A Study on Optimistic & Pessimistic Pareto-fronts in Multiobjective Bilevel Optimization via δ-Perturbation

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Abstract. In bilevel optimization, an upper level (UL) decision maker seeks to optimize an objective function while considering the optimal solutions of a lower level (LL) optimization problem. This hierarchical structure poses modeling and solution challenges, especially when the LL problem has multiple solutions. In such a case, the UL needs to make assumptions about the LL reaction. In the optimistic approach, the UL assumes that the LL reaction will be favorable, while in the pessimistic approach the opposite is true. In this study, we consider the case of a multiobjective bilevel optimization problem, where the UL has multiple objectives, while the LL has a single objective, but multiple optimal solutions for any given UL decision. Given that the LL can choose any solution from its optimal set, and in case the UL is not aware of the LL choice function, it leads to the possibility of two Pareto-optimal fronts at the UL, i.e. the optimistic and the pessimistic frontiers. To approximate both Pareto-optimal fronts, a δ-perturbation approximation is proposed in this paper, where the LL objective is perturbed by a small δ by utilizing the UL objectives at the LL. The perturbed reformulated bilevel problem is then solved via an extension of m-BLEAQ, an evolutionary bilevel algorithm that can deal with bilevel problems with multiple objectives at the UL and a single objective at the LL. The application of the m-BLEAQ algorithm to the reformulated bilevel problem leads to the identification of the optimistic and pessimistic frontiers for the multiobjective bilevel problem. In this proof-of-concept study, the proposed reformulation strategy is demonstrated on two test problems, showing the effectiveness of the proposal in accurately finding the optimistic and pessimistic frontiers.

Keywords: Multiobjective bilevel optimization · Optimistic approach · Pessimistic approach · Evolutionary multiobjective optimization

The Version of Record of this contribution is published in Lecture Notes in Computer Science (LNCS, volume 15512), and is available online at https://doi.org/10.1007/978-981-96-3506-1_8.

1 Introduction

A bilevel optimization problem (BOP) constitutes a distinctive category of optimization problems, characterized by two interconnected optimization levels: the upper level (UL) and the lower level (LL). The two levels control different sets of variables and have their own objective functions and constraints. Due to its hierarchical nested nature, the LL problem is part of the UL constraints, which requires that every feasible UL solution satisfies the optimality conditions of the LL optimization problem. Related overviews for bilevel optimization can be found in [10, 26, 2]. In case both or one of the levels have multiple objectives, we have a multiobjective bilevel optimization problem (MBOP). The readers can find a comprehensive review on these problems in [11].

In bilevel optimization problems, when the LL has multiple solutions for each UL solution, it implies that a set of efficient LL decision vectors exists for each given UL vector. This means that for a given UL decision, the optimal LL solution is not a singleton, but a set. This creates ambiguity about which LL solution should be used by the UL while searching for the bilevel optimal solution(s). To address this uncertainty, two approaches are commonly taken: the optimistic and pessimistic approach [26]. When taking an optimistic perspective, the UL assumes that the LL will choose that solution from its optimal set, which is the best for the UL. In the pessimistic approach, the UL aims to optimize their objective by assuming that the LL may make a worst-case choice for the UL.

For most situations, the UL does not have knowledge of the choice function of the LL, meaning the LL may select any solution from its set of optimal solutions for a given UL decision. In the case of multiobjective bilevel optimization, this can result in two distinct Pareto-optimal fronts for the UL and also raises a number of decision making difficulties. These fronts correspond to the best and worst-case scenarios, respectively, for the objectives of the UL, and finding these frontiers is the focus of this study.

In recent years, evolutionary approaches (EAs) have been developed to address the multiobjective bilevel problem, such as [12, 14, 9]. The authors of [9] suggested an evolutionary local-search algorithm for multiobjective bilevel problems that handles multiple objectives at both levels. The work was an improvement over the older versions of their algorithm [8, 22]. Further improvements in multiobjective bilevel solution methods were made in [20] by approximating the LL Pareto-optimal set.

The multiobjective bilevel evolutionary algorithm (m-BLEAQ) was proposed in [24, 27] to solve problems with multiple objectives at the UL and a single objective at the LL. A value function, representing the preferences of the LL decision maker, models LL optimization when both levels have multiple objectives. This value function selects a single preferred solution from the LL Pareto front, effectively reducing the multiobjective LL problem to a single-objective one. This reduction assumes that the preference structure of the LL decision maker is known to the UL decision maker. The algorithm's principle relies on estimating LL decisions as a function of UL decisions, utilizing local quadratic approximations to reduce computational costs. Similar bilevel mapping-approximation ideas are used to solve single objective bilevel optimization problems as well [21, 19]. In cases of single objective at the LL, if the LL returns a singleton, an optimistic or pessimistic position does not arise.

Most multiobjective methods [9, 12] assume an optimistic approach, mostly due to the ease in finding the Pareto-optimal front compared to the pessimistic case. In the context of the pessimistic approach, additional levels of complexity arise from the intricate decision-making interactions between UL and LL. However, the solutions on an optimistic Pareto-optimal frontier are not always realistic since the UL cannot control the choices of the LL. Moreover, the pessimistic Pareto front is essential as it provides a bound for worst-case LL responses. By identifying this front, the decision maker ensures robustness, i.e., by choosing a point from the pessimistic frontier, the UL decision maker's actual realized solution can never be in the region dominated by the frontier, irrespective of the choices of the LL decision maker. In [1], the authors focused on semivectorial [4] bilevel problems and suggested that, apart from optimistic and pessimistic solutions, an UL may also aim for intermediate (moderate) solutions based on the level of risk they are willing to take.

Recently, Deb et al. [7] introduced an evolutionary approach for such hierarchical problem solving, allowing UL decision makers to minimize deviations from their expectations caused by independent LL decisions by identifying a set of pseudopessimistic solutions. To our best knowledge, there is no study that approximates both the optimistic and pessimistic Pareto-optimal fronts of the multiobjective bilevel optimization problem, where the UL contains multiple objectives and the LL contains a single objective with multiple optimal solutions.

In the classical bilevel optimization literature, approximation methods exist for addressing problems with specific mathematical structures- e.g. linearity or convexity- where pessimistic formulations are perturbed and reformulated to facilitate solvability [17, 18]. Using an appropriate perturbation strategy for optimistic or pessimistic formulations, it is possible to ensure that the LL problem contains a unique (approximate) optimal solution for any given UL decision. In our previous work [3], we introduced the δ-perturbed formulation for bilevel optimization, designed to handle the challenge of multiple optimal solutions at the LL. This approach modifies the LL objective by incorporating the UL objective by introducing a small perturbation parameter δ. We provided theoretical proofs that establish error bounds and computational results that validate the theoretical findings. The perturbation approach is able to solve bilevel problems with both optimistic and pessimistic formulations by forcing a single optimal solution from the LL optimization problem. The approach thus enables standard bilevel algorithms to be applied to the optimistic as well as the pessimistic formulations.

Building on this foundation, in the present work we extend the δ-perturbed approach for multiobjective bilevel optimization problems that contain multiple objectives at the UL and single objective with multiple optimal solutions at the LL. Our contributions lie in reformulating such a bilevel optimization problem in

[4] In semivectorial bilevel optimization, the UL problem is single objective, while the LL problem is multiobjective.

a form so that the LL returns a singleton for optimistic and pessimistic positions. This enables our approach to identify both the optimistic and pessimistic Pareto-frontiers for the problem. In this proof-of-concept study, we use the m-BLEAQ algorithm [24] to solve the reformulated problem, demonstrating the effectiveness of the proposed δ-perturbation method in identifying optimistic and pessimistic Pareto-optimal fronts.

The remainder of this paper is organized as follows: In Section 2, we provide definitions of the MBOP and its optimistic and pessimistic Pareto-optimal fronts. Section 3 introduces the proposed δ-perturbation approach for MBOPs. Section 4 presents the evaluation of the proposed method on two test problems. Finally, Section 5 concludes the paper and outlines potential directions for future research.

2 Background

In this section, we present the mathematical formulation of the multiobjective bilevel optimization problem. Moreover, we define the optimistic and pessimistic Pareto-optimal fronts.

2.1 Multiobjective Bilevel Optimization Problem

The general mathematical definition of a multiobjective bilevel optimization problem is as follows:

$$
\begin{aligned}
&\text{“}\min_{x,y}\text{”}\; F(x,y) = (F_1(x,y), \ldots, F_M(x,y)) \\
&\text{subject to} \\
&\quad y \in \operatorname*{argmin}_{y} \{f(x,y) = (f_1(x,y), \ldots, f_N(x,y)) : g_j(x,y) \le 0,\ j = 1, \ldots, J\} \\
&\quad G_k(x,y) \le 0,\ k = 1, \ldots, K
\end{aligned}
$$

$G_k : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $k = 1, \ldots, K$, denotes the k^{th} UL constraint, and $g_j : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$, $j = 1, \ldots, J$, denotes the j^{th} LL constraint. For simplicity, we do not include equality constraints. Any variable bounds on x and y are included in the inequality constraint set.

In this study, we are interested in the following case of multiobjective bilevel optimization problems, where the UL has multiple objectives while the LL has a single objective:

$$
\text{“}\min_{x,y}\text{”}\; F(x,y) = (F_1(x,y), \ldots, F_M(x,y)) \tag{1}
$$

subject to

$$
y \in \operatorname*{argmin}_{y} \{f(x,y) : g_j(x,y) \le 0,\ j = 1, \ldots, J\} \tag{2}
$$

$$
G_k(x,y) \le 0,\ k = 1, \ldots, K \tag{3}
$$

The problem under consideration in this paper can also be expressed in terms of set-valued mapping, $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ as follows, where $g(x,y)$ and $G(x,y)$ represent vector-valued constraints:

$$\Psi(x) = \underset{y}{\operatorname{argmin}}\{f(x,y) : g(x,y) \leq 0\}$$

The multiobjective bilevel optimization problem, in terms of $\Psi(x)$, can be formulated as a constrained optimization problem in the following manner:

$$\begin{aligned}
&\text{“}\min_{x,y}\text{”}\, F(x,y) = (F_1(x,y), \ldots, F_M(x,y)) \\
&\text{subject to} \\
&\quad y \in \Psi(x) \\
&\quad G(x,y) \leq 0
\end{aligned}$$

Note that we are using quotations in our formulations as we have not yet defined the position that we take, i.e. optimistic or pessimistic, while solving the problem. Without stating the position, the above formulations are ill-posed.

2.2 Optimistic vs. Pessimistic Pareto Front

As discussed, when the LL contains multiple optimal solutions for a given UL decision, we have to make some assumptions. If the UL anticipates that the LL will select that optimal solution from its set that suits the UL, we state that the UL is taking an optimistic position. In such cases, the UL desires to identify an optimistic Pareto-optimal front. The optimistic Pareto-front corresponds to the solution of the following problem.

$$\begin{aligned}
&\min_{x,y} F(x,y) = (F_1(x,y), \ldots, F_M(x,y)) \\
&\text{subject to} \\
&\quad y \in \Psi^O(x) \\
&\quad G(x,y) \leq 0
\end{aligned}$$

where $\Psi^O(x)$ is a subset of $\Psi(x)$, which provides the optimistic UL feasible solutions and is defined as follows:

$$\Psi^O(x) = \underset{y}{\operatorname{argmin}}\{(F_1(x,y), \ldots, F_M(x,y)) \mid y \in \Psi(x)\}$$

Optimistic Pareto Front Definition: The Pareto front P^O consists of those solutions (x^*, y^*) from $\Omega^O = \{(x,y) : y \in \Psi^O(x), G(x,y) \leq 0\}$ for which there is no solution (x', y') that simultaneously satisfies the following conditions:

$$\begin{aligned}
&\forall\ i = 1 \text{ to } M,\ F_i(x',y') \leq F_i(x^*,y^*) \quad \text{for } (x^*,y^*) \in \Omega^O, (x',y') \in \Omega^O \\
&\exists\ j \text{ such that } F_j(x',y') < F_j(x^*,y^*) \quad \text{for } (x^*,y^*) \in \Omega^O, (x',y') \in \Omega^O
\end{aligned}$$

If the UL anticipates the worst-case from LL, we take the pessimistic approach and converge to the pessimistic Pareto front. The solutions here are:

$$\begin{aligned}
&\min_{x,y} F(x,y) = (F_1(x,y), \ldots, F_M(x,y)) \\
&\text{subject to} \\
&\quad y \in \Psi^P(x) \\
&\quad G(x,y) \leq 0
\end{aligned}$$

where $\Psi^P(x)$ is a subset of $\Psi(x)$, which provides the pessimistic UL feasible solutions and is defined as follows:

$$\Psi^P(x) = \underset{y}{\operatorname{argmax}}\{(F_1(x,y), \ldots, F_M(x,y)) \mid y \in \Psi(x)\}$$

Pessimistic Pareto Front Definition: The Pareto front P^P consists of those solutions (x^*, y^*) from $\Omega^P = \{(x,y) : y \in \Psi^P(x), G(x,y) \leq 0\}$ for which there is no solution (x', y') that simultaneously satisfies the following conditions:

$$\begin{aligned}
&\forall\ i = 1 \text{ to } M,\ F_i(x', y') \leq F_i(x^*, y^*) \quad \text{for } (x^*, y^*) \in \Omega^P, (x', y') \in \Omega^P \\
&\exists\ j \text{ such that } F_j(x', y') < F_j(x^*, y^*) \quad \text{for } (x^*, y^*) \in \Omega^P, (x', y') \in \Omega^P
\end{aligned}$$

The relationship between the objective space (F-space) and the decision space $\Psi(x) - x$ for a multiobjective bilevel optimization problem (two UL objectives and one LL objective with multiple optimal solutions) is shown through Fig. 1. For simplicity, assume that there are no UL constraints, $G(x,y) \leq 0$. On the left, the decision space $\Psi(x) - x$ is shown, where the shaded gray region represents the full set of feasible y for each x. The blue curve represents a subset of optimistic feasible solutions, $\Psi^O(x)$, that contribute to the optimistic Pareto front, while the red curve represents a subset of pessimistic feasible solutions, $\Psi^P(x)$, that contribute to the pessimistic Pareto front. Optimistic and pessimistic solutions for a few x are shown as blue and red circles, respectively. The gray points denote solutions that are part of $\Psi(x)$ but do not lie on either the optimistic or the pessimistic Pareto fronts.

On the right, the objective space is illustrated, where the functions F_1 and F_2 represent the UL objectives. Once again, the blue curve represents the optimistic Pareto front (P^O), and the red curve represents the pessimistic Pareto front (P^P) for a minimization case. The red, blue, and gray points from the left-hand side figure map on to the corresponding points in the right-hand side figure.

The left figure also shows a special UL solution, x^0, for which all the LL optimal solutions are shown in green. On the green line, the blue point corresponds to a solution on the optimistic Pareto front, the red point corresponds to a solution on the pessimistic Pareto front, and the gray point corresponds to neither of the two. All the solutions on the green line in the left figure are mapped to the green curve in the right figure.

It is important to note that in some sense the pessimistic Pareto front can be regarded as an upper bound (in the case of minimization) for the UL decision maker. If a UL decision maker chooses a decision (say (x^0, y')) from the pessimistic Pareto front, then no matter which solution among the multiple optimal solutions is picked up by the LL decision maker (say y'' for x^0), the solution (x^0, y'') will lie somewhere between the pessimistic and optimistic frontiers. The figure illustrates one such scenario using green lines. However, note that it is not necessary that there will always be an optimistic Pareto point corresponding to a UL decision made from the pessimistic Pareto front (as in the case of UL decision x^1 for which LL decisions are shown with yellow lines). Since the realized solution may lie anywhere between the two frontiers, the problem considered in this paper raises a number of decision making difficulties.

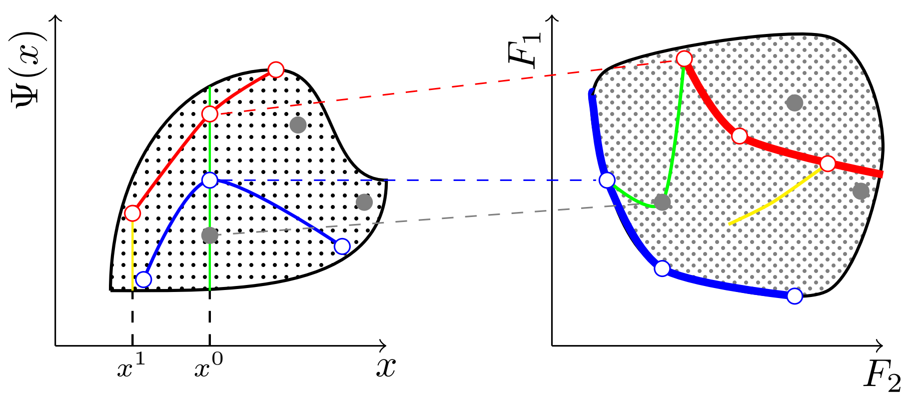


Fig. 1: Illustration of the UL decision space $\Psi(x) - x$ (left) and the UL objective space F (right).

3 δ-perturbation of MBOPs

In this section, we explain the δ-perturbed formulation of the MBOP. For better understanding, let us first introduce the δ-perturbed formulation of the single-objective bilevel problem,

$$\begin{aligned} \text{“}\min_{x,y}\text{”} \quad & F(x, y) \\ \text{s.t.} \quad & G(x, y) \leq 0 \\ & y \in \operatorname{argmin}\{f(x, y) \,|\, g(x, y) \leq 0\} \end{aligned}$$

which has been thoroughly analyzed in [3]. For a small positive δ, the perturbed optimistic and pessimistic bilevel formulation is obtained by adding or subtracting $\delta F(x,y)$, respectively, as follows:

$$\min_{x,y} \quad F(x,y) \tag{4}$$

$$\text{s.t.} \quad G(x,y) \le 0 \tag{5}$$

$$y \in \operatorname*{argmin}_{y}\{f(x,y) \pm \delta F(x,y) \,|\, g(x,y) \le 0\} \tag{6}$$

When δ is extremely small, the above formulation leads to an approximate optimistic/pessimistic solution depending on the sign that is chosen. In this paper, we extend this idea for multiobjective bilevel problems where the UL has multiple objectives, while the LL has a single objective but multiple optimal solutions corresponding to a UL decision.

Theorem 1. *For a bilevel optimization problem with multiple objectives at the UL and a single objective at the LL, approximate optimistic and pessimistic Pareto-fronts can be obtained by solving the following bilevel problem with a '+' and '-' sign, respectively.*

$$\min_{x,w,y} \quad F(x,y) = (F_1(x,y),\ldots,F_M(x,y)) \tag{7}$$

$$s.t. \quad G(x,y) \le 0 \tag{8}$$

$$y \in \operatorname*{argmin}_{y}\{f(x,y) \pm \delta V(w, F(x,y)) \mid g(x,y) \le 0\} \tag{9}$$

where $V(.)$ represents a suitable scalarization scheme such as weighted sum scalarization [13], weighted Chebyshev [28], conic scalarization [16] or Benson's scalarization [5] techniques with w being the preference vector, which is considered to be an UL variable.

Proof. Let us write the problem in (1)-(3) into a single objective problem at the UL for a given preference vector w.

$$\text{``}\min_{x,y}\text{''}\, V(w, F_1(x,y),\ldots,F_M(x,y)) \tag{10}$$

subject to

$$y \in \operatorname*{argmin}_{y}\{f(x,y) : g_j(x,y) \le 0,\ j = 1,\ldots,J\} \tag{11}$$

$$G_k(x,y) \le 0,\ k = 1,\ldots,K \tag{12}$$

Using (4)-(6), one can find the approximated optimistic and the pessimistic solutions for (10)-(12) by solving the following problem.

$$\begin{aligned}
\min_{x,w,y} \quad & V(w, F_1(x,y),\ldots,F_M(x,y)) \\
\text{s.t.} \quad & G(x,y) \le 0 \\
& y \in \operatorname*{argmin}_{y}\{f(x,y) \pm \delta V(w, F(x,y)) \mid g(x,y) \le 0\}
\end{aligned}$$

If the above problem is solved for all possible values of w, it leads to all the (approximate) optimistic and (approximate) pessimistic solutions corresponding to w. Therefore, one can solve the multiobjective problem in (7)-(9) to get an approximation of the entire optimistic and pessimistic Pareto-fronts by choosing the appropriate signs (+/-).

4 Proof-of-Concept Results

In this section, we briefly describe the m-BLEAQ algorithm, the test problems, and the results obtained from applying the proposed approach. It is important to note that the focus of this work is not on developing or testing a new algorithm but rather on reformulating bilevel optimization problems using the δ-perturbation method. The m-BLEAQ algorithm is utilized as an existing solver to demonstrate the feasibility and effectiveness of the reformulated problem.

4.1 m-BLEAQ Algorithm and Experimental Settings

The multiobjective BiLevel Evolutionary Algorithm based on Quadratic approximations, known as m-BLEAQ [24], is a model-based evolutionary method that utilizes insights from parametric optimization theory to reduce computational costs. It is an extension of BLEAQ [21], which was developed for solving single objective bilevel problems and was later extended to BLEAQ-II [19]. BLEAQ utilizes only the reaction set mapping in its implementation, while BLEAQ-II utilizes both the reaction set mapping and value function mapping to solve bilevel problems. In this paper, we employ both the reaction set mapping and value function mapping in our m-BLEAQ implementation, following BLEAQ-II's approach.

With approximations of the mappings in bilevel optimization, the algorithm quickly converges to the Pareto-optimal solutions of the multiobjective bilevel problem. The algorithm has been applied and found to approximate successfully the Pareto fronts to problems with known LL value function in both test problems [24] and applications, such as transportation policy [25] and agricultural policy [4]. Given its capability to handle bilevel problems with multiobjective UL problems and single-objective LL problems, it can serve as a baseline algorithm to validate our proposal in this paper.

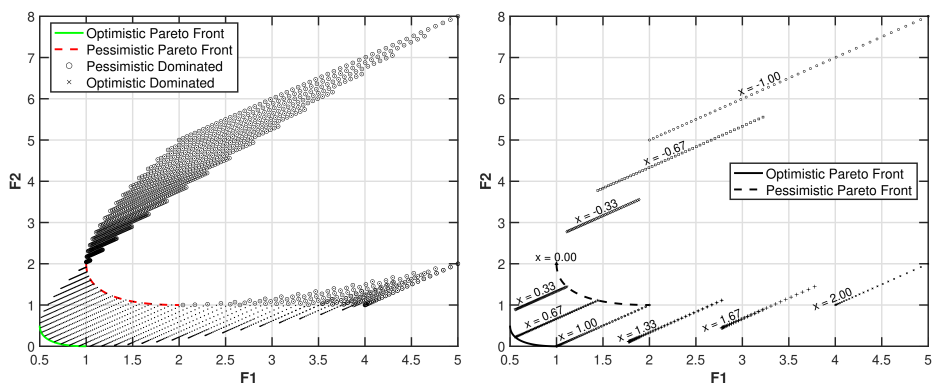
The parameters used in m-BLEAQ were maintained at their default settings, with the exception of the population size and generation size, both of which were set to 500. These values were chosen to ensure a sufficiently large population for testing the perturbation method, without any parameter tuning. The perturbation value δ was set to 0.01, chosen as a small enough value based on our previous work [3]. The weighted-sum scalarization technique was used, with the weight parameter w sampled from $[0, 1]$. For each test problem, we conducted 20 independent runs on an Intel(R) Core(TM) i5-8365U CPU @ 1.60GHz with 16 GB of RAM running Windows 10 and MATLAB R2023b.

4.2 Test Problems

The first problem in our study, referred to as Test Problem 1, is taken from [8], where it originally includes two objectives at both the UL and LL. In our formulation, we modify the LL problem to have a single objective with multiple optimal solutions, as shown below:

$$
\begin{aligned}
\min_{x,y}\quad & F_1(x,y) = (y_1-1)^2 + y_2^2 + x^2 \\
& F_2(x,y) = (y_1-1)^2 + y_2^2 + (x-1)^2 \\
\text{s.t.}\quad & y \in \operatorname*{argmin}_{y}\{\sqrt{y_1^2+y_2^2} + \sqrt{(y_1-x)^2+y_2^2} : -1 \le y_1, y_2 \le 2\} \\
& -1 \le x \le 2
\end{aligned}
$$

The optimistic PF is obtained when $x \in [0.5, 1]$ and $y_1 = x, y_2 = 0$. In contrast, the pessimistic PF is derived for $x \in [0, 1]$ and $y_1 = y_2 = 0$.



(a) Optimistic and pessimistic Pareto fronts in the UL objective space.

(b) Optimal LL solutions for various x by varying y_1 and keeping $y_2 = 0$.

Fig. 2: Illustration of the optimistic and pessimistic feasible solutions in the UL objective space for Test Problem 1.

We illustrate the characteristics of Test Problem 1 through Fig. 2, which shows the relationship between optimistic and pessimistic solutions in the UL objective space. If for any fixed x, the LL has multiple optimal solutions to choose from, then we refer to the non-dominated solution(s) with respect to the UL objectives as optimistic solution(s). Conversely, for that x, the solutions in the subset of pessimistic feasible space are referred to as pessimistic solution(s), with the non-dominated solutions among them forming the pessimistic Pareto front. For certain problems, for any given x, the optimistic/pessimistic solution(s) may coincide. Our objective is to find the best (with respect to UL objectives) optimistic and pessimistic fronts, i.e. optimistic and pessimistic Pareto fronts. Fig. 2a shows the optimistic and pessimistic feasible solutions in the UL objective space. The solid line represents the optimistic Pareto front, while the dashed line corresponds to the pessimistic Pareto front. The black points represent pessimistic-dominated solutions, while the 'x' markers denote optimistic-dominated solutions. Naturally, all pessimistic solutions, including both the Pareto front and the dominated solutions, are dominated by the optimistic Pareto front. Fig. 2b illustrates the range of LL optimal solutions for a fixed value of x in the UL decision space. The multiple LL optimal solutions for any x can be generated by fixing $y_2 = 0$ and varying y_1.

The second problem considered in our study is Test Problem 2, which combines modified SMD1 at the UL and SMD6 at the LL [23]. SMD6 is selected for the LL due to its multimodal structure. The description of Test Problem 2 is as follows:

$$
\begin{aligned}
\min_{x,y} \quad & F_1(x,y) = x_1^2 + y_1^2 + y_2^2 + x_2^2 + (x_2 - y_3)^2 \\
& F_2(x,y) = 1 - |x_1| + x_2^2 + y_1^2 + y_2^2 + (x_2 - y_3)^2 \\
\text{s.t.} \quad & y \in \operatorname*{argmin}_y \{x_1^2 + (y_2 - y_1)^2 + (x_2 - y_2)^2 : -5 \leq y_1, y_2 \leq 10\} \\
& -5 \leq x_1, x_2, x_3 \leq 10
\end{aligned}
$$

The optimistic PF is obtained when $x_1 \in [-5, 10]$ and $x_2 = y_1 = y_2 = y_3 = 0$, while the pessimistic PF is derived for $x_1 \in [-5, 10]$ and $x_2 = y_3 = 0, y_1 = y_2 = 10$.

4.3 Results and Discussion

To evaluate the performance of the algorithm, it is essential to choose an appropriate performance indicator to assess the quality and diversity of the solutions obtained. Commonly used indicators in multiobjective EAs include hypervolume (HV)[29], inverted generational distance (IGD)[6], and others. While using EAs with a perturbation strategy, the Pareto-optimal solutions obtained are an approximation of the true optima. Therefore, a better Pareto front in terms of HV may not necessarily indicate superior algorithm performance, as an inaccurate solution at the LL may provide an artificially better UL solution. Instead, IGD calculates the sum of distances between each point of the true Pareto front and the nearest non-dominated set found by the algorithm. Smaller IGD values indicate that the approximated points are closer to the Pareto front of the problem. IGD+ [15], a Pareto-compliant alternative, modifies IGD by incorporating dominance relationships to exclude dominated solutions, ensuring a more accurate assessment. Both IGD and IGD+ are used in this study to analyze the quality of the approximated Pareto front.

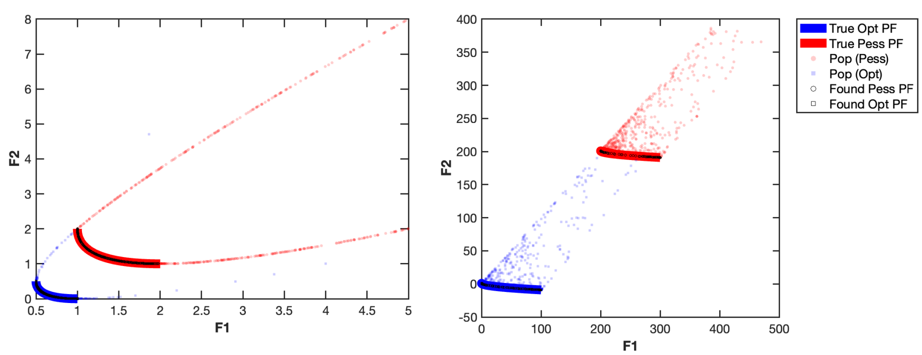
We report four IGD values, IGD_O, IGD_{O+}, IGD_P, and IGD_{P+}, for the optimistic and pessimistic Pareto fronts in Table 1, calculated as the median of 20 runs. For the first problem, all values are of the order 0.001, while for the second problem, they are of the order 0.01. This difference can be attributed to the higher dimensionality and complexity of the second problem. In all cases,

$IGD+$ values are slightly higher than IGD due to the exclusion of dominated solutions, which can artificially improve IGD by reducing the apparent distance to the true Pareto front. Both results demonstrate that the algorithm effectively approximates the true Pareto fronts.

Table 1: Median IGD and IGD+ values for the Test Problems over 20 runs.

	IGD$_O$	IGD$_{O+}$	IGD$_P$	IGD$_{P+}$
Test Problem 1	2.218e-03	2.699e-03	2.652e-03	2.653e-03
Test Problem 2	6.097e-02	6.30e-02	5.930e-02	6.091e-02

To further illustrate the performance, we display in Fig. 3 the true optimistic (blue line) and pessimistic (red line) Pareto fronts, alongside the final population solutions (blue and red) and the Pareto fronts derived from these solutions for the two Test Problems after one run. The black circles indicate the pessimistic-nondominated solutions from the final population, while the black crosses show the optimistic-nondominated solutions. In Fig. 3a, which corresponds to Test Problem 1, the solutions found are densely distributed and spread well, closely approximating the true Pareto fronts. In Fig. 3b, which corresponds to Test Problem 2, the solutions are less densely distributed, which aligns with the slightly higher IGD values observed above.



(a) Test Problem 1. (b) Test Problem 2.

Fig. 3: True and algorithm-obtained Pareto fronts for optimistic and pessimistic cases and algorithm's final population.

For both test problems, the optimistic and pessimistic Pareto fronts can become valuable tools for decision-makers. For example, in both test problems, the optimistic and pessimistic solutions deviate significantly. From a decision-maker's perspective, the risk of adopting the optimistic assumption may result in surprises and substantial costs to the UL decision maker. This makes such a kind of analysis critically important. Also note that identifying the optimistic and pessimistic Pareto fronts does not completely solve the decision making challenges for these problems.

As a final remark, it is important to highlight that, as previously mentioned, we used the weighted sum scalarization, and it worked well since the specific Pareto fronts are convex. In the case of nonconvexity, other suitable methods should be selected.

5 Conclusion and Future Work

In this paper, we investigated the multiobjective bilevel optimization problem, where the UL has multiple objectives, and the LL has a single-objective but multiple optimal solutions for any given UL decision. We first defined the problem, introducing both optimistic and pessimistic Pareto-fronts in the context of multiobjective bilevel problems. To approximate these fronts, we developed a perturbation technique that perturbs the lower-level objective by a small δ based on the weighted values of the UL objectives. The reformulated bilevel problem was then addressed using m-BLEAQ, an evolutionary algorithm capable of handling multiple objectives at the UL. The effectiveness of the proposed method was demonstrated through tests on two benchmark problems, successfully capturing both optimistic and pessimistic Pareto fronts.

As part of our ongoing research, we plan to extend this work by testing the proposed approach on a wider range of benchmark problems and exploring real-world applications. Additionally, a theoretical investigation of the δ-perturbation method, including the derivation of error bounds, will be valuable for further refinement. We also aim to expand this approach to scenarios where both levels involve multiobjective optimization, potentially offering decision-makers richer, more informed solution landscapes in multiobjective bilevel problems.

Acknowledgements

The authors acknowledge the financial support from the Slovenian Research Agency (research core funding No. P2-0098). This work was also partially funded by the European Union's Horizon Europe research an innovation programme under Grant Agreement No 101077049 (CONDUCTOR).

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