Theoretical Computer Science 1057 (2025) 115559

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Complexity of the game connected domination problem

Vesna Iršič Chenoweth [a,b,*]

[a] Faculty of Mathematics and Physics, University of Ljubljana, Ljubljana, Slovenia
[b] Institute of Mathematics, Physics and Mechanics, Ljubljana, Slovenia

ARTICLE INFO

Section Editor: Pinyan Lu

AMS subject classifications:
05c57
05C69
68Q17

Keywords:
Connected domination game
Complexity
PSPACE-complete

ABSTRACT

The connected domination game is a variant of the domination game where the played vertices must form a connected subgraph at all stages of the game. In this paper we prove that deciding whether the game connected domination number is smaller than a given integer is PSPACE-complete using log-space reductions for both Dominator- and Staller-start connected domination game.

1. Introduction

The *connected domination game* was introduced in 2019 by Borowiecki, Fiedorowicz, and Sidorowicz [1], and has been studied afterwards in [2–4]. The game is played on a graph G by two players: Dominator and Staller. They alternately select vertices of the graph. A move is legal if the selected vertex dominates at least one vertex which is not already dominated by previously played vertices, and if the set of vertices selected so far induces a connected subgraph of G. More precisely, choosing the vertex v_i in the ith move is legal if for the vertices $v_1, \dots, v_{i-1}$ chosen so far we have $N[v_i] \setminus \bigcup_{j=1}^{i-1} N[v_j] \neq \emptyset$, and the vertices $v_1, \dots, v_i$ induce a connected subgraph of G. Thus we only consider the connected domination game played on connected graphs. The game ends when there are no legal moves, so when the set of played vertices is a connected dominating set of G. The goal of Dominator is to finish the game with minimum number of moves, while the aim of Staller is to maximize the number of moves.

If both players play optimally, the number of moves on a graph G is the *game connected domination number* $\gamma_{\rm cg}(G)$ if Dominator starts the game on G (D-game). If Staller starts the game (S-game), the number of moves is the *Staller-start game connected domination number* $\gamma_{\rm cg}'(G)$. Note that the terminology "connected game domination number" is also used in the literature, but we try to make it consistent with the terminology of other domination games (as in [5]). The moves in the D-game are denoted by $d_1, s_1, d_2, s_2, \dots$, where d_i are Dominator's moves and s_i are Staller's moves, while the moves in the S-game are denoted by $s_1', d_1', \dots$. For brevity, we sometimes refer to Dominator as he/him, and to Staller as she/her.

The relation between the Dominator- and Staller-start game connected domination number has been investigated in [1,4] and differs drastically from the analogous result for the (total) domination game. If G is a connected graph, then

$$\gamma_{\rm cg}(G) - 1 \le \gamma_{\rm cg}'(G) \le 2\gamma_{\rm cg}(G)$$

* Corresponding author.
E-mail address: vesna.irsic@fmf.uni-lj.si

https://doi.org/10.1016/j.tcs.2025.115559
Received 12 April 2024; Received in revised form 30 May 2025; Accepted 19 September 2025
Available online 24 September 2025

and the bounds are tight. Recall that on the other hand, the difference between the Dominator- and Staller-start game (total) domination number is at most 1 [6–8].

The connected domination game is a variant of the classical domination game which was introduced by Brešar, Klavžar and Rall in [6]. For a survey of existing results (and the variants of the game) we refer the reader to [5]. The famous $\frac{3}{5}$- and $\frac{3}{4}$-conjectures have recently been resolved [9,10], and progress towards solving the $\frac{1}{2}$-conjecture has been made [11]. The domination game is known to be difficult from the algorithmic point of view; the decision version of the game is PSPACE-complete using log-space reductions [12,13]. The game total domination problem is also PSPACE-complete using log-space reductions [14]. The connected variant of the coloring game has also been studied [15–17].

A useful tool for proving bounds for the domination games is the so-called imagination strategy and we will use it several times in this paper. It was first used in [6]; see also [5, Section 2.2]. The general idea of the strategy is that while a domination game is played on a graph, one of the players imagines another game is played in parallel on a modified graph. The player imagining the game plays optimally in the imagined game and copies their moves to the real game, while they also copy the moves of the other player from the real to the imagined game. Care needs to be taken to make this copied moves legal. In our case, we will also use the POS-CNF game as the imagined game, thus we will need to describe even more precisely how the moves are copied between the games.

In this paper we prove that the connected domination game is of the same complexity. In Section 2, we recall known results that are needed in the rest of the paper. The proof that the game connected domination problem is log-complete in PSPACE is given in Section 3, while an analogous result for the Staller-start game is presented in Section 4.

2. Preliminaries

For an integer n, let $[n] = \{1, \dots, n\}$. Let G be a graph. A set $S \subseteq V(G)$ is a *connected dominating set* of G if every vertex in $V(G) \setminus S$ has a neighbor in S and the subgraph of G induced on S is connected. The smallest size of a connected dominating set is the *connected domination number* $\gamma_c(G)$ of G. Each connected dominating set of G of size $\gamma_c(G)$ is called a γ_c*-set* of G. We also recall the following result.

Theorem 1 ([1, Theorem 1]). *If G is a connected graph, then $\gamma_c(G) \le \gamma_{cg}(G) \le 2\gamma_c(G) - 1$.*

A decision problem is *PSPACE-complete* if it can be solved using working space of polynomial size with respect to the input length, and every other problem solvable in polynomial space can be reduced to it in polynomial time. If a problem is PSPACE-complete using log-space reductions, we say that it is *log-complete in PSPACE*.

We consider the following decision problems.

GAME CONNECTED DOMINATION PROBLEM
Input: A graph G and an integer m.
Question: Is $\gamma_{cg}(G) \le m$?

STALLER-START GAME CONNECTED DOMINATION PROBLEM
Input: A graph G and an integer m.
Question: Is $\gamma'_{cg}(G) \le m$?

POS-CNF PROBLEM
Input: A positive CNF formula $\mathcal{F}$ with k variables and n clauses.
Question: Does Player 1 win on $\mathcal{F}$?

The POS-CNF PROBLEM is known to be log-complete in PSPACE [18]. In the POS-CNF game, we are given a formula $\mathcal{F}$ with k variables that is a conjunction of n disjunctive clauses in which only positive variables appear. Two players alternate turns, Player 1 setting a previously unset variable TRUE, and Player 2 setting one FALSE. When all k variables are set, Player 1 wins if the formula $\mathcal{F}$ is TRUE, otherwise Player 2 wins.

Recall the properties of the following graph from [3,4]. Let H_n, $n \ge 2$, be a graph with vertices $V(G_n) = \{u_0, \dots, u_{n+1}\} \cup \{x_1, \dots, x_{n-1}\} \cup \{y_1, \dots, y_{n-1}\}$ and edges u_iu_{i+1} for $i \in \{0, \dots, n\}$, u_ix_i, x_iy_i, y_iu_{i+1}, and $u_{i+1}x_i$ for $i \in [n-1]$. For example, see Fig. 1.

Lemma 1 ([3]). *If $n \ge 2$, then $\gamma_{cg}(H_n) = n$ and $\gamma'_{cg}(H_n) = 2n$.*

Dominator's strategy for the D-game on H_n is to play $d_1 = u_n$ which makes all the remaining moves unique and the game finishes in n moves. As in [3], we call this strategy FAST. Note that exactly vertices $u_n, \dots, u_1$ are played during the game.

Staller's strategy in the S-game is to start on u_0 and to play vertices $x_1, \dots, x_{n-1}$ whenever she can. She is able to force $n-1$ additional moves on $V(H_n) \setminus \{u_0, \dots, u_{n+1}\}$. Thus, apart from the vertices $u_1, \dots, u_n$, exactly n additional moves are played (counting the move u_0 as well). We call this strategy of Staller SLOW.

Observe that since $\gamma_c(H_n) = n$, there are always at least n moves played on H_n (even if players do not alternate taking moves). It also follows from the above that n moves are played on H_n (again, even if players do not alternate taking moves) if and only if exactly vertices $u_1, \dots, u_n$ are played.

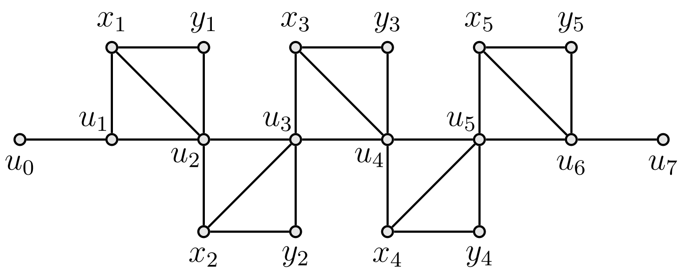

Fig. 1. The graph H_6.

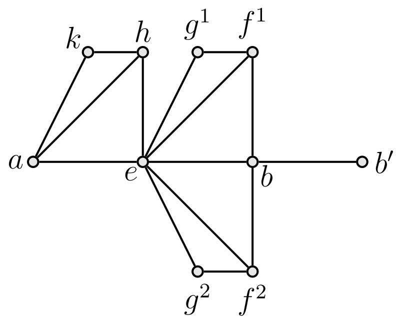

Fig. 2. The graph B.

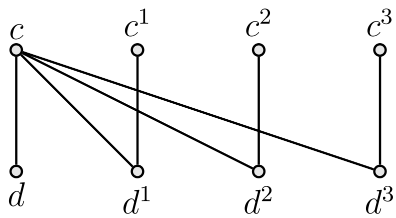

Fig. 3. The graph $C(3)$.

3. Complexity of the dominator-start game

In this section we provide a reduction from the POS-CNF PROBLEM to the GAME CONNECTED DOMINATION PROBLEM. The construction needed is described in Section 3.1, while the properties of the reduction and the final results are presented in Section 3.2. Note that the complexities of the domination and total domination games are also determined using a reduction from the POS-CNF PROBLEM, but the constructions used are different for each game; see [12,14].

3.1. Construction

Let the graph B be as in Fig. 2. When we refer to a copy B_i of the graph B, its vertices are labeled w_i for every $w \in \{a, e, b, b', h, k, f^1, g^1, f^2, g^2\}$.

Lemma 2. *If B is the graph defined above, then $\gamma_c(B) = 3$. The same holds even if vertices a and b are predominated.*

For $m \geq 1$, let the graph $C(m)$ have the vertex set $\{c, c^1, \dots, c^m, d, d^1, \dots, d^m\}$ and edges cd, cd^i for every $i \in [m]$ and $c^i d^i$ for every $i \in [m]$. For example, see Fig. 3. When we refer to a copy $C(m)_i$ of the graph $C(m)$, its vertices are labeled as w_i for every $w \in V(C(m))$. When m is clear from the context, we simply write C instead of $C(m)$.

Let A be the graph in Fig. 4. Observe that if the first move played on A is p_1 then four moves are needed to end the game if Dominator starts and three moves if Staller starts.

Given a formula $\mathcal{F}$ with k variables and n disjunctive clauses, we built a graph $G_{\mathcal{F}}$ in the following way. For each variable X_i, $i \in [k]$, we add to the graph a copy B_i of the graph B (called a gadget B_i). For each clause C_j, $j \in [n]$, we add to the graph a copy $C(n)_j = C_j$ of the graph $C(n)$ (called a gadget C_j) and make all vertices $\{c_j, c_j^1, \dots, c_j^n\}$ adjacent to a_i if and only if the variable X_i appears in the clause C_j. We add to the graph a disjoint copy of the graph H_{2n+7} and make u_0 adjacent to vertices a_i and b_i for all $i \in [k]$. If k is odd, we also add to the graph a disjoint copy of the graph A and make p_1 adjacent to u_0. For example, see Fig. 5.

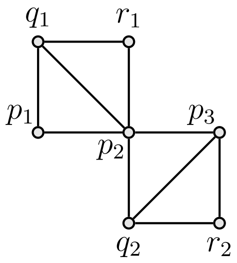

Fig. 4. The graph A.

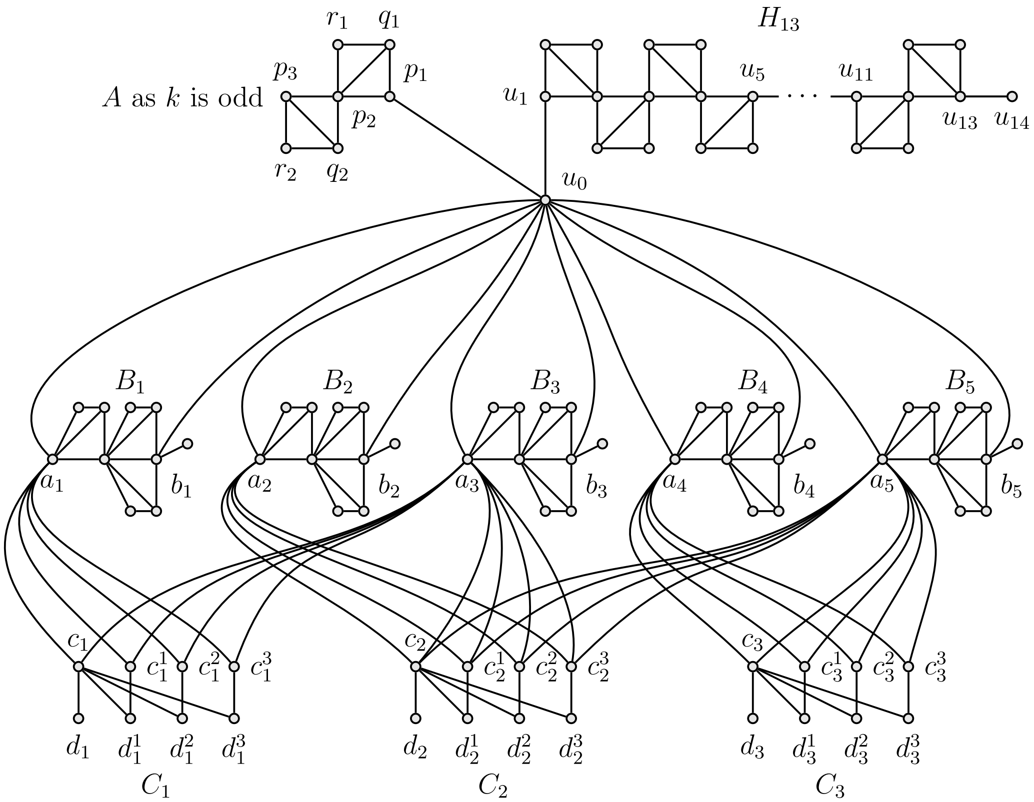

Fig. 5. The graph $G_{\mathcal{F}}$ obtained from the formula $\mathcal{F} = (X_1 \vee X_3) \wedge (X_2 \vee X_3 \vee X_5) \wedge (X_4 \vee X_5)$.

Lemma 3. *If $\mathcal{F}$ is a forumla with k variables and n disjunctive clauses, and $G_{\mathcal{F}}$ is the graph defined above, then*

$$\gamma_c(G_{\mathcal{F}}) = \begin{cases} 3k+3n+8; & k \textit{ even}, \\ 3k+3n+11; & k \textit{ odd}, \end{cases}$$

and every γ_c-set of $G_{\mathcal{F}}$ contains vertices $\{u_0, u_1, \dots, u_{2n+7}\}$, vertices $\{c_1, \dots, c_n\}$, exactly three vertices from each gadget B_i, $i \in [k]$, and if k is odd also exactly three vertices from A.

Proof. Let $D = \{u_0, u_1, \dots, u_{2n+7}\} \cup \bigcup_{i\in[k]}\{a_i, e_i, b_i\} \cup \bigcup_{j\in[n]}\{c_j\}$. Clearly, $|D| = 3k+3n+8$ and if k is even, then D is a connected dominating set of $G_{\mathcal{F}}$. If k is odd, $D \cup \{p_1, p_2, p_3\}$ is of order $3k+3n+11$ and is a connected dominating set of $G_{\mathcal{F}}$. Thus the desired upper bound for $\gamma_c(G_{\mathcal{F}})$ follows.

By Lemma 2, three vertices are needed to dominate each gadget B_i (even if a_i and b_i are already dominated). At least one vertex from each C_j, $j \in [n]$, is needed to dominate C_j, and c_j is the unique such vertex. By Lemma 1, $2n+7$ vertices are needed to dominate H_{2n+7} and the unique γ_c-set of H_{2n+7} of size $2n+7$ is $\{u_1, \dots, u_{2n+7}\}$. But since the connected dominating set of $G_{\mathcal{F}}$ must be connected, u_0 must be included in any connected dominating set of $G_{\mathcal{F}}$. If k is odd, at least three vertices are needed to dominate A. Thus the desired lower bound for $\gamma_c(G_{\mathcal{F}})$ follows, as does the description of a minimum connected dominating set. $\square$

The following simple observations about the connected domination game on $G_{\mathcal{F}}$ that easily follow from Lemma 3 will also be useful:

- At least $2n+8$ moves must be played in the copy of H_{2n+7} in $G_{\mathcal{F}}$, including the vertex u_0.
- At least three moves must be played in each gadget B_i.
- If Staller is the first to play on C_j, then at least two moves are played on C_j.

3.2. Proof of the PSPACE-completeness

We consider two auxiliary lemmas that are needed for the main result.

Lemma 4. *If Player 1 has a winning strategy for the POS-CNF game played on $\mathcal{F}$, then*

$$\gamma_{\rm cg}(G_{\mathcal{F}}) \leq \begin{cases} 3k+4n+8; & k \text{ even}, \\ 3k+4n+11; & k \text{ odd}. \end{cases}$$

Proof. We describe Dominator's strategy that ensures that the game ends in at most $3k+4n+8$ moves if k is even and in at most $3k+4n+11$ moves if k is odd. Most of the proof is the same for both parities of k.

Dominator starts the game by playing $d_1 = u_{2n+7}$. Thus he forces the strategy FAST to be played on H_{2n+7}, so the moves $s_1 = u_{2n+6}, d_2 = u_{2n+5}, \dots, d_{2n+7} = u_1, s_{2n+8} = u_0$ are forced. As $2n+8$ is even, Staller plays u_0. Observe that no vertex d_j^m can be a legal move in the rest of the game.

Dominator's next move is to play $d_{2n+9} = a_i$ if setting X_i to TRUE is the optimal first move of Player 1 in the POS-CNF game on $\mathcal{F}$. For the rest of the game, Dominator will imagine a game is being played on $\mathcal{F}$, specifying certain moves of Staller and himself as moves in the POS-CNF game. More precisely, in the game on $\mathcal{F}$, Player 1 will be playing optimally, and their moves will determine some of the corresponding moves of Dominator on $G_{\mathcal{F}}$. As Player 1 has a winning strategy on $\mathcal{F}$, they can win no matter how Player 2 plays. Player2′s moves on $\mathcal{F}$ will be determined by some of Staller's moves on $G_{\mathcal{F}}$. Which moves of Staller mean that Player 2 makes a move on $\mathcal{F}$ and how does Player1′s reply on $\mathcal{F}$ translate to Dominator's reply on $G_{\mathcal{F}}$ is explained in (6) below.

We will prove that Dominator can ensure on average at most three moves on each B_i, at most two moves on each C_j and at most three moves on A. By saying on average we mean that the moves can be counted as at most three on each B_i and at most two on each C_j, as Dominator's strategy ensures that if four moves are played on some B_i, there is a C_j with only one move played on it.

Dominator follows the rules listed below (each rule is roughly summarized first, followed by the exact Dominator's strategy) with the following addition: when we say that Dominator plays *any legal move*, we mean that he plays any c_j, a_i or e_i if possible (in this order of preference), and otherwise he plays any vertex. This is to ensure at most two moves on each C_j and at most three moves on each B_i even when Dominator does not have a more precise strategy. As a result, it might sometimes happen that Dominator's strategy below wants him to play a vertex that is not a legal move anymore. In this case, he plays any legal move (with the rules given above), but he still performs potential other actions associated with the move that is not legal anymore as if he played it now. It is thus necessary for Dominator to keep track of the vertices he played due to an exact strategy and the vertices he played as any legal moves.

(1) If Staller plays p_1, Dominator replies by playing p_2.
This ensures that if Staller is the first to play on A, then exactly three moves are made on A during the game.
(2) If Staller plays p_3 or q_2, then Dominator plays any legal move.
Note that this means that on $G - H_{2n+7} - A$, Dominator makes two consecutive moves which we have to consider when analyzing some of the cases below.
(3) If Staller plays on C_j, Dominator plays c_j.
If Dominator played first on C_j (before this move of Staller), then his strategy ensures he played c_j, thus no more moves are legal on C_j. So we know that the current move of Staller is in fact the first move on C_j. If there are still undominated vertices on C_j after Staller's move, c_j is a legal reply for Dominator that ensures that at most two moves are made on C_j during the game. Otherwise, so if Staller's move on C_j leaves no undominated vertices in C_j, then Staller played c_j, Dominator plays any legal move, and just one move is made on C_j during the game.
(4) If Staller plays on B_i and not all vertices of B_i are dominated after her move, Dominator replies on B_i as well.
More precisely, if Staller is the first to play on some B_i, then Dominator replies by playing e_i. If Staller is not the first to play on B_i, but B_i is still not entirely dominated, then this is only possible if Dominator played the first move on this B_i, so by (6), he played a_i before. If Staller now plays e_i, he replies on b_i, and if she plays b_i, he selects e_i (no other move on B_i is a legal move for Staller at this stage of the game). Note that if Dominator made two consecutive moves on B_i, they were a_i and e_i, thus after Staller's move on B_i, there are no more undominated vertices left and we are not in this case. Similarly if Dominator made three consecutive moves on B_i.
Notice that with this strategy, Dominator ensures that no more than three moves are played on each B_i unless Staller plays a move from (5).
(5) If Staller plays a_i and all vertices of B_i are dominated already before her move, Dominator replies by playing c_j in the appropriate gadget C_j.
Note that if such a move of Staller was legal, it must have newly dominated vertices $c_j, c_j^1, \dots, c_j^n$ for some $j \in [n]$. Thus Dominator's reply on c_j is a legal move. Additionally, notice that while four moves were played on B_i, only one move was played on C_j (and no more moves on B_i or C_j are possible during the game). For counting purposes, we consider this as three moves on B_i and two on C_j.

(6) If Staller plays on B_i that was not dominated before her move and all vertices of B_i are dominated after her move, Dominator selects an appropriate gadget B_j to play on.
If X_i already has an assigned value in the imagined POS-CNF game on $\mathcal{F}$, then Dominator plays any legal move. Otherwise, Dominator imagines that Player 2 set X_i to FALSE in the POS-CNF game on $\mathcal{F}$ now. If this ends the POS-CNF game on $\mathcal{F}$, Dominator plays any legal move. Otherwise, Dominator considers Player1's optimal strategy in the POS-CNF game, and plays on B_j such that Player1's strategy is to set X_j to TRUE. Dominator's strategy is to play a_j if possible, otherwise he plays b_j if this is the last move on B_j, or he plays any legal move (in this order of preference). Once Dominator sets X_j to TRUE as Player1's move on $\mathcal{F}$ and wants to play on B_j, the following options are possible for B_j: no move has been played on it so far (Dominator now plays a_j), Dominator already played one or more moves on it (Dominator now plays b_j if this is the last move on B_j, or he plays any legal move, but we know he played a_j before), Staller and Dominator both played on B_j before (Dominator now plays a_j if possible, or b_j if this is the last move on B_j, or he plays any legal move, but again we know that a_j has been played before). In all cases we notice that Dominator is ensuring that at most three moves have been or will be played on B_j, a_j is one of them, and that if Staller was the first to play on B_j, all vertices of B_j are dominated after this move.
Notice that with this strategy, Dominator ensures that for all indices $i \in I$ that Player 1 sets to TRUE in his winning strategy in the POS-CNF game on $\mathcal{F}$, the vertices a_i, $i \in I$, are played. Thus by the end of the POS-CNF game, all vertices $c_1, \dots, c_n, \dots, c_1^n, \dots, c_n^n$ will be dominated in $G_{\mathcal{F}}$ (since Player 1 wins on $\mathcal{F}$) and so all vertices from $C_1, \dots, C_n$ can be dominated by using at most three moves on each B_i and at most two on each C_j (on average).

With the described strategy, Dominator ensures exactly $2n + 8$ moves on H_{2n+7}, and on average at most three moves on each B_i and at most two moves on each C_j. Observe also that the imagined POS-CNF game either ends with the win of Player 1 (meaning that all vertices $c_1, \dots, c_n, \dots, c_1^n, \dots, c_n^n$ are dominated and are thus legal moves can be played on any $C_1, \dots, C_n$ that is not dominated yet) or the POS-CNF game does not end as some variables never get assigned values. By (6) we know that Dominator played the third and last move on all the remaining (unassigned) variables. If the first player playing on such B_i did no play a_i, then we know Staller made the move on b_i and Dominator replied by playing e_i. Thus by Dominator's strategy, he can play a_i as the last move on B_i. If Dominator now finished playing the POS-CNF game with the remaining variables, selecting moves of Player 2 arbitrarily, he would still win, and by the argument above, it still holds that for all indices $i \in I$ that Player 1 sets to TRUE in the POS-CNF game on $\mathcal{F}$, the vertices a_i, $i \in I$, are played. Hence, in this case, all vertices $c_1, \dots, c_n, \dots, c_1^n, \dots, c_n^n$ are dominated as well. Thus Dominator also ensures that by keeping at most three moves played on each B_i and at most two on each C_j, all vertices from $G_{\mathcal{F}} - H_{2n+7} - A$ can be dominated (i.e. it cannot happen that no move on some C_j would be legal).

If k is even, there are no moves on A. If k is odd and Staller made the first move on A, then by (1), three moves are played on A. Thus the only remaining case to consider is if k is odd and Dominator is forced to play the first move on A. This happens if only vertices on A are undominated, p_1 has not been played before, and it is Dominator's turn now, Dominator must play p_1 (it is the only legal move in the game), thus possibly four moves are made on A. (His consequent move is p_2 or p_3.) If at most $2n + 8 + 3k + 2n - 1 = 4n + 3k + 7$ moves were made during the game before this move, the game ends after at most $4n + 3k + 11$ moves. Otherwise, so if exactly $2n + 8 + 3k + 2n = 4n + 3k + 8$ moves were made before (by arguments above Dominator ensures at most three moves on each B_i and at most two on each C_j), as k is odd, $4n + 3k + 8$ is also odd, so it cannot be Dominator's turn now. Thus if four moves are played on A, then at most $4n + 3k + 7$ moves were played on the rest of the graph.

It follows from the above that the game ends in at most $2n + 8 + 3k + 2n = 4n + 3k + 8$ moves if k is even, and in at most $2n + 8 + 3k + 2n + 3 = 4n + 3k + 11$ moves if k is odd. ☐

Lemma 5. *If Player 2 has a winning strategy for the POS-CNF game played on $\mathcal{F}$, then*

$$\gamma_{\mathrm{cg}}(G_{\mathcal{F}}) \geq \begin{cases} 3k + 4n + 9; & k \text{ even}, \\ 3k + 4n + 12; & k \text{ odd}. \end{cases}$$

Proof. We describe Staller's strategy that ensures that the game ends in at least $3k + 4n + 9$ moves if k is even and in at least $3k + 4n + 12$ moves if k is odd. Most of the proof is the same for both parities of k. Staller's strategy is composed of two phases and a series of general rules that Staller follows during both phases. Phase 1 lasts until too many vertices have been played on H_{2n+7}, A, some gadget B_i or some gadget C_j. Afterwards, the game is in Phase 2 till the end. The general rules and both phases are described in detail later.

If Dominator's first move is a vertex from $G_{\mathcal{F}} - H_{2n+7}$, then Staller's strategy is as follows. She partitions the graph into two subgraphs, $H_{2n+7} - u_0$ and $G_{\mathcal{F}} - (H_{2n+7} - u_0)$, and when Dominator plays on one of the parts, she plays on the same part if possible. Additionally, she follows the strategy Slow on H_{2n+7}, i.e. she plays on $x_1, \dots, x_{2n+6}$ whenever possible. By Lemma 3, at least $3k + n + 1$ moves are played on $G_{\mathcal{F}} - (H_{2n+7} - u_0)$ (including u_0). As Staller is not able to reply to Dominator's move on $G_{\mathcal{F}} - (H_{2n+7} - u_0)$ at most once, she is able to ensure that at least $2n + 5$ vertices among $x_1, \dots, x_{2n+6}$ are played during the game. Thus the game will last for at least $(3k + n + 1) + (2n + 7) + (2n + 5) = 5n + 3k + 13$ which is more than the desired lower bound.

If Dominator's first move is a vertex from $H_{2n+7} - u_{2n+7}$, at least $2n + 9$ moves are needed to dominate H_{2n+7}. If Staller plays u_0, the rest of the game proceeds as in the next case (i.e. if Dominator started on u_{2n+7}), except that the game is in Phase 2 from the start. If Dominator plays u_0, then Staller plays b_1 in her next move. For the rest of the game (which is also in Phase 2 already), she uses the same strategy as if Dominator started on u_{2n+7}, and pretends that she has not played b_1 yet. Thus during the strategy described below, it may happen that Staller's prescribed move is not legal anymore. In this case she pretends that she played the move now, follows any additional part of her strategy related to this move, but in reality she plays *any legal move* with the following restrictions:

if possible she never plays vertices $c_1, \ldots, c_n$ or $a_1, \ldots, a_k$. Additionally, if she plays two moves on the same B_i while Dominator plays no moves there, she plays b_i and f_i^1, thus the game enters Phase 2.

From now on, assume that $d_1 = u_{2n+7}$. The next few moves are forced: $s_1 = u_{2n+6}, d_2 = u_{2n+5}, \ldots, d_{2n+7} = u_1, s_{2n+8} = u_0$. As $2n+8$ is even, Staller plays u_0. Observe again that no vertex d_j^m can be a legal move in the rest of the game.

For the rest of the game, Staller will imagine a game is being played on $\mathcal{F}$, specifying certain moves of Dominator and herself as moves in the POS-CNF game. More precisely, in the game on $\mathcal{F}$, Player 2 will be playing optimally, and their moves will determine some of the corresponding moves of Staller on $G_{\mathcal{F}}$. As Player 2 has a winning strategy on $\mathcal{F}$, they can win no matter how Player 1 plays. Player1′s moves on $\mathcal{F}$ will be determined by some of Dominator's moves on $G_{\mathcal{F}}$. Which moves of Dominator mean that Player 1 makes a move on $\mathcal{F}$ and how does Player2′s reply on $\mathcal{F}$ translate to Staller's reply on $G_{\mathcal{F}}$ is explained in (2) below.

Staller's staretegy consists of two phases, but some part of her strategy is the same for both. Again we list rules with a concise but simplified description first.

General rules If Dominator's move is among these rules, Staller plays according to them no matter which phase the game is in.

(A) If Dominator plays p_1, Staller plays q_1.
This ensures at least four moves are played on A if Dominator is the first player to play on it.

(B) If Dominator plays p_2, Staller plays q_2.

(C) If Dominator plays some a_i in B_i, then Staller plays on appropriate gadgets C_j for the next sequence of moves, until the game enters Phase 2 or she is directed to play on B_i.
As a_i was a legal move for Dominator, it newly dominated some $c_{j_1}, \ldots, c_{j_m}$, $m \geq 1$. Staller's strategy is to first play $c_{j_1}^1$. If Dominator replies by playing c_{j_1}, then Staller plays $c_{j_2}^1$. If Dominator replies to Staller's move on $c_{j_r}^1$ by playing c_{j_r}, then Staller plays $c_{j_{r+1}}^1$ next. If it exists, let $p \in [m]$ be the first index where Staller played $c_{j_p}^1$ and Dominator did not reply by playing c_{j_p}. If Dominator plays on A, Staller follows (A) or (B). If Dominator plays some other $a_{i'}$, she follows (C) from the start (each $C_{j'}$ on which moves can be played thus has an associated gadget $B_{i'}$ and set of gadgets C_j). Otherwise, Staller plays a vertex from $\{c_{j_p}^2, \ldots, c_{j_p}^n\}$. If Dominator is playing only on vertices c_j, then this strategy of Staller ensures that on average at least two moves are played on each gadget C_j (only one on some, but accordingly more on C_{j_p}). If Dominator plays any other vertex, Staller's strategy ensures that at least three vertices will be played on C_{j_p} (even after averaging the counting as before), thus the game enters Phase 2.
If no move on $C_{j_1}, \ldots, C_{j_m}$ is legal and the game is still in Phase 1, then Staller plays on B_i according to the rules given in (1) below (the game stays in Phase 1 only if Dominator is playing only on vertices $c_{j_1}, \ldots, c_{j_m}$ until no more move on $C_{j_1}, \ldots, C_{j_m}$ is legal, thus no player made a move on the rest of the graph in the meantime). If no move on $C_{j_1}, \ldots, C_{j_m}$ is legal and the game is now in Phase 2, then Staller follows the rules of Phase 2.

Phase 1 Exactly $2n+8$ moves were played on H_{2n+7}, at most three moves were played on A, and on average, at most three moves were played on every B_i, $i \in [k]$, and at most two moves on every C_j, $j \in [n]$.

Staller plays according to the general rules in addition to the rules given below. By saying on average when counting moves we mean that possibly there are four moves on some B_i, but then this B_i is associated with a gadget C_j on which only one move was played, or that possibly there are more moves played on some C_j but then there is only one move played on accordingly many other gadgets $C_{j'}$.

(1) If Dominator plays on B_i and not all vertices of B_i are dominated after his move, Staller plays on B_i.
More precisely, if Dominator's move was the first on the gadget, then he played a_i or b_i. If he played a_i, Staller follows the general rule (C) and if it eventually directs her to play on B_i, she plays b_i. If Dominator played b_i, she selects f_i^1. In the latter case, at least four moves will be needed to dominate B_i, thus the game enters Phase 2. See Fig. 6.
If Dominator's move was not the first move on B_i, then Staller made the first move on B_i by playing b_i (by (2) and her restrictions when playing any legal move). If Dominator plays e_i, Staller replies on h_i. If he plays f_i^ℓ for some $\ell \in \{1, 2\}$, then she plays $f_i^{3-\ell}$. Observe that in this case, at least one additional move is needed to dominate B_i, thus the game already enters Phase 2. If he plays a_i, she follows the general rule (C); if the rule eventually directs her to play on B_i, she selects f_i^1. In this case, at least one more move is needed on B_i, thus the game also enters Phase 2. See Fig. 7.
Notice that if Dominator makes the first move on B_i, then exactly three moves will be played on that B_i or the game will enter Phase 2. If Staller makes the first move on B_i, then either three moves are played on it and a_i is not one of them, or the game enters Phase 2.

(2) If Dominator plays on B_i, all vertices of B_i are dominated after his move and the POS-CNF game has not ended yet, Staller selects an appropriate gadget B_j to play on.
If X_i already has a value assigned in the POS-CNF game, then Staller plays any legal move. Otherwise, Staller sets X_i to TRUE as the move of Player 1 in the POS-CNF game. If this ends the POS-CNF game, Staller plays as in (3). Otherwise, she lets Player 2 play optimally on $\mathcal{F}$, setting some X_j to FALSE. Staller now tries to play on B_j. If no move was made before on B_j, Staller plays b_j. If exactly one move was made on B_j before we know by (1) that it was a move of Staller, so she played b_j, and can play f_j^1 now, thus the game enters Phase 2. If two moves were played on B_j so far, then as we are in Phase 1 and by (1), we know that Dominator played a_j and Staller played b_j. Now Staller plays f_j^1, thus the game enters Phase 2. It follows from the above that more moves on B_j could not have been played yet as we are in Phase 1.
Note that this means that while we are in Phase 1, a variable X_j is set to FALSE only if Staller played first on B_j, a_j has not been played, and at most three moves were played on B_j. As long as the game is in Phase 1, we also know that a_j will not be played.

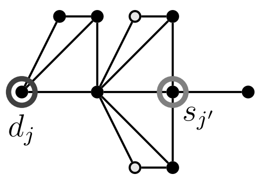

Fig. 6. Staller's strategy in (1) if Dominator played first on B_i. Dominated vertices are colored black.

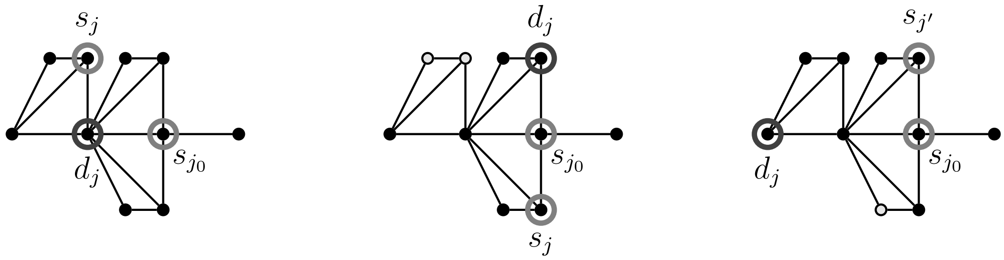

Fig. 7. Staller's strategy in (1) if Dominator played second on B_i. Dominated vertices are colored black.

(3) Otherwise, so if Dominator played on B_i, all vertices of B_i are dominated after his move and the POS-CNF game has ended already, Staller does the following.
The POS-CNF game has ended with the win of Player 2. If not all gadgets B_i are dominated yet, Staller plays on one of them, but not playing a_i (which she can as if a_i is legal, so is h_i). If all gadgets B_i are dominated already, then as Player 2 won the POS-CNF game, we know that there is at least one C_j that is not dominated yet. As we are in Phase 1, we know that no move has been made on A yet (by (A)). As all B_i are dominated and we are in Phase 1, exactly three moves were made on each gadget B_i and exactly zero or two moves were made on each gadget C_j (on average). If k is even, an even number of moves was played in the game so far, so it cannot be Staller's turn now. If k is odd, it can indeed be Staller's turn now, and she plays p_1 (which is present as k is odd). If Dominator replies by playing q_1, the game enters Phase 2. Otherwise, Dominator plays p_2, Staller plays p_3, and now it is Dominator's turn. The only legal moves for him are to play some a_i that newly dominates some c_j. Staller now plays c_j^1 (as in (C)). This ensures four moves on B_i and at least two on C_j, thus the game enters Phase 2.

Phase 2 At least $2n+9$ moves were played on H_{2n+7}, four moves have been or will be played on A, or on average, at least four moves have been or will be played on some gadget B_i or at least three moves have been or will be played on some gadget C_j.
Staller plays according to the general rules or if they are not applicable, she plays any legal move until the game ends with the following restrictions.

- If possible, Staller never plays vertices $c_1, \dots, c_n$.
  Staller is forced to play on $c_1, \dots, c_n$ only if some of these are the only legal moves in the game. But as c_j is the only legal move in C_j only if d_j is the only undominated vertex in C_j, and so if $c_j^1, \dots, c_j^n$ have all been played already, this move maintains the property that at least two moves are played on each C_j.
- If possible, Staller never plays vertices $a_1, \dots, a_k$.
  Suppose that a_i is a legal move for Staller. If not all vertices of B_i are already dominated, she can play h_i instead (which is also legal as a_i is legal). Otherwise, so if a_i is a legal move only because it newly dominates vertices of some C_j, observe that at least three moves were already played on B_i as it is already dominated. Staller now plays a_i as the fourth move on B_i, allowing Dominator to dominate C_j with only one move. But we can still count this as three moves on B_i and two moves on C_j.

Notice that by above rules, Staller is always ensuring that at least three moves are played on each B_i and at least two on each C_j, on average. It also follows from the rules that the game always enters Phase 2. Thus Staller's strategy ensures that at least $(2n+8)+3k+2n+1 = 4n+3k+9$ moves are made if k is even and at least $(2n+8)+3k+2n+4 = 4n+3k+12$ moves are made if k is odd. $\square$

We remark that a more detailed analysis of the game might yield that the graph H_{2n+7} in $G_{\mathcal{F}}$ could be replaced with some H_i, $i \leq 2n+6$, but feel that a simpler proof better illustrates the general idea.

Theorem 2. *The* Game connected domination problem *is PSPACE-complete.*

Proof. It is easy to see that the Game connected domination problem is in PSPACE. (For example, use the fact that NP $\subseteq$ PSPACE.)

To prove that the problem is PSPACE-complete, we use a reduction from the POS-CNF problem. Given a POS-CNF formula $\mathcal{F}$, the graph $G_{\mathcal{F}}$ is obtained as described in Section 3.1. Combining Lemmas 4 and 5 implies the following. If k is even, $\gamma_{\rm cg}(G_{\mathcal{F}}) \leq 3k+4n+8$ if and only if Player 1 wins the POS-CNF game on $\mathcal{F}$. If k is odd, $\gamma_{\rm cg}(G_{\mathcal{F}}) \leq 3k+4n+11$ if and only if Player 1 wins the POS-CNF

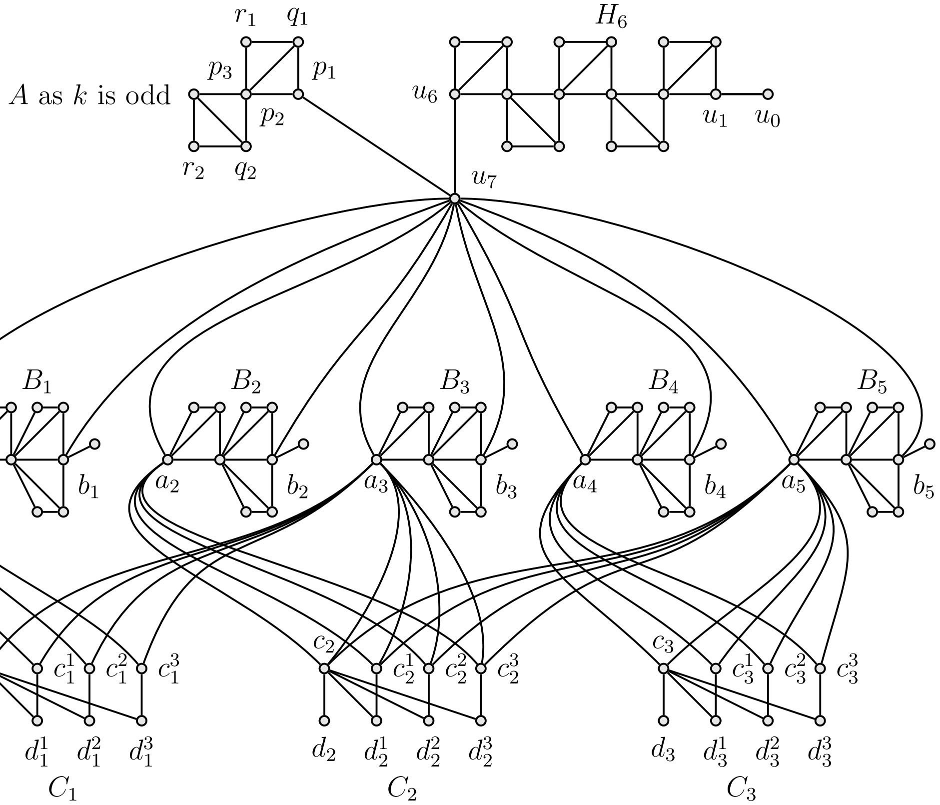

Fig. 8. The graph $G'_{\mathcal{F}}$ obtained from the formula $\mathcal{F} = (X_1 \vee X_3) \wedge (X_2 \vee X_3 \vee X_5) \wedge (X_4 \vee X_5)$.

game on $\mathcal{F}$. Since POS-CNF problem is known to be PSPACE-complete and the described reduction from POS-CNF PROBLEM to GAME CONNECTED DOMINATION PROBLEM can be computed with a polynomial size working space, the desired result follows. □

Since the reduction in Theorem 2 can be computed with a logarithmic size working space, and since POS-CNF PROBLEM is known to be log-complete in PSPACE [18], we obtain an even stronger result.

Corollary 1. *The GAME CONNECTED DOMINATION PROBLEM is log-complete in PSPACE.*

4. Complexity of the Staller-start game

In this section we consider the Staller-start connected domination game. The main idea of the proof of the PSPACE-completeness of the STALLER-START GAME CONNECTED DOMINATION PROBLEM is similar as in the Dominator-start game. However, the graph H_{2n+7} in the construction is replaced with a smaller graph and connected to the rest of the graph in a slightly different way.

Given a formula $\mathcal{F}$ with k variables and n disjunctive clauses, we built a graph $G'_{\mathcal{F}}$ in the following way. For each variable X_i, $i \in [k]$, we add to the graph a copy B_i of the graph B (called a gadget B_i). For each clause C_j, $j \in [n]$, we add to the graph a copy $C(n)_j = C_j$ of the graph $C(n)$ (called a gadget C_j) and make all vertices $\{c_j, c_j^1, \dots, c_j^n\}$ adjacent to a_i if and only if the variable X_i appears in the clause C_j. We add to the graph a disjoint copy of the graph H_6 and make u_7 adjacent to vertices a_i and b_i for all $i \in [k]$. If k is odd, we also add to the graph a disjoint copy of the graph A and make p_1 adjacent to u_7. For example, see Fig. 8.

Lemma 6. *If Player 1 has a winning strategy for the POS-CNF game played on $\mathcal{F}$, then*

$$\gamma'_{\rm cg}(G'_{\mathcal{F}}) \le \begin{cases} 3k + 2n + 13; & k \text{ even}, \\ 3k + 2n + 16; & k \text{ odd}. \end{cases}$$

Proof. Dominator's strategy consists of two phases. Phase 1 is the first phase of the game and Dominator's goal is to reach Phase 2 with as few moves as possible. Phase 2 starts with the first move of Dominator after the vertex u_7 has been played (so either with the move d'_i if $s'_i = u_7$ or with the move d'_{i+1} if $d'_i = u_7$). In Phase 2, Dominator follows the same strategy as in Lemma 4 (after Staller played u_0 on $G_{\mathcal{F}}$). If during this strategy, if some vertex is not a legal move (because it was played before), Dominator considers the imagined POS-CNF game as if the move was played in the current step, but plays an arbitrary legal move in $G'_{\mathcal{F}}$, again following the rules for any legal move from the proof of Lemma 4. By the Connected Game Continuation Principle from [3] this is not a loss for

Dominator. However, we may need to take these additional moves into account at the final count of moves. For this sake, let

$$M = \begin{cases} \{s'_1, d'_1, \dots, s'_{i-1}, d'_{i-1}\} \cap \left(V(G'_{\mathcal{F}}) \setminus V(H_6)\right); & s'_i = u_7, \\ \left(\{s'_1, d'_1, \dots, s'_{i-1}, d'_{i-1}\} \cup \{s'_i, s'_{i+1}\}\right) \cap \left(V(G'_{\mathcal{F}}) \setminus V(H_6)\right); & d'_i = u_7, \end{cases}$$

denote the set of possibly additional moves in $V(G'_{\mathcal{F}}) \setminus V(H_6)$ made during the game. If $s'_1 \in V(H_6)$, then $|M| \le 1$. If $s'_1 \in V(G'_{\mathcal{F}}) \setminus V(H_6 \cup A)$, then since every vertex $x \in V(G'_{\mathcal{F}}) \setminus V(H_6 \cup A)$ is at distance at most 3 from u_7, Dominator can ensure that $|M| \le 6$ by playing the vertices on the shortest path between s'_1 and u_7. If $s'_1 \in V(A)$, then it is not hard to see that $|M| \le 6$ as well (only the vertex r_2 is at distance 4 from u_7 and has to be considered separately).

To finalize the proof, consider the following cases. Let

$$f(n,k) = \begin{cases} 3k + 2n; & k \text{ even}, \\ 3k + 2n + 3; & k \text{ odd}. \end{cases}$$

Observe that this equals the upper bound given in Lemma 4 without the $2n + 8$ moves made on H_{2n+7}, so the moves played in the game after u_0 is played by Staller.

Case 1. $s'_1 = u_0$.
Dominator's strategy is to play only vertices from $\{u_1, \dots, u_7\}$, thus at most 13 moves are played on H_6. If indeed 13 moves are played on H_6, then Staller plays u_7 and $M = \emptyset$. However, if Dominator is forced to play u_7, then at most 12 moves were made on H_6 and $|M| \le 1$. In Phase 2, Dominator can ensure at most $f(n,k)$ moves are played on $V(G'_{\mathcal{F}}) \setminus V(H_6)$ since Player 1 has a winning strategy in the POS-CNF game on $\mathcal{F}$, using the same strategy as in the proof of Lemma 4. Altogether, either at most $13 + f(n,k)$ moves are played, or at most $12 + f(n,k) + |M| \le 13 + f(n,k)$ moves are made in the game.

Case 2. $s'_1 \in V(H_6) \setminus \{u_0\}$.
Dominator's strategy is to play only vertices from $\{u_1, \dots, u_7\}$, thus at most 12 moves are played on H_6. Combining Dominator's strategy from Phase 2 and the fact that $|M| \le 1$, we conclude that at most $13 + f(n,k)$ moves are played in the game.

Case 3. $s'_1 \in V(G'_{\mathcal{F}}) \setminus V(H_6)$.
Dominator's strategy is to ensure that u_7 is played as soon as possible. Afterwards, he utilizes the strategy Fast on H_6. More precisely, whenever Staller plays a vertex on H_6, Dominator replies by playing a vertex on H_6 as well. Since the remaining moves on H_6 can be paired $((u_6, u_5), (u_4, u_3), (u_2, u_1))$ the development of the game on H_6 is independent of Dominator's strategy on $V(G'_{\mathcal{F}}) \setminus V(H_6)$, and exactly seven moves are played on H_6. Combining this with Dominator's strategy from Phase 2 and the fact that $|M| \le 6$ yields that the number of moves in the game is at most $13 + f(n,k)$.

This concludes the proof of Lemma 6. □

We remark that a more detailed analysis of the game might yield that the graph H_6 in $G'_{\mathcal{F}}$ could be replaced with some H_i, $i \le 5$, but feel that a simpler proof better illustrates the general idea.

Lemma 7. *If Player 2 has a winning strategy for the POS-CNF game played on $\mathcal{F}$, then*

$$\gamma'_{\rm cg}(G'_{\mathcal{F}}) \ge \begin{cases} 3k + 2n + 14; & k \text{ even}, \\ 3k + 2n + 17; & k \text{ odd}. \end{cases}$$

Proof. It suffices to provide a strategy for Staller that ensures that at least $14 + f(n,k)$ moves are played on $G'_{\mathcal{F}}$ where

$$f(n,k) = \begin{cases} 3k + 2n; & k \text{ even}, \\ 3k + 2n + 3; & k \text{ odd}. \end{cases}$$

Staller's strategy is to start the game by playing u_0 and using her strategy Slow on H_6. This ensures that 13 moves are played on H_6 and that Staller plays u_7. Now, using the same argument as in the proof of Lemma 5 (after Staller played u_0) we can see that since Player 2 wins the POS-CNF game on $\mathcal{F}$, Staller can force at least $f(n,k) + 1$ moves on $V(G'_{\mathcal{F}}) \setminus V(H_6)$. Thus the game lasts at least $14 + f(n,k)$ moves. □

Using analogous arguments as for the Dominator-start game, we obtain the following.

Theorem 3. *The Staller-start game connected domination problem is PSPACE-complete. Moreover, the problem is log-complete in PSPACE.*

CRediT authorship contribution statement

Vesna Iršič Chenoweth: Writing – original draft, Investigation.

Data availability

No data was used for the research described in the article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

I would like to thank B. Brešar, Cs. Bujtás, S. Cabello, S. Klavžar and M. Konvalinka for pointing out the problem studied in this paper, and acknowledge the financial support from the Slovenian Research and Innovation Agency (ARIS) under the grants Z1-50003, P1-0297, N1-0218, N1-0285 and N1-0355, and from the European Union (ERC, KARST, 101071836). I would also like to thank the anonymous reviewers for their close reading and helpful suggestions.

References

[1] M.a. Borowiecki, A. Fiedorowicz, E. Sidorowicz, Connected domination game, Appl. Anal. Discrete Math. 13 (1) (2019) 261–289. https://doi.org/10.2298/AADM171126020B
[2] C. Bujtás, P. Dokyeesun, V. Iršič, S. Klavžar, Connected domination game played on cartesian products, Open Math. 17 (2019) 1269–1280. https://doi.org/10.1515/math-2019-0111
[3] C. Bujtás, V. Iršič, S. Klavžar, Predominating a vertex in the connected domination game, Graphs Combin. 38 (3) (2022) 77, 21. https://doi.org/10.1007/s00373-022-02478-y
[4] V. Iršič, Connected domination game: predomination, staller-start game, and lexicographic products, Appl. Anal. Discrete Math. 16 (1) (2022) 66–82. https://doi.org/10.2298/aadm190206002i
[5] B. Brešar, M.A. Henning, S. Klavžar, D.F. Rall, Domination games played on graphs, SpringerBriefs in Mathematics, Springer International Publishing, 2021. https://books.google.si/books?id=_34nzgEACAAJ.
[6] B. Brešar, S. Klavžar, D.F. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (3) (2010) 979–991. https://doi.org/10.1137/100786800
[7] M.A. Henning, S. Klavžar, D.F. Rall, Total version of the domination game, Graphs Combin. 31 (5) (2015) 1453–1462. https://doi.org/10.1007/s00373-014-1470-9
[8] W.B. Kinnersley, D.B. West, R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (4) (2013) 2090–2107. https://doi.org/10.1137/120884742
[9] J. Portier, L.V. Versteegen, A proof of the 3/4-conjecture for the total domination game, SIAM J. Discrete Math. 39 (1) (2025) 1–18. https://doi.org/10.1137/23M1551584
[10] L. Versteegen, A proof of the 3/5-conjecture in the domination game, European J. Combin. 122 (2024) 104034, 33. https://doi.org/10.1016/j.ejc.2024.104034
[11] J. Portier, L. Versteegen, Progress towards the 1/2-conjecture for the domination game, Discrete Appl. Math. 351 (2024) 36–43. https://doi.org/10.1016/j.dam.2024.03.001
[12] B. Brešar, P. Dorbec, S. Klavžar, G. Košmrlj, G. Renault, Complexity of the game domination problem, Theoret. Comput. Sci. 648 (2016) 1–7. https://doi.org/10.1016/j.tcs.2016.07.025
[13] S. Klavžar, G. Košmrlj, S. Schmidt, On the computational complexity of the domination game, Iran. J. Math. Sci. Inform. 10 (2) (2015) 115–122, 134.
[14] B. Brešar, M.A. Henning, The game total domination problem is log-complete in PSPACE, Inform. Process. Lett. 126 (2017) 12–17. https://doi.org/10.1016/j.ipl.2017.05.007
[15] C. Charpentier, H. Hocquard, E. Sopena, X. Zhu, A connected version of the graph coloring game, Discrete Appl. Math. 283 (2020) 744–750. https://doi.org/10.1016/j.dam.2020.03.012
[16] C.V.G.C. Lima, T. Marcilon, N. Martins, R. Sampaio, The connected greedy coloring game, Theoret. Comput. Sci. 940 (2023) 1–13. https://doi.org/10.1016/j.tcs.2022.10.034
[17] P. Bradshaw, A note on the connected game coloring number, Discrete Appl. Math. 333 (2023) 13–19. https://doi.org/10.1016/j.dam.2023.02.017
[18] T.J. Schaefer, On the complexity of some two-person perfect-information games, J. Comput. System Sci. 16 (2) (1978) 185–225. https://doi.org/10.1016/0022-0000(78)90045-4