Applied Mathematics and Computation 510 (2026) 129700

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Full Length Article

Induced cycles vertex number and (1, 2)-domination in cubic graphs

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ARTICLE INFO

2020 MSC:
05C38
05C69

Keywords:
Cubic graphs
(1, 2)-Domination
Induced 2-regular subgraphs
Induced cycles vertex number

ABSTRACT

A $(1,2)$-dominating set in a graph G is a set S such that every vertex outside S has at least one neighbor in S, and each vertex in S has at least two neighbors in S. The $(1,2)$-domination number, $\gamma_{1,2}(G)$, is the minimum size of such a set, while $c_{\text{ind}}(G)$ is the cardinality of the largest vertex set in G that induces one or more cycles. In this paper, we initiate the study of a relationship between these two concepts and discuss how establishing such a connection can contribute to solving a conjecture on the lower bound of $c_{\text{ind}}(G)$ for cubic graphs. We also establish an upper bound on $c_{\text{ind}}(G)$ for cubic graphs and characterize graphs that achieve this bound.

1. Introduction

A cubic graph is a simple graph where each vertex is adjacent to exactly three other vertices. These graphs have been a central object of study in graph theory for many years due to their balanced structure and wide applicability across various fields. Cubic graphs appear in numerous contexts, including network design, chemistry (such as in molecular structures like fullerene graphs), and in problems related to graph coloring, Hamiltonicity, and domination theory.

In this paper, we explore an open and intriguing question: the relationship between the induced cycles vertex number and $(1,2)$-domination in cubic graphs. Understanding this relationship not only sheds light on the fundamental interplay between induced cycles and $(1,2)$-domination in graphs, but also offers potential application in solving a conjecture in the area of induced subgraph problems for cubic graphs.

The task of identifying the largest induced r-regular subgraph within a given graph has been a topic of interest for some time, dating back to work by Erdős et al. [2]. Cardoso et al. [1] demonstrated that determining the maximum induced r-regular subgraph is NP-hard. Further algorithmic aspects were studied in [4,7,8].

One can note that the case when $r = 0$ corresponds to the well-known problem of finding a maximum independent set, and for $r = 1$ it involves finding a maximum induced matching, both of which are well-studied problems. However, when $r = 2$, the problem becomes much more complex as demonstrated in [5] and [6]. Therein, the *induced cycles vertex number*, denoted as $c_{\text{ind}}(G)$, is defined as the maximum number of vertices in G that induce a 2-regular subgraph, or equivalently, the cardinality of a largest set of vertices that induces one or more cycles. In [5], Henning et al. established NP-completeness of $c_{\text{ind}}(G)$ for graphs G of maximum degree 4. For an r-regular graph G of order n, they proved that

$$c_{\text{ind}}(G) \geq \frac{n}{2(r-1)} + \frac{1}{(r-1)(r-2)}, \tag{1}$$

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https://doi.org/10.1016/j.amc.2025.129700
Received 28 January 2025; Received in revised form 10 June 2025; Accepted 18 August 2025
Available online 31 August 2025

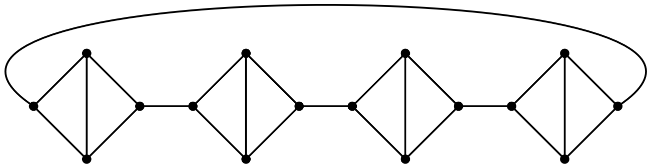

Fig. 1. Diamond-necklace graph N_4.

which, for cubic graphs, simplifies to $c_{\text{ind}}(G) \geq \frac{n+2}{4}$. In the case of claw-free cubic graphs, they derived an asymptotically tight bound of $c_{\text{ind}}(G) > \frac{13n}{20}$. Moreover, they proposed that the general lower bound could be improved, and conjectured the following.

Conjecture 1. *[5] If G is a cubic graph of order n, then $c_{\text{ind}}(G) \geq \frac{n}{2}$.*

We also need to mention that if Conjecture 1 holds, the suggested lower bound is tight as demonstrated by an example of a graph in [5].

For an integer $k \geq 3$, a graph G is called *k-chordal* if it does not contain any induced cycle of length greater than k. Henning et al. established a lower bound for $c_{\text{ind}}(G)$ for k-chordal cubic graphs. Furthermore, based on a precise structural description of 4-chordal cubic graphs, the following result was derived in [6].

Theorem 1. *[6] If G is a connected cubic 4-chordal graph not isomorphic to K_4, $K_{3,3}$ nor $K_2\square K_3$, then*

$$c_{\text{ind}}(G) \geq \frac{5n}{8} + \frac{3}{4}. \tag{2}$$

It is easy to verify that $c_{\text{ind}}(K_4) = 3$, $c_{\text{ind}}(K_{3,3}) = 4$, and $c_{\text{ind}}(K_2\square K_3) = 4$. This together with Theorem 1 implies that Conjecture 1 holds for all connected cubic 4-chordal graphs.

It is somewhat surprising that the upper bound on $c_{\text{ind}}(G)$ had not been previously addressed. However, the authors of [5] encountered a class of graphs, known as diamond-necklace graphs (see Fig. 1 for an example), where the induced cycles vertex number equals $\frac{3}{4}$ of the graph's order. In this paper, we prove that this value serves as the upper bound for $c_{\text{ind}}(G)$ in connected cubic graphs.

In a related line of research, Fakhran et al. considered the notion of $(1,2)$-domination in cubic graphs [3]. A $(1,2)$-*dominating set* in a graph G is a set of vertices S such that every vertex not in S has at least one neighbor in S, and every vertex in S has at least two neighbors in S. The $(1,2)$-*domination number*, $\gamma_{1,2}(G)$, is the minimum cardinality of such a set. The authors proved that for any connected cubic graph G of order n,

$$\frac{n}{2} \leq \gamma_{1,2}(G) \leq \frac{3n}{4}. \tag{3}$$

Moreover, they characterized graphs for which the lower bound is obtained, and provided examples where the upper bound is tight, such as already mentioned diamond-necklace graphs.

It is easy to observe (as demonstrated below in Observation 1 in Section 2) that if S is a set that induces a 2-regular subgraph of a graph G and every vertex in $V(G) \setminus S$ is adjacent to a vertex in S, then

$$\gamma_{1,2}(G) \leq c_{\text{ind}}(G). \tag{4}$$

This observation, combined with the fact that $\frac{n}{2}$ and $\frac{3n}{4}$ serve as the lower and upper bounds for $\gamma_{1,2}(G)$, respectively, as well as our finding that $\frac{3n}{4}$ is also the tight upper bound for $c_{\text{ind}}(G)$ (see Theorem 2 in Section 3), together with Conjecture 1, prompted us to explore whether there exists a relationship between $\gamma_{1,2}(G)$ and $c_{\text{ind}}(G)$ for a cubic graph G. Namely, if G is a graph for which inequality (4) holds, then for such a graph the Conjecture 1 is valid, as by the above results we have

$$\frac{n}{2} \leq \gamma_{1,2}(G) \leq c_{\text{ind}}(G) \leq \frac{3n}{4}. \tag{5}$$

For graphs satisfying (4) the above sequence of inequalities also yields an alternative argument for the upper bound on $\gamma_{1,2}(G)$.

In the subsequent sections, we outline our findings as follows. Section 2 establishes the foundational concepts and terminology necessary for the discussion. In Section 3, we determine the upper bound on the induced cycles vertex number of a graph and characterize the graphs that achieve this bound. In Section 4, we present some sufficient conditions that guarantee the inequality $\gamma_{1,2}(G) \leq c_{\text{ind}}(G)$. Section 5 introduces the family of *trilobite* graphs T_n, where n represents the order of the graph, and demonstrates that $\gamma_{1,2}(T_n) > c_{\text{ind}}(T_n)$. Finally, in the concluding section we summarize the families of graphs supporting Conjecture 1 and conjecture that for each $n \geq 10$, the trilobite T_n is the unique graph for which the $(1,2)$-domination number exceeds the induced cycles vertex number.

2. Preliminaries

We begin by reviewing a few essential definitions. Since in the case of disconnected graphs, both invariants are obtained as the sum of the corresponding invariants of all connected components, we restricted our study to connected cubic graphs.

The notation $[k]$ will represent the set $\{1, 2, \dots, k\}$. We will use n to denote the order of a graph $G = (V, E)$, i.e. $n = |V(G)|$, and we set $m = |E(G)|$. In a graph G, the *degree* of a vertex $v \in V(G)$, denoted by $\deg_G(v)$, is the number of edges incident to v, or equivalently, the number of vertices adjacent to v. The *closed neighborhood* of a vertex $v \in V(G)$, denoted $N_G[v]$, is the set consisting of the vertex v and all vertices adjacent to v, i.e., $N_G[v] = \{v\} \cup N_G(v)$, where $N_G(v)$ is the open neighborhood of v (the set of all vertices adjacent to v). Similarly, the *closed neighborhood* of a set of vertices $S \subseteq V(G)$, denoted by $N_G[S]$, is the union of the closed neighborhoods of all the vertices in S, i.e., $N_G[S] = \bigcup_{v \in S} N_G[v]$.

In a *cubic graph*, each vertex has a degree of 3. As a result, every cubic graph contains an even number of vertices. For our purposes, we will distinguish two cases: if G has order n, then n is either $4k$ or $4k + 2$ for some integer $k \geq 1$. Also recall that the number of edges in a cubic graph equals $m = \frac{3n}{2}$.

A $(1,2)$-*dominating* set is a set of vertices in a graph such that every vertex outside this set is connected to at least one vertex inside it, and every vertex within the set is connected to at least 2 other vertices from the same set. The cardinality of a smallest such set is called the $(1,2)$-*domination number*, and any set that achieves this minimum size is referred to as a $\gamma_{1,2}$-*set* of the graph. With $c_{\text{ind}}(G)$ we denote the cardinality of the largest vertex set in G that induces one or more cycles.

Let S be a set of vertices in G that induces a 2-regular subgraph $\langle S \rangle$ of G. If $N_G[S] = V(G)$, meaning that every vertex outside of S has a neighbor in S, then S is a $(1,2)$-dominating set of G and it follows that $\gamma_{1,2}(G) \leq |S| \leq c_{\text{ind}}(G)$. Therefore, in this case, the Conjecture 1 clearly holds, as explained in the introduction. For easier reference in the following sections, let us formally state the above observation.

Observation 1. *Let G be a cubic graph. If G contains a 2-regular subgraph induced by $S \subseteq V(G)$ such that $N_G[S] = V(G)$, then $\gamma_{1,2}(G) \leq c_{\text{ind}}(G)$.*

For an integer $k \geq 2$, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies $D_1, D_2, \dots, D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and the only missing edge between vertices of D_i is $a_i d_i$. The graph N_k, referred to as a *diamond-necklace* with k diamonds, is formed by joining these k diamonds through the edges $\{a_{i+1} d_i, i \in [k-1]\}$, with the additional edge $a_1 d_k$, see Fig. 1 where N_4 is depicted. It is known, see [5], that if G is a diamond-necklace of order n, then $c_{\text{ind}}(G) = \frac{3n}{4}$.

3. Tight upper bound on $c_{\text{ind}}(G)$

In this section, we establish the upper bound for the induced cycles vertex number of a graph and identify the graphs that achieve this bound. These graphs are found to have a specific structure, described as follows. Let G_1 be a 2-regular graph with $3k$ vertices and let G_2 be an edgeless graph of order $k \geq 1$. To the disjoint union of G_1 and G_2 we add $3k$ edges having one endvertex in G_1 and the other in G_2, so that each vertex in G_1 has exactly one neighbor in G_2, and each vertex in G_2 has exactly three neighbors in G_1. It is easy to verify that this construction yields a 3-regular graph. We denote the family of graphs constructed in this way by Γ.

Theorem 2. *Let G be a cubic graph of order n. Then $c_{\text{ind}}(G) \leq \frac{3n}{4}$, where the equality is attained if and only if $G \in \Gamma$.*

Proof. Let G be a cubic graph of order n. Since G is not a tree, it contains at least one cycle, meaning that G contains an induced 2-regular subgraph. Let S be the set of vertices that induces a 2-regular subgraph of order r in G. Each vertex of S has exactly one neighbor in $V(G) \setminus S$. Furthermore, no more than three vertices of S can share a common neighbor $a \in V(G) \setminus S$, as this would imply $\deg_G(a) > 3$, contradicting the assumption that G is cubic. Therefore, it follows that $|N_G(S)| \geq \frac{r}{3}$. Thus, we have:

$$|N_G[S]| = |S| + |N_G(S)| \geq \frac{4}{3} r.$$

Since $|N_G[S]| \leq n$, it follows that $r \leq \frac{3}{4} n$. Hence, $c_{\text{ind}}(G) \leq \frac{3}{4} n$. Now let us consider the graphs that attain this bound.

Assume that $G \in \Gamma$. Then, by the above construction, G is of order $n = 4k$, $k \geq 1$, and contains an induced 2-regular subgraph G_1 of order $3k$. Thus $c_{\text{ind}}(G) \geq 3k = \frac{3n}{4}$, and consequently $c_{\text{ind}}(G) = \frac{3n}{4}$.

To prove the converse, let $c_{\text{ind}}(G) = \frac{3n}{4}$. Since $c_{\text{ind}}(G)$ is an integer, it follows that $n = 4k$, $k \geq 1$. Let S be a set of vertices inducing a 2-regular subgraph of G with $|S| = c_{\text{ind}}(G) = 3k$. Since G is a cubic graph, each vertex in S has exactly one neighbor in $V(G) \setminus S$, i.e. there are exactly $3k$ edges between vertices of S and $V(G) \setminus S$. Since the number of edges in G is precisely $\frac{3n}{2} = 6k$, we infer that the subgraph induced by vertices in $V(G) \setminus S$ has no edges. Thus $G \in \Gamma$. $\square$

As shown, the upper bound established in Theorem 2 is achieved by graphs in Γ of order $n = 4k$. Now, consider the case where $n = 4k + 2$, with $k \geq 1$. Then we have $\frac{3n}{4} = 3k + \frac{3}{2}$. Since $c_{\text{ind}}(G)$ must be an integer, the upper bound in this case is adjusted to $\lfloor \frac{3n}{4} \rfloor = 3k + 1 = \frac{3n-2}{4}$.

4. Graphs with $\gamma_{1,2}(G) \leq c_{\text{ind}}(G)$

As outlined in Introduction, our focus is on graphs where the inequality $\gamma_{1,2}(G) \leq c_{\text{ind}}(G)$ holds. In such cases, we immediately obtain graphs G of order n for which $\frac{n}{2} \leq \gamma_{1,2}(G) \leq c_{\text{ind}}(G)$, thus validating Conjecture 1.

Let G be a connected cubic graph of order n. If $n = 4$, then $G = K_4$, and in this case we have $\gamma_{1,2}(K_4) = c_{\text{ind}}(K_4) = 3$. If $n = 6$, then $G \in \{K_2 \Box K_3, K_{3,3}\}$. It is easy to verify that $\gamma_{1,2}(K_2 \Box K_3) = 3 < c_{\text{ind}}(K_2 \Box K_3) = 4$, and $\gamma_{1,2}(K_{3,3}) = c_{\text{ind}}(K_{3,3}) = 4$. There are 5 connected cubic graphs of order 8. It turns out that the inequality $\gamma_{1,2}(G) \leq c_{\text{ind}}(G)$ holds for all of them as well. We will easily justify this after considering the following.

By Theorem 2, we have that $c_{\text{ind}}(G)$ is at most $\lfloor \frac{3n}{4} \rfloor$ for any cubic graph G. However, the same upper bound holds for $\gamma_{1,2}(G)$, as shown by Fakhran et al. [3]. Thus we immediately arrive at the following conclusion.

Proposition 1. *Let G be a cubic graph of order n. If $c_{\text{ind}}(G) = \lfloor \frac{3n}{4} \rfloor$, then $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$.*

Note that if $c_{\text{ind}}(G) = \lfloor \frac{3n}{4} \rfloor$, then either $c_{\text{ind}}(G) = \frac{3n}{4}$ (if $n = 4k$, $k \ge 1$), or $c_{\text{ind}}(G) = \frac{3n-2}{4}$ (if $n = 4k+2$, $k \ge 1$). So these two cases are settled. In what follows we consider the case when $c_{\text{ind}}(G) = \frac{3n}{4} - 1$.

Proposition 2. *Let G be a cubic graph of order $n \ge 8$. If $c_{\text{ind}}(G) = \frac{3n}{4} - 1$, then the inequality $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$ holds.*

Proof. Let G be a cubic graph of order $n \ge 8$ such that $c_{\text{ind}}(G) = \frac{3n}{4} - 1$. This implies that $n = 4k$, where $k \ge 2$, and $c_{\text{ind}}(G) = 3k - 1$.

Let S be a set of vertices that induces a 2-regular subgraph of G such that $|S| = c_{\text{ind}}(G) = 3k - 1$. First assume that $N_G[S] \ne V(G)$. Thus there exists $v \notin S$ that has no neighbor in S. This implies that G has at least $2|S| + 3$ edges, namely, $|S|$ edges between two vertices in S, $|S|$ edges between one vertex in S and another in $V(G) \setminus S$, and three edges incident to v. Thus, the total number of edges is

$$m = \frac{3n}{2} = 6k \ge 2 \cdot |S| + 3 = 2 \cdot (3k - 1) + 3 = 6k + 1, \tag{6}$$

a contradiction. Thus $N_G[S] = V(G)$, and the result follows by Observation 1. □

Now it is easy to verify the following.

Proposition 3. *If G is a cubic graph on $n \le 8$ vertices, then $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$.*

Proof. We already considered cases when $n \in \{4, 6\}$, thus let G be a cubic graph of order 8. By Theorem 2 we have $c_{\text{ind}}(G) \le 6$. If $c_{\text{ind}}(G) \le 4$, then G is 4-chordal, thus by Theorem 1 we have $c_{\text{ind}}(G) \ge 6$, a contradiction. If $c_{\text{ind}}(G) \in \{5, 6\}$, the desired inequality holds by Propositions 1 and 2, respectively. □

Fakhran et al. [3] constructed the family of connected cubic graphs $\mathcal{G}_{\text{cubic}}$ as follows. For an integer $k \ge 3$, let G_1 and G_2 be vertex-disjoint 2-regular graphs (not necessarily connected), each on k vertices. Hence, for $i \in \{1, 2\}$, G_i is either a cycle C_k or a disjoint union of cycles whose total order is k. The graph G of order $n = 2k$ is obtained by taking the disjoint union of G_1 and G_2, and adding a perfect matching between the vertices of G_1 and G_2. They showed that if G is a connected cubic graph of order n, then $\gamma_{1,2}(G) \ge \frac{n}{2}$, where equality holds if and only if $G \in \mathcal{G}_{\text{cubic}}$. From the construction it is clear that G_1 and G_2 are induced 2-regular subgraphs of G, thus the following immediately follows.

Proposition 4. *Let G be a cubic graph of order $n \ge 6$ such that $\gamma_{1,2}(G) = \frac{n}{2}$. Then the inequality $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$ holds.*

5. Graphs with $\gamma_{1,2}(G) > c_{\text{ind}}(G)$

In this section, we construct a family of graphs, denoted by $\mathcal{T}$, such that for every graph $T \in \mathcal{T}$, the inequality $\gamma_{1,2}(T) > c_{\text{ind}}(T)$ holds. Graphs in this family will be referred to as *trilobites*. We believe that trilobites might be the only graphs satisfying this inequality.

We begin by constructing the trilobite on 10 vertices, denoted by T_{10}. Let $x_1x_2x_3$, $y_1y_2y_3$, and $z_1z_2z_3$ be three vertex-disjoint paths. Add edges so that vertices in $\{x_1, y_1, z_1\}$ and $\{x_3, y_3, z_3\}$, respectively, induce a triangle. Finally, add a vertex c_2 and connect it to x_2, y_2, and z_2.

Let $k \ge 3$. To define the next two families of trilobites, let X, Y, and Z denote three vertex-disjoint paths on k vertices: $x_1\, x_2 \dots x_k$, $y_1\, y_2 \dots y_k$ and $z_1\, z_2 \dots z_k$, respectively.

The first family of graphs, which we denote by T_{4k}, consists of graphs with $4k$ vertices. These graphs are formed by starting with X, Y, and Z and adding three edges so that x_1, y_1, and z_1 induce a triangle. For each $i \in \{2, 3, \dots, k\}$, a vertex c_i is added and connected to x_i, y_i, and z_i. Finally, a vertex d_k is added and connected to x_k, y_k, and z_k.

The second family, whose graphs are denoted by T_{4k+2}, consists of graphs on $4k+2$ vertices. These graphs include the union of paths X, Y, and Z, with a vertex c_i added and connected to x_i, y_i, and z_i for each $i \in \{1, 2, \dots, k\}$. Additionally, a vertex d_1 is connected to x_1, y_1, and z_1, and a vertex d_k is connected to x_k, y_k, and z_k.

We define $\mathcal{T}$ as the family of graphs that includes T_{10}, T_{4k}, and T_{4k+2} for all $k \ge 3$. Fig. 2 displays the first five smallest trilobites. For a graph $T \in \mathcal{T}$, let L_i denote the set of vertices sharing the same subscript i. Then $\langle L_i \rangle$ is called the ith *layer* of T. Note that in T_{4k}, L_1 induces a triangle, while in T_{4k+2}, L_1 induces a $K_{2,3}$. For any trilobite on at least 12 vertices, L_k induces a $K_{2,3}$, and for each $i \in \{2, 3, \dots, k-1\}$, L_i induces a claw graph $K_{1,3}$.

Proposition 5. *If T is a trilobite of order n, then $c_{\text{ind}}(T) = \frac{n}{2} + 1$.*

Proof. It is easy to see, that every trilobite of order n contains an induced cycle of order $\frac{n}{2} + 1$. Specifically, in T_{10} such cycle contains all vertices of X and Y, in T_{4k} it contains all vertices of X and Y along with c_k, and in T_{4k+2} it contains all vertices of X and Y together with c_1 and c_k. Thus, for every graph $T \in \mathcal{T}$, we conclude that $c_{\text{ind}}(T) \ge \frac{n}{2} + 1$.

To prove the opposite inequality, let S be the set of vertices that induces a 2-regular subgraph of $T \in \mathcal{T}$ of maximal cardinality. For $i \in [k]$, where k denotes the number of layers in T, let s_i be the number of vertices in the ith layer that belong to S. Then we have $c_{\text{ind}}(T) = |S| = \sum_{i=1}^{k} s_i$.

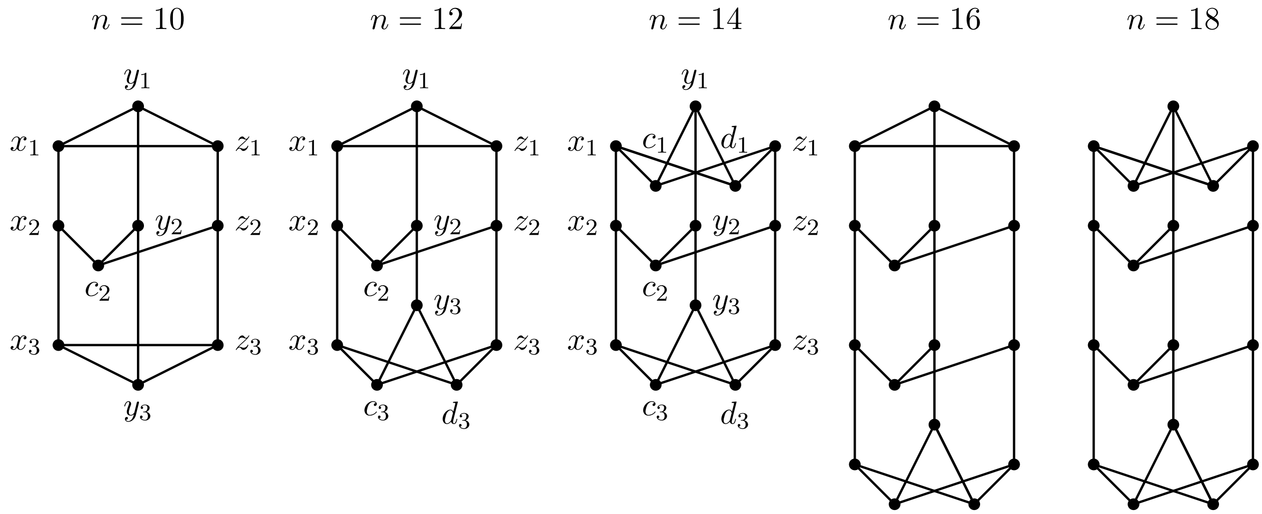

Fig. 2. Trilobites T_n.

Note that for every $i \in [2, k-1]$, $s_i \le 3$, for otherwise there is a vertex of degree 3 in $\langle S\rangle$, a contradiction. For the same reason, $s_i \le 4$, if the ith layer, $i \in \{1, k\}$, is isomorphic to $K_{2,3}$. Let i be an arbitrary integer from $[2, k-1]$. Assume that $s_i = 3$ and $c_i \notin S$. Then intersections of both neighboring layers with S contain exactly the vertices in $X \cup Y \cup Z$. Repeating the argument we derive that $X \cup Y \cup Z \subseteq S$, which implies the existence of a vertex of degree 3 in $\langle S\rangle$, a contradiction. Therefore, if $s_i = 3$ for $i \notin \{1, k\}$, then $c_i \in S$.

Let s_i^* denote the maximal possible value of s_i, i.e. $s_i^* = 4$ if ith layer is isomorphic to $K_{2,3}$, otherwise $s_i^* = 3$. If for every $i \in [k]$ we have $s_i < s_i^*$, then $s_i \le 2$ if the ith layer is not isomorphic to $K_{2,3}$ and $s_i \le 3$ if the ith layer is isomorphic to $K_{2,3}$. Consequently, $|S| \le s_1 + s_k + 2(k-2)$. The latter expression equals $2k+1$ if $T = T_{4k}$, and $2k+2$ if $T = T_{4k+2}$, thus $|S| \le \frac{n}{2} + 1$ in both cases. Therefore, in what follows we assume that there exists $i \in [k]$, such that $s_i = s_i^*$.

If $s_1 = s_1^*$, then vertices of S in the first layer induced a cycle, and therefore $s_2 = 0$. Similarly, if $s_k = s_k^*$ then $s_{k-1} = 0$. Assume now, that $s_i = 3$ for some $i \in [2, k-1]$, then $c_i \in S$, and the neighboring layers must collectively contain at least two vertices of S. Also note that $s_j \ne 1$ for every $j \in [k]$, as a single vertex from S in a layer does not lie on any cycle in $\langle S\rangle$, which is a contradiction. Thus, if $s_i = 3$, one neighboring layer contains at least two vertices of S, and the other neighboring layer contains no vertices of S. Now, for each $i \in [k]$, let

$$a_i = \begin{cases} 2, & s_i = 0,\ s_{i-1} \ge 3,\ s_{i+1} \ge 3; \\ 1, & s_i = 0,\ s_p \ge 3,\ s_r \le 2,\ \{p, r\} = \{i-1, i+1\}; \\ s_i - 1, & s_i = s_i^*; \\ s_i, & \text{otherwise}. \end{cases} \tag{7}$$

Since for every $i \in [k]$ such that $s_i = s_i^*$, there exists exactly one $j \in \{i-1, i+1\}$ such that $s_j = 0$, we have $\sum_{i=1}^k a_i = \sum_{i=1}^k s_i = |S|$. Furthermore, for every $i \in [k]$, $a_i \le 2$ if the ith layer is not isomorphic to $K_{2,3}$, and $a_i \le 3$ otherwise. Therefore, $|S| \le a_1 + a_k + 2(k-2) \le \frac{n}{2} + 1$, which completes the proof. □

Proposition 6. *If T is a trilobite of order n, then $\gamma_{1,2}(T) = \frac{n}{2} + 2$.*

Proof. Let T be a trilobite of order n, and $k \ge 3$ the number of layers in T. First we will prove that $\gamma_{1,2}(T) \ge \frac{n}{2} + 2$. Let S be a $\gamma_{1,2}$-set of T and let w_i, $i \in [k]$, denote the number of vertices in S within the ith layer. Then $\gamma_{1,2}(T) = |S| = \sum_{i=1}^k w_i$. For an integer a, let $\overline{a}$ represent any integer b such that $b \ge a$, and let $S(T) = (w_1, w_2, \ldots, w_k)$.

Observe that for the first (or the last) layer, we have $w_1 \in \{0, 2, 3\}$ if the vertices in the layer induce a triangle, and $w_1 \in \{3, 4, 5\}$ if they induce a $K_{2,3}$. For intermediate layers, where $i \in [2, k-1]$, it holds that $w_i \in \{1, 2, 3, 4\}$. Furthermore, if $T \in \{T_{10}, T_{4k}\}$, then $(w_1, w_2) \in \{(0, 4), (2, \overline{2}), (3, \overline{1})\}$, and if $T = T_{4k+2}$, then $(w_1, w_2) \in \{(3, \overline{2}), (4, \overline{1}), (5, \overline{1})\}$. Due to the symmetry, in the latter case, we also have $(w_k, w_{k-1}) \in \{(3, \overline{2}), (4, \overline{1}), (5, \overline{1})\}$.

Let $k = 3$. On can check that (up to the symmetry) we have

$$S(T_{10}) \in \{(0,4,3), (2,2,3), (2,3,2), (3,1,3)\}, \text{ and} \tag{8}$$

$$S(T_{12}) \in \{(0,4,4), (2,3,3), (3,1,4), (3,2,3)\}. \tag{9}$$

Assume that for the trilobite T_{14}, we have $|S| \le 8$. Since $w_1 + w_2 \ge 5$ and $w_3 \ge 3$, it follows that $w_1 + w_2 = 5$ and $w_3 = 3$. Consequently, $w_1 = 3$ and $w_2 = 2$, which is not possible because in this case there exists a vertex in L_2 without a neighbor in S. Thus, for a trilobite T with 3 layers, we have $|S| \ge \frac{n}{2} + 2$.

Let $k \geq 4$, and let $R = \{(3,2,4),(3,3,\overline{1}),(3,4,\overline{1}),(4,1,\overline{3}),(5,1,\overline{1})\}$. Note that is not possible that $(w_k, w_{k-1}) \in \{(4,\overline{2}),(5,\overline{2})\}$, as in either case omitting a vertex in $\{c_k, d_k\}$ from S would result in a $(1,2)$-dominating set with the cardinality less than $|S|$, leading to a contradiction. Thus $(w_k, w_{k-1}, w_{k-2}) \in R$. If $T \in T_{4k+2}$, then we also have $(w_1, w_2, w_3) \in R$. Let $r = w_k + w_{k-1} + w_{k-2}$, and note that $r \geq 7$.

Let $k = 4$, i.e., $T \in \{T_{16}, T_{18}\}$. In this case we have $|S| = w_1 + r$. If $r = 7$ (which corresponds to two possible sequences from R), then $w_2 = 1$. Consequently, $w_1 = 3$ if $T = T_{16}$, and $w_1 \geq 4$ if $T = T_{18}$. Thus, the desired inequality holds in this case. Now consider $r \geq 8$. As $w_1 \geq 3$, it is evident that for $T = T_{18}$, the inequality holds true. Similarly, for $T = T_{16}$, the inequality holds if $w_1 \neq 0$. If $w_1 = 0$, it follows that $(w_1, w_2, w_3, w_4) \in \{(0,4,3,4),(0,4,4,3)\}$. Therefore, for trilobites T with 4 layers, it holds that $|S| \geq \frac{n}{2} + 2$.

Finally, let $k \geq 5$. First assume that for every $i \in [2, k-1]$ it holds $w_i > 1$. Then $r \geq 8$. If $T = T_{4k+2}$, then we have $r + w_1 \geq 11$, which implies $|S| \geq 2(k-4) + 11 = 2k + 3$. If $T = T_{4k}$, then $r + w_1 + w_2 \geq 12$, leading to $|S| \geq 2(k-5) + 12 = 2k + 2$. Thus, the claim holds in both cases.

Now, assume that there exists an index $i \in [2, k-1]$ such that $w_i = 1$. For such layer, we know that the neighboring layers satisfy $w_{i-1} \neq 2$ and $w_{i+1} \neq 2$. If $w_{i+1} = 1$, then all vertices in L_{i-1} and L_{i+2} must belong to S. This implies: $w_{i-1} = w_{i+2} = 4$ if $i \in [3, k-3]$, or, in the edge cases, it holds that $w_k = 5$ and $w_1 \in \{3,5\}$. Additionally, if $w_j = 4$ and $w_{j-1} = 1$ for some $j \in [3, k-1]$, then it must hold that $w_{j+1} \geq 2$ to satisfy the $(1,2)$-domination condition. To further analyze this case, define a sequence $A = (a_i)$, $i \in [k]$, as follows:

$$a_i = \begin{cases} 2, & w_i = 1; \\ w_i - 1, & w_i > 1 \text{ and } w_{i+1} = 1; \\ w_i - 1, & w_{i-1} = w_{i-2} = 1; \\ w_i, & \text{otherwise}. \end{cases} \tag{10}$$

For each $i \in [2, k-1]$, if $w_i = 1$, then one of the following holds for the neighboring layers:

- $w_{i-1} \geq 3$, in which case $a_{i-1} = w_{i-1} - 1 \geq 2$, or
- $w_{i-1} = w_i = 1$, which implies $w_{i+1} \geq 4$. Here, we decrease w_{i+1}, resulting in $a_{i+1} = w_{i+1} - 1 \geq 3$.

Note that any layer with 4 vertices belonging to S, can have at most one neighboring layer with exactly 1 vertex in S. Furthermore, there are at most two consecutive layers having exactly 1 vertex in S. Therefore, $|S| = \sum_{i=1}^{k} a_i$ and $a_i \geq 2$ for all $i \in [2, k]$.

In addition, it is straightforward to verify that for every sequence $(w_k, w_{k-1}, w_{k-2}) \in R$, it holds that $a_k + a_{k-1} + a_{k-2} \geq 8$. Moreover, if $T = T_{4k+2}$, recall that $w_1 \geq 3$. Note that its value decreases by 1 only when $w_2 = 1$. In this case, $w_1 \geq 4$, implying $a_1 \geq 3$. Consequently, we have $|S| = \sum_{i=1}^{k} a_i \geq 3 + 2(k-4) + 8 = 2k + 3$. For $T = T_{4k}$, recall that $(w_1, w_2) \in \{(0,4),(2,\overline{2}),(3,\overline{1})\}$. Thus $a_1 + a_2 \geq 4$, and consequently $\sum_{i=1}^{k} a_i \geq 4 + 2(k-5) + 8 = 2k + 2$.

With this we have proved that $\gamma_{1,2}(T) \geq \frac{n}{2} + 2$. To prove the opposite inequality, let $S(T_{10}) = \{x_1, y_1, z_1, x_2, x_3, y_3, z_3\}$ and for $T \in \{T_{4k}, T_{4k+2}\}$, let $S(T) = W_1 \cup W_k \cup W_i$, where $i \in [2, k-1]$, and

$$W_1 = \begin{cases} \{x_1, y_1, z_1\}, & T \in T_{4k}; \\ \{x_1, c_1, z_1, d_1\}, & T \in T_{4k+2}; \end{cases} \tag{11}$$

$$W_i = \begin{cases} \{x_i\}, & i \equiv 2 \pmod 6; \\ \{x_i, c_i, y_i\}, & i \equiv 3 \pmod 6; \\ \{y_i\}, & i \equiv 4 \pmod 6; \\ \{y_i, c_i, z_i\}, & i \equiv 5 \pmod 6; \\ \{z_i\}, & i \equiv 0 \pmod 6; \\ \{x_i, c_i, z_i\}, & i \equiv 1 \pmod 6; \end{cases} \tag{12}$$

$$W_k = \begin{cases} \{v_k;\ v_{k-1} \in W_{k-1}\}, & k \equiv 0 \pmod 2; \\ \{x_k, c_k, y_k, d_k\}, & k \equiv 3 \pmod 6; \\ \{y_k, c_k, z_k, d_k\}, & k \equiv 5 \pmod 6; \\ \{x_k, c_k, z_k, d_k\}, & k \equiv 1 \pmod 6\ . \end{cases} \tag{13}$$

Now it is a straightforward task (see for instance Fig. 3) to check that $S(T)$ as defined above is a $(1,2)$-dominating set of T of order $\frac{n}{2} + 2$, which completes the proof. □

Propositions 5 and 6 confirm that for every trilobite T, the inequality $c_{\text{ind}}(T) < \gamma_{1,2}(T)$ is satisfied. In addition to this result, the following straightforward observation will provide a useful insight for the concluding discussion.

Corollary 1. *Conjecture 1 holds for the family of trilobite graphs.*

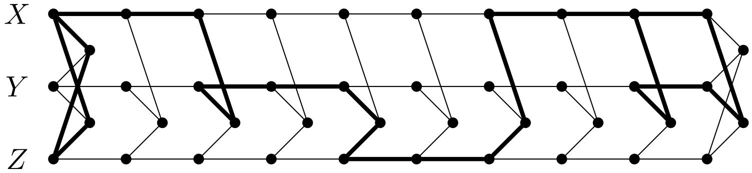

Fig. 3. Trilobite T_{42} and its subgraph induced by a $\gamma_{1,2}$-set.

6. Conclusion

In this paper, we determined the exact upper bound on the induced cycles vertex number for cubic graphs and began investigating its connection with the $(1,2)$-domination number. This connection has the potential to serve as a tool for supporting Conjecture 1. Expanding on the established results that this conjecture holds for cubic 4-chordal graphs and for cubic claw-free graphs, we further demonstrated that the following additional families of cubic graphs satisfy the inequality $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$:

- all connected graphs of order $n \le 8$,
- all graphs containing a subset of vertices S that induces a 2-regular subgraph with the property $N_G[S] = V(G)$,
- graphs G with the property $c_{\text{ind}}(G) = \lfloor \frac{3n}{4} \rfloor$, i.e. graphs attaining the upper bound for $c_{\text{ind}}(G)$,
- graphs G with $c_{\text{ind}}(G) = \frac{3n}{4} - 1$,
- graphs G with $\gamma_{1,2}(G) = \frac{n}{2}$, i.e. graphs attaining the lower bound for $\gamma_{1,2}(G)$.

Our experiences with some further examples suggest that the following conjecture might hold.

Conjecture 2. *If a cubic graph G satisfies $c_{\text{ind}}(G) \ge \frac{n}{2} + 2$, then $\gamma_{1,2}(G) \le c_{\text{ind}}(G)$.*

Moreover, we believe that even stronger result might hold, that for each $n \ge 10$ inequality (4) does not hold only for the trilobite T_n.

Conjecture 3. *For a connected cubic graph G we have $\gamma_{1,2}(G) > c_{\text{ind}}(G)$ if and only if G is a trilobite.*

Note that if Conjecture 3 holds, then the chain of inequalities in (5), together with Corollary 1, guarantees the validity of Conjecture 1. Therefore, further exploration of the relationship between the $(1,2)$-domination number and the induced cycles vertex number of a graph is well justified.

Statements and Declarations

Authors have no conflict of interest to declare.

Data availability

No data was used for the research described in the article.

Acknowledgments

The second researcher is partially supported by Slovenian research agency ARIS, program no. P1–0297.

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