ and thus $H = G$. Moreover, $\sigma_2\rho$ is the unique non-trivial element of $\mathrm{Stab}_G(vw)$ and therefore $\rho^2 = 1$, showing that $\mathrm{Stab}_G(e') = \langle\sigma_2\sigma_1, \sigma_2\rho\rangle = \langle\sigma_2\sigma_1, \rho\sigma_1\rangle$. We shall use these facts throughout the rest of the proof whenever we assume that the hypothesis of (b) holds.

Let $v_i = v\sigma_1^i$ for $i \in \mathbb{Z}$ and consider the sequence of vertices $f = (v_0, v_1, \ldots, v_{p-1}, v_p)$. Clearly, $v_0 = v = v_p$, $w = v_1$ and $u = v_{p-1}$. Since w is adjacent to v by assumption, the above sequence is a closed walk in Γ. Let us now show that under the hypothesis of (a) or (b), the walk f is in fact a cycle. If this were not the case, then $v = v\sigma_1^k$ for some

$k \in \{1, \ldots, p-1\}$ and thus $\sigma_1^k \in \mathrm{Stab}_G(vw)$. Since σ_1 is of order p, only the hypothesis of (b) can hold, in which case σ_1^k is the unique non-trivial element of $\mathrm{Stab}_G(vw)$ and thus $\sigma_1^k = \sigma_2\rho$. But then $\sigma_2 = \sigma_1^k\rho^{-1}$, contradicting the fact that σ_2 does not invert σ_1 while ρ^{-1} does. This shows that f is indeed a cycle of Γ.

Let us now consider the action of ρ on the vertices of f, where $\rho \in \mathrm{Aut}(\Gamma)$, $v\rho = v$ and $\sigma_1^\rho = \sigma_1^{-1}$, as in (a) and (b). Let $i \in \{0, \ldots, p-1\}$. Then $v_i\rho = v\sigma_1^i\rho = v\rho\sigma_1^{-i} = v\sigma_1^{-i} = v_{p-i}$. In short, ρ preserves f and acts on it as a reflection through v.

Now let $\mathcal{C} = \{f\gamma \mid \gamma \in G\}$ be the orbit of the cycle f under G. We shall show that $\mathcal{C}$ is a double cycle cover of Γ and that the vertex figure with respect to $\mathcal{C}$ at every vertex is connected. Since G is edge- and vertex-transitive and $\mathcal{C}$ is an orbit under the action of G, it suffices to show that the edge $e' = \{v, w\}$ belongs to exactly two cycles in $\mathcal{C}$ and that the vertex-figure at v is connected.

Suppose that the edge e' lies on a cycle $f' \in \mathcal{C}$. Choose $\alpha \in G$ such that $f'\alpha = f$. Since $\langle\sigma_1\rangle$ is transitive on the edges of f, we see that for an appropriate $i \in \mathbb{Z}_p$ the element $\gamma = \alpha\sigma_1^i \in G$ maps f' to f and preserves e'. That is, $f'\gamma = f$ and $\gamma \in \mathrm{Stab}_G(e')$. Now recall that $\mathrm{Stab}_G(e')$ is either $\langle\sigma_2\sigma_1\rangle$ (in case that the hypothesis of (a) holds) or $\langle\sigma_2\sigma_1, \rho\sigma_1\rangle$ (in the case of (b)). Since both ρ and σ_1 preserve f and since $\gamma \in \mathrm{Stab}_G(e')$, this implies that $f' \in \{f, f\sigma_2\sigma_1\}$. In particular, e' belongs to at most two cycles in $\mathcal{C}$, namely f and $f\sigma_2\sigma_1$, and it belongs to exactly two cycles in $\mathcal{C}$ unless $f = f\sigma_2\sigma_1$. If the latter happens, then $f = f\sigma_2\sigma_1 = f\sigma_1^{-1}\sigma_2^{-1} = f\sigma_2^{-1}$, implying that $(u, v, w)\sigma_2^{-1} = (w, v, w\sigma_2^{-1})$ is a path of length 2 of f centred at v. But then $w\sigma_2^{-1} = u = w\sigma_2$, contradicting the assumption that $\langle\sigma_2\rangle$ transitively permutes the q neighbours of v and that $q \geq 3$. This shows that e' lies in precisely two cycles in $\mathcal{C}$, and thus, that $\mathcal{C}$ is a cycle double cover of Γ.

Now consider the vertex figure at v with respect to $\mathcal{C}$. For $i \in \mathbb{Z}_q$, let $z_i = w\sigma_2^i$ and observe that $\{z_i \mid i \in \mathbb{Z}_q\}$ is the neighbourhood of v in Γ. Moreover, the edges $\{z_0, v\} = \{w, v\}$ and $\{z_1, v\} = \{u, v\}$ are two consecutive edges on a cycle in $\mathcal{C}$, implying that they are adjacent in the vertex figure at v. But then the edges $\{w, v\}\sigma_2^i = \{z_i, v\}$ and $\{u, v\}\sigma_2^i = \{z_{i+1}, v\}$ are also adjacent in the vertex figure, implying that the vertex figure is connected.

We are now in a position to apply Lemma 2.1 to conclude that the graph Γ is the skeleton of a map $\mathcal{M}$ with $\mathrm{Aut}^+(\mathcal{M}) = G$. The existence of σ_1 and σ_2 implies that the map $\mathcal{M}$ is rotary. Now recall that whenever ρ exists (as in (a) or (b)), it preserves f and normalises G, implying that it preserves $\mathcal{C}$ and is thus an automorphism of the map $\mathcal{M}$. Moreover, the automorphism $\sigma_2\rho$ is a non-trivial element of the group $\mathrm{Stab}_H(vw)$, where $H = \langle\sigma_1, \sigma_2, \rho\rangle \leq \mathrm{Aut}(\mathcal{M})$. In particular, $\sigma_2\rho$ is an involution (and so is ρ), implying that $\sigma_2^\rho = \sigma_2^{-1}$. The rest of the claims of the theorem now follow directly from Lemma 2.2. □

We conclude the section with a lemma that, given a polytopal reflexible map $\mathcal{M}$, allows us to obtain yet another map with the same skeleton, often referred to as the Petrie dual of $\mathcal{M}$ (see, for example, [3]).

Lemma 2.4. *Let $\mathcal{M}$ be a polytopal reflexible map of type $\{p, q\}$ with $q \geq 3$. Suppose that the skeleton Γ of $\mathcal{M}$ is simple. Let σ_1, σ_2 be the distinguished generators of $\mathrm{Aut}^+(\mathcal{M})$ with respect to some base flag Φ, and let ρ be an involutory automorphism of $\mathcal{M}$ fixing the vertex v of Φ and inverting σ_1 and σ_2. Then there exists a reflexible polytopal map $\mathcal{M}^\pi$ with the skeleton Γ such that $\eta_1 := \sigma_1\sigma_2\rho$ and $\eta_2 := \sigma_2$ are the distinguished generators of $\mathrm{Aut}^+(\mathcal{M}^\pi)$ with respect to some base flag. The map $\mathcal{M}^\pi$ is non-orientable if and only if $\rho \in \langle \eta_1, \eta_2 \rangle$.*

Proof. We need to show that the conditions for σ_1 and σ_2 of Theorem 2.3 are satisfied by η_1 and η_2. Condition (i) of Theorem 2.3 is trivially satisfied by η_2. Next, since $\sigma_1\sigma_2$ is an involution and ρ inverts σ_1, we see that $v\eta_1 = v\sigma_1\sigma_2\rho = v\sigma_2^{-1}\sigma_1^{-1}\rho = v\rho\sigma_1 = v\sigma_1$, showing that condition (ii) of Theorem 2.3 is satisfied by η_1. Since ρ also inverts σ_2, we see that $(\eta_1\eta_2)^2 = (\sigma_1\sigma_2\rho\sigma_2)^2 = (\sigma_1\rho)^2 = 1$, implying that condition (iii) of Theorem 2.3 is satisfied by η_1 and η_2.

It is also straightforward to see that ρ inverts η_1: $\eta_1^\rho = \rho\sigma_1\sigma_2 = \rho\sigma_2^{-1}\sigma_1^{-1} = (\sigma_1\sigma_2\rho)^{-1} = \eta_1^{-1}$. Set $H = \langle \eta_1, \eta_2 \rangle$ and observe that $H \leq \mathrm{Aut}(\mathcal{M})$, implying that the stabiliser of an arc in the group H has order at most 2. In fact, since clearly $H = \mathrm{Aut}(\mathcal{M})$ if and only if $\rho \in H$, the stabiliser of an arc in the group H has order 2 or 1, depending on whether $\rho \in H$ or not, respectively. Theorem 2.3 thus implies that η_1 and η_2 are the distinguished generators (with respect to some base flag) of a reflexible polytopal map $\mathcal{M}^\pi$ with skeleton Γ, and that $\mathcal{M}^\pi$ is non-orientable if and only if $\rho \in H$. $\square$

3. The graphs $C_n[mK_1]$

Let m and s be positive integers where $m \geq 3$ is odd, and let $n = ms$. We let $\Gamma = C_n[mK_1]$ be the lexicographic product of the n-cycle C_n by the edgeless graph mK_1 of order m. That is, the vertices of Γ are of the form (v, w), where $v \in C_n$ and $w \in mK_1$, and there is an edge between the vertices (v_1, w_1) and (v_2, w_2) whenever there is an edge between v_1 and v_2 in C_n. Hence, Γ is a graph with mn vertices and each vertex has valency $2m$, implying that Γ has $2m^2n$ arcs.

By labelling the vertices of C_n and mK_1 appropriately, we may assume that the vertex set of $C_n[mK_1]$ is the Cartesian product $\{1, \ldots, n\} \times \{1, \ldots, m\}$ with the vertices (i_1, j_1) and (i_2, j_2) adjacent if and only if $i_1 = i_2 \pm 1$, where the sum is taken mod n and where we use n instead of 0 (this convention is taken throughout the paper).

It is well known and easy to see that the automorphism group $\mathrm{Aut}(\Gamma)$ of Γ is the wreath product $S_m \wr D_n$ of the symmetric group S_m by the dihedral group D_n (recall that $n \neq 4$). In other words, it is equal to the semidirect product

$$(S_m \times S_m \times \cdots \times S_m) \rtimes D_n$$

in its imprimitive action, where we have n copies of S_m in the direct product and where D_n acts on this direct product by permuting the coordinates. We write the elements of this group in the form $(\alpha_1, \alpha_2, \ldots, \alpha_n)x$, where $\alpha_i \in S_m$ for each i, $1 \leq i \leq n$, and $x \in D_n$. The action of $\mathrm{Aut}(\Gamma)$ on the vertex-set of Γ is then given by

$$(i,j)(\alpha_1, \alpha_2, \ldots, \alpha_n)x = (ix, j\alpha_i) \tag{1}$$

for every vertex $(i,j) \in \{1,\ldots,n\} \times \{1,\ldots,m\}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n)x \in S_m \wr D_n$. By a slight abuse of notation, we shall view elements $x \in D_n$ and $\alpha \in S_m$ also as elements of $S_m \wr D_n$ by identifying them with $(1,1,\ldots,1)x$ and $(\alpha,\alpha,\ldots,\alpha)1$.

Let $c \in S_m$ be the m-cycle $c = (1\,2\,\cdots\,m)$ and let $t \in S_m$ be the involution fixing 1 and interchanging each j, $2 \leq j \leq m$, with $m - j + 2$. Furthermore let $r \in D_n$ be the n-cycle $r = (1\,2\,\cdots\,n)$ and let $z \in D_n$ be the reflection fixing 1 and interchanging each j, $2 \leq j \leq n$, with $n-j+2$. We point out that for each $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in S_m \times S_m \times \cdots \times S_m$ the following holds:

$$\begin{aligned} r(\alpha_1, \alpha_2, \ldots, \alpha_n) &= (\alpha_2, \alpha_3, \ldots, \alpha_n, \alpha_1)r \\ &\text{and} \\ z(\alpha_1, \alpha_2, \ldots, \alpha_n) &= (\alpha_1, \alpha_n, \alpha_{n-1}, \ldots, \alpha_3, \alpha_2)z. \end{aligned} \tag{2}$$

4. Chiral maps

In this section we construct a family of chiral maps with the skeletons $C_n[mK_1]$, where $m \geq 3$ is an odd integer and $n = sm$, with s a positive integer. Throughout this section, we let Γ denote the graph $C_n[mK_1]$ and let the elements $c, t \in S_m$, $r, z \in D_n$ be as in Section 3. We start by the following definitions and a lemma:

$$\begin{aligned} &\ell_i = 1 + 1 + 2 + 3 + \cdots + (i-2) = 1 + \frac{(i-2)(i-1)}{2} \text{ for each } i \in \{2, 3, \ldots, n\}, \\ &\sigma_1 = (c, 1, 1, \ldots, 1)r, \\ &\sigma_2 = (t, tc^{\ell_2}, tc^{\ell_3}, tc^{\ell_4}, \ldots, tc^{\ell_n})z \;=\; (t, tc, tc^2, tc^4, \ldots, tc^{1+(n-2)(n-1)/2})z, \text{ and} \\ &G = \langle \sigma_1, \sigma_2 \rangle. \end{aligned} \tag{3}$$

Lemma 4.1. *Let σ_1, σ_2 and G be as in (3). Then the following holds:*

$$|\sigma_1| = mn, \qquad |\sigma_2| = 2m, \qquad |\sigma_1\sigma_2| = 2 \qquad \textit{and} \qquad |G| = 2m^2n.$$

Proof. Using (2) we first note that:

$$\sigma_1^n = (c, c, c, \ldots, c)1, \tag{4}$$

which implies that the order of σ_1^n is the same as that of c, and therefore the order of σ_1 is mn. Observe that for each i with $3 \leq i \leq n$, one of n and $n + 3 - 2i$ is even. Since m is odd and $n = ms$, the product $n(n + 3 - 2i)$ is an even multiple of m, implying that

$$\ell_{n-i+3} = 1 + \frac{(n-i+1)(n-i+2)}{2} = 1 + \frac{n(n+3-2i)+(i-1)(i-2)}{2} \equiv \ell_i \pmod{m}.$$

Therefore,

$$\sigma_2 = (t, tc, tc^2, tc^4, tc^7, \dots, tc^7, tc^4, tc^2)z. \tag{5}$$

Using (2) again it now easily follows that

$$\sigma_2^2 = (1, c, c^2, c^3, \dots, c^{n-1})1. \tag{6}$$

In particular, σ_2 is of order $2m$. By (2), (3) and (5), it follows that

$$\sigma_1\sigma_2 = (t, tc^2, tc^4, \dots, tc^4, tc^2, t)rz,$$

and one then easily verifies that this is an involution, settling the first three claims of the lemma.

We shall now determine the order of the group G. To do so, start by noticing that from (3) and (6) we also easily deduce that

$$\sigma_2^{-2}\sigma_1\sigma_2^2 = (c^2, c, c, \dots, c)r = \sigma_1^{n+1} \in \langle\sigma_1\rangle.$$

Therefore, since the intersection $\langle\sigma_1\rangle \cap \langle\sigma_2\rangle$ clearly coincides with the intersection $\langle\sigma_1^n\rangle \cap \langle\sigma_2^2\rangle$, which by (4) and (6) is trivial, this shows that the subgroup $H = \langle\sigma_1, \sigma_2^2\rangle$ is of order m^2n. Now, since $\sigma_1\sigma_2$ is an involution, it follows that $\sigma_2\sigma_1 = \sigma_1^{-1}\sigma_2^{-1}$. Therefore,

$$(\sigma_2\sigma_1)\sigma_1(\sigma_2\sigma_1) = \sigma_1^{-1}\sigma_2^{-2} \in H$$

and

$$(\sigma_2\sigma_1)\sigma_2^2(\sigma_2\sigma_1) = \sigma_1^{-1}\sigma_2^2\sigma_1 \in H,$$

implying that $\sigma_2\sigma_1$ normalises H. Since $\langle\sigma_1, \sigma_2\rangle = \langle\sigma_1, \sigma_2^2, \sigma_2\sigma_1\rangle$, this finally shows that the group G has order $2m^2n$. □

The above lemma has the following immediate consequence.

Proposition 4.2. *Let $m \geq 3$ be an odd integer and let $n = sm$, where s is a positive integer. Let σ_1, σ_2 and G be as in (3). Then $\Gamma = C_n[mK_1]$ is the skeleton of a polytopal chiral map $\mathcal{M}$ of type $\{mn, 2m\}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) = G$.*

Proof. Let v be the vertex $(1, 1)$ of Γ and note that $v\sigma_2 = (1, 1)\sigma_2 = (1z, 1t) = (1, 1)$, showing that σ_2 fixes v. Observe that the neighbourhood of v consists of the vertices of the form $(2, j)$ and (n, k), for $1 \leq j, k \leq m$. Note that $(2, j)\sigma_2 = (2z, jtc) =$

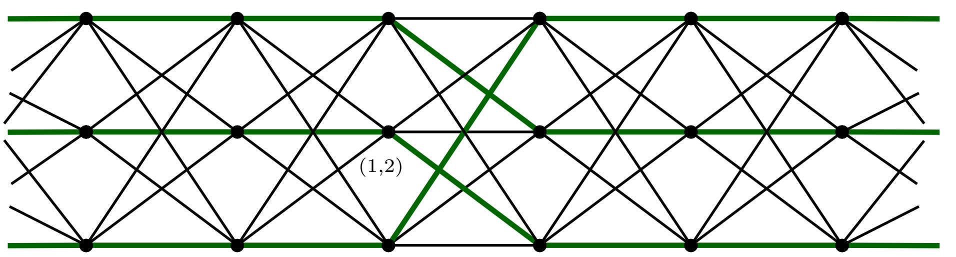

Fig. 1. The graph $C_6[3K_1]$ and the base face of the chiral map constructed in Proposition 4.2.

$(n, m-j+3)$ and $(n,k)\sigma_2 = (nz, ktc^2) = (2, m-k+4)$, where the second component is computed modulo m if necessary. Therefore, σ_2 cyclically permutes the vertices $(2,2),(n,1),(2,3),(n,m),(2,4),(n,m-1),(2,5),\ldots,(n,2)$ around v, implying that $\langle\sigma_2\rangle$ acts transitively on the neighbours of v. Hence, σ_2 satisfies condition (i) of Theorem 2.3.

Further, $v\sigma_1 = (2,2)$, which is a neighbour of v (in fact, the cycle through v induced by σ_1 is $((1,1),(2,2),(3,2),\ldots,(n,2),(1,2),(2,3),\ldots,(1,3),(2,4),\ldots,(n,1))$, and is depicted in Fig. 1), showing that σ_1 satisfies condition (ii) of Theorem 2.3.

Moreover, as shown in Lemma 4.1, $(\sigma_1\sigma_2)^2 = 1$, implying that condition (iii) of Theorem 2.3 is fulfilled. Theorem 2.3 then implies that G is arc-transitive. Since $|G| = 2m^2n$, which equals the number of arcs of Γ, we see that the arc-stabiliser in G is trivial. By part (a) of Theorem 2.3, it follows that Γ is the skeleton of a polytopal map $\mathcal{M}$ of type $\{mn, 2m\}$.

Finally, to show that $\mathcal{M}$ is chiral, it suffices to show that $\mathrm{Aut}(\Gamma)$ contains no involution ρ fixing v, normalizing G and inverting σ_1. By way of contradiction suppose such a ρ exists and note that by Theorem 2.3 it inverts σ_2. Moreover, since it fixes v and inverts σ_1 it must be of the form

$$\rho = (\alpha_1, \alpha_2, \ldots, \alpha_n)z.$$

Since ρ is an involution, we deduce that

$$\alpha_1^2 = 1 \text{ and } \alpha_{n-i+2} = \alpha_i^{-1} \text{ for all } i,\ 2 \le i \le n. \tag{7}$$

By (3), we see that $\rho\sigma_1 = (\alpha_1 c, \alpha_2, \alpha_3, \ldots, \alpha_n)zr$, and so

$$(\rho\sigma_1)^2 = (\alpha_1 c\alpha_2, \alpha_2\alpha_1 c, \alpha_3\alpha_n, \alpha_4\alpha_{n-1}, \ldots, \alpha_n\alpha_3)1.$$

Since ρ inverts σ_1, we also deduce that $\alpha_1 c\alpha_2 = 1$ and $\alpha_{n-i+3} = \alpha_i^{-1}$ for all i with $3 \le i \le n$. Together with (7), this thus implies that

$$\alpha_1 c\alpha_2 = 1,\ \alpha_2 = \alpha_3 = \ldots = \alpha_n, \text{ and } \alpha_2^2 = 1. \tag{8}$$

Finally, by (5) we now see that

$$\rho\sigma_2 = (\alpha_1 t, \alpha_2 tc^2, \alpha_2 tc^4, \alpha_2 tc^7, \dots, \alpha_2 tc)1,$$

which can thus only be an involution if α_2 is centralised by tc and tc^2. But then it is also centralised by c, and so the fact that the centraliser of c in S_m is $\langle c\rangle$, forces $\alpha_2 = 1$ (recall that m is odd). Then (8) implies that $\alpha_1 c = 1$, which contradicts $\alpha_1^2 = 1$. □

5. Reflexible maps

In this section we construct non-orientable reflexible and orientable reflexible maps whose skeleton is $C_n[mK_1]$, with $n = sm$ where $m \geq 3$ is an odd integer and where s is a positive integer not divisible by 4. The constructions for s odd and s even are a bit different (even though there are many similarities).

Throughout this section we let $\Gamma = C_n[mK_1]$, where $m \geq 3$ is odd, and where $n = sm$ for some $s \geq 1$. We represent $\mathrm{Aut}(\Gamma)$ just as we did in Section 3 and we also let c, t, r and z have the same meaning as in Section 3.

5.1. The examples with s odd

Throughout this subsection let s be odd. We first construct a non-orientable reflexible map. To this end, set

$$\begin{aligned} \sigma_1 &= (1, 1, \dots, 1, tc^{-1})r, \\ \sigma_2 &= (t, c^{-1}, c^3, c^{-5}, \dots, c^{(-1)^{i-1}(2i-3)}, \dots, c^5, c^{-3})z, \text{ and} \\ G &= \langle \sigma_1, \sigma_2\rangle. \end{aligned} \tag{9}$$

Note that the assumptions that n is odd and $c^n = (c^m)^s = 1$ indeed yield $c^{(-1)^{n-1}(2n-3)} = c^{-3}$.

Lemma 5.1. *Let σ_1, σ_2 and G be as in (9). Then the following holds:*

$$|\sigma_1| = 2n, \qquad |\sigma_2| = 2m, \qquad |\sigma_1\sigma_2| = 2, \qquad \textit{and} \qquad |G| = 4m^2 n.$$

Proof. We first note that

$$\sigma_1^n = (tc^{-1}, tc^{-1}, \dots, tc^{-1})1, \tag{10}$$

implying that $|\sigma_1| = 2n$. Similarly,

$$\sigma_2^2 = (t, c^{-1}, \dots, c^{(-1)^{i-1}(2i-3)}, \dots, c^{-3})(t, c^{-3}, \dots, c^{(-1)^{n-i+1}(2n-2i+1)}, \dots, c^{-1})1,$$

and so setting $\varphi = \sigma_2^2$, we see that

$$\varphi = (1, c^{-4}, c^8, \ldots, c^{(-1)^{i-1}4(i-1)}, \ldots, c^8, c^{-4})1. \tag{11}$$

This of course implies that $|\sigma_2| = 2m$, as claimed. Next, note that

$$\sigma_1\sigma_2 = (c^{-1}, c^3, \ldots, c^{(-1)^i(2i-1)}, \ldots, c^{-3}, c)rz, \tag{12}$$

from which it is easy to verify that $\sigma_1\sigma_2$ is indeed an involution.

Let $\psi = \sigma_1^{-1}\varphi\sigma_1$ and note that

$$\psi = (tc^{-1}, 1, \ldots, 1)(c^{-4}, 1, c^{-4}, c^8, \ldots, c^{(-1)^{i-2}4(i-2)}, \ldots, c^8)(tc^{-1}, 1, \ldots, 1)1,$$

that is

$$\psi = (c^4, 1, c^{-4}, \ldots, c^{(-1)^i 4(i-2)}, \ldots, c^8)1. \tag{13}$$

Since φ and ψ commute and are both of order m (recall that m is odd), the group $H_1 = \langle \varphi, \psi \rangle \cong C_m \times C_m$ is abelian and is of order m^2. In a completely analogous way as we computed ψ we can verify that

$$\sigma_1^{-1}\psi\sigma_1 = (c^{-8}, c^4, 1, \ldots, c^{(-1)^{i+1}4(i-3)}, \ldots, c^{-12})1.$$

It follows that $\sigma_1^{-1}\psi\sigma_1 = \varphi^{-1}\psi^{-2} \in H_1$, which implies that σ_1 normalises H_1. Similarly, σ_2 commutes with φ and since (recall that $\sigma_1\sigma_2$ is an involution)

$$\sigma_2^{-1}\psi\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2 = \sigma_1\sigma_2^2\sigma_1^{-1} = \sigma_1\varphi\sigma_1^{-1} \tag{14}$$

and σ_1 normalises H_1, we see that in fact σ_2 also normalises H_1, proving that H_1 is normal in G. Clearly, $\langle \sigma_1 \rangle \cap H_1 = 1$, and so $K_1 = \langle \sigma_1, \varphi, \psi \rangle$ is a group of order $2m^2n$. Now, σ_1 normalises K_1, and so the fact that

$$\sigma_2^{-1}\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_2^{-1}\sigma_1^{-1} = \varphi^{-1}\sigma_1^{-1} \in K_1 \tag{15}$$

implies that K_1 is also a normal subgroup of G. Since $\sigma_2 \notin K_1$ and $\sigma_2^2 \in K_1$, this finally proves that G is of order $4m^2n$, as claimed. □

Proposition 5.2. *Let s and m be odd integers with $m \geq 3$. Let σ_1, σ_2 and G be as in (9). Then $\Gamma = C_n[mK_1]$ is the skeleton of a polytopal non-orientable reflexible map $\mathcal{M}$ of type $\{2n, 2m\}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) = G$.*

Proof. Let $v := (1, 1)$ and observe that σ_1 maps v to the neighbour $w := (2, 1)$ of v. In fact, the orbit of v under $\langle \sigma_1 \rangle$ constitutes the cycle f of length $2n$ with vertices $(1, 1), (2, 1), (3, 1), \ldots, (n, 1), (1, m), (2, m), \ldots, (n, m)$, in that order (see Fig. 2).

Moreover, $v\sigma_2 = (1z, 1t) = (1, 1) = v$, while for each j and k with $1 \leq j, k \leq m$ we see that $(2, j)\sigma_2 = (n, j-1)$ and $(n, k)\sigma_2 = (2, k-3)$, implying that σ_2 stabilises v and

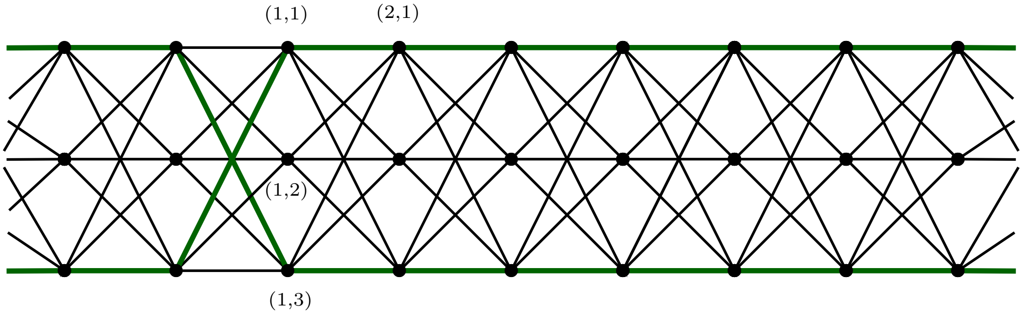

Fig. 2. The graph $C_9[3K_1]$ and the base face of the reflexible non-orientable map constructed in Proposition 5.2.

cyclically permutes the vertices $(2,1),(n,m),(2,m-3),(n,m-4),\dots,(n,4)$. Since m is odd, this implies that $\langle\sigma_2\rangle$ acts transitively on the neighbourhood of v. We can thus use Theorem 2.3 to deduce that G acts transitively on the arcs of Γ.

Moreover, by Lemma 5.1, the order of G is twice the number of arcs in Γ, implying that $|\mathrm{Stab}_G(uw)|=2$. Let us now find an involution $\rho\in G$ fixing v and inverting σ_1; the result will then follow by Theorem 2.3.

Note that $c^{(m-1)(m+1)}=c^{-1}$ and $c^{(m+1)^2}=c$. Therefore, defining φ,ψ as in the proof of Lemma 5.1, one can verify that (11) and (13) imply that

$$\varphi^{(m-1)(m+1)/4}\psi^{(m+1)^2/4}=(c,c,c^{-3},c^5,\dots,c^{(-1)^{i-1}(3-2i)},\dots,c^3)1.$$

Thus (10) implies that setting $\rho=\sigma_1^n\varphi^{(m-1)(m+1)/4}\psi^{(m+1)^2/4}\sigma_2$, we have $\rho\in G$ and

$$\rho=(1,tc^{-1},tc^{-1},\dots,tc^{-1})z. \tag{16}$$

Observe that $(1,1)\rho=(1,1)$ and that ρ is an involution. Since

$$\rho\sigma_1=(1,tc^{-1},tc^{-1},\dots,tc^{-1})(1,tc^{-1},1,\dots,1)zr=(1,1,tc^{-1},\dots,tc^{-1})zr, \tag{17}$$

it follows that $\rho\sigma_1$ is also an involution, showing that ρ inverts σ_1, as desired. □

Let us now use the Petrie dual construction described in Lemma 2.4 to deduce existence of an orientable reflexible map with the skeleton Γ. Let σ_1, σ_2 be as in (9), and let ρ be as in (16). Let

$$\eta_1=\sigma_1\sigma_2\rho, \qquad \eta_2=\sigma_2 \qquad \text{and} \qquad H=\langle\eta_1,\eta_2\rangle. \tag{18}$$

Proposition 5.3. *Let s and m be odd integers with $m\geq 3$, let $n=sm$ and let η_1, η_2 and H be as in (18). Then $\Gamma=C_n[mK_1]$ is the skeleton of a polytopal orientable reflexible map $\mathcal{M}$ of type $\{n,2m\}$ with $\mathrm{Aut}^+(\mathcal{M})=H$.*

Proof. By Lemma 2.4 and Proposition 5.2, we already know that Γ is the skeleton of a reflexible map $\mathcal{M}^\pi$ with $\mathrm{Aut}^+(\mathcal{M}^\pi) = H$ with the distinguished generators of a base flag being η_1 and η_2. To conclude the proof we only need to prove that $|\eta_1| = n$ and that $\mathcal{M}^\pi$ is orientable.

Let σ_1 and σ_2 be as in (9), and let ρ be as in (16). From (12), (16) and (18) we easily see that

$$\eta_1 = (t, tc^{-4}, tc^4, \ldots, tc^{(-1)^{i+1}(2i-1)-1}, \ldots, tc^2, c)r.$$

Observe that multiplying the components of $(t, tc^{-4}, tc^4, \ldots, tc^{(-1)^{i+1}(2i-1)-1}, \ldots, tc^2, c)$ in this order, we obtain

$$c^{-4}c^{-12}c^{-20} \cdots c^{-4(n-2)}c = c^\ell,$$

where

$$\ell = -4(1 + 3 + 5 + \cdots + (n-2)) + 1 = -4((n-1)/2)^2 + 1 = n(2-n).$$

Therefore, $c^\ell = 1$, and it now easily follows that $\eta_1^n = 1$, proving that $|\eta_1| = n$, as claimed.

Let us now show that $\mathcal{M}^\pi$ is orientable. Observe that it suffices to show that H is of order $2m^2n$. Let $\zeta = \eta_2^2$ and $\xi = \eta_1^{-1}\zeta\eta_1$. Since $\sigma_1\sigma_2 = \sigma_2^{-1}\sigma_1^{-1}$ and since ρ inverts σ_1 and σ_2, we find that

$$\xi = \rho\sigma_2^{-1}\sigma_1^{-1}\sigma_2^2\sigma_1\sigma_2\rho = \rho\sigma_1\sigma_2^2\sigma_1^{-1}\rho = \sigma_1^{-1}\sigma_2^{-2}\sigma_1.$$

Therefore, $\langle \zeta, \xi \rangle = H_1 \cong C_m \times C_m$, where H_1 is as in the proof of Lemma 5.1. Setting $K_2 = \langle \zeta, \xi, \eta_1 \rangle$, we thus see that K_2 is a group of order m^2n (recall that H_1 is normal in $\langle \sigma_1, \sigma_2 \rangle$). Moreover, as in the proof of Proposition 5.2, the fact that this time $\eta_1\eta_2$ is an involution implies that

$$\eta_2^{-1}\eta_1\eta_2 = \eta_2^{-1}\eta_2^{-1}\eta_1^{-1} = \zeta^{-1}\eta_1^{-1} \in K_2,$$

showing that K_2 is normal in H. Since $\eta_2^2 \in K_2$, this implies that K_2 is of index 2 in H, and consequently $|H| = 2m^2n$, as claimed (Fig. 3). $\square$

5.2. The examples with s even

Throughout this subsection let s be an even number not divisible by 4. It turns out that this time the orientable and non-orientable reflexible maps that we construct are not the Petrie duals of one another (in fact, the non-orientable one is self-Petrie). Thus, we construct each of them in a rather different way.

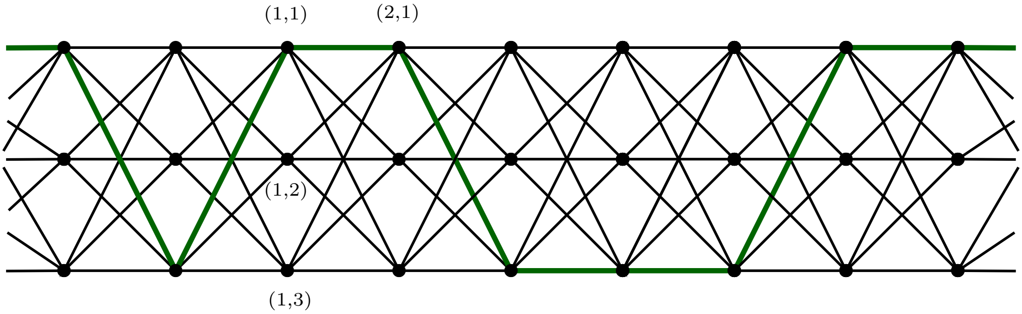

Fig. 3. The graph $C_9[3K_1]$ and the base face of the reflexible orientable map constructed in Proposition 5.3.

We start with the construction for the non-orientable maps by defining

$$\begin{aligned} \sigma_1 &= (1, 1, \dots, 1, tc^{-1})r, \\ \sigma_2 &= (t, c^{-1}, c, c, c^{-1}, c^{-1}, \dots, c, c, c^{-1}, c^{-1})z, \text{ and} \\ G &= \langle \sigma_1, \sigma_2 \rangle, \end{aligned} \tag{19}$$

where for $3 \leq i \leq n$ the i-th component of the "n-tuple" for σ_2 is c or c^{-1}, depending on whether i is congruent to one of 0 and 3 modulo 4, or one of 1 and 2 modulo 4, respectively. Note that the σ_1 from (19) has the exact same form as the σ_1 defined in (9).

Lemma 5.4. *Let σ_1, σ_2 and G be as in (19). Then the following holds:*

$$|\sigma_1| = 2n, \qquad |\sigma_2| = 2m, \qquad |\sigma_1\sigma_2| = 2 \qquad \textit{and} \qquad |G| = 4m^2 n.$$

Proof. As in the proof of Lemma 5.1, we can deduce that $|\sigma_1| = 2n$. Set $\varphi = \sigma_2^2$. Since n is twice an odd number it is easy to see that

$$\varphi = (1, c^{-2}, 1, c^2, 1, \dots, c^2, 1, c^{-2})1, \tag{20}$$

and so the fact that m is odd implies that $|\sigma_2| = 2m$, as claimed. We also see that

$$\sigma_1\sigma_2 = (1, 1, \dots, 1, tc^{-1})(c^{-1}, c, c, \dots, c^{-1}, c^{-1}, t)rz = (c^{-1}, c, c, \dots, c^{-1}, c^{-1}, c)rz,$$

where the i-th component of the n-tuple is c^{-1} whenever i is congruent to 0 or 1 modulo 4, and is c otherwise. It is now easy to verify that $\sigma_1\sigma_2$ is an involution. As in the proof of Lemma 5.1, set $\psi = \sigma_1^{-1}\varphi\sigma_1$. A straightforward calculation shows that

$$\psi = (c^2, 1, c^{-2}, 1, \dots, c^{-2}, 1, c^2, 1)1,$$

where the i-th components of the n-tuple are all equal to 1 for i even, while for i odd they alternate between c^2 and c^{-2}. In view of (20) it now follows that $H_1 = \langle \varphi, \psi \rangle \cong C_m \times C_m$. Clearly, $\sigma_1^{-1}\varphi\sigma_1, \sigma_2^{-1}\varphi\sigma_2 \in H_1$. Moreover,

$$\sigma_1^{-1}\psi\sigma_1 = (tc^{-1}, 1, \dots, 1)(1, c^2, 1, c^{-2}, \dots, 1, c^2)(tc^{-1}, 1, \dots, 1)1,$$

and so in fact $\sigma_1^{-1}\psi\sigma_1 = \varphi^{-1}$. Repeating the computation from (14) we also see that $\sigma_2^{-1}\psi\sigma_2 \in H_1$, showing that H_1 is normal in G. It is clear that $H_1 \cap \langle \sigma_1 \rangle = 1$, and so $K_1 = \langle \sigma_1, \varphi, \psi \rangle$ is a group of order $2m^2n$. Just like in (15) we see that $\sigma_2^{-1}\sigma_1\sigma_2 \in K_1$, proving that K_1 is normal in G. Finally, since $\sigma_2 \notin K_1$ and $\sigma_2^2 \in K_1$, we see that G is of order $4m^2n$, as claimed. □

Proposition 5.5. *Let $m \geq 3$ be an odd integer and let $n = sm$, with s an even integer not divisible by 4. Let σ_1, σ_2 and G be as in (19). Then $\Gamma = C_n[mK_1]$ is the skeleton of a polytopal non-orientable reflexible map $\mathcal{M}$ of type $\{2n, 2m\}$ with $\mathrm{Aut}(\mathcal{M}) = \mathrm{Aut}^+(\mathcal{M}) = G$.*

Proof. In the same way as in the proof of Proposition 5.2, we observe that σ_1 maps the vertex $v := (1, 1)$ to its neighbour $w := (2, 1)$ and that the orbit of v under $\langle \sigma_1 \rangle$ constitutes the cycle f of length $2n$ with vertices $(1, 1), (2, 1), \dots, (n, 1), (1, m), (2, m), \dots, (n, m)$, in that order. Further, note that $v\sigma_2 = v$, and that for every j and k with $1 \leq j, k \leq m$, the action of σ_2 on the neighbours $(2, j)$ and (n, k) of v is given by $(2, j)\sigma_2 = (n, j - 1)$ and $(n, k)\sigma_2 = (2, k - 1)$. Since m is odd, this shows that σ_2 cyclically permutes the neighbours $(2, 1), (n, m), (2, m - 1), (n, m - 2), \dots, (n, 2)$ of v. By Theorem 2.3 and Lemma 5.4, the group G acts transitively on the arcs of Γ and has twice as many elements as there are arcs of Γ, implying that $|\mathrm{Stab}_G(vw)| = 2$.

To complete the proof set $\rho = \sigma_1^n\varphi^{(m-1)/2}\psi^{(m+1)/2}\sigma_2$, where φ and ψ are as in the proof of Lemma 5.4, and note that $\rho \in G$. It is easy to see that

$$\sigma_1^n\varphi^{(m-1)/2}\psi^{(m+1)/2} = (t, t, tc^{-2}, tc^{-2}, t, t, \dots, tc^{-2}, tc^{-2}, t, t)1,$$

from which one easily verifies that ρ in fact has the same form as in (16), that is

$$\rho = (1, tc^{-1}, tc^{-1}, \dots, tc^{-1})z,$$

which is an involution. That $\rho\sigma_1$ is also an involution can be verified just as in (17). Therefore, ρ inverts σ_1. Moreover, it is clear that ρ fixes v. The result now follows by part (b) of Theorem 2.3. □

As already remarked it turns out that the Petrie dual of the map corresponding to σ_1 and σ_2 from Proposition 5.5 results in the same map. We thus cannot simply use Lemma 2.4 in order to obtain a reflexible map on an orientable surface (as was done in the previous section). We thus need a different construction. To this end we set

$$\begin{aligned}\eta_1 &= (t,t,\ldots,t)r,\\ \eta_2 &= (t,t,tc,tc,tc^2,tc^2,\ldots,tc^{-1},tc^{-1})z, \text{ and}\\ H &= \langle \eta_1,\eta_2\rangle,\end{aligned} \tag{21}$$

where for η_2, the $(2i+1)$-th and $(2i+2)$-th component of the "n-tuple" is tc^i for each i with $0 \le i < n/2$. Note that n is even (since s is), and thus for $i = \frac{n}{2} - 1$ we see that $c^i = c^{\frac{n}{2}-1} = c^{-1}$, since m divides $\frac{n}{2}$. Therefore, the last two components are indeed tc^{-1}.

Lemma 5.6. *Let η_1, η_2 and H be as in (21). Then the following holds:*

$$|\eta_1| = n, \qquad |\eta_2| = 2m, \qquad |\eta_1\eta_2| = 2 \qquad \textit{and} \qquad |H| = 2m^2 n.$$

Proof. That $|\eta_1| = n$ is clear (recall that n is even). Next,

$$\eta_2^2 = (t,t,tc,tc,tc^2,tc^2,\ldots,tc^{-1},tc^{-1})(t,tc^{-1},tc^{-1},tc^{-2},tc^{-2},\ldots,tc,tc,t)1,$$

and so setting $\zeta = \eta_2^2$ we deduce

$$\zeta = (1,c^{-1},c^{-2},\ldots,c^{1-i},\ldots,c)1, \tag{22}$$

which clearly shows that $|\eta_2| = 2m$. It is also easy to see that

$$\eta_1\eta_2 = (1,c,c,c^2,c^2,\ldots,c^{-1},c^{-1},1)rz,$$

which is easily seen to be an involution.

Let $\xi = \eta_1^{-1}\zeta\eta_1$ and note that

$$\xi = (t,t,\ldots,t)(c,1,c^{-1},\ldots,c^{2-i},\ldots,c^2)(t,t,\ldots,t)1 = (c^{-1},1,c^1,\ldots,c^{i-2},\ldots,c^{-2})1. \tag{23}$$

It follows that $H_2 = \langle \zeta, \xi\rangle \cong C_m \times C_m$. Since

$$\begin{aligned}\eta_1^{-1}\xi\eta_1 &= (t,t,\ldots,t)(c^{-2},c^{-1},1,\ldots,c^{i-3},\ldots,c^{-3})(t,t,\ldots,t)1\\ &= (c^2,c^1,1,\ldots,c^{3-i},\ldots,c^3)1,\end{aligned}$$

we see that $\eta_1^{-1}\xi\eta_1 = \zeta^{-1}\xi^{-2} \in H_2$, showing that η_1 normalises H_2. Analogously to (14) we can verify that $\eta_2^{-1}\xi\eta_2 = \eta_1\zeta\eta_1^{-1} \in H_2$, showing that H_2 is normal in H. Since $H_2 \cap \langle \eta_1\rangle = 1$, we thus find that $K_2 = \langle \eta_1, \zeta, \xi\rangle$ is of order $m^2 n$ and just as in (15) we see that $\eta_2^{-1}\eta_1\eta_2 = \zeta^{-1}\eta_1^{-1} \in K_2$, showing that K_2 is also normal in H. Finally, since clearly $\eta_2 \notin K_2$ but $\eta_2^2 \in K_2$, H is a group of order $2m^2 n$, as claimed. □

Proposition 5.7. *Let $m \ge 3$ be an odd integer, let $s \ge 2$ be even but not divisible by 4, and let $n = sm$. Furthermore, let η_1, η_2 and H be as in (21). Then $\Gamma = C_n[mK_1]$ is the skeleton of a polytopal orientable reflexible map $\mathcal{M}$ of type $\{n, 2m\}$ with $\mathrm{Aut}^+(\mathcal{M}) = H$.*

Proof. Note that η_1 maps $v := (1,1)$ to its neighbour $w := (2,1)$, while η_2 permutes the neighbours of v cyclically (in the order $(2,1),(n,1),(2,m),(n,2),(2,m-1),(n,3),\ldots,(n,m)$). Theorem 2.3 and Lemma 5.6 thus imply that H acts regularly on the arcs of Γ. Consequently, part (a) of Theorem 2.3 in fact implies that the graph Γ is the skeleton of a polytopal rotary map $\mathcal{M}$ of type $\{n, 2m\}$ on an orientable surface with $\mathrm{Aut}^+(\mathcal{M}) = H$.

To complete the proof, set $\rho = z$ and observe that it is an involution fixing v. Since the automorphisms $\eta_1\rho$ and $\eta_2\rho$ are clearly both involutions, ρ inverts each of η_1 and η_2 and therefore normalises H. Part (a) of Theorem 2.3 thus implies that $\mathcal{M}$ is reflexible. □

6. Proof of the main theorem and concluding remarks

The proof of Theorem 1.2 follows directly from Propositions 4.2, 5.2, 5.3, 5.5 and 5.7; the genus of the maps can of course be calculated from the order of the skeleton and the type of the map.

Let us conclude the paper by the following remarks.

1. A recently computed census of rotary maps [2] shows that there are precisely 282 simple graphs with at most 3000 edges that embed as skeletons of all three classes of rotary maps. The constructions provided in this paper cover about 60% of them. A natural question that arises is whether one can characterise all simple graphs of this type.
2. Further, recall that a polytopal map is *polyhedral* provided that every two distinct faces intersect in a vertex, an edge or not at all (see [1]). While all the maps constructed in this paper are polytopal, not all are polyhedral. For example, the faces of the chiral maps constructed in Section 4, are all hamiltonian cycles, implying that any two faces meet in all the vertices of their skeletons. As the census [2] shows, there are only two graphs with at most 3000 edges that embed as polyhedral rotary maps in all three possible ways. Both have 576 vertices, are of valence 6 and are non-bipartite. They are denoted as PlhSk(576; 27) and PlhSk(576; 32) in the list of all graphs that embed as polyhedral rotary maps, provided at [2]. This raises an obvious question whether one can find an infinite family of such graphs.

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Data availability

Data will be made available on request.

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