

SPREADING IN CLAW-FREE CUBIC GRAPHS

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Abstract. Let $p \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$. We study a dynamic coloring of the vertices of a graph G that starts with an initial subset S of blue vertices, with all remaining vertices colored white. If a white vertex v has at least p blue neighbors and at least one of these blue neighbors of v has at most q white neighbors, then by the spreading color change rule the vertex v is recolored blue. The initial set S of blue vertices is a (p, q) -spreading set for G if by repeatedly applying the spreading color change rule all the vertices of G are eventually colored blue. The (p, q) -spreading set is a generalization of the well-studied concepts of k -forcing and r -percolating sets in graphs. For $q \geq 2$, a $(1, q)$ -spreading set is exactly a q -forcing set, and the $(1, 1)$ -spreading set is a 1-forcing set (also called a zero forcing set), while for $q = \infty$, a (p, ∞) -spreading set is exactly a p -percolating set. The (p, q) -spreading number, $\sigma_{(p,q)}(G)$, of G is the minimum cardinality of a (p, q) -spreading set. In this paper, we study (p, q) -spreading in claw-free cubic graphs. While the zero-forcing number of claw-free cubic graphs was studied earlier, for each pair of values p and q that are not both 1 we either determine the (p, q) -spreading number of a claw-free cubic graph G or show that $\sigma_{(p,q)}(G)$ attains one of two possible values.

Keywords: bootstrap percolation, zero forcing set, k -forcing set, spreading.

Mathematics Subject Classification: 05C35, 05C75.

1. INTRODUCTION

In this paper we continue the study of dynamic graph colorings and explore the concept of (p, q) -spreading in claw-free cubic graph. The concept of spreading in graphs is a generalization of the well-studied concepts of k -forcing and r -percolating sets in graphs.

Consider a dynamic coloring of the vertices of a graph G that starts with an initial subset S of blue vertices, with all remaining vertices colored white. For $k \in \mathbb{N}$, the color change rule in k -forcing is defined as follows: *if a blue vertex u has at most k*

white neighbors, then all white neighbors of u are recolored blue. In particular, when $k = 1$, this is the color change rule for zero forcing in graphs. The initial set S of blue vertices is a k -forcing set for G if by repeatedly applying the color change rule in k -forcing all the vertices of G are eventually colored blue. The k -forcing number of G , denoted $F_k(G)$, is the minimum cardinality among all k -forcing set of G . When $k = 1$, the k -forcing number of G is called the *zero forcing number* and is denoted by $Z(G)$, and so $Z(G) = F_1(G)$.

The concept of bootstrap percolation has a similar definition to that of zero forcing, yet a different motivation. It was introduced as a simplified model of a magnetic system in 1979 [10], and was later studied on random graphs and also on deterministic graphs. In particular, already some early papers considered bootstrap percolation in grids [7, 8].

As before, we consider a dynamic coloring of the vertices of a graph G that starts with an initial subset S of blue vertices, with all remaining vertices colored white. We refer to blue vertices as “infected” and white vertices as “uninfected”. For $r \in \mathbb{N}$, the color change rule in r -percolation is defined as follows: *if a white (uninfected) vertex u has at least r blue (infected) neighbors, then the vertex u is recolored blue*. The initial set S of blue vertices is an r -neighbor bootstrap percolating set, or simply an r -percolating set, of G if by repeatedly applying the color change rule in r -percolation all the vertices of G are eventually colored blue. The r -neighbor bootstrap percolation number, or simply the r -percolation number, of G , denoted $m(G, r)$, is the minimum cardinality among all r -neighbor bootstrap percolating sets of G .

Motivated by the similarity of the definitions of the above two concepts, a common generalization of k -forcing and r -bootstrap percolation was introduced in [9]. Let $p \in \mathbb{N}$ and $q \in \mathbb{N} \cup \{\infty\}$. As before, we consider a dynamic coloring of the vertices of a graph G that starts with an initial subset S of blue vertices, with all remaining vertices colored white. The color change rule in (p, q) -spreading is defined as follows: *if a white vertex w has at least p blue neighbors, and one of the blue neighbors of w has at most q white neighbors, then the vertex w is recolored blue*. The initial set S of blue vertices is a (p, q) -spreading set of G , if by repeatedly applying the (p, q) -spreading color change rule all the vertices of G are eventually colored blue. The (p, q) -spreading number, denoted $\sigma_{(p, q)}(G)$, of a graph G is the minimum cardinality among all (p, q) -spreading sets.

The (p, q) -spreading set is a generalization of the well-studied concepts of k -forcing and r -percolating sets in graphs. For $q \geq 2$, a $(1, q)$ -spreading set is exactly a q -forcing set [3], and the $(1, 1)$ -spreading set is a 1-forcing set (also called a zero forcing set [2, 21]), while if $q \geq \Delta(G)$ (including $q = \infty$), then a (p, q) -spreading set is exactly a p -percolating set. In the foundational paper [9], the complexity of the decision version of the (p, q) -spreading number was proved to be NP-complete for all p and q , while efficient algorithms for determining these numbers were found in trees. In addition, for almost all values of p and q , the (p, q) -spreading numbers of Cartesian grids $P_n \square P_m$ were established in [9].

The following trivial observation will be (at least implicitly) used several times in the paper.

Observation 1.1. *Let G be a graph, $p \geq 2$ and $q \in \mathbb{N} \cup \{\infty\}$. If P is a (p, q) -spreading set in G , then P is also a $(p, q+1)$ -spreading set in G as well as a $(p-1, q)$ -spreading set in G . In particular,*

$$\sigma_{(p,q)}(G) \geq \sigma_{(p,q+1)}(G) \text{ and } \sigma_{(p,q)}(G) \geq \sigma_{(p-1,q)}(G).$$

A graph is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. A *cubic graph* (also called a *3-regular graph*) is a graph in which every vertex has degree 3. In this paper we study spreading in claw-free cubic graphs.

1.1. GRAPH THEORY NOTATION AND TERMINOLOGY

For notation and graph theory terminology, we in general follow [17]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. A *neighbor* of a vertex v in G is a vertex u that is adjacent to v , that is, $uv \in E(G)$. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v , while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N(v)$. We denote the *degree* of v in G by $\deg_G(v) = |N_G(v)|$, and $\Delta(G) = \max\{\deg_G(v) : v \in V(G)\}$. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v \in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$.

For a set $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Further, the subgraph of G obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G - S$; that is, $G - S = G[V(G) \setminus S]$. If $S = \{v\}$, then we also denote $G - S$ simply by $G - v$. If F is a graph, then an *F-component* of G is a component isomorphic to F . We denote by $\alpha(G)$ the independence number of G , and so $\alpha(G)$ is the maximum cardinality among all independent sets of G . A *vertex cover* of G is a set S of vertices such that every edge of G is incident with at least one vertex in S . The *vertex covering number* $\beta(G)$ (also denoted $\tau(G)$ in the literature), equals the minimum cardinality among all vertex covers of G . By the well-known Gallai theorem, $\alpha(G) + \beta(G) = n(G)$ holds in any graph G .

We denote the *path*, *cycle*, and *complete graph* on n vertices by P_n , C_n , and K_n , respectively, and we denote the *complete bipartite graph* with partite sets of cardinality n and m by $K_{n,m}$. A *triangle* in G is a subgraph isomorphic to K_3 , whereas a *diamond* in G is an induced subgraph of G isomorphic to K_4 with one edge missing, denoted by $K_4 - e$. A graph is *diamond-free* if it does not contain $K_4 - e$ as an induced subgraph. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \dots, k\}$.

1.2. MAIN RESULTS AND ORGANIZATION OF THE PAPER

Davila and Henning studied zero forcing in claw-free cubic graphs [11]. They established two upper bounds, which are summarized in the following result (for the definition of diamond-necklace N_k , see Section 2).

Theorem 1.2 ([11]). *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n , then the following properties hold:*

- (a) $Z(G) \leq \alpha(G) + 1$,
- (b) $Z(G) \leq \frac{n}{3} + 1$, unless $G = N_2$ in which case $Z(G) = \frac{1}{3}(n + 4)$.

It is proved in [11] that both upper bounds in Theorem 1.2 are sharp. He *et al.* [18] recently characterized the connected, claw-free cubic graphs G for which $Z(G) = \alpha(G) + 1$. Notably, there are only three sporadic graphs that attain this value, namely N_2 , N_3 and the Hamming graph $K_3 \square K_2$. Since these graphs have 8, 12 and 6 vertices, respectively, the following result immediately follows.

Theorem 1.3 ([18]). *If $G \neq K_4$ is a connected, claw-free, cubic graph of order at least 14, then $Z(G) \leq \alpha(G)$.*

Note that $Z(G) = \sigma_{(1,1)}(G)$, and the mentioned result (Theorem 1.2(a)) is placed in the (1, 1)-entry of Table 1.

From properties of connected, claw-free, cubic graphs G which we discuss in Section 2, if $G \neq K_4$ then there is a unique partition of the vertex set $V(G)$ into subsets each of which induces either a triangle or a diamond. The number of these subsets, called units, in G is denoted by $u(G)$. Table 1 summarizes the values of (p, q) -spreading numbers in claw-free cubic graphs $G \neq K_4$ for all possible p and q in terms of the parameters $\alpha(G)$, $\beta(G)$, $u(G)$ and $n(G)$. Besides the (1, 1)-entry giving the mentioned upper bound on $\sigma_{(1,1)}(G)$ all other values in Table 1 are obtained in this paper. It is easy to see that $\sigma_{(1,2)}(G) = 2$ for any claw-free cubic graph G . Indeed, any adjacent pair of vertices in G is a (1, 2)-spreading set in G . The results $\sigma_{(1,q)}(G) = 1$ for any $q \geq 3$ and $\sigma_{(p,q)}(G) = n(G)$ for any $p \geq 4$ and any $q \in \mathbb{N} \cup \{\infty\}$ are trivial consequences of definitions.

Table 1
(p, q)-spreading numbers of claw-free cubic graphs G , $G \neq K_4$

$\begin{matrix} q \\ p \end{matrix}$	1	2	≥ 3
1	$\leq \alpha(G) + 1$, $n \geq 14$ [Thm. 1.3] ([Thm. 1.2])	2	1
2	$u(G) + 1$ or $u(G) + 2$ [Thm. 4.9]	$u(G)$ or $u(G) + 1$ [Cor. 4.8]	$u(G)$ or $u(G) + 1$ [Cor. 4.5]
3	$\beta(G)$ or $\beta(G) + 1$ [Prop. 3.12]	$\beta(G)$ [Prop. 3.8]	$\beta(G)$ [Prop. 3.8]
≥ 4	$n(G)$	$n(G)$	$n(G)$

The paper is organized as follows. In Section 3, we consider $(3, q)$ -spreading numbers in claw-free cubic graphs; their values can be found in Line 3 of Table 1. (Note that 3-percolation coincides with $(3, q)$ -spreading when $q \geq \Delta(G)$.) In Section 4, we deal with $(2, q)$ -spreading numbers of claw-free cubic graphs; see Line 2 in Table 1. In the case of $(2, q)$ -spreading, where $q \geq 3$, $\sigma_{(2,q)}(G) = u(G) + 1$ precisely when G is a diamond necklace N_k , while in all other claw-free cubic graphs $\sigma_{(2,q)}(G) = u(G)$. Thus, the 2-percolation number of claw-free cubic graphs is fully determined. On the other hand, when $q \leq 2$ a claw-free cubic graph can attain two values: $\sigma_{(2,2)}(G) \in \{u(G), u(G) + 1\}$, and $\sigma_{(2,1)}(G) \in \{u(G) + 1, u(G) + 2\}$, but we have not established exactly which

claw-free cubic graphs attain which of the two possible values. We conclude the paper with some open problems that arise from this study.

2. CLAW-FREE CUBIC GRAPHS

The following property of connected, claw-free, cubic graphs is established in [20].

Lemma 2.1 ([20]). *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n , then the vertex set $V(G)$ can be uniquely partitioned into sets each of which induces a triangle or a diamond in G .*

By Lemma 2.1, the vertex set $V(G)$ of connected, claw-free, cubic graph $G \neq K_4$ can be uniquely partitioned into sets each of which induce a triangle or a diamond in G . Following the notation introduced in [20], we refer to such a partition as a *triangle-diamond partition* of G , abbreviated Δ -D-partition. We call every triangle and diamond induced by a set in our Δ -D-partition a *unit* of the partition. A unit that is a triangle is called a *triangle-unit* and a unit that is a diamond is called a *diamond-unit*. (We note that a triangle-unit is a triangle that does not belong to a diamond.) Two units in the Δ -D-partition are *adjacent* if there is an edge joining a vertex in one unit to a vertex in the other unit. In what follows, we define several structures in claw-free, cubic graphs that we will need when proving our main results. Some of these definitions have already appeared in the literature, see, for example, [4, 5, 20].

For $k \geq 2$ an integer, let N_k be the connected cubic graph constructed as follows. Take k disjoint copies $D_1, D_2, \dots, D_k$ of a diamond, where $V(D_i) = \{a_i, b_i, c_i, d_i\}$ and where $a_i b_i$ is the missing edge in D_i . Let N_k be obtained from the disjoint union of these k diamonds by adding the edges $\{a_i b_{i+1} : i \in [k-1]\}$ and adding the edge $a_k b_1$. We call N_k a *diamond-necklace* with k diamonds. Let $\mathcal{N}_{\text{cubic}} = \{N_k \mid k \geq 2\}$. The diamond-necklace, N_4 , with four diamonds is illustrated in Figure 1.

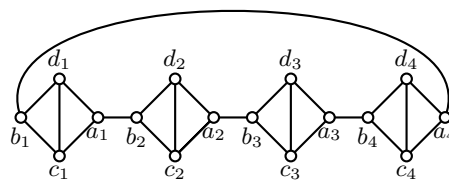


Fig. 1. The diamond-necklace N_4

For $k \geq 1$ an integer, let F_{2k} be the connected cubic graph constructed as follows. Take $2k$ disjoint copies $T_1, T_2, \dots, T_{2k}$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2k]$. Let

$$\begin{aligned} E_a &= \{x_{2i-1}x_{2i} : i \in [k]\}, \\ E_b &= \{y_{2i-1}y_{2i} : i \in [k]\}, \\ E_c &= \{z_{2i}z_{2i+1} : i \in [k]\}, \end{aligned}$$

where addition is taken modulo $2k$ (and so, $z_1 = z_{2k+1}$). Let F_{2k} be obtained from the disjoint union of these $2k$ triangles by adding the edges $E_a \cup E_b \cup E_c$. The resulting graph F_{2k} we call a *triangle-necklace* with $2k$ triangles. Let $\mathcal{T}_{\text{cubic}} = \{F_{2k} : k \geq 1\}$. The triangle-necklace F_6 is shown in Figure 2.

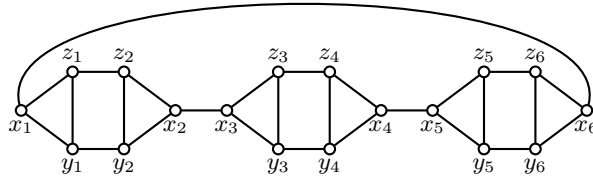


Fig. 2. The triangle-necklace F_6

For $k \geq 2$ an integer, take $2k$ disjoint copies $T_1, T_2, \dots, T_{2k}$ of a triangle, where $V(T_i) = \{x_i, y_i, z_i\}$ for $i \in [2k]$, and take k disjoint copies $D_1, D_2, \dots, D_k$ of a diamond, where $V(D_j) = \{a_j, b_j, c_j, d_j\}$ and where $a_j b_j$ is the missing edge in D_j for $j \in [k]$. Let

$$\begin{aligned} E_1 &= \{x_{2i-1} a_i : i \in [k]\}, \\ E_2 &= \{x_{2i} b_i : i \in [k]\}, \\ E_3 &= \{y_{2i-1} z_{2i+1} : i \in [k-1]\} \cup \{y_{2k-1} z_1\}, \\ E_4 &= \{y_{2i} z_{2i+2} : i \in [k-1]\} \cup \{y_{2k} z_2\}. \end{aligned}$$

Let H_{2k} be obtained from the disjoint union of these $2k$ triangles and k diamonds by adding the edges $E_1 \cup E_2 \cup E_3 \cup E_4$. The resulting graph H_{2k} we call a *triangle-diamond-necklace* with k diamonds. Let $\mathcal{H}_{\text{cubic}} = \{H_{2k} : k \geq 2\}$. The triangle-diamond-necklace H_6 is shown in Figure 3.

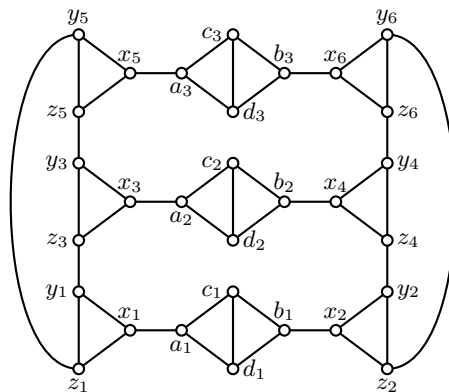


Fig. 3. The triangle-diamond-necklace H_6

3. 3-PERCOLATION IN CLAW-FREE, CUBIC GRAPHS

In this section, we study 3-percolation in claw-free, cubic graphs. The following lemma is already known in the literature and follows readily from the definition of r -bootstrap percolation.

Lemma 3.1 ([19]). *For $r \geq 2$ if H is a subgraph of a graph G such that every vertex in H has strictly less than r neighbors in G that belong to $V(G) \setminus V(H)$, then every r -percolating set of G contains at least one vertex of H .*

As a consequence of Lemma 3.1, we establish a relationship between the r -percolation number and the independence number of an r -regular graph. Recall that $\beta(G)$ is the cardinality of a minimum vertex cover of G .

Proposition 3.2. *Let G be an r -regular graph of order n , where $r \geq 2$. A set $S \subset V(G)$ is an r -percolating set in G if and only if S is a vertex cover in G . In particular, $m(G, r) = \beta(G) = n - \alpha(G)$.*

Proof. Let S be an r -percolating set of an r -regular graph G . Suppose that $G - S$ contains an edge uv . Let $H = G[\{u, v\}]$, and so $H = K_2$ is a subgraph of G such that every vertex in H has strictly less than r neighbors in G that belong to $V(G) \setminus V(H)$. By Lemma 3.1, we infer that every r -percolating set of G contains at least one vertex of H , which is a contradiction with our assumption. Thus, S is a vertex cover of G . Conversely, suppose that S is a vertex cover of G , and let $I = V(G) \setminus S$. Since the set I is an independent set of G , every vertex in I has all its r neighbors in the set $V(G) \setminus I$. The set S is therefore an r -percolating set of G . Since $m(G, r)$ is the cardinality of a minimum r -percolating set, we infer that $m(G, r) = \beta(G)$. Consequently, by the famous Gallai theorem ($\alpha(G) + \beta(G) = n(G)$ in any graph G) we also have $m(G, r) = n - \alpha(G)$. $\square$

Let us recall the statement of the well-known Brooks' Coloring Theorem.

Theorem 3.3 (Brooks' Coloring Theorem). *If G is a connected graph, which is not a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

As a consequence of Theorem 3.3, we have the following trivial lower bound on the independence number of a graph.

Theorem 3.4. *If $G \neq K_n$ is a connected graph of order n with maximum degree $\Delta \geq 3$, then $\alpha(G) \geq \frac{n}{\Delta}$.*

Proof. As a consequence of Theorem 3.3, if $G \neq K_n$ is a connected graph of order n with maximum degree $\Delta \geq 3$, then the chromatic number of G is at most Δ , that is, $\chi(G) \leq \Delta$. Alternatively, viewing a Δ -coloring of G as a partitioning of its vertices into Δ independent sets, called color classes, we infer by the Pigeonhole Principle that G contains an independent set of cardinality at least n/Δ , implying that $\alpha(G) \geq n/\Delta$. $\square$

The following upper bound on the independence number of a claw-free graph is given by several authors.

Theorem 3.5 ([14, 23]). *If G is a claw-free graph of order n with minimum degree δ , then*

$$\alpha(G) \leq \left(\frac{2}{\delta + 2} \right) n.$$

As a consequence of Theorems 3.4 and 3.5, we have the following bounds on the independence number of a claw-free graph.

Theorem 3.6. *If $G \neq K_n$ is a connected, claw-free graph of order n with minimum degree δ and maximum degree $\Delta \geq 3$, then*

$$\frac{n}{\Delta} \leq \alpha(G) \leq \left(\frac{2}{\delta + 2} \right) n.$$

As a consequence of Proposition 3.2 and Theorem 3.6, we have the following result.

Theorem 3.7. *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n , then the following properties hold.*

- (a) $\frac{1}{3}n \leq \alpha(G) \leq \frac{2}{5}n$.
- (b) $\frac{3}{5}n \leq m(G, 3) \leq \frac{2}{3}n$.

We show next that the bounds of Theorem 3.7 are tight (in the sense that they hold for connected graphs of arbitrarily large orders). Suppose that $G \in \mathcal{T}_{\text{cubic}}$. Thus, G is a triangle-necklace F_{2k} for some $k \geq 1$, and so G has order $n = 6k$ and contains $2k$ vertex disjoint triangles. Moreover, every unit of the (unique) Δ -D-partition of G is a triangle-unit, implying that $\alpha(G) \leq \frac{1}{3}n$. By Theorem 3.7(a), $\alpha(G) \geq \frac{1}{3}n$. Consequently, $\alpha(G) = \frac{1}{3}n$. For example, the white vertices in the triangle-necklace F_6 of order $n = 18$ shown in Figure 4 form an α -set of F_6 (of cardinality $6 = \frac{1}{3}n$). By Proposition 3.2, we infer that $m(G, 3) = \frac{2}{3}n$, and the corresponding set of shaded vertices in Figure 4 is a β -set of G . This shows that the lower bound of Theorem 3.7(a) is tight, as is the upper bound of Theorem 3.7(b).

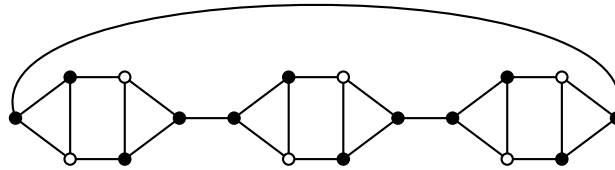


Fig. 4. A β -set in the triangle-necklace F_6

Suppose next that $G \in \mathcal{H}_{\text{cubic}}$. Thus, G is a triangle-diamond-necklace H_{2k} for some $k \geq 2$, and so G has order $n = 10k$. We can choose an independent set of G to contain one vertex from every triangle-unit and two vertices from every diamond-unit, implying that $\alpha(G) \geq 4k = \frac{2}{5}n$. For example, the white vertices in the triangle-diamond-necklace H_6 of order $n = 30$ shown in Figure 5 form an α -set of H_6 (of cardinality $12 = \frac{2}{5}n$). By Proposition 3.2, we infer that $m(G, 3) = \frac{3}{5}n$, and the corresponding set of shaded vertices in Figure 5 is a β -set of H_6 . This shows that the upper bound of Theorem 3.7(a) is tight, as is the lower bound of Theorem 3.7(b).

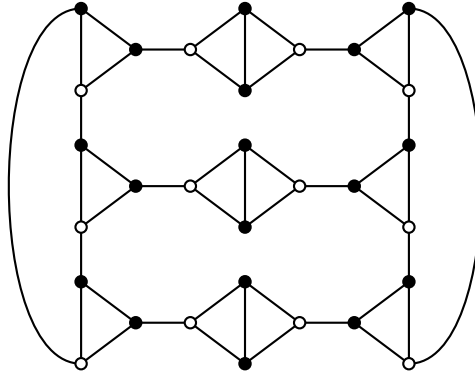


Fig. 5. A β -set in the triangle-diamond-necklace H_6

Note that $m(G, 3)$ in (claw-free) cubic graphs coincides with $\sigma_{(3,q)}(G)$ for all $q \geq 3$, establishing the (3, 3)-entry of Table 1. Hence, by Proposition 3.2, we have $\sigma_{(3,q)}(G) = m(G, 3) = \beta(G) = n(G) - \alpha(G)$ for all $q \geq 3$. The result can be extended to $\sigma_{(3,2)}(G)$, which yields the following result.

Proposition 3.8. *If G is a connected, claw-free cubic graph of order n , then $\sigma_{(3,q)}(G) = \beta(G)$ for every $q \geq 2$.*

Proof. We have already established that $\sigma_{(3,q)}(G) = \beta(G)$ for every $q \geq 3$, so it remains to resolve the case $q = 2$. Consider a 3-percolating set P of a connected, claw-free cubic graph G , and color all its vertices blue. By Proposition 3.2, P is a vertex cover and so $V(G) \setminus P$ is an independent set in G . Now, since an independent set in G contains at most one vertex from each triangle, a vertex cover contains at least two vertices in each triangle. This implies that every vertex in P has a neighbor in P . Therefore, since $\deg_G(x) = 3$ for all $x \in V(G)$, we infer that every (blue) vertex in P has at most two (white) neighbors in $V(G) \setminus P$. Thus, P is a (3, 2)-spreading set. Since this is true for every 3-percolating set P of G , we infer that $\sigma_{(3,2)}(G) \leq m(G, 3) = \beta(G)$.

Conversely, by Observation 1.1, $\sigma_{(3,2)}(G) \geq \sigma_{(3,3)}(G)$. Combining the obtained inequalities, we infer

$$\beta(G) \geq \sigma_{(3,2)}(G) \geq \sigma_{(3,3)}(G) = m(G, 3) = \beta(G). \quad \square$$

We remark that $\sigma_{(3,1)}(G) > \beta(G)$ if $G \in \mathcal{H}_{\text{cubic}}$ is a triangle-diamond necklace. Indeed, in any vertex cover P of G , every vertex in P has a neighbor in P , and so it does not have at most one neighbor in $V(G) \setminus P$.

Recall that two units in the Δ -D-partition are adjacent if there exists at least one edge joining a vertex in one unit to a vertex in the other unit. We say that a unit in the Δ -D-partition of G is *infected* if all vertices in the unit are infected.

Lemma 3.9. *If G is a connected, claw-free cubic graph that contains only triangle-units, then there exists an independent set in G that contains a vertex from every triangle of G .*

Proof. By supposition, every unit in the Δ -D-partition of the connected, claw-free cubic G is a triangle-unit. Since G is diamond-free, we note that the triangle-units in G correspond to the triangles in G . The graph G has order $n = 3t$ where t denotes the number of triangle-units in G . By Theorem 3.7(a), we have $t = \frac{1}{3}n \leq \alpha(G)$. Since every independent set in G contains at most one vertex from every triangle-unit in G , we infer that $\alpha(G) \leq t = \frac{1}{3}n$. Consequently, $\alpha(G) = \frac{1}{3}n = t$ and every maximum independent set in G contains a vertex from every triangle of G . $\square$

Lemma 3.10. *If G is a connected, claw-free cubic graph, then there exists an independent set in G that contains a vertex from every triangle of G .*

Proof. If the connected, claw-free cubic G that contains only triangle-units, then the result follows from Lemma 3.9. Hence, we may assume that G contains at least one diamond-unit. We now construct an independent set I of G as follows. Initially, we set $I = \emptyset$. For each diamond-unit D of G , we add to the set I exactly one of the two vertices of degree 3 in the diamond-unit D . Thereafter, we delete all diamond-units from the graph G . If G contains only diamond-units, then the resulting set I has the desired property that it contains a vertex from every triangle of G . Hence, we may assume that G contains at least one triangle-unit.

Let G' be the graph obtained by deleting all diamond-units from G . We note that every component, C , of G' has the following properties: (1) the component C contains no diamond, (2) every vertex in C belongs to a triangle in the component C and (3) the component C has minimum degree 2 (and maximum degree at most 3). We now select an arbitrary vertex u_1 of degree 2 in C , add the vertex u_1 to the set I , and delete the triangle that contains the vertex u_1 . In the resulting graph G_2 , once again properties (1), (2) and (3) hold, and we select an arbitrary vertex u_2 of degree 2 in G_2 , add the vertex u_2 to the set I , and delete from G_2 the triangle that contains the vertex u_2 . Upon completion of this process, the resulting set I is an independent set in G that contains a vertex from every triangle of G . $\square$

We note that the statement in Lemma 3.10 is equivalent to the following statement.

Lemma 3.11. *If G is a connected, claw-free cubic graph, then there exists a vertex cover P such that every triangle of G contains exactly two vertices from P .*

Proposition 3.12. *If G is a connected, claw-free cubic graph, then*

$$\sigma_{(3,1)}(G) \leq \beta(G) + 1,$$

and this bound is sharp.

Proof. Let G be a connected, claw-free cubic graph, and let P be a 3-percolating set of G satisfying $|P| = m(G, 3) = \beta(G)$. We note that P is a vertex cover of G and $V(G) \setminus P$ is a maximum independent set of G . Clearly, every vertex, which is not in P , has three neighbors in P , and so the first condition of the $(3, 1)$ -spreading rule is always satisfied with respect to P . In addition, due to Lemma 3.11 we may assume that P contains exactly two vertices from every triangle in G , or, equivalently, so that the independent set $V(G) \setminus P$ contains a vertex from every triangle in G . Now,

consider an arbitrary triangle T in G , and let $P^* = P \cup \{v\}$, where v is the unique vertex in $V(T) \setminus P$. Let the vertices of P^* be initially infected, and note that (with respect to P^*) the triangle T is completely infected.

Let u be a vertex that is not yet infected and is adjacent to an infected triangle T' (with all three vertices of T' infected). In particular, we note that $u \notin P^*$. Suppose firstly that u is adjacent to two vertices of T' . Thus, the vertices in $V(T') \cup \{u\}$ induce a diamond-unit. In this case, the vertex u immediately becomes infected, since both of its neighbors in T' have no other uninfected neighbor. As a result, the (unique) triangle containing the vertex u becomes an infected triangle, and so the diamond-unit containing u is infected. Suppose next that u is adjacent to exactly one vertex w' of T' . Once again in this case, the vertex u immediately becomes infected, since its neighbor w' in T' has no other uninfected neighbor. On the other hand, suppose that u is infected and is adjacent to an infected triangle T' (with all three vertices of T' infected), and there exists an uninfected neighbor v of u that belongs to the same unit as u . Then v becomes infected, because u has only one uninfected neighbor, namely v .

In all of the above cases, the unit, which is adjacent to an infected unit, becomes infected. Since G is connected, and initially there is a triangle infected (making the unit in which the triangle lies also infected), we infer that due to the above arguments all vertices in G become infected. Thus, $\sigma_{(3,1)}(G) \leq |P^*| = |P| + 1 = \beta(G) + 1$, thereby proving the desired upper bound. The case of a triangle-diamond necklace $G \in \mathcal{H}_{\text{cubic}}$ shows that the bound is sharp. $\square$

4. 2-PERCOLATION IN CLAW-FREE, CUBIC GRAPHS

In this section, we study 2-percolation in claw-free, cubic graphs. It is easy to see that $m(K_4, 2) = 2$. In what follows we therefore restrict our attention to 2-percolation in connected, claw-free, cubic graphs G different from K_4 . Thus, by Lemma 2.1, such a graph G has a Δ -D-partition (in which every unit is a triangle-unit or a diamond-unit). Recall that $u(G)$ is defined as the number of units in this (unique) Δ -D-partition. In what follows, if D is a diamond-unit in the Δ -D-partition of G , then a *dominating vertex* in D is a vertex that is adjacent to the three other vertices in the diamond-unit D .

Proposition 4.1. *If $G \in \mathcal{N}_{\text{cubic}}$, then $m(G, 2) = u(G) + 1$.*

Proof. Suppose that $G \in \mathcal{N}_{\text{cubic}}$, and so G is a diamond-necklace N_k for some $k \geq 2$. Thus, G has $u(G) = k$ units, each of which is a diamond-unit. Let S be a 2-percolating set of G of minimum cardinality, and so S is a 2-percolating set of G satisfying $|S| = m(G, 2)$. By Lemma 3.1, we infer that the set S contains at least one vertex from every diamond-unit of G . Moreover, if D is a diamond-unit of G and the set S contains exactly one vertex from D , then by Lemma 3.1 we infer that such a vertex of S is a dominating vertex of D . In particular, since S contains at least one vertex from every unit in the Δ -D-partition of G , we note that $m(G, 2) = |S| \geq u(G) = k$.

Suppose that $|S| = k$. By our earlier observations, the set S contains exactly one vertex from every diamond-unit, namely a vertex from each diamond-unit that dominates that unit. The resulting set S is a 2-packing in G ; that is, $d_G(u, v) \geq 3$ for

every two distinct vertices u and v that belong to S . Moreover in this case, letting $H = G - S$, we note that H is a cycle C_{3k} and every vertex in H has exactly one neighbor that belongs to $V(G) \setminus V(H) = S$. Thus by Lemma 3.1, we infer that every 2-percolating set of G contains at least one vertex of H . However, this contradicts our supposition that S is a 2-percolating set of G that contains no vertex of H . Hence, $m(G, 2) = |S| \geq k + 1$.

To establish an upper bound on $m(G, 2)$ in this case when $G = N_k$, if S^* consists of a dominating vertex from $k - 1$ diamond-units and two dominating vertices from the remaining diamond-unit, then S^* is a 2-percolating set of G and $|S^*| = k + 1$. For example, if $G = N_4$ (here $k = 4$), then such a set S^* illustrated in Figure 6 by the shaded vertices is a 2-percolating set of G and $|S^*| = 5$.

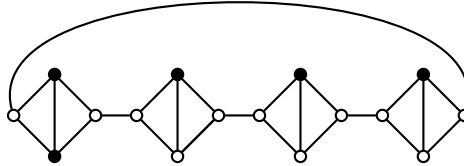


Fig. 6. The diamond-necklace N_4 and a 2-percolating set of size 5

This implies that $m(G, 2) \leq |S^*| = k + 1$. Consequently, $m(G, 2) = k + 1$. Thus in this case when $G \in \mathcal{N}_{\text{cubic}}$, we have shown that $m(G, 2) = u(G) + 1$. $\square$

If two distinct units are joined by at least two edges, then we say that these two units are *double-bonded*.

Lemma 4.2. *If $G \neq K_4$ is a connected, claw-free, cubic graph and $G \notin \mathcal{N}_{\text{cubic}}$, then there exists a triangle-unit T' , such that the graph $G - T'$ has at most two components.*

Proof. Let $\mathcal{T}$ be the set of all triangle-units in G . Suppose to the contrary that for every $T \in \mathcal{T}$ the graph $G - T$ consists of three components. Let $T_1 \in \mathcal{T}$ be a triangle-unit such that at least one of the three components of the graph $G - T_1$ does not contain a triangle-unit. Denote this component by G_1 and let D_1 be the diamond-unit in G_1 adjacent to T_1 . Then in G_1 , D_1 is adjacent to exactly one diamond-unit D_2 , which is in turn adjacent to another diamond-unit D_3 and so on, which yields a contradiction since G is finite. Therefore, G_1 contains a triangle-unit $T' \notin \mathcal{T}$, which is the final contradiction. $\square$

The following observation will be necessary for proving the subsequent theorem.

Observation 4.3. *Let $G \neq K_4$ be a connected, claw-free, cubic graph, and U, V adjacent units in the Δ - D -partition of G . Also, let uv be an edge connecting units U and V , where $u \in U$ and $v \in V$. If u is infected and another vertex $v_1 \in V$, where $d(v_1, u) = 2$, is also infected, then the whole unit V becomes infected.*

Proof. Vertex v has two infected neighbors, namely u and v_1 . After that, if V is a triangle unit, the remaining vertex of V is also infected, and if V is a diamond unit, the remaining two vertices of V become infected after two steps. $\square$

Theorem 4.4. *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n that contains $u(G)$ units, then*

$$m(G, 2) = \begin{cases} u(G) + 1, & \text{if } G \in \mathcal{N}_{\text{cubic}}, \\ u(G), & \text{otherwise.} \end{cases}$$

Proof. By Lemma 3.1, we infer that every 2-percolating set of G contains at least one vertex from every triangle-unit and every diamond-unit of G . Moreover, if D is a diamond-unit of G and a 2-percolating set contains exactly one vertex from D , then by Lemma 3.1 we infer that such a vertex is a dominating vertex of D . In particular, since every 2-percolating set of G contains at least one vertex from every unit in the Δ -D-partition of G , we note that $m(G, 2) \geq u(G)$. If $G \in \mathcal{N}_{\text{cubic}}$, then by Proposition 4.1, $m(G, 2) = u(G) + 1$. Hence, we may assume that $G \notin \mathcal{N}_{\text{cubic}}$, for otherwise the desired result follows.

Since every triangle-unit contributes 3 to the order of the graph and every diamond-unit contributes 4 to the order of the graph, we observe that if G has order n with u_t triangle-units and u_d diamond-units, then $u(G) = u_t + u_d$ and $n = 3u_t + 4u_d$. By assumption, $u_t \geq 1$. Thus, since n is even, $u_t \geq 2$, that is, G contains at least two triangle-units. According to Lemma 4.2, there exists a triangle-unit T_1 of G , where $V(T_1) = \{t_1, t_2, t_3\}$, such that $G - T_1$ consists of at most two components. This implies that at most one of the edges incident with exactly one vertex of T_1 is a bridge. Since G is connected, the triangle-unit T_1 is adjacent to at least one other unit. Let U_1 be a unit that is adjacent to T_1 such that the edge between U_1 and T_1 is not a bridge, or T_1 and U_1 are double-bonded. Finally denote as G_1 the component of $G - T_1$ containing U_1 .

As observed earlier, $m(G, 2) \geq u(G)$. Hence, it suffices for us to show that $m(G, 2) \leq u(G)$. For this purpose, we construct a 2-percolating set S of G that contains exactly one vertex from each unit of G . Initially, we let $S = \emptyset$. We consider three cases. First, suppose that the units T_1 and U_1 are not double-bonded.

Case 1. The unit U_1 is a triangle-unit. Let $V(U_1) = \{a_1, b_1, c_1\}$ and where t_1c_1 is an edge. In this case, we add to S the vertices t_1 and a_1 , and so $S = \{t_1, a_1\}$. The vertices in S are indicated by the shaded vertices in Figure 7(a). Due to Observation 4.3, every vertex in the triangle-unit U_1 becomes infected.

Case 2. The unit U_1 is a diamond-unit. Let $V(U_1) = \{a_1, b_1, c_1, d_1\}$ and where a_1b_1 is the missing edge in U_1 and where t_1b_1 is an edge of G . In this case, we add to S the vertices t_1 and c_1 , and so $S = \{t_1, c_1\}$. The vertices in S are indicated by the shaded vertices in Figure 7(b). Due to Observation 4.3, every vertex in the diamond-unit U_1 becomes infected.

Case 3. The units T_1 and U_1 are double-bonded. Thus, there is a vertex $u \in U_1$ that is joined to a vertex of T_1 different from t_1 . Renaming the vertices t_2 and t_3 if necessary, we may assume that ut_2 is such an edge between the units T_1 and U_1 . In particular,

we note that $t_2 \notin S$ and let $S = \{t_1, u\}$. The vertices in S are indicated by the shaded vertices in Figure 7(c). Again, due to Observation 4.3, every vertex in both units T_1 and U_1 becomes infected. In this case, the set S consists of two vertices, one vertex from each of the units U_1 and T_1 .

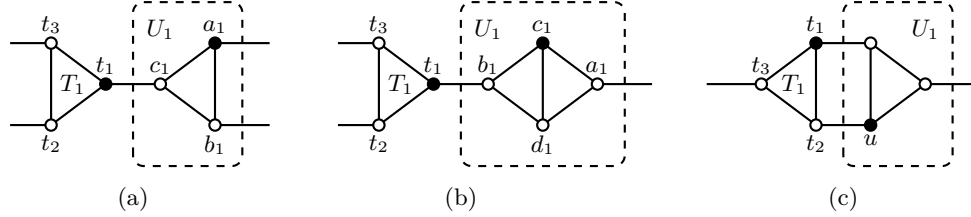


Fig. 7. Possible adjacent units in the proof of Theorem 4.4

Cases 1, 2 and 3 occur in Step 1 of our procedure to construct the set S . Thus after Step 1, the initial set S consists of two vertices, namely a vertex t_1 from the triangle-unit T_1 and one vertex from the unit U_1 that is adjacent to T_1 . Moreover in all three cases, every vertex in the unit U_1 becomes infected. We now proceed to Step 2 of our procedure to construct the set S .

Let U_2 be a unit different from T_1 that is adjacent to the infected unit U_1 , and let uv be an edge that joins a vertex $u \in U_1$ and a vertex $v \in U_2$. Note that if there is no such unit U_2 , then G consist only of units T_1 and U_1 and is now fully infected, yielding the desired result. Since $U_2 \neq T_1$, we note that $v \neq t_1$. In particular, we note that $v \notin S$ since at this stage of the construction the set S only contains the vertex t_1 and one other vertex, namely a vertex from U_1 .

Suppose that U_2 is a triangle-unit. Let $V(U_2) = \{v, v_1, v_2\}$. We now add to the set S exactly one of the vertices v_1 and v_2 . By symmetry, we may assume that v_1 is added to the set S . Once again, due to Observation 4.3, every vertex in the triangle-unit U_2 becomes infected.

Suppose next that U_2 is a diamond-unit. Let $V(U_2) = \{v, v_1, v_2, v_3\}$, where v_1 and v_2 are the dominating vertices of U_2 (and so, vv_3 is the missing edge in D_v). We now add to the set S exactly one of the dominating vertices of U_2 . By symmetry, we may assume that dominating vertex v_1 of U_2 is added to S . We again invoke Observation 4.3, thus every vertex in the diamond-unit U_2 becomes infected.

After Step 2 of our procedure to construct the set S , every vertex in the unit U_2 becomes infected, and the set S contains exactly one vertex from each of the units T_1, U_1 and U_2 .

Suppose that $i \geq 2$ and after Step i of our procedure to construct the set S , the units $U_1, U_2, \dots, U_i$ are all infected, and the set S contains exactly one vertex from each of the units $U_1, U_2, \dots, U_i$. Moreover, if $T_1 = U_j$ for some $j \in [i] \setminus \{1\}$, then after Step i we have $|S| = i$ and the set S contains exactly one vertex from each of the units $U_1, U_2, \dots, U_i$, while if $T_1 \neq U_j$ for any $j \in [i]$, then $|S| = i + 1$ and S contains one vertex from each of the units $U_1, U_2, \dots, U_i, T_1$.

In Step $i + 1$, let U_{i+1} be a unit different the units $U_1, U_2, \dots, U_i$ that is adjacent to an infected unit U_j for some $j \in [i]$, and let uv be an edge that joins a vertex $u \in U_j$ and a vertex $v \in U_{i+1}$.

Suppose that U_{i+1} is a triangle-unit. Let $V(U_{i+1}) = \{v, v_1, v_2\}$. In this case, either $U_{i+1} = T_1$ and renaming vertices of T_1 if necessary we may assume that $v_1 = t_1$, or $U_{i+1} \neq T_1$, and we add to the set S the vertex v_1 . Suppose next that U_{i+1} is a diamond-unit. Let $V(U_{i+1}) = \{v, v_1, v_2, v_3\}$, where v_1 and v_2 are the dominating vertices of U_{i+1} (and so, vv_3 is the missing edge in D_v). We now add to the set S the dominating vertex v_1 of U_{i+1} . In either of these cases the vertices u and v_1 satisfy the conditions of Observation 4.3, therefore the whole unit U_{i+1} becomes infected.

Continuing in the described way, the above process continues until every unit of G_1 and also the unit T_1 become infected. If $G - T_1$ is connected, then all vertices of G have become infected and S contains exactly one vertex from each unit of G , thus the proof is complete. Otherwise, if $G - T_1$ is not connected, then it has at most two components, and let G_2 be the component of $G - T_1$ different from G_1 . Let U'_1 be the unit in G_2 that is adjacent to T_1 , and let v_2a be the bridge of G connecting T_1 with U'_1 . Now, adding a dominating vertex b of U'_1 , different from a , we infer that all vertices of U'_1 become infected (using the analogous arguments as in the previous cases). Now, we can continue with the same process, continuing by infecting the units U'_i in G_2 that are adjacent to an already infected unit by adding to S appropriately selected dominating vertex of U'_i . Since G_2 is connected, we infer that all vertices of G_2 become infected, where we put in S exactly one vertex of each unit. Thus, S is a 2-percolating set of G , implying that $m(G, 2) \leq u(G)$. As observed earlier, $m(G, 2) \geq u(G)$. Consequently, $m(G, 2) = u(G)$. This completes the proof of Theorem 4.4. $\square$

By definition of a (p, q) -spreading set in a graph, if G is cubic graph, then a set is a $(2, 3)$ -spreading set if and only if it is 2-percolating set, implying that $\sigma_{(2,3)}(G) = m(G, 2)$. As an immediate consequence of Theorem 4.4, we infer the following result.

Corollary 4.5. *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n that contains $u(G)$ units, then*

$$\sigma_{(2,3)}(G) = \begin{cases} u(G) + 1, & \text{if } G \in \mathcal{N}_{\text{cubic}}, \\ u(G), & \text{otherwise.} \end{cases}$$

We are now in a position to establish the following relationships between the $(2, 2)$ -spreading number and the $(2, 3)$ -spreading number of a connected, claw-free, cubic graph.

Proposition 4.6. *If $G = K_4$ or if $G \in \mathcal{N}_{\text{cubic}}$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$.*

Proof. If $G = K_4$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G) = 2$. Hence, we may assume that $G \neq K_4$. Suppose that $G \in \mathcal{N}_{\text{cubic}}$. By Corollary 4.5, we have $\sigma_{(2,3)}(G) = u(G) + 1$. Moreover, as shown in the proof of Proposition 4.1, if $G = N_k$ for some $k \geq 2$, then we can choose the $(2, 3)$ -spreading set S of G to consists of a dominating vertex from $k - 1$

diamond-units and two dominating vertices from the remaining diamond-unit in G (as illustrated in Figure 6 in the case when $k = 4$). However, such a set S is also a $(2, 2)$ -spreading set S of G , and so $\sigma_{(2,2)}(G) \leq |S| = \sigma_{(2,3)}(G)$. By Observation 1.1, $\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G)$. Consequently, if $G \in \mathcal{N}_{\text{cubic}}$, then $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$. $\square$

Proposition 4.7. *If G is a connected, claw-free, cubic graph, then*

$$\sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G) + 1.$$

Proof. Let G be a connected, claw-free, cubic graph. If $G = K_4$ or if $G \in \mathcal{N}_{\text{cubic}}$, then by Proposition 4.6 we have $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$. Hence, we may assume that $G \neq K_4$ and $G \notin \mathcal{N}_{\text{cubic}}$. To prove that $\sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G) + 1$, let S be a $(2, 3)$ -spreading set of G as constructed in the proof of Theorem 4.4. Adopting the notation from the proof of Theorem 4.4, in Step 1 of our procedure to construct the set S we add one additional vertex to the set S as follows. If the unit U_1 is a triangle-unit (see Figure 7(a)), then we add the common neighbor of t_1 and a_1 , namely the vertex c_1 , to the set S . If the unit U_1 is a diamond-unit (see Figure 7(b)), then we add the common neighbor of t_1 and c_1 , namely the vertex b_1 , to the set S . Let S^* denote the resulting set upon completion of the construction of the set S , and so $|S^*| = |S| + 1 = \sigma_{(2,3)}(G) + 1$. With the addition of the vertex c_1 in Case 1 or the vertex b_1 in Case 2, the set S^* is a $(2, 2)$ -spreading set S of G , implying that $\sigma_{(2,2)}(G) \leq |S^*| = |S| + 1 = \sigma_{(2,3)}(G) + 1$. $\square$

Every $(2, 3)$ -spreading set of G is by definition also a $(2, 2)$ -spreading set of G , and so $\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G)$. Hence, as a consequence of Proposition 4.7 and applying also Corollary 4.5, we have the following result.

Corollary 4.8. *If G is a connected, claw-free, cubic graph, then*

$$\sigma_{(2,3)}(G) \leq \sigma_{(2,2)}(G) \leq \sigma_{(2,3)}(G) + 1.$$

In addition,

$$u(G) \leq \sigma_{(2,2)}(G) \leq u(G) + 1.$$

If $G = F_{2k} \in \mathcal{T}_{\text{cubic}}$ then it is not difficult to see that $S = \{z_i, : i \in [2k-1]\} \cup \{x_{2k}\}$ is a $(2, 2)$ -spreading set. Notably, the process starts with the infection of vertex x_1 , and then continues with y_1, y_2 and so on. Therefore, $\sigma_{(2,2)}(G) = u(G)$. On the other hand, if $G \in \mathcal{N}_{\text{cubic}}$, then $\sigma_{(2,2)}(G) = u(G) + 1$. In both of these cases it holds that $\sigma_{(2,2)}(G) = \sigma_{(2,3)}(G)$, however this does not hold in general. For instance, see the graph G in Figure 8 for which we have $\sigma_{(2,3)} = u(G)$, where the shaded vertices indicated in the figure form a $(2, 3)$ -spreading set of G . Yet, one can verify that $\sigma_{(2,2)} = u(G) + 1$.

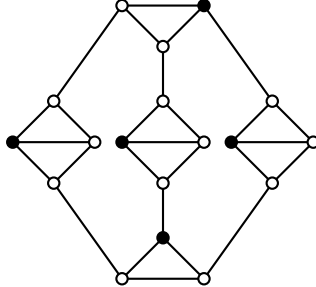


Fig. 8. A graph G with $\sigma(2, 3) = u(G)$ and $\sigma(2, 2) = u(G) + 1$

Finally, we consider $(2, 1)$ -spreading, and bound the corresponding invariant in claw-free cubic graphs. Again we see that it can achieve only two possible values.

Theorem 4.9. *If $G \neq K_4$ is a connected, claw-free, cubic graph of order n that contains $u(G)$ units, then*

$$u(G) + 1 \leq \sigma_{(2,1)}(G) \leq u(G) + 2.$$

Proof. First note that due to Lemma 3.1 a $(2, 1)$ -spreading set S needs to contain at least one vertex in each unit of $G \neq K_4$. Now, if S is a set of vertices that contains exactly one vertex from each unit of G , then every vertex $v \in S$ has at least two neighbors in $V(G) \setminus S$, and therefore such a set S cannot be a $(2, 1)$ -spreading set. Hence, $u(G) + 1 \leq \sigma_{(2,1)}(G)$.

We show next that $\sigma_{(2,1)}(G) \leq u(G) + 2$ by constructing a $(2, 1)$ -spreading set S of G satisfying $|S| \leq u(G) + 2$. Let T be an arbitrary triangle in G (that may also be part of a diamond-unit). Initially, we let $S = V(T)$, and so the vertices in T form the set of initially infected vertices. We now enlarge the set S as follows. Suppose that T belongs to a diamond-unit D . In this case, we let v be the vertex in the diamond-unit D that is not in T . The vertex v immediately becomes infected in the $(2, 1)$ -spreading process, since both of its neighbors in T have no other uninfected neighbor. As a result, all vertices in the diamond-unit D become infected, that is, D becomes an infected unit. Suppose next that T is a triangle-unit of G . Then T is already an infected unit.

Finally let T' be the infected unit containing T (that is, $T' = D$ or $T' = T$). We now extend the set S in the same manner as in the proof of Theorem 4.4. Let U be a unit adjacent to T' . If U is a triangle-unit, then we add to S a vertex in U that is not adjacent to any vertex in the unit T' . If U is a diamond-unit, then we add to S a vertex in U that is a dominating vertex of the unit U . Proceeding analogously as in the proof of Theorem 4.4, this results in all vertices of the unit U becoming infected in the $(2, 1)$ -spreading process.

Continuing in this way by considering a vertex not yet infected that is adjacent to an infected triangle, we obtain a $(2, 1)$ -spreading set S of G starting with the set $V(T)$ and adding exactly one vertex from every unit of G that does not contain the triangle T . Thus, $\sigma_{(2,1)}(G) \leq |S| = u(G) + 2$. $\square$

5. CONCLUDING REMARKS

We end this paper with some remarks concerning the computational complexity of determining $\sigma_{(p,q)}(G)$ in claw-free cubic graphs G . We have already mentioned that it is unclear whether one can determine the zero forcing number of a claw-free cubic graph efficiently. For some of the values of p and q , where $p > 1$ or $q > 1$, there exists a polynomial-time algorithm to determine $\sigma_{(p,q)}(G)$. First, determining the independence number in claw-free graphs can be done in polynomial time (see [24], where a polynomial-time algorithm is presented even for the weighted version of the problem). Therefore, the problem of determining $\sigma_{(3,q)}(G)$ is polynomial in claw-free cubic graphs G for all $q \geq 2$. Similarly, one can efficiently determine the number of units in a claw-free cubic graph. Thus, the problem of determining $\sigma_{(2,q)}(G)$ is also polynomial in claw-free cubic graphs G for any $q \geq 3$.

The following problems remain open:

Problem 5.1. Provide a structural characterization of the connected claw-free cubic graphs G for which $\sigma_{(2,2)} = u(G)$ (resp., $u(G)+1$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

Problem 5.2. Provide a structural characterization of the connected claw-free cubic graphs G for which $\sigma_{(2,1)} = u(G) + 1$ (resp., $u(G) + 2$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

Problem 5.3. Provide a structural characterization of the connected claw-free cubic graphs G for which $\sigma_{(3,1)} = \beta(G)$ (resp., $\beta(G) + 1$). Is there a polynomial-time algorithm to recognize the corresponding classes of connected claw-free cubic graphs?

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