Applied Mathematics and Computation 510 (2026) 129695

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On reduced Hamilton walks

Aleksander Malnič [a,b,c,1], Rok Požar [d,b,c,2,*]

[a] University of Ljubljana, Faculty of Education, Kardeljeva ploščad 16, Ljubljana, 1000, Slovenia
[b] University of Primorska, Andrej Marušič Institute, Muzejski trg 2, Koper, 6000, Slovenia
[c] Institute of Mathematics, Physics and Mechanics, Jadranska 19, Ljubljana, 1000, Slovenia
[d] University of Primorska, Faculty of Mathematics, Natural Sciences and Information Technologies, Glagoljaška 8, Koper, 6000, Slovenia

ARTICLE INFO

Keywords:
Algorithm
Hamilton walk
Nonstandard metric
Reduced walk

ABSTRACT

A Hamilton walk in a finite graph is a walk, either open or closed, that traverses every vertex at least once. Here, we introduce Hamilton walks that are reduced in the sense that they avoid immediate backtracking: a reduced Hamilton walk never traverses the same edge forth and back consecutively.

While every connected graph admits a Hamilton walk, existence of a reduced Hamilton walk is not guaranteed for all graphs. However, we prove that a reduced Hamilton walk does exist in a connected graph with minimal valency at least 2.

Furthermore, given such a graph on n vertices, we present an $O(n^2)$-time algorithm that constructs a reduced Hamilton walk of length at most $n(n+3)/2$. Specifically, for a graph belonging to a family of regular expander graphs, we can find a reduced Hamilton walk of length at most $c(6n-2)\log n + 2n$, where c is a constant independent of n.

1. Introduction

A *Hamilton walk*, either open or closed, is a walk that traverses every vertex of a finite graph at least once. Such walks exist between any two vertices, whether distinct or not. Finding minimal-length Hamilton walks is a fundamental problem with numerous real-world applications. While in the literature it is typically assumed that Hamilton walks are closed and of minimal length, our work considers a broader definition.

Goodman and Hedetniemi [1] established first upper bounds for the minimal length of closed Hamilton walks. Bermond [2] provided a different estimate based on a generalization of Ore's and Pósa's theorems. Takamizawa, Nithizeki, and Saito [3] improved the result of Bermond, and also provided an algorithm for finding a closed Hamilton walk of length less than a prescribed value in cases when graphs satisfy a certain Chvátal's type condition.

Subsequent studies have explored closed Hamilton walks in specific graph classes, such as planar graphs [4–6], cubic graphs [7], directed graphs [8–10], and Möbius double loop networks [11]. In certain cases, Hamilton walks are simple, leading to the classic problem of determining whether a graph contains a Hamilton path or cycle.

In what follows we consider finite graphs that may have loops and multiple edges. We could as well consider just simple graphs, however, allowing loops and multiple edges will make certain induction arguments easier to handle. Formally, a *walk* $W : u \to v$

* Corresponding author.
E-mail addresses: aleksander.malnic@guest.arnes.si (A. Malnič), rok.pozar@upr.si (R. Požar).

[1] This work is supported in part by the Slovenian Research Agency (research program P1-0285).
[2] This work is supported in part by the Slovenian Research Agency (research program P1-0285 and research projects N1-0159, J1-2451, N1-0209 and J5-4596).

https://doi.org/10.1016/j.amc.2025.129695
Received 25 February 2025; Received in revised form 2 August 2025; Accepted 16 August 2025
Available online 2 September 2025

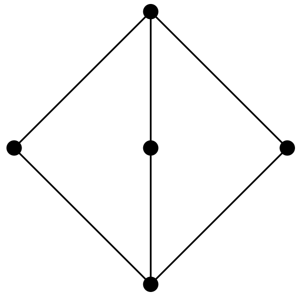

Fig. 1. The Theta graph.

of length $k \geq 0$ from a vertex u to a vertex v is a sequence $W = ue_1e_2 \cdots e_kv$, where $e_1, e_2, \ldots, e_k$ are edges such that for each index j $(1 \leq j < k)$ the edges e_j and e_{j+1} share a common endvertex. A walk of length $k = 0$ is *trivial* and is denoted $u = uu$. The walk $W^{-1} = ve_ke_{k-1} \ldots e_1u$ is the *inverse walk* of W. A walk is *closed* if $u = v$, and is *open* otherwise. For walks $W_1 : u \to v$ and $W_2 : v \to w$ there is naturally defined concatenation of walks $W_1 + W_2 : u \to w$.

In this paper we introduce the concept of a reduced Hamilton walk. An open walk $W = ue_1e_2 \ldots e_kv$ is *reduced* whenever it avoids traversing the same edge forth and back consecutively, that is, if the following condition holds:

$$e_j \neq e_{j+1} \text{for each } j = 1, 2, \ldots, k-1. \tag{R}$$

A closed walk $W = ue_1e_2 \ldots e_ku$ is *reduced* whenever apart from condition (R) we also require that $e_k \neq e_1$. Reduced Hamilton walks have potential applications in real-world transportation problems where U-turns or backtracking are undesirable or prohibited. It is important to note that existing literature on Hamilton walks is not applicable to the problem of finding reduced Hamilton walks. For instance, a shortest reduced Hamilton closed walk in the Theta graph (see Fig. 1) has length 8, while a shortest Hamilton closed walk is of length 6.

While every connected graph admits a Hamilton walk, a reduced Hamilton walk is not guaranteed for all connected graphs. Trees, for example, do not possess reduced Hamilton walks. However, if every vertex in a graph has valency of at least two, the situation changes significantly. The next section formally proves the existence of a reduced Hamilton walk in such graphs (Theorem 2.1). Section 3 offers an alternative proof by presenting an algorithm that explicitly constructs an appropriate walk (Theorem 3.4). Furthermore, we prove an upper bound on the length of a shortest reduced Hamilton walk, demonstrating that it is at most a quadratic function of the number of vertices (Theorem 3.5). Section 4 provides empirical results. The paper concludes with some problems for future research.

2. Reduced Hamilton walks

We now prove our first main result that a graph in which every vertex has valency of at least two has a reduced Hamilton walk.

Theorem 2.1. *Let G be a connected graph (loops and multiple edges allowed) on $n \geq 2$ vertices such that each vertex has valency at least 2. Then for any pair of (not necessary distinct) vertices u and v there exists a reduced Hamilton walk $u \to v$ in G.*

Proof. The proof proceeds by induction on the number of vertices. The claim is obviously true for graphs on two vertices. For the inductive step, consider a connected graph G on $n + 1 \geq 3$ vertices with minimal valency at least 2, and assume that for each such graph on at most n vertices reduced Hamilton walks, closed and open, do exist.

We first show that G has a reduced closed Hamilton walk. Let u be an arbitrary vertex, and let $e_1, e_2, \ldots, e_k$, $k \geq 1$, be the edges incident with u that are not loops. For each $i = 1, 2, \ldots, k$, let $P_i : u \to v_i$ starting with e_i be the longest path with all interior vertices of valency 2 within G. Note that if P_i has no interior vertices, then $P_i = u\, e_i\, v_i$. We consider three cases.

Case 1: $k = 1$. In this case, there is a loop f at u. Let G' be the graph obtained from G by deleting u and all interior vertices of P_1. Then G' is connected, has at most n vertices, and all its vertices have valency at least 2. By induction, G' has a reduced closed Hamilton walk W_1 at v_1. Then, $P_1 + W_1 + P_1^{-1} + f$ is a reduced closed Hamilton walk at u in G.

Case 2: $k = 2$. Let G' be the graph obtained from G by deleting u and all interior vertices of P_1 and P_2. Then G' is a graph on at most n vertices, and all its vertices have valency at least 2. If G' is connected, then, by induction, it has a reduced Hamilton walk W_1 from v_1 to v_2. Thus, $P_1 + W_1 + P_2^{-1}$ is a reduced closed Hamilton walk at u in G. If G' is not connected, it has two components: $G(v_1)$ containing v_1 and $G(v_2)$ containing v_2. By induction, each component has a reduced closed Hamilton walk, W_1 at v_1 and W_2 at v_2, respectively. Consequently, $P_1 + W_1 + P_1^{-1} + P_2 + W_2 + P_2^{-1}$ is a reduced closed Hamilton walk at u in G.

Case 3: $k \geq 3$. We consider two subcases, depending on whether there exists a path P_i with interior vertices or not.

Subcase 3.1: Without loss of generality, suppose that P_1 has interior vertices. Let G' be the graph obtained from G by deleting all interior vertices of P_1. Then G' is a graph on at most n vertices, and all its vertices have valency at least 2. If G' is connected, then, by induction, it has a reduced Hamilton walk W_1 from u to v_1. Hence, $W_1 + P_1^{-1}$ is a reduced closed Hamilton walk at u in G. If G' is not connected, it has two components: $G(u)$ containing u and $G(v_1)$ containing v_1. By induction, each component has a reduced closed Hamilton walk, W_1 at u and W_2 at v_1, respectively. It follows that $W_1 + P_1 + W_2 + P_1^{-1}$ is a reduced closed Hamilton walk at u in G.

Subcase 3.2: Suppose none of the P_i has interior vertices. Let G' be the graph obtained from G by deleting the vertex u. Then G' is a graph on at most n vertices, and all its vertices have valency at least 2. If G' is connected, by induction, it has a reduced Hamilton walk W_1 from v_1 to v_2. So, $e_1 + W_1 + e_2$ is a reduced closed Hamilton walk at u in G. If G' is not connected, it has components $G(v_{i_1}), G(v_{i_2}), \dots, G(v_{i_m})$ containing $v_{i_1}, v_{i_2}, \dots, v_{i_m}$, respectively. By induction, each component $G(v_{i_j})$ has a reduced closed Hamilton walk $W_{i_j} : v_{i_j} \to v_{i_j}$. Then, the walk

$$(P_{i_1} + W_{i_1} + P_{i_1}^{-1}) + (P_{i_2} + W_{i_2} + P_{i_2}^{-1}) + \cdots + (P_{i_m} + W_{i_m} + P_{i_m}^{-1})$$

is a reduced closed Hamilton walk at u in G.

It remains to prove that for any pair of distinct vertices u and v in G there is a reduced Hamilton walk from u to v. By the first part, there is a closed reduced Hamilton walk W at u. But then we can construct a reduced Hamilton walk from u to v by concatenating W at u with the subwalk W_v^u that goes from u to v within W. This completes the proof. □

Remark 2.2. If G has vertices of valency 1, then no reduced Hamilton closed walks exist. As for the open walks, the following holds. If G has at least three vertices of valency 1, then no such walks exist. If G has exactly two vertices of valency 1, say u and v, then there exists a reduced Hamilton walk $u \to v$, however, no reduced Hamilton walks exist between a pair of vertices $\{x, y\} \neq \{u, v\}$. If G has exactly one vertex of valency 1, say u, then for any vertex $v \neq u$ there is a reduced Hamilton walk $u \to v$, and no such walks between any other pair of vertices.

Remark 2.3. While the preceding proof is existential using inductive arguments, in the next section we provide a constructive reproof of Theorem 2.1 by presenting an algorithm that explicitly constructs the desired walk (see the proof of Theorem 3.4).

For an arbitrary pair of distinct vertices u, v in $V(G)$, denote the length of the shortest reduced Hamilton walk $u \to v$ by $d_{\mathrm{sp}}^{\mathrm{r}}(u, v)$. Additionally, for each vertex $v \in V(G)$, set $d_{\mathrm{sp}}^{\mathrm{r}}(v, v) = 0$. Interestingly, it turns out that $d_{\mathrm{sp}}^{\mathrm{r}}$ is a nonstandard discrete metric.

Theorem 2.4. *If G is a connected graph such that each vertex has valency at least 2, then $d_{\mathrm{sp}}^{\mathrm{r}}$ is a metric on G.*

Proof. It is enough to consider graphs on at least 3 vertices, and we only need to prove the triangle inequality

$$d_{\mathrm{sp}}^{\mathrm{r}}(u, w) \leq d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w),$$

where u, v, w are arbitrary pairwise distinct vertices of G.

Let $W_1 : u \to v$ and $W_2 : v \to w$ be shortest reduced Hamilton walks. Let Q be the largest subwalk on W_1 such that W_1 and W_2 are of the form $W_1 = P + Q$ and $W_2 = Q^{-1} + R$.

Case 1: Q is trivial. In this case, the walk $W = W_1 + W_2 : u \to w$ is Hamilton and reduced. Hence $d_{\mathrm{sp}}^{\mathrm{r}}(u, w) \leq |W| = |W_1| + |W_2| = d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$.

Case 2: Q is not trivial. In this case, the walk $W_1 + W_2$ is Hamilton but not reduced. Since all vertices have valency at least 2, there is an edge e incident to v such that e is not the last edge traversed by W_1. Let x be the other endvertex of e. We consider two subcases.

Subcase 2.1: e is not a loop. Since the walk W_1 is Hamilton, it must traverse x. Let $W_1' \subset W_1$ be the initial part of W_1 up to the first visit of the vertex $x \neq v$. Clearly, $|W_1'| \leq |W_1| - 1$. The walk $W = W_1' + e + W_2 : u \to w$ is Hamilton and reduced, and $|W| = |W_1'| + 1 + |W_2| \leq |W_1| + |W_2| = d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$. Hence $d_{\mathrm{sp}}^{\mathrm{r}}(u, w) \leq d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$.

Subcase 2.2: e is a loop. Let $S : v \to w$ be a shortest path. If S itself is not a Hamilton path, consider the walk $W = W_1 + e + S$. Then, W is a reduced Hamilton walk. Since S leaves out at least one vertex distinct from v and w, we have $|S| \leq |W_2| - 2$. Consequently, $|W| = |W_1| + 1 + |S| \leq |W_1| + 1 + |W_2| - 2 < d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$. This implies $d_{\mathrm{sp}}^{\mathrm{r}}(u, w) < d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$, and we are done.

Finally, suppose that S is a Hamilton path. This implies that W_2 is also a Hamilton path, and so $|W_2| = |S|$. Let $S' : u \to v$ be the subpath of S^{-1}. Then $W = S' + e + S : u \to w$ is a reduced Hamilton walk. Since S' leaves out at least one vertex, possibly just w, and since W_1 might be yet another Hamilton path, we have $|S'| \leq |S| - 1 = |W_2| - 1 \leq |W_1| - 1$. Hence $|W| \leq |W_1| - 1 + 1 + |W_2| \leq d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$. Consequently, $d_{\mathrm{sp}}^{\mathrm{r}}(u, w) \leq d_{\mathrm{sp}}^{\mathrm{r}}(u, v) + d_{\mathrm{sp}}^{\mathrm{r}}(v, w)$, as before.

This shows that $d_{\mathrm{sp}}^{\mathrm{r}}$ is indeed a discrete metric on G. □

The minimal length among all reduced Hamilton walks between distinct pair of vertices is given by the parameter

$$\overset{\mathrm{r}}{\underset{\mathrm{sp}}{\min}}(G) = \min_{\substack{u,v \in V \\ u \neq v}} d_{\mathrm{sp}}^{\mathrm{r}}(u, v),$$

while

$$\overset{\mathrm{r}}{\underset{\mathrm{sp}}{\max}}(G) = \max_{\substack{u,v \in V \\ u \neq v}} d_{\mathrm{sp}}^{\mathrm{r}}(u, v)$$

tells that between any pair of distinct vertices there is a reduced Hamilton walk of length at most $\max_{\mathrm{sp}}^{\mathrm{r}}(G)$. Additionally, let

$$c_{\mathrm{r}}^{\mathrm{sp}}(G)$$

denote the length of a shortest reduced Hamilton closed walk in G. This number is clearly uniquely defined. In Table 1, we illustrate these parameters for some standard graphs.

Consider a connected graph G on n vertices such that each vertex has valency at least 2. Then

$$n - 1 \leq \overset{\mathrm{r}}{\underset{\mathrm{sp}}{\min}}(G) \leq \overset{\mathrm{r}}{\underset{\mathrm{sp}}{\max}}(G).$$

Table 1
Parameters $\min^{r}_{sp}(G)$, $\max^{r}_{sp}(G)$ and $c^{sp}_{r}(G)$ for some standard graphs.

Graph G	$\min^{r}_{sp}(G)$	$\max^{r}_{sp}(G)$	$c^{sp}_{r}(G)$
Cyclic graph C_n	$n-1$	$\lfloor 3n/2\rfloor$	n
Complete graph K_n	$n-1$	$n-1$	n
Petersen graph P	9	10	12
Complete bipartite graph $K_{2,n}$	$2n-2$	$2n$	$2n+2$

Moreover, $\min^{r}_{sp}(G) = n-1$ is equivalent to G having a Hamilton path while $\max^{r}_{sp}(G) = n-1$ is equivalent to G being Hamilton-connected. Also, $c^{sp}_{r}(G) \geq n$, where the equality holds if and only if G has a Hamilton cycle.

In general, these parameters are related by the following inequalities.

Theorem 2.5. *If G be a connected graph such that each vertex has valency at least 2, then*
$\max^{r}_{sp}(G) \leq \left\lfloor \frac{3}{2}\, c^{sp}_{r}(G)\right\rfloor$ *and* $\max^{r}_{sp}(G) \leq 2\min^{r}_{sp}(G) + 1$.

Proof. To prove the first inequality, let W be a reduced closed Hamilton walk of minimal length $c^{sp}_{r}(G)$, and let $u \neq v$ be arbitrary vertices. Denote by W^u_v, the subwalk of W that goes from u to v. Without loss of generality, we may assume that the length of W^u_v is less than or equal to $c^{sp}_{r}(G)/2$. The length of the walk $u \to v$ consisting of W at u and the walk W^u_v is bounded above by $3c^{sp}_{r}(G)/2$, which proves the claim.

To prove the second inequality, we may without loss of generality assume that $n \geq 4$. Let $u \neq v$ be arbitrary vertices, and let $W^u_v : u \to v$ be a reduced Hamilton walk of minimal length $\min^{r}_{sp}(G)$. Next, let $x \neq y$ be arbitrary vertices in G. Without loss of generality, we may assume that the vertices u, x, y, v appear in this order when traversing the chosen walk W^u_v. Next, let e be an edge with endvertices u, α which is not the first edge traversed by W^u_v, and let f be an edge with endvertices β, v which is not the last edge traversed by W^u_v. We show that there is a reduced Hamilton walk from x to y of length less than $2\min^{r}_{sp}(G)$ in the subgraph of G formed by W^u_v together with the edges e and f. To this end, we consider the 12 cases according to where on W^u_v the vertices α and β are located.

Case 1: $u, \beta, \alpha, x, y, v$. If $\beta = u$, $\alpha = x$ and $y = v$, take $W^x_y + f + W^u_y$ and the case is clear. Otherwise, take the walk $W^x_v + f + W^\beta_\alpha + e + W^u_y$. Its length is equal to $|W^u_v + e + f + W^\beta_\alpha + W^x_y| = |W^u_v| + 2 + |W^\beta_\alpha + W^x_y| \leq 2|W^u_v| + 1$.

Case 2: $u, \alpha, \beta, x, y, v$. If $\alpha = u$, $\beta = x$ and $y = v$, take $W^x_u + e + W^u_v$, and the case is clear. Otherwise, take $W^x_v + f + W^\beta_\alpha + e + W^u_y$. This case is similar to the general case in Case 1.

Case 3: $u, \alpha, x, \beta, y, v$. If $\alpha = u$, $\beta = x$ and $y = v$, take $W^x_u + e + W^u_v$, and the case is clear. Otherwise, take $W^x_\alpha + e + W^u_v + f + W^\beta_y$. Its length is equal to $|W^u_v + e + f + W^\alpha_x + W^\beta_y| = |W^u_v| + 2 + |W^\alpha_x + W^\beta_y| \leq 2|W^u_v| + 1$.

Case 4: $u, \alpha, x, y, \beta, v$. Take $W^x_\alpha + e + W^u_v + f + W^\beta_y$. Its length is equal to $|W^u_v + e + f + W^\alpha_x + W^\beta_y| = |W^u_v| + 2 + |W^\alpha_x + w^\beta_y| \leq 2|W^u_v| + 1$, since $x \neq y$.

Case 5: $u, \beta, x, \alpha, y, v$. If $\beta = u$, $\alpha = x$ and $y = v$, take $W^x_y + f + W^u_v$, and the case is clear. Otherwise, take the walk $W^x_\beta + f + W^v_\alpha + e + W^u_y$. Its length is equal to $|W^u_v + e + f + W^\beta_x + W^\alpha_y| = |W^u_v| + 2 + |W^\beta_x + W^\alpha_y| \leq 2|W^u_v| + 1$.

Case 6: $u, x, \beta, \alpha, y, v$. If $x = u$, $\alpha = \beta$ and $y = v$, take W^u_v, and the case is clear. Otherwise, take $W^x_\beta + f + W^v_y + W^y_\alpha + e + W^u_y$. Its length is equal to $|W^u_v + e + f + W^x_\beta + W^\alpha_y| = |W^u_v| + 2 + |W^x_\beta + W^\alpha_y| \leq 2|W^u_v| + 1$.

Case 7: $u, x, \alpha, \beta, y, v$. If $x = u$, $\alpha = \beta$ and $y = v$, take W^u_v, and the case is clear. Otherwise, take $W^x_\alpha + e + W^u_v + f + W^\beta_y$. Its length is equal to $|W^u_v + e + f + W^x_\alpha + W^\beta_y| = |W^u_v| + 2 + |W^x_\alpha + W^\beta_y| \leq 2|W^u_v| + 1$.

Case 8: $u, x, \alpha, y, \beta, v$. If $x = u$, $\alpha = y$ and $\beta = v$, take $W^u_v + f + W^v_y$, and the case is clear. Otherwise, take $W^x_\alpha + e + W^u_v + f + W^\beta_y$. This case is similar to the general case in Case 7.

Case 9: $u, \beta, x, y, \alpha, v$. Take $W^x_\beta + f + W^v_u + e + W^\alpha_y$. Its length is equal to $|W^v_u + e + f + W^x_\beta + W^\alpha_y| = |W^u_v| + 2 + |W^x_\beta + W^\alpha_y| \leq 2|W^u_v| + 1$, since $x \neq y$.

Case 10: $u, x, \beta, y, \alpha, v$. If $x = u$, $\alpha = v$ and $\beta = y$, take $W^u_v + f + W^\beta_y$, and the case is clear. Otherwise, take $W^x_\beta + f + W^v_u + e + W^\alpha_y$. Its length is equal to $|W^v_u + e + f + W^\alpha_y + W^\beta_x| = |W^u_v| + 2 + |W^\alpha_y + W^\beta_x| \leq 2|W^u_v| + 1$.

Case 11: $u, x, y, \beta, \alpha, v$. If $x = u$, $\alpha = v$ and $\beta = y$, take $W^u_v + e$, and the case is clear. Otherwise, take $W^x_v + f + W^\beta_\alpha + e + W^u_y$. Its length is equal to $|W^v_u + e + f + W^x_y + W^\beta_\alpha| = |W^u_v| + 2 + |W^x_y + W^\beta_\alpha| \leq 2|W^u_v| + 1$.

Case 12: $u, x, y, \alpha, \beta, v$. If $x = u$, $\alpha = y$ and $\beta = v$, take $W^u_v + f + W^v_y$, and the case is clear. Otherwise, take $W^x_v + f + W^\beta_\alpha + e + W^u_y$. This case is similar to the general case in Case 11. □

In particular, by Theorem 2.5, if G on n vertices has a Hamilton cycle, then $\max^{r}_{sp}(G) \leq \lfloor 3n/2\rfloor$. When G has a Hamilton path, then $\max^{r}_{sp}(G) \leq 2n-1$.

3. Construction of a reduced Hamilton walk

Let G be a finite connected graph with at least two vertices, where all vertices have valency at least 2. For a pair of not necessarily distinct vertices u and v in G, we present in Section 3.1 an algorithm for constructing a reduced Hamilton walk $W : u \to v$ in G. An illustrative example is provided in Section 3.2, followed by analysis of the algorith's time complexity in Section 3.4. In Section 3.5, we establish upper bounds on the length of the shortest reduced Hamilton walks.

Before describing the algorithm, we introduce some definitions and notation. Let T be a tree. We denoted the unique path from a vertex α to a vertex β in T by T^{α}_{β}. If $\alpha = \beta$, $T^{\alpha}_{\alpha} = \alpha$ is the trivial walk at α. If T is a spanning tree of G, and edge e of $G \setminus T$ is called a *T-cotree edge*. A *fundamental cycle* in G determined by a T-cotree edge e is the cycle formed by T and e. (Note that this is not necessarily a fundamental closed walk rooted at u.)

3.1. Algorithm description

The algorithm, presented in the procedure Reduced-Spanning-Walk, begins by constructing a spanning tree T of G, rooted at u (line 1). The algorithm iteratively extends W (lines 9-18) from a trivial walk at root (line 7).

At each step, an unvisited leaf α of T is selected (lines 9-10). The function Extend-Walk extends W to α using a path in T (and possibly edges of the fundamental cycle determined by the last T-cotree edge f in W) while maintaining condition (R) (line 12). A T-cotree edge e incident to α (and some β) is added to W, and the walk is further extended along a path in T such that: first, it includes all edges of the fundamental cycle determined by e, and second, it ends at a vertex on this cycle (lines 13-17). This ensures that all vertices of T^{u}_{α} and T^{u}_{β} are visited, and condition (R) can be maintained for the next extension.

Algorithm 1: Extend-Walk(G, T, W, f, α).

Input: A finite graph G represented by the adjacency list Adj[w] for each $w \in V(G)$; a spanning tree T of G represented by the parent array, a walk W of G, the last T-cotree edge f in W – if it exists, or NIL otherwise; a vertex α in G.

Output: An extension of W to α, using a path in T – and, if necessary, edges from the fundamental cycle induced by the last T-cotree edge f traversed in W – while preserving condition (R).

Let y be the last vertex in W;
if $f =$ NIL **then**
  return $W + T^{y}_{\alpha}$;
Let γ and δ be end-vertices of f;
Suppose that W traverses f from γ to δ;
Let r be the last common vertex shared by T^{y}_{δ} and T^{y}_{α};
if $y \neq r$ **then**
  $W := W + T^{y}_{\gamma} + f + T^{\delta}_{\alpha}$;
else
  $W := W + T^{y}_{\alpha}$;
return W;

Once all leaves have been visited, W becomes a Hamilton walk satisfying condition (R). Finally, W is extended to v to create a reduced Hamilton walk $W : u \to v$ (19-27). When constructing a closed walk, T is first rerooted to ensure a T-cotree edge is incident to the root (lines 2-5). This is necessary to fulfill the condition that the first edge of the walk differs from the last.

3.2. An illustrative example

To better understand the Reduced-Spanning-Walk algorithm, we illustrate its execution on the Theta graph shown in Fig. 1, aiming to construct a reduced Hamilton walk $1 \to 2$. We begin by building a breadth-first search spanning tree T, rooted at $u = 1$. The tree T has three leaves: 3, 4 and 5. The two cotree-edges in the graph are $\{3, 5\}$ and $\{4, 5\}$. See Fig. 2.

Since the Theta graph is simple, we represent walks as sequences of vertices traversed. In the first iteration, we select vertex 3 as the first unvisited leaf and extend the trivial walk $W = 1$ to this leaf using only tree edges (line 12). This yields the walk $W = 1, 3$. See Fig. 2(a). The T-cotree edge incident to 3 is $e = \{3, 5\}$. We extend the walk W via this edge and then follow the path in T back to the root $x = 1$ (line 17), resulting in the walk $W = 1, 3, 5, 2, 1$. See Fig. 2(b). This completes the first iteration, during which leaves 3 and 5 are visited.

In the second iteration, the remaining unvisited leaf is vertex 4. We extend the walk to this leaf (line 12), giving $W = 1, 3, 5, 2, 1, 4$. See Fig. 2(c). The T-cotree edge incident to 4 is $e = \{4, 5\}$, so we extend the walk through this edge to reach vertex $x = 5$ (line 17), resulting in $W = 1, 3, 5, 2, 1, 4, 5$. See Fig. 2(d).

At this point, the core of the algorithm is complete, and W is a reduced Hamilton walk $W : 1 \to 5$. The final step is to extend this walk to the target vertex $v = 2$ (line 24), yielding the required reduced Hamilton walk $W = 1, 3, 5, 2, 1, 4, 5, 2$.

3.3. Algorithm correctness

Before proving the correctness of the algorithm Reduced-Spanning-Walk, we establish some essential properties of the function Extend-Walk. The proofs of the following two lemmas are straightforward.

Algorithm 2: REDUCED-SPANNING-WALK(G, u, v).

Input: A finite graph G satisfying conditions from Theorem 2.1, represented by the adjacency list Adj$[w]$ for each $w \in V(G)$; vertices u, v in G.
Output: A reduced Hamilton walk $u \to v$ in G, if $u \neq v$, or a reduced Hamilton closed walk otherwise.

Build a u-rooted spanning tree T, represented by the parent array;
if $u = v$ **then**
  Let e' be any T-cotree edge and let w be one of its endvertices;
  Reroot T so that w becomes the new root of T;
  $u := w, v := w$;
Mark every vertex in T as *univsited*;
$W := u$; // trivial walk at u
$f :=$ NIL; // the last T-cotree edge traversed by W
foreach leaf α in T **do**
  if α is *unvisited* **then**
    Mark α as *visited*;
    $W :=$ EXTEND-WALK(G, T, W, f, α);
    Pick any T-cotree edge e incident to α;
    Let β be the other endvertex of e;
    Mark β as *visited*;
    Let x be the last vertex on the path T_β^u that belongs to W;
    $W := W + e + T_x^\beta$;
    $f := e$;
if $u = v$ **then**
  Let β be the other endvertex of e';
  $\alpha := \beta$;
else
  $\alpha := v$;
$W :=$ EXTEND-WALK(G, T, W, f, α);
if $u = v$ **then**
  $W := W + e'$;
return W;

Lemma 3.1. *Let $W : u \to y$ be a walk with the following property:*

If W traverses a vertex s, it traverses all vertices of T_s^u. (C)

Then the walk $u \to \alpha$ returned by EXTEND-WALK(G, T, W, f, α) *has the same property (C).*

Lemma 3.2. *If W does not end at vertex α, then the last edge of the walk returned by* EXTEND-WALK(G, T, W, f, α) *belongs to T.*

In general, the walk returned by the algorithm EXTEND-WALK may not satisfy condition (R). It does, however, if W possesses a certain property. A sufficient condition (and also necessary) is provided in the following lemma.

Lemma 3.3. *Let W be a walk that satisfies condition (R) and includes at least one T-cotree edge. If f is the last T-cotree edge traversed by W, and W ends at a vertex on the fundamental cycle determined by f, then the walk returned by* EXTEND-WALK(G, T, W, f, α) *also satisfies condition (R).*

Proof. Let W start at u and end at y (line 1), and let f be the last T-cotree edge traversed by W. By assumption, y lies on the fundamental cycle determined by f. Let γ and δ be the end-vertices of edge f, and assume W traverses f from γ to δ (lines 4-5). Next, let r be the last common vertex shared by T_δ^y and T_α^y (line 6). To prove that the returned walk satisfies condition (R), we distinguish two cases: $y \neq r$ and $y = r$.

Case 1: $y \neq r$ (line 8). In this case, $\delta \neq y$, and hence f is not a loop. Combined with the fact that f is the last T-cotree edge traversed by W, this implies that W can be expressed as $W = W_\gamma^u + f + T_y^\delta$, where W_γ^u denotes the subwalk of W that goes from u to γ. See Fig. 3(a). Consequently, the returned walk takes the form $W + T_\gamma^y + f + T_\alpha^\delta = W_\gamma^u + f + T_y^\delta + T_\gamma^y + f + T_\alpha^\delta$. See Fig. 3(b). To show that this walk satisfies condition (R), it suffices to show that its subwalk $T_y^\delta + T_\gamma^y + f$ is reduced. However, this subwalk is precisely the fundamental cycle determined by f (and hence reduced), as y is assumed to lie on this cycle.

Case 2: $y = r$ (line 10). We further divide this case into two subcases: $\delta \neq y$ and $\delta = y$.

Subcase 2.1: $\delta \neq y$. In this subcase, W can again be expressed as $W = W_\gamma^u + f + T_y^\delta$, using the same reasoning as in *Case 1*. See Fig. 4(a). Consequently, the returned walk takes the form $W + T_\alpha^y = W_\gamma^u + f + T_y^\delta + T_\alpha^y$. See Fig. 4(b). To show that this walk satisfies condition (R), it suffices to show that its subwalk $T_y^\delta + T_\alpha^y$ is reduced. This is indeed true, as y is the only common part of T_y^δ and T_α^y.

Subcase 2.2: $\delta = y$. In this subcase, W can be expressed as $W = W_\gamma^u + f$, and the returned walk becomes $W + T_\alpha^y = W_\gamma^u + f + T_\alpha^y$, which clearly satisfies condition (R). See Fig. 5. This completes the proof. □

We now prove the correctness of the algorithm Reduced-Spanning-Walk.

Theorem 3.4. *The algorithm Reduced-Spanning-Walk(G, u, v) correctly finds a reduced Hamilton walk from u to v in a connected finite graph G with at least two vertices, where all vertices have valency at least two.*

Proof. We first show the following loop invariant: The **foreach** loop in lines 6-18 maintains the following three-part invariant at the end of each iteration of the loop:

(i) The walk W satisfies condition (R).
(ii) The walk W includes at least one T-cotree edge and ends at a vertex on the fundamental cycle determined the last T-cotree edge traversed by W.

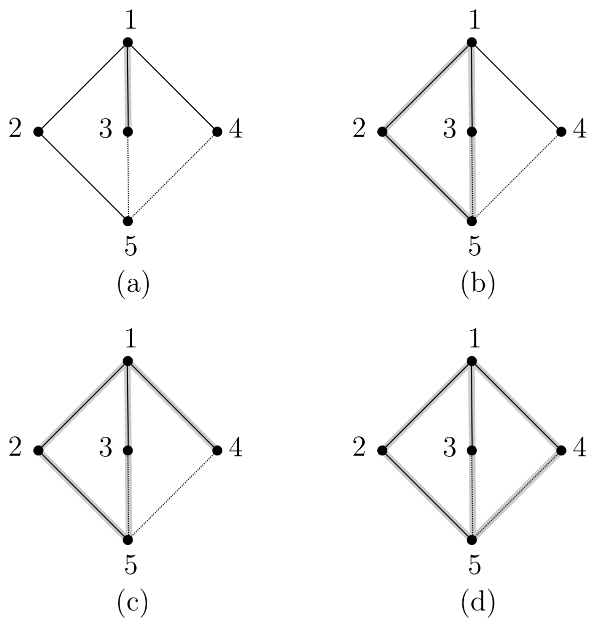

Fig. 2. Illustration of the core of the Reduced-Spanning-Walk algorithm (lines 9-18). Cotree edges are shown as dashed lines, while the walk constructed so far is highlighted with shaded edges. (a) First iteration: The walk starts at vertex 1 and is extended to the unvisited leaf 3 (line 12), resulting in $W = 1, 3$. (b) First iteration (continued): The walk is further extended through the cotree edge $e = \{3, 5\}$ to the root 1 (line 17), yielding $W = 1, 3, 5, 2, 1$. (c) Second iteration: The walk continues to the next unvisited leaf 4 (line 12), resulting in $W = 1, 3, 5, 2, 1, 4$. (d) Second iteration (continued): The walk is further extended through the cotree edge $e = \{4, 5\}$ to vertex 5 (line 17), producing $W = 1, 3, 5, 2, 1, 4, 5$.

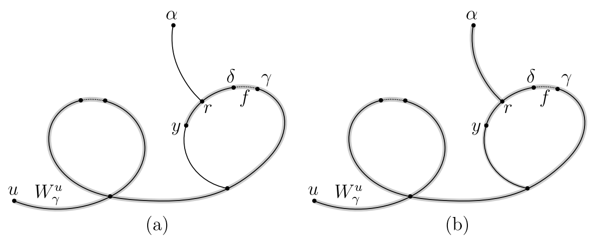

Fig. 3. Case 1: $y \neq r$. Cotree edges are dashed. (a) The walk $W = W_\gamma^u + f + T_y^\delta$ shown by shaded walks. (b) The returned walk $W_\gamma^u + f + T_y^\delta + T_\gamma^y + f + T_\alpha^\delta$.

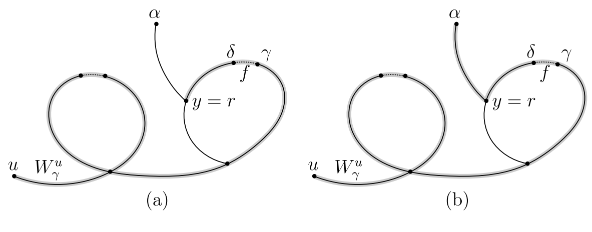

Fig. 4. Subcase 2.1: $y = r$ and $\delta \neq y$. Cotree edges are dashed. (a) The walk $W = W_\gamma^u + f + T_y^\delta$ shown by shaded walks. (b) The returned walk $W_\gamma^u + f + T_y^\delta + T_\alpha^y$ shown by shaded walks.

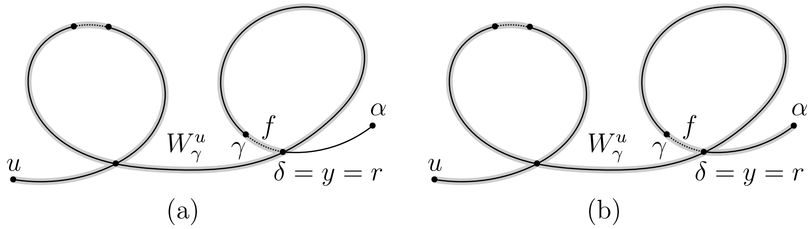

Fig. 5. Subcase 2.2: $\delta = y = r$. Cotree edges are dashed. (a) The walk $W = W^u_\gamma + f$ shown by shaded walks. (b) The returned walk $W^u_\gamma + f + T^y_\alpha$.

(iii) The walk W satisfies condition (C).

After the first iteration, $W = T^u_\alpha + e + T^\beta_x$, where e is a T-cotree edge with endvertices α and β. Clearly, W satisfies property (i). As for (ii), note that x is the last common vertex shared by T^u_α and T^u_β and hence the walk $T^x_\alpha + e + T^\beta_x$ represents the fundamental cycle determined by e, proving (ii). Property (iii) is evident.

We now show that the loop invariant holds after each iteration. We use W to denote the walk at the end of the previous iteration, and W^* to denote the walk at the end of the current iteration. Inductively we assume that properties (i)-(iii) hold for W before the current iteration.

Let α denote the unvisited leaf in the current iteration (lines 9-10), and let W^u_α be the walk returned by the function Extend-Walk(G, T, W, f, α) (line 12). By assumption, W satisfies condition (R), includes at least one T-cotree edge, and ends at a vertex on the fundamental cycle determined the last T-cotree edge f traversed by W. Consequently, W^u_α satisfies condition (R) by Lemma 3.3. Additionally, since α was previously unvisited vertex it differs from the last vertex of W, Lemma 3.2 implies that the last edge of W^u_α belongs to T. Furthermore, by Lemma 3.1, W^u_α satisfies condition (C), as W is assumed to satisfy condition (C). Let e be a T-cotree edge with endvertices α and β, selected in the current iteration (lines 13-14).

Let x be the last vertex on the path T^u_β that belongs to W^u_α (line 16). See Fig. 6(a). Then, $W^* = W^u_\alpha + e + T^\beta_x$ (line 17). See Fig. 6(b). Since W^u_α satisfies condition (R) and its last edge belongs to T, the walk W^* also satisfies condition (R), and property (i) holds. To prove (ii), we first note that x is the last vertex of W^*. We also observe that e is the last T-cotree edge traversed by W. The fundamental cycle determined by e takes the form $T^r_\alpha + e + T^\beta_r$, where r is the last common vertex shared by T^u_α and T^u_β. As W^u_α traverses α and satisfies condition (C), it covers all vertices of T^u_α. Combined with the fact that x is the last vertex on the path T^u_β that belongs to W^u_α, the path T^β_x is a subgraph of the path T^β_r. Hence, x lies on the fundamental cycle determined by e, proving (ii). Finally, since W^u_α satisfies condition (C), it follows that W^* also satisfies condition (C), proving (iii).

We now show that the walk returned by Reduced-Spanning-Walk is a reduced Hamilton walk $u \to v$. Let W denoted the walk after the loop terminates. By construction, this walk starts at u, traverses all the leaves of T, and, by the above loop invariant, satisfies property (iii). This implies that W is a Hamilton walk starting at u, and hence the walk returned by Reduced-Spanning-Walk is also such a walk.

Finally, we prove that the walk returned by Reduced-Spanning-Walk ends at v and is reduced. We distinguish two cases: $u \neq v$ and $u = v$.

Case 1: $u \neq v$. In this case, the walk returned by the algorithm Reduced-Spanning-Walk is identical to the walk returned by the function Extend-Walk(G, T, W, f, v) (line 24), where f represents the last T-cotree edge traversed by W. This latter walk evidently terminates at v. Furthermore, given that by the loop invariant W fulfills properties (i) and (ii), Lemma 3.3 implies that the walk returned by Extend-Walk(G, T, W, f, v) satisfies condition (R), and is therefore reduced (since $u \neq v$).

Case 2: $u = v$. Considering that the initial edge of the walk returned by the algorithm Reduced-Spanning-Walk is part of T while the terminal edge is not, this case is addressed in a manner analogous to the preceding case. This completes the proof. □

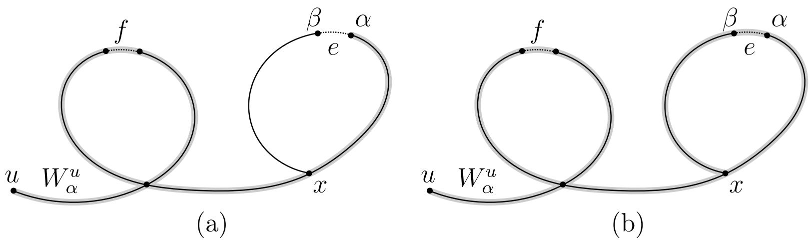

Fig. 6. The construction of W^*. Cotree edges are dashed. (a) The walk W^u_α shown by shaded walks. (b) The resulting walk $W^* = W^u_\alpha + e + T^\beta_x$.

3.4. Time complexity analysis

Theorem 3.5. *Let G be a connected graph with n vertices and m edges, where each vertex has valency of at least 2. Then, the time complexity of* REDUCED-SPANNING-WALK(G, u, v) *is* $O(n^2 + m)$.

Proof. The construction of the spanning tree T (represented by the parent array) in line 1 requires $O(n + m)$, which can be achieved using breadth-first search. Lines 2-8 take $O(n)$. Each call to EXTEND-WALK depends on the depth of a spanning tree T, and hence takes $O(n)$ time. Since T is represented by the parent array, determining if an edge belongs to T can be done in constant time. Therefore, each iteration of the loop takes $O(n)$ time. With $l \le n - 1$ loop iterations, the overall loop execution time is $O(n^2)$ (lines 9-18). Lines 19-23 take constant time, line 24 takes $O(n)$ time, and lines 26-27 require constant time. Thus, the total running time of the algorithm is $O(n^2 + n + m)$. □

3.5. Upper bounds on the length of a shortest reduced Hamilton walk

The following result establishes an upper bound on the length of a shortest reduced Hamilton walk in terms of the number of vertices.

Proposition 3.6. *In a connected graph on $n \ge 2$ vertices, where each vertex has valency at least 2, the length of a shortest reduced Hamilton walk, either closed or open, is at most $n(n + 3)/2$.*

Proof. For a graph G satisfying the conditions in Proposition 3.6, we show that REDUCED-SPANNING-WALK(G, u, v) returns a walk $u \to v$ with a length of at most $n(n + 3)/2$. Let W denote the walk at the beginning of the current iteration, and let $W + W'$ be the walk at the end of the iteration. Observe that W' without the last edge is a path, and so at most the last vertex in W' is repeated. Furthermore, in the worst case, W' includes only one previously unvisited leaf. Therefore, if there are i unvisited leaves before the current iteration, then W' without the last edge has a length of at most $n - i$. Summing over all $l \le n - 1$ iterations, the total length of the walks added during the loop is at most

$$\sum_{i=1}^{l}(n - i + 1) \le \sum_{i=1}^{n-1}(n - i + 1) < n(n + 1)/2.$$

Lines 24 and 26 together increase the length of the walk by at most n. Therefore, the final length of the walk is at most $n(n + 3)/2$. □

An alternative upper bound on the length of a shortest reduced Hamilton walk can be expressed in terms of a graph diameter. To this end we first prove the following lemma.

Lemma 3.7. *Let G be a connected graph with diameter Δ, and let T be a breadth-first search spanning tree. Then the diameter Δ' of T is bounded above by 2Δ.*

Proof. Let T be rooted at a vertex u, and let $P : v \to w$ be the longest path in T. Note that P must traverse the vertex u, and so $P = T_u^v + T_w^u$. Since T is a breadth-first search spanning tree, the paths T_u^v and T_w^u are shortest paths in G. Hence, the lengths $|T_u^v|$ and $|T_w^u|$ are bounded by Δ. As $\Delta' = |T_u^v| + |T_w^u|$, the result follows. □

Proposition 3.8. *Let G be a connected graph on $n \ge 2$ vertices such that each vertex has valency at least 2, and let Δ denote the diameter of G. Then the length of a shortest reduced Hamilton walk, either closed or open, is at most $(6n - 2)\Delta + 2n$.*

Proof. We show that REDUCED-SPANNING-WALK(G, u, v) returns a walk $u \to v$ of the desired length. To this end, we assume that a breadth-first search is used to construct the spanning tree T in line 1. Let Δ' denote the diameter of T. By Lemma 3.7, $\Delta' \le 2\Delta$. Each call to EXTEND-WALK increases the length of the walk by at most $2\Delta' + 1 \le 4\Delta + 1$ (line 8). Line 17 can further increase the length by at most $\Delta' + 1 \le 2\Delta + 1$. Therefore, each loop iteration increases the length by at most $6\Delta + 2$. With at most $n - 1$ loop iterations, the overall length of the walk after the for loop is $(n - 1)(6\Delta + 2)$. Lines 19-27 can further increase the length by at most $2\Delta' + 2 \le 4\Delta + 2$. Thus, the final walk length is at most $(6n - 2)\Delta + 2n$. □

For certain classes of graphs we can provide a tighter bound.

Corollary 3.9. *Let G be a graph on n vertices belonging to a family of regular expander graphs. Then the length of a shortest reduced Hamilton walk, either closed or open, in G is at most $c(6n - 2)\log n + 2n$, where c is a constant independent of n.*

Proof. The diameter of G is at most $c \log n$ for some constant c, independent of n [12, Corollary 4.8]. The result then directly follows from Proposition 3.8. □

4. Experimental evaluation

This section presents empirical results for the REDUCED-SPANNING-WALK algorithm applied to real-world networks. To compute the spanning tree T in line 1, we used a breadth-first search approach. We implemented the algorithm in the GAP system [13], and all experiments were conducted on a computer equipped with an Apple M3 Pro processor, 18 GB of RAM, running macOS Sonoma 14.6.1.

We performed two experiments on real-world networks sourced from the *Stanford Large Network Dataset Collection* (SNAP) [14]. The first experiment used graphs from the *Collaboration Networks* collection, which model scientific collaborations between authors

Table 2
Results on collaboration networks. For each graph, all presented structural and computational metrics are based on the 2-core of its largest connected component.

Graph	n	m	Length	Time (ms)
Astro Physics	16,910	195,978	88,086	1488
Condensed Matter	19,606	89,529	92,653	2029
General Relativity	3413	12,677	13,859	76
High Energy Physics	10,051	116,466	46,063	360
High Energy Physics Theory	7059	23,227	33,138	195

Table 3
Results on road networks. For each graph, all presented structural and computational metrics are based on the 2-core of its largest connected component.

Graph	n	m	Length	Time (min)
California	1,589,938	2,393,299	12,111,576	649
Pennsylvania	873,219	1,327,171	7,168,493	246
Texas	1,068,728	1,596,792	7,939,195	295

[15]. In these graphs, if author i co-authored a paper with author j, there is an edge between i and j. The second experiment used graphs from the *Road Networks* collection, which represent road infrastructure [16]. Here, vertices correspond to intersections or endpoints, and edges represent the roads connecting them.

To ensure that each evaluated graph was connected and that every vertex had valency at least 2, we applied a two-step preprocessing procedure to each original graph from the dataset. First, we extracted the largest connected component of the graph. Next, we computed the 2-core of this component, which is the maximal subgraph in which every vertex has degree at least 2 [17]. The resulting 2-core subgraphs were used in all experimental evaluations.

The results are shown in Table 2 for the Collaboration Networks and in Table 3 for the Road Networks. In each table, the first column lists the names of the tested graphs, followed by two columns that represent their structural properties (number of vertices and edges). The final two columns report the length of the computed reduced closed Hamiltonian walk and the CPU time required for its computation-expressed in milliseconds for the Collaboration Networks and in minutes for the Road Networks.

5. Concluding remarks

We have shown that the length of a shortest reduced Hamilton walk in connected graph on n vertices with minimal valency at least 2 is at most $n(n+3)/2$. Interestingly, our experimental results suggest that, in practice, the observed lengths are substantially shorter than this worst-case upper bound. It is worth noting that the graphs used in our experiments were sparse, which may have influenced the results. To better understand whether the practical behavior of the walk length scales more closely with a quadratic or linear function of n, a more comprehensive experimental analysis is needed.

We propose some problems that may be of interest for future research.

Problem 5.1. Is there an infinite family of graphs with shortest reduced Hamilton walks of lengths that are quadratic in the number of vertices?

Problem 5.2. Is the length of a shortest reduced Hamilton walk in a planar graph linear in the number of vertices?

Data availability

No data was used for the research described in the article.

Acknowledgments

The authors thank the anonymous referees for their valuable and constructive comments, which greatly improved the overall quality of the paper.

References

[1] S. Goodman, S. Hedetniemi, On Hamilton walks in graphs, SIAM J. Comput. 3 (1974) 214–221.
[2] J.C. Bermond, On Hamilton walks, Congr. Numer. 15 (1976) 41–51.
[3] K. Takamizawa, T. Nishizeki, N. Saito, An algorithm for finding a short closed spanning walk in a graph, Networks 10 (1980) 249–263.
[4] T. Asano, T. Nishizeki, T. Watanabe, An upper bound on the length of a Hamilton walk of a maximal planar graph, J. Graph Theory 4 (1980) 315–336.
[5] T.M. Lewis, On the Hamilton number of a plane graph, Discussiones Math. Graph Theory 39 (2019) 171–181.

[6] T. Nishizeki, T. Asano, T. Watanabe, An approximation algorithm for the Hamilton walk problem on maximal planar graphs, Discrete Appl. Math. 5 (1983) 211–222.
[7] S. Thaithae, N. Punnim, The Hamilton number of cubic graphs, Lect. Notes Comput. Sci. 4535 (2008) 213–223.
[8] T.P. Chang, L.D. Tong, The Hamilton numbers in digraphs, J. Comb. Optim. 25 (2014) 694–701.
[9] L.D. Tong, H.Y. Yang, Hamiltonian numbers in oriented graphs, J. Comb. Optim. 34 (2017) 1210–1217.
[10] H. Yang, J. Liu, J. Meng, Antidirected spanning closed trail in tournaments, Appl. Math. Comput. 462 (2024) 128336.
[11] G.J. Chang, T.P. Chang, L.D. Tong, Hamilton numbers of Möbius double loop networks, J. Comb. Optim. 23 (2012) 462–470.
[12] M. Krebs, A. Shaheen, Expander Families and Cayley Graphs: A Beginner's Guide, Oxford, Oxford University Press, 2011.
[13] The GAP Group, GAP - groups, algorithms, and programming, Version 4.11.0, 2020. http://www.gap-system.org.
[14] J. Leskovec, A. Krevl, SNAP datasets: Stanford large network dataset collection, 2014. https://snap.stanford.edu/data/#p2p.
[15] J. Leskovec, J. Kleinberg, C. Faloutsos, Graph evolution: densification and shrinking diameters, ACM Trans. Knowl. Discov. Data (ACM TKDD) 1 (2007) Article 2, 41 pp.
[16] J. Leskovec, K. Lang, A. Dasgupta, M. Mahoney, Community structure in large networks: natural cluster sizes and the absence of large well-defined clusters, Internet Math. 6 (2009) 29–123.
[17] V. Batagelj, M. Zaveršnik, An $o(m)$ algorithm for cores decomposition of networks, 2003. https://arxiv.org/abs/cs/0310049.