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Research Paper

Edge-transitive cubic graphs: analysis, cataloguing and enumeration

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ABSTRACT

This paper deals with finite cubic (3-regular) graphs whose automorphism group acts transitively on the edges of the graph. Such graphs split into two broad classes, namely arc-transitive and semisymmetric cubic graphs, and then these divide respectively into 7 types (according to a classification by Djoković and Miller (1980) [17]) and 15 types (according to a classification by Goldschmidt (1980) [23]), in terms of certain group amalgams. Such graphs of small order were previously known up to orders 2048 and 768, respectively, and we have extended each of the two lists of all such graphs up to order 10000. Before describing how we did that, we carry out an analysis of the 22 amalgams, to show which of the finitely-presented groups associated with the 15 Goldschmidt amalgams can be faithfully embedded in one or more of the other 21 (as subgroups of finite index), complementing what is already known about such embeddings of the 7 Djoković-Miller groups in each other. We also give an example of a graph of each of the 22 types, and in most cases, describe the smallest such graph, and we then use regular coverings to prove that there are infinitely many examples of each type. Finally, we discuss the asymptotic enumeration of the graph orders, proving that if $f_{\mathcal{C}}(n)$ is the number of cubic edge-

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transitive graphs of type $\mathcal{C}$ on at most n vertices, then there exist positive real constants a and b and a positive integer n_0 such that $n^{a\log(n)} \leq f_{\mathcal{C}}(n) \leq n^{b\log(n)}$ for all $n \geq 0$.

1. Introduction

This paper deals with *cubic edge-transitive graphs*, or more precisely, 3-regular graphs whose automorphism group acts transitively on the edges of the graph. The reader may assume throughout the paper that all the graphs discussed are simple and connected, and moreover, with the exception of the (infinite) 3-valent tree $\mathcal{T}_3$, all of them are finite.

In general, every edge-transitive regular graph with odd valency has one of two symmetry types, depending on whether its automorphism group acts transitively on the vertices or not. If the action of the automorphism group on the vertices is transitive, then it is also transitive on the arcs (where an *arc* is defined as an ordered pair of adjacent vertices), and in that case the graph is said to be *arc-transitive*, or sometimes *symmetric*. On the other hand, if the action is intransitive on the vertices, then the graph is said to be *semisymmetric*, and in that case the graph is bipartite with the two vertex-orbits forming the bipartition.

The study of cubic edge-transitive graphs is one the oldest topics in algebraic graph-theory, going back to Tutte's ingenious discovery and proof of an upper bound of 32 on the order of a finite edge-stabiliser in the automorphism group of an arc-transitive cubic graph [32], and an analogous theorem of Goldschmidt [23] for semisymmetric cubic graphs. Tutte's and Goldschmidt's work had a profound impact not only on this particular branch of graph theory, but also on the development of local analysis and amalgams in group theory. The latter also played (and still do play) an important role in the Classification of Finite Simple Groups project.

Research on cubic edge-transitive graphs has in large part been driven and facilitated by the amazing and famous Foster census [5], which contained information on almost all of the cubic arc-transitive graphs on up to 512 vertices, and its later completion and extension up to 768 vertices, by Conder and Dobcsányi [9], later complemented by the authors of the current paper jointly with Malnič and Marušič [11] in producing a list of all semisymmetric cubic graphs on up to 768 vertices. The list of symmetric graphs in [9] was extended further in 2006 by the first author of the current paper, up to order 2048, and later in 2011 up to order 10000 (see [6]). These lists contained several previously undiscovered small graphs. The third smallest graph on the list of semisymmetric graphs in [11], on 112 vertices, was the subject of a separate paper [12].

These lists of graphs have subsequently been used to test conjectures and to search for potential counterexamples to others, as well as to make new discoveries, look for patterns, and formulate new conjectures, as for example in [16,18–22,27,29,31].

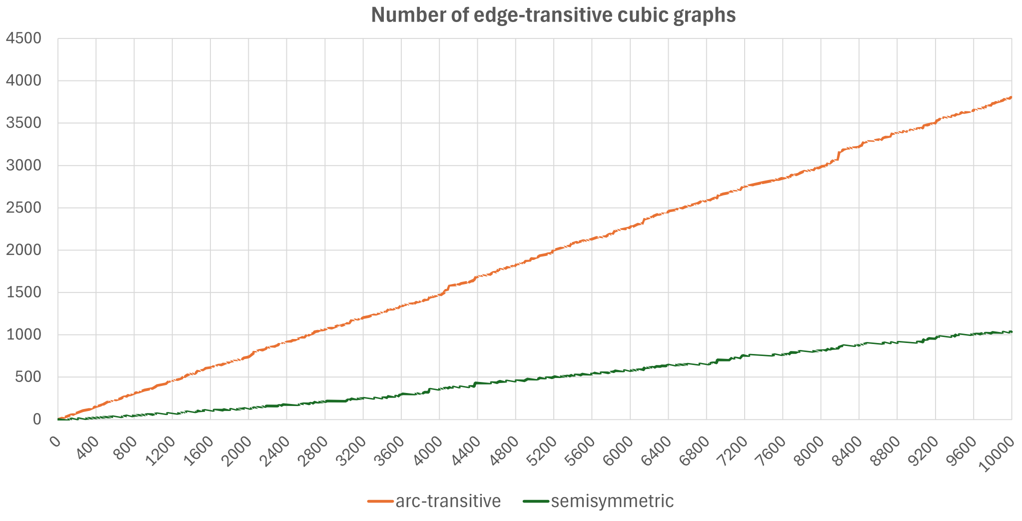

Fig. 1. Numbers of edge-transitive cubic graphs of up to given order ($\leq$ 10000).

One of the main aims of the current paper is to announce, document and explain the construction of a much larger census, namely one of all cubic edge-transitive on at most 10000 vertices. We can now state the following new theorem (accompanied by Fig. 1 which illustrates the growth in the number of them), while postponing a more detailed description until Section 3.

Theorem 1. *Up to isomorphism there are precisely* 4858 *connected finite edge-transitive cubic graphs on up to* 10000 *vertices, with* 3815 *of them being arc-transitive, and the other* 1043 *being semisymmetric.*

Remark 2. The 4858 cubic edge-transitive graphs themselves are stored online at [14], with summary data also available at [6,7]. At [14], for each admissible n we provide files 'CAT_n.s6' and 'CSS_n.s6' containing the sparse6 codes [25] of cubic arc-transitive and cubic semisymmetric graphs of order n, respectively, with one graph per line. For future reference, we denote the graph in the kth row of the file CAT_n.s6 as CAT(n, k) and the one in the kth row of the file CSS_n.s6 as CSS(n, k). Additionally, several graph invariants have been pre-computed, and are presented at [14] in a tabular format. Moreover, hamiltonicity of each of these graphs was tested using the `hamheuristic` command from the `gtools` part of `nauty` [26]. Apart from two well-known exceptions (the Petersen graph CAT(10,1) and the Coxeter graph CAT(28,1)), each of the cubic edge-transitive graphs in the list was found to contain a hamilton cycle. Hence in particular, all of them have a hamilton path.

Let us now turn our attention to the only infinite graph that we will be interested in, namely the 3-regular tree $\mathcal{T}_3$.

The automorphism group of $\mathcal{T}_3$, when equipped with the topology of point-wise convergence, is a locally compact totally disconnected topological group. A subgroup G of $\mathrm{Aut}(\mathcal{T}_3)$ is discrete in this topology if and only if the stabiliser in G of every vertex $v \in \mathrm{V}(\mathcal{T}_3)$ is finite. By theorems of Tutte [32], Djoković and Miller [17] and Goldschmidt [23], there are precisely seven conjugacy classes of discrete arc-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$, and precisely fifteen conjugacy classes of discrete edge- but not vertex-transitive (and therefore semisymmetric) subgroups of $\mathrm{Aut}(\mathcal{T}_3)$.

The seven classes of arc-transitive groups were determined and explicitly described in [17], with alternative descriptions as finitely-presented groups given in [10], and will be denoted here by DjM_1, DjM_2^1, DjM_2^2, DjM_3, DjM_4^1, DjM_4^2 and DjM_5. The fifteen classes of semisymmetric groups were determined in [23] and explicitly described as finitely-presented groups in [11], and will be denoted here by G_1, G_1^1, G_1^2, G_1^3, G_2, G_2^1, G_2^2, G_2^3, G_2^4, G_3, G_3^1, G_4, G_4^1, G_5 and G_5^1. (Incidentally, in the presentation for G_3 in [11], the final relator should be $cdydy$, not $cdyd$. Curiously, it could also be replaced by $cd^{-1}ydy$, under an isomorphism that inverts each of the generators c, d, x and y, which we will mention later in the paper.)

Each of the $7 + 15 = 22$ classes of edge-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$ comes from an 'amalgam' of finite groups, as we explain briefly below.

It follows from Bass-Serre theory that every discrete edge-transitive subgroup of $\mathrm{Aut}(\mathcal{T}_3)$ can be obtained as an amalgamated free product $A *_C B$ of a triple of groups (A, B, C) with $C = A \cap B$. The latter is called an *amalgam* of groups. In our context, for some connected finite cubic graph Γ, the following hold (with $u \neq v$ in each case):

(1) in the arc-transitive case: $A = G_v$, $C = G_{uv}$ and $B = G_{\{u,v\}}$ are the stabilisers of an incident vertex-arc-edge triple in an arc-transitive group G of automorphisms of Γ, while
(2) in the semisymmetric case: $A = G_v$, $C = G_{uv}$ and $B = G_u$ are the stabilisers of an incident vertex-arc-vertex triple in a semisymmetric group G of automorphisms of Γ.

In all of the 22 amalgams associated with edge-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$, the groups A, B and C are finite. Moreover, no non-trivial subgroup of C is normal in both A and B, by faithfulness of the action of the group G on the edges of the graph Γ, and also $(|A:C|, |B:C|) = (3, 3)$ or $(3, 2)$, depending on whether the action of G on Γ is semisymmetric or arc-transitive. Amalgams satisfying these properties are said to be *finite* and *simple*, with *index* $(3, 3)$ or $(3, 2)$ respectively.

Conversely, if (A, B, C) is a finite simple amalgam with index $(3, 3)$ or $(3, 2)$, then the corresponding amalgamated product $A *_C B$ is obtainable from a representative of exactly one of the 22 conjugacy classes of discrete edge-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$.

In fact, this establishes a bijective correspondence between the isomorphism classes of simple amalgams of index $(3, 3)$ and $(3, 2)$ and the 22 conjugacy classes of edge-transitive discrete subgroups of $\mathrm{Aut}(\mathcal{T}_3)$, with the 15 amalgams of index $(3, 3)$ corresponding to semisymmetric subgroups, and the 7 amalgams of index $(3, 2)$ to arc-transitive subgroups.

(Here an *isomorphism* of two amalgams is to be understood in an appropriate category of diagrams $A \leftarrow C \rightarrow B$.)

We will sometimes abuse notation and use the above symbols for the 22 classes of edge-transitive discrete subgroups of $\mathrm{Aut}(\mathcal{T}_3)$ to denote specific representatives of these classes, rather than the classes themselves, and for reasons that will become apparent later, we will also call the seven classes of discrete arc-transitive groups the *Djoković-Miller classes* (or *Djoković-Miller amalgams*), and the fifteen classes of discrete semisymmetric groups the *Goldschmidt classes* (or *Goldschmidt amalgams*).

The importance for finite cubic edge-transitive graphs of the above classification of discrete edge-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$ is revealed by the following folklore theorem, which again follows from Bass-Serre theory. We state it in the context of cubic edge-transitive graphs, but of course it also holds in a more general setting.

Also we note in advance that if N is a normal subgroup of the class representative group $\tilde{G}$, then $\mathcal{T}_3/N$ is the *quotient graph*, whose vertices are the orbits of N on $\mathrm{V}(\mathcal{T}_3)$, with two such orbits adjacent whenever they contain a pair of vertices that are adjacent in $\mathcal{T}_3$.

Theorem 3. *Let* Γ *be a connected finite cubic graph, and let* G *be a group that acts transitively on the edges of* Γ. *Then there exists a unique conjugacy class of discrete edge-transitive subgroups of* $\mathrm{Aut}(\mathcal{T}_3)$ *such that for one (and hence every) representative* $\tilde{G}$ *of that class, there exists a normal subgroup* $N \trianglelefteq \tilde{G}$ *with the following properties:*

(1) *N is a free group of finite rank β, where $\beta = |\mathrm{V}(\Gamma)| - |\mathrm{E}(\Gamma)| + 1$ is the Betti number of Γ;*
(2) *N acts semiregularly on the vertices and on the edges of $\mathcal{T}_3$, and the quotient graph $\mathcal{T}_3/N$ is isomorphic to Γ;*
(3) *the quotient group $\tilde{G}/N$ is isomorphic to G, and what is more, any isomorphism between Γ and $\mathcal{T}_3/N$, when taken together with an appropriate isomorphism from G to $\tilde{G}/N$, yields an isomorphism between the action of G on $\mathrm{V}(\Gamma)$ and the obvious action of $\tilde{G}/N$ on $\mathrm{V}(\mathcal{T}_3/N)$.*

The above theorem reduces the problem of finding all connected cubic edge-transitive graphs with at most m edges to the problem of finding all normal subgroups of index up to cm in a representative $\tilde{G}$ from each of the Djoković-Miller classes and each of the Goldschmidt classes, where c is the order of the associated edge-stabiliser in $\tilde{G}$.

Of course the automorphism group of the graph $\Gamma \cong \mathcal{T}_3/N$ might be larger than $G \cong \tilde{G}/N$. (In particular, if G acts semisymmetrically on Γ, it can often happen that Γ is vertex-transitive and hence arc-transitive.) In that case $\mathrm{Aut}(\Gamma)$ is a quotient of some amalgam 'larger' than $\tilde{G}$, containing $\tilde{G}$ as a subgroup of finite index.

Accordingly, although the 22 classes of discrete edge-transitive of subgroups of $\mathrm{Aut}(\mathcal{T}_3)$ are well understood, and descriptions of their representatives as finitely-presented groups are known, it is important to understand the possible inclusions of the associated amal-

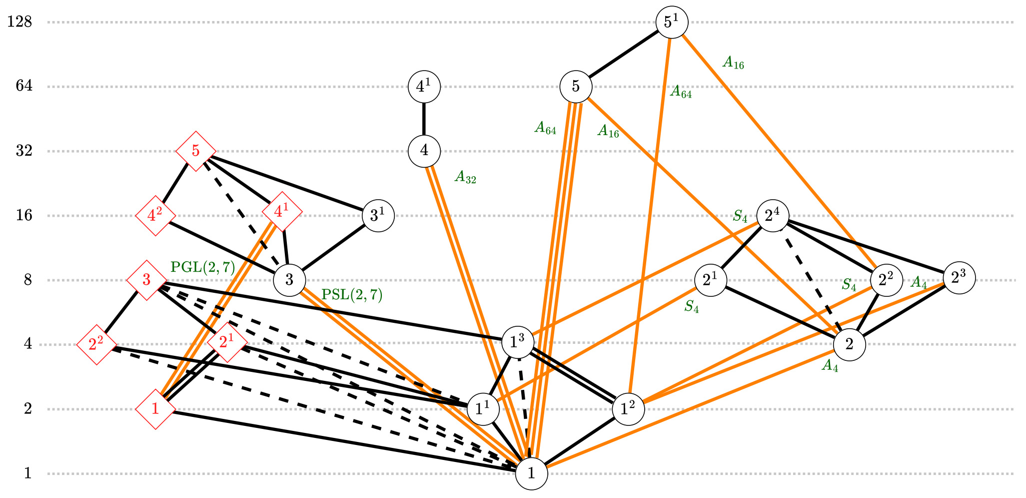

Fig. 2. The 22 classes of discrete edge-transitive subgroups of $\mathrm{Aut}(\mathcal{T}_3)$.

gams in each other. Such inclusions are already known for the seven Djoković-Miller classes in the arc-transitive case, as explained in [17] and taken further in [10] and [13], and we complement that work by dealing with inclusions of each of the 15 Goldschmidt classes in one or more of the other 21.

To be more precise, suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two of the 22 classes. If a representative of $\mathcal{C}_1$ is a subgroup of a representative of $\mathcal{C}_2$, then we say that $\mathcal{C}_1$ is *included* in $\mathcal{C}_2$, and write $\mathcal{C}_1 \le \mathcal{C}_2$. Moreover, if $\mathcal{P}$ is a property that is meaningful for a pair of groups, one contained in the other, such as 'normal', or 'non-normal', or 'maximal', or 'non-maximal', then we say that $\mathcal{C}_1$ is $\mathcal{P}$-included in $\mathcal{C}_2$ provided that representatives $X \in \mathcal{C}_1$ and $Y \in \mathcal{C}_2$ can be chosen so that the pair $X \le Y$ has property $\mathcal{P}$. It is not entirely obvious, but we will indirectly show that if $\mathcal{C}_1$ is normally included in $\mathcal{C}_2$, then it cannot be non-normally included in $\mathcal{C}_2$. There are cases, however, in which $\mathcal{C}_1$ is both maximally and non-maximally included in $\mathcal{C}_2$. For example, a representative G of G_5 contains representatives of G_1 as maximal subgroups, but also other representatives of G_1 that are not maximal in G, being contained in a representative of G_2, which in turn is maximal in G. Similarly, representatives of G_1^2 are included in representatives of G_5^1 both maximally as well as non-maximally via G_2^2. Moreover, as can be seen from Fig. 2, the are no other examples of simultaneous maximal and non-maximal inclusions among Goldschmidt and Djoković-Miller classes. Details can be found in Subsection 2.1.

We can now state the second of our main new theorems:

Theorem 4. *The mutual inclusions of the* 22 *conjugacy classes of discrete edge-transitive subgroup of* $\mathrm{Aut}(\mathcal{T}_3)$ *are as indicated in Fig. 2.*

In Fig. 2, the classes are arranged in horizontal layers, with each layer corresponding to the order (indicated by the number left of each layer) of the edge-stabiliser in a representative of the class.

The seven Djoković-Miller classes are represented by diamond shapes of red colour, while the fifteen Goldschmidt classes are encircled and written in black. Solid lines represent 'maximal inclusions' (inclusions of a representative subgroup as a maximal subgroup of a representative subgroup of another). In particular, black solid lines represent normal maximal inclusions, while orange solid lines represent non-normal maximal inclusions. Of course all index 2 subgroups are maximal.

We have also indicated all normal inclusions that are not maximal: they are depicted by dashed black lines. The double black line between the Djoković-Miller classes DjM_1 and DjM_2^1 and between Goldschmidt classes G_1^2 and G_1^3 means that the larger group of one type contains two smaller normal subgroups of the other type. Similarly, double and triple orange lines (such as between DjM_1 and DjM_4^1, or between G_1 and G_5) indicate that the larger group contains two or three conjugacy classes of maximal subgroups of a given type.

The green text next to the orange lines gives information about the quotient of the top group, say K, by the core of the (non-normal) subgroup, say J. When the core is the alternating group A_n or symmetric group S_n, then the conjugacy class always has size n. On the other hand, the group in the case of the group $\mathrm{PSL}(2,7)$, the size of the conjugacy class is 8 and the action on the corresponding coset space is as on the projective line $\mathrm{PG}(1,7)$.

More details of the inclusions and further explanations are given in Section 2.

A quick look at Fig. 2 reveals that among the Goldschmidt classes, only the classes G_1, G_1^j for $j \in \{1,2,3\}$, G_3 and G_3^1 are included in any of the Djoković-Miller classes. This has the following immediate consequence:

Corollary 5. *If a finite cubic graph admits a semisymmetric group of automorphisms of type* G_2, G_2^j *for* $j \in \{1,2,3,4\}$, G_4, G_4^1, G_5 *or* G_5^1, *then the graph is semisymmetric.*

Let us now turn our attention back to the graphs. As we pointed out in Theorem 3, each group G acting edge-transitively on a finite cubic graph is a quotient of a representative of exactly one of the 22 Djoković-Miller and Goldschmidt classes, and in that case we will say that this class is the *type* of the group G. In fact, each class $\mathcal{C}$ can be realised in a finite edge-transitive cubic graph, in the sense that there exists such a graph Γ and an edge-transitive group G of automorphisms of Γ such that G is of type $\mathcal{C}$.

This can be seen directly, by exhibiting a carefully chosen sample of finite cubic graphs whose automorphism groups contain edge-transitive subgroups. In particular, the Djoković-Miller classes DjM_1, DjM_2^1, DjM_2^2 and DjM_2^3 occur in the complete bipartite graph $\mathrm{K}_{3,3}$, denoted by $\mathrm{CAT}(6,1)$ in our census, while the types DjM_4^1, DjM_4^2 and DjM_5 occur in Tutte's 8-cage on 30 vertices, denoted by $\mathrm{CAT}(30,1)$. Moreover, $\mathrm{K}_{3,3}$ and Tutte's 8-cage contain semisymmetric groups of types G_1, G_1^1, G_1^2, G_1^3, and G_3^1, G_3^2, respectively.

Similarly, types G_2, G_2^1, G_2^2, G_2^3 and G_2^4 are realised in the smallest cubic semisymmetric graph, namely Marion Gray's graph on 54 vertices, denoted by $CSS(54,1)$, while the types G_4 and G_4^1 occur in the incidence graph of the generalised hexagon, denoted by $CSS(126,1)$, and the types G_5 and G_5^1 occur in the unique semisymmetric graph on 990 vertices (of girth 16 and diameter 12), denoted by $CSS(990,1)$.

Alternatively, one can prove the same fact using a theorem of Baumslag [1, Theorem 2], which states that an amalgamated free product of finite groups is residually finite.

It is not so immediately clear, however, that (or how) each of the 22 classes can be realised in terms of the (full) automorphism group of some cubic edge-transitive graph.

Let us say that a conjugacy class $\mathcal{C}$ of discrete edge-transitive subgroup of $\mathrm{Aut}(\mathcal{T}_3)$ has a *strong realisation* in a graph Γ, provided that $\mathrm{Aut}(\Gamma)$ is of type $\mathcal{C}$. When that happens, we shall say that the graph Γ has *type* $\mathcal{C}$. As we will show in this paper, all Djoković-Miller and Goldschmidt classes can indeed be realised strongly. We can prove this is in two different ways: first constructively, by finding examples among the graphs of order at most 10000, or by ad-hoc constructions when no such small examples exist, and then also non-constructively, using the theory of lifting groups along regular covering projections of graphs. The latter approach proves for each type $\mathcal{C}$ the existence of not just one but an infinite family of examples of graphs strongly realising $\mathcal{C}$. To summarise, we prove the following:

Theorem 6. *For each conjugacy class $\mathcal{C}$ of discrete edge-transitive subgroups of* $\mathrm{Aut}(\mathcal{T}_3)$*, there exists an infinite family of finite cubic edge-transitive graphs whose automorphism groups are of type $\mathcal{C}$.*

Existence of an infinite family of graphs strongly realising each Djoković-Miller or Goldschmidt class raises a question about the growth of the number of such graphs with respect to their order. For a conjugacy class $\mathcal{C}$ of discrete edge-transitive subgroup of $\mathrm{Aut}(\mathcal{T}_3)$, let $f_{\mathcal{C}}(n)$ denote the number of cubic edge-transitive graphs of type $\mathcal{C}$ on at most n vertices. The sums of all $f_{\mathcal{C}}(n)$ for $n \leq 10000$ over all Djoković-Miller classes $\mathcal{C}$, as well as the sums over all Goldschmidt classes, are shown in Fig. 1.

A quick look might suggest that the growth of these sums is roughly linear. A similar conclusion might be drawn from the plots of each of the corresponding 22 functions $f_{\mathcal{C}}(n)$; see Fig. 3.

The lines on the left-hand part of Fig. 3, from the top to bottom, represent the classes DjM_1, DjM_2^1, DjM_3, DjM_4^1, DjM_2^2, DjM_5 and DjM_4^2, with the last four essentially indistinguishable. On the right-hand part, the Goldschmidt classes appear in the order G_1, G_1^1, G_1^3, G_1^2, then G_2^1 and G_2^4 almost overlapping, similarly, G_4^1, G_3^1, and G_5^1 lying just above the horizontal axis, and the lines corresponding to G_2, G_2^2, G_2^3, G_3, G_4 and G_5 coinciding with the horizontal axis (since there are no graphs of these types having order at most 10000).

But such conclusions about linear growth of the sums of all $f_{\mathcal{C}}(n)$ for a given class $\mathcal{C}$ would be far from true, because we prove the following in Section 6:

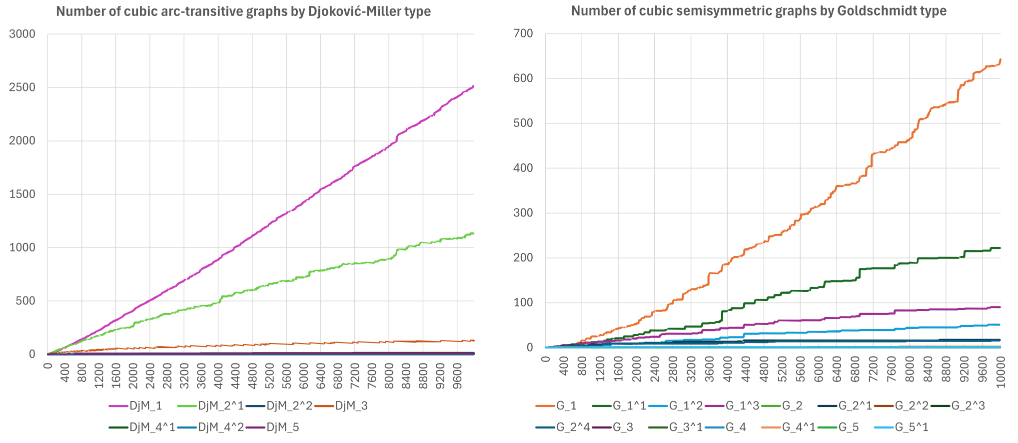

Fig. 3. Numbers of arc-transitive and semisymmetric cubic graphs of each type.

Theorem 7. *For each conjugacy class* $\mathcal{C}$ *of discrete edge-transitive subgroups of* $\mathrm{Aut}(\mathcal{T}_3)$, *there exist positive real constants* a *and* b *and a positive integer* n_0 *such that*

$$n^{a\log(n)} \leq f_{\mathcal{C}}(n) \leq n^{b\log(n)} \quad \text{for all } n \geq n_0.$$

This asymptotic enumeration theorem counters not only an impression that one might gain by considering graphs of orders up to 10000, but also a widely spread belief that cubic semisymmetric graphs (or even those of a given class) are rare.

The fact that the number of cubic edge-transitive graphs grows faster than any polynomial function might also suggest that the set of their orders represents a significant proportion of the positive integers. This expectation is to some extent supported by the empirical data that can be derived from our lists of cubic edge-transitive graphs, as depicted in Fig. 4. This figure exhibits the proportion of positive integers n that are the orders of arc-transitive cubic graphs, and the same for semisymmetric cubic graphs, which we call the *order density* in each case.

Looking at Fig. 4, one might conjecture that this ratio stabilises at about 15% for arc-transitive graphs, and at about 3% for semisymmetric graphs. Such a conjecture would also be wrong, however, because the natural density of the set of orders of connected finite cubic edge-transitive graphs is 0, as was proved recently by Conder, Verret and Young (see [15, Theorem 4.3]).

We now return to the matter of determining edge-transitive cubic graphs of up to a given order.

In the arc-transitive case, we have already mentioned the Foster census [5] (for order up to 512) and its completion and extension up to order 768 in [9] and later to order 2048. The two extensions were obtained by finding all suitable quotients of each of the seven Djoković-Miller groups. The construction of the list of semisymmetric cubic graphs on up to 768 vertices in [11] was achieved using Goldschmidt's classification [23] of finite primitive amalgams of index $(3, 3)$.

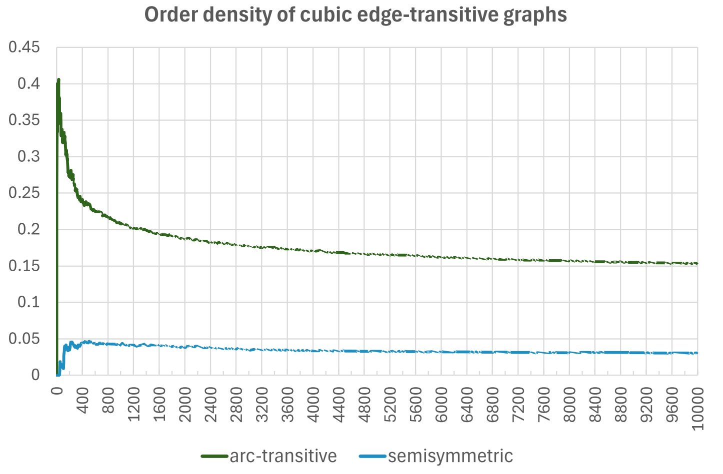

Fig. 4. The order density of cubic arc-transitive and semisymmetric graphs.

The Goldschmidt classification implies that every edge- but not vertex-transitive group of automorphisms of a connected finite cubic graph is obtainable as a homomorphic image of the universal completion of one of the 15 Goldschmidt amalgams we described earlier. In all 15 of these finite primitive amalgams of index $(3,3)$, the orders of the two subgroups A $(\cong G_u)$ and B $(\cong G_v)$ are at most $384 = 3 \cdot 2^7$, and it follows that in the automorphism group of any connected finite semisymmetric cubic graph Γ, the stabiliser of any given vertex has order $3 \cdot 2^{s-1}$ for some $s \leq 8$. Clearly this is the analogue for semisymmetric cubic graphs of the now classical theorem of Tutte [32], which states that $s \leq 5$ in the case where the graph Γ is arc-transitive, and in particular, the word 'semisymmetric' can be replaced by 'edge-transitive' in the previous sentence.

This paper is organised as follows.

In Section 2 we consider the fifteen Goldschmidt classes and the seven Djoković-Miller classes in more detail, determining all possible inclusions of such amalgams among each other. Then in Section 3 we describe how we used the Djoković-Miller and Goldschmidt amalgams to help find all the edge-transitive cubic graphs of order up to 10000, and in Section 4 we give some information about examples of connected finite edge-transitive cubic graphs of each type, including the smallest one in all but a few cases. Then we use regular covering projections in Section 5 to prove our Theorem 6 about there being infinitely many examples of each type, and finally we justify our comments above about asymptotic enumeration of edge-transitive graphs of each type in Section 6.

Just before continuing, however, we comment on notation used in various parts of the paper. We use C_n to denote the cyclic group of order n (or degree n when considered as a permutation group), and D_n, A_n and S_n to denote the dihedral, alternating and symmetric groups of degree n. (This contrasts with the notation in [11], where D_{2n} was used for the dihedral group of order $2n$.)

2. Conjugacy classes of discrete edge-transitive subgroups of Aut($\mathcal{T}_3$)

Recall that the fifteen isomorphism classes of finite simple amalgams of index $(3,3)$ were determined by Goldschmidt in [23], and explicit descriptions of the corresponding semisymmetric subgroups of Aut($\mathcal{T}_3$) as finitely-presented groups were given in [11]. Similarly, the seven isomorphism classes of finite simple amalgams of index $(3,2)$ were determined by Djoković and Miller in [23] in [17].

In the rest of this section, we will consider each of the Goldschmidt and Djoković-Miller classes of subgroups of Aut($\mathcal{T}_3$), one by one, determining inclusions of other ('smaller') amalgams within them. This was already done in [17], [10] and [13] for the inclusions among the seven Djoković-Miller classes, but we repeat the information here for the sake of completeness. We consider the inclusions of Goldschmidt amalgams among each other in Subsection 2.1, and the inclusion of both kinds of amalgams in Djoković-Miller amalgams in Subsection 2.2.

2.1. Inclusions among Goldschmidt classes

To find out which of the Goldschmidt classes are included in other such classes, we made extensive use of the Magma system [3].

To explain this in more detail, we suppose that we fix a Goldschmidt class $\mathcal{G}$ and a semisymmetric subgroup $G \le \mathrm{Aut}(\mathcal{T}_3)$ representing it. Then $G = A *_C B$, where $A = G_v$, $C = G_{uv}$ and $B = G_u$ for two adjacent vertices v and u of $\mathcal{T}_3$, and then look inside G for an edge-transitive subgroup H from some 'smaller' Goldschmidt class $\mathcal{H}$. Note that the amalgam (H_v, H_{uv}, H_u) is then a representative of the class $\mathcal{H}$, and that edge-transitivity of H implies that $G = G_{uv}H$, and hence that $|G{:}H| = |G_{uv}|/|H_{uv}|$.

In most cases (and whenever $|G_{uv}| \le 16$), we used the `LowIndexSubgroups` procedure to find all conjugacy classes of subgroups of the relevant index (such as index $|G_{uv}|/2 = 4$ when determining edge-transitive subgroups of a group G of type G_2^1 that belong to the class G_1^1). Once all conjugacy classes of subgroups of up to desired index are found, edge-transitivity of such a subgroup H could be tested by checking whether G_{uv} acts transitively on the coset space $(G{:}H)$.

When the relevant index was 32 or more, the latter approach was too slow, and so instead we needed to take a different approach, namely as follows.

If (u,v) is a given arc, and the automorphisms x and y of order 3 in Aut($\mathcal{T}_3$) are chosen to fix the vertices u and v respectively, then any copy of G_1 is conjugate to a subgroup generated by two elements x' and y' of order 3 fixing u and v (respectively), such that $x'x^{-1}$ and $y'y^{-1}$ fix the arc (u,v). Here both $x'x^{-1}$ and $y'y^{-1}$ lie in the stabiliser C of (u,v), making $x' = rx$ and $y' = sy$ for some r and s in C, and as C is small, it is easy to run through the few possibilities for the pair (r,s), and for each one, test the subgroup generated by $x' = rx$ and $y' = sy$. This helps to find all possibilities for a conjugacy class of edge-transitive subgroups of type G_1 with the required index in G, and can be adapted to find edge-transitive subgroups of other types (with smaller index in G) as necessary.

In what follows, we consider each of the fifteen Goldschmidt classes in turn. We first describe each class $\mathcal{G}$ by specifying a finite presentation for a representative $G \in \mathcal{G}$, and for the purposes of this exposition, we will call this presentation a *standard presentation* for the corresponding Goldschmidt class, and call the generators in this presentation the *standard generators* for that class. We also list generators for the stabilisers $A = G_u$, $B = G_v$ and $C = A \cap B = G_{uv}$. We then list all edge-transitive subgroups of G (one per each G-conjugacy class) and provide information about the sizes of their G-conjugacy classes and their Goldschmidt types. Note that if a subgroup $H \leq G$ is of Goldschmidt type $\mathcal{H}$, then it can be presented in terms of the standard presentation of the class $\mathcal{H}$. The inclusion of H to G is then described by specifying the images of the standard generators of $\mathcal{H}$ as words in the standard generators of $\mathcal{G}$.

For example, if the standard presentations of $\mathcal{H}$ and $\mathcal{G}$ are $H = \langle\, c, x, y \mid c^2, x^3, y^3, (cx)^2, (cy)^2 \,\rangle$ and $G = \langle\, c, d, x, y \mid c^2, d^2, [c, d], x^3, y^3, (cx)^2, [d, x], [c, y], (dy)^2 \,\rangle$, then by $(c, x, y)_H \mapsto (cd, x, y)_G$ we mean a homomorphism from H into G that maps the generators c, x and y of H to the elements represented by the words cd, x and y in the generators c, d, x and y of G.

Finally, for each Goldschmidt class we also provide a parameter called its *local arc-transitivity*, defined as follows. For a positive integer s, an *s-arc* in a graph Γ is a sequence of s vertices, where any two consecutive vertices are adjacent and any three are pairwise distinct. Clearly, if $G \leq \mathrm{Aut}(\Gamma)$, then the vertex-stabiliser G_u acts on the set of all s-arcs whose initial vertex is u, in an obvious way. If this action is transitive, then we say that G is locally s-arc-transitive at u, and the maximum value of s such that G is locally s-arc-transitive at u is denoted by $s_u(G)$. If G is an edge-transitive group of automorphisms of Γ and (u, v) is an arc of Γ, then the ordered pair $(s_u(G), s_v(G))$ is called the *local s-arc-transitivity* of G. (Note that this parameter does not depend on the choice of the given edge $\{u, v\}$.) The values $s_u(G)$ and $s_v(G)$ are both given, with the former referring to the vertex u (whose stabiliser G_u is the group A), and the latter referring to the vertex v (whose stabiliser G_v is the group B). Of course the roles of u and v can be interchanged (in a form of duality), provided that the roles of A and B (and their generating sets) are interchanged at the same time. Also to avoid possible notational confusion, note that in the case of G_5^1, the vertex v is not the same as the generator v used in the presentation for the representative group G.

Type: G_1, $G = \langle\, x, y \mid x^3, y^3 \,\rangle$
$A = \langle x \rangle \cong C_3$ and $B = \langle y \rangle \cong C_3$, of order 3; $C = \{1\}$ (trivial), $(s_u, s_v) = (1, 1)$

A group of type G_1 contains no proper edge-transitive subgroup of $\mathrm{Aut}(\mathcal{T}_3)$.

Type: G_1^1, $G = \langle\, c, x, y \mid c^2, x^3, y^3, (cx)^2, (cy)^2 \,\rangle$
$A = \langle c, x \rangle \cong S_3$ and $B = \langle c, y \rangle \cong S_3$, of order 6; $C = \langle c \rangle \cong C_2$, $(s_u, s_v) = (2, 2)$

List of edge-transitive subgroups of G:

- Index 2 normal subgroup H of type G_1, with standard generator inclusion $(x,y)_H \mapsto (x,y)_G$, and conjugation by c in G inducing an automorphism of H that takes $(x,y)_H$ to $(x^{-1},y^{-1})_H$.

Type: G_1^2, $G = \langle\, c,x,y \mid c^2, x^3, y^3, (cx)^2, [c,y]\,\rangle$
$A = \langle c,x\rangle \cong S_3$ and $B = \langle c,y\rangle \cong C_6$, of order 6; $C = \langle c\rangle \cong C_2$, $(s_u,s_v) = (1,2)$

List of edge-transitive subgroups of G:

- Index 2 normal subgroup H of type G_1, with standard generator inclusion $(x,y)_H \mapsto (x,y)_G$, and conjugation by c in G inducing an automorphism of H that takes $(x,y)_H$ to $(x^{-1},y)_H$.

Type: G_1^3, $G = \langle\, c,d,x,y \mid c^2, d^2, [c,d], x^3, y^3, (cx)^2, [d,x], [c,y], (dy)^2\,\rangle$
$A = \langle c,d,x\rangle \cong D_6$ and $B = \langle c,d,y\rangle \cong D_6$, of order 12; $C = \langle c,d\rangle \cong C_2 \times C_2$, $(s_u,s_v) = (3,3)$

List of edge-transitive subgroups of G:

- Index 4 normal (and non-maximal) subgroup H of type G_1, with standard generator inclusion $(x,y)_H \mapsto (x,y)_G$, and conjugation by c and d in G inducing automorphisms of H that take $(x,y)_H$ to $(x^{-1},y)_H$ and $(x,y^{-1})_H$.
- Index 2 normal subgroup H of type G_1^1, with standard generator inclusion $(c,x,y)_H \mapsto (cd,x,y)_G$, and conjugation by d in G inducing an automorphism of H that takes $(cd,x,y)_H$ to $(cd,x,y^{-1})_H$.
- Index 2 normal subgroup H of type G_1^2, with standard generator inclusion $(c,x,y)_H \mapsto (c,x,y)_G$, and conjugation by d in G inducing an automorphism of H that takes $(c,x,y)_H$ to $(c,x,y^{-1})_H$.
- Another index 2 normal subgroup H of type G_1^2, with standard generator inclusion given by $(c,x,y)_H \mapsto (d,y,x)_G$, and conjugation by c in G inducing an automorphism of H that takes $(c,x,y)_H$ to $(c,x^{-1},y)_H$.

Note: The last two subgroups (of type G_1^2) are conjugate to each other within a group of type DjM_3.

Type: G_2, $G = \langle\, c,d,x,y \mid c^2, d^2, [c,d], x^3, y^3, (cx)^2, [d,x], (cy)^3, dy^{-1}cy\,\rangle$
$A = \langle c,d,x\rangle \cong D_6$ and $B = \langle c,d,y\rangle \cong A_4$, of order 12; $C = \langle c,d\rangle \cong C_2 \times C_2$, $(s_u,s_v) = (1,2)$

List of edge-transitive subgroups of G:

- Conjugacy class of four index 4 maximal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x,y)_H \mapsto (x,y)_G$, and quotient of G by the core of each subgroup being isomorphic to A_4.

Type: G_2^1, $G = \langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, (cx)^2, [d, x], y^3, (dy^{-1})^2, cydy \,\rangle$
$A = \langle c, d, x\rangle \cong D_{12}$ and $B = \langle c, d, y\rangle \cong S_4$, of order 24;
$C = \langle c, d\rangle \cong D_4$, $(s_u, s_v) = (3, 4)$

List of edge-transitive subgroups of G:

- Conjugacy class of four index 8 non-normal (and non-maximal) subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Conjugacy class of four index 4 non-normal maximal subgroups of type G_1^1, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (c, x, d^2y)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Index 2 normal subgroup H of type G_2, with standard generator inclusion given by $(c, d, x, y)_H \mapsto (cd, d^2, x, y^{-1})_G$, and conjugation by c in G inducing an automorphism of H that takes $(c, d, x, y)_H$ to $(cd, d, x^{-1}, cy^{-1})_H$.

Type: G_2^2, $G = \langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, [c, x], [d^2, x], d^{-1}xdx, y^3, (dy^{-1})^2, cydy \,\rangle$
$A = \langle c, d, x\rangle \cong C_3 \times D_4$ and $B = \langle c, d, y\rangle \cong S_4$, of order 24;
$C = \langle c, d\rangle \cong D_4$, $(s_u, s_v) = (3, 4)$

List of edge-transitive subgroups of G:

- Conjugacy class of four index 8 non-normal (and non-maximal) subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Conjugacy class of four index 4 non-normal maximal subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (c, d^2y, x)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Index 2 normal subgroup H of type G_2, with standard generator inclusion given by $(c, d, x, y)_H \mapsto (cd, d^2, x, y^{-1})_G$, and conjugation by c in G inducing an automorphism of H that takes $(c, d, x, y)_H$ to $(cd, d, x, cy^{-1})_H$.

Type: G_2^3, $G = \langle\, c, d, e, x, y \mid c^2, d^2, e^2, [c, d], [c, e], [d, e], x^3, (cx)^2, [d, x], [e, x],$ $y^3, [c, y], y^{-1}cdye, (ye)^3 \,\rangle$
$A = \langle c, d, e, x\rangle \cong D_6 \times C_2$ and $B = \langle c, d, e, y\rangle \cong A_4 \times C_2$, of order 24;
$C = \langle c, d, e\rangle \cong C_2 \times C_2 \times C_2$, $(s_u, s_v) = (1, 2)$

List of edge-transitive subgroups of G:

- Conjugacy class of four index 8 non-normal (and non-maximal) subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to $A_4 \times C_2$.
- Conjugacy class of four index 4 non-normal maximal subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (c, x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to A_4.

- Index 2 normal subgroup H of type G_2, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (cd,e,x,y)_G$, and conjugation by c in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(c,d,x^{-1},y)_H$.

Type: G_2^4, $G = \langle\, c,d,e,x,y \mid c^2, d^4, e^2, (cd)^2, [c,e], [d,e], x^3, [ce,x], [d,x], (ex)^2, y^3, (dy^{-1})^2, cydy, [y,e]\,\rangle$
$A = \langle c,d,e,x\rangle \cong S_3 \times D_4$ and $B = \langle c,d,e,y\rangle \cong S_4 \times C_2$, of order 48; $C = \langle c,d,e\rangle \cong D_4 \times C_2$, $(s_u, s_v) = (3,4)$

List of edge-transitive subgroups of G:

- Conjugacy class of four index 16 non-normal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x,y)_H \mapsto (x,y)_G$, and quotient of G by the core of each subgroup being isomorphic to $S_4 \times C_2$.
- Conjugacy class of four index 8 non-normal subgroups of type G_1^1, with standard generator inclusion for one of them (say H) given by $(c,x,y)_H \mapsto (c,x,d^2y)_G$, and quotient of G by the core of each subgroup being isomorphic to $S_4 \times C_2$.
- Conjugacy class of four index 8 non-normal (and non-maximal) subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c,x,y)_H \mapsto (e,x,y)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Another conjugacy class of four index 8 non-normal subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c,x,y)_H \mapsto (cd^2e, cdy, x)_G$, and quotient of G by the core of each subgroup being isomorphic to $S_4 \times C_2$.
- Conjugacy class of four index 4 maximal non-normal subgroups of type G_1^3, with standard generator inclusion for one of them (say H) given by $(c,d,x,y)_H \mapsto (e, ce, x, d^2y)_G$, and quotient of G by the core of each subgroup being isomorphic to S_4.
- Index 4 normal (and non-maximal) subgroup H of type G_2, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (cd, d^2, x, y^{-1})_G$, and conjugation by c in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(cd, d, x^{-1}, cy^{-1})_H$.
- Index 2 normal subgroup H of type G_2^1, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (c,d,x,y)_G$, and conjugation by e in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(c,d,x^{-1},y)_H$.
- Index 2 normal subgroup H of type G_2^2, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (ce, de, c, y)_G$, and conjugation by e in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(c,d,x^{-1},y)_H$.
- Index 2 normal subgroup H of type G_2^3, with standard generator inclusion given by $(c,d,e,x,y)_H \mapsto (e, cde, d^2, x, y^{-1})_G$, and conjugation by c in G inducing an automorphism of H that takes $(c,d,e,x,y)_H$ to $(c, de, e, x^{-1}, cdy^{-1})_H$.

Type: G_3, $G = \langle\, c, d, x, y \mid c^2, d^4, (cd)^2, x^3, (dx^{-1})^2, cxdx, y^3, (dy^{-1})^2, cdydy \,\rangle$
$A = \langle c, d, x\rangle \cong S_4$ and $B = \langle c, d, y\rangle \cong S_4$, of order 24;
$C = \langle c, d, e\rangle \cong D_4$, $(s_u, s_v) = (4, 4)$

Again here we point out that the final relator *cdydy* corrects a typographical error in the description of this group in [11], where it was given as *cdyd*, as noted in the Introduction.

List of edge-transitive subgroups of G:

- Two conjugacy classes of eight index 8 non-normal maximal subgroups isomorphic to G_1, with standard generator inclusions for a representative H of each class given by $(x, y)_H \mapsto (x, y)_G$ and $(x, y)_H \mapsto (x, d^{-1}y^{-1})_G$, and quotient of G by the core of each subgroup being isomorphic to $\mathrm{PSL}(2, 7)$.

Note: These two conjugacy classes of subgroups (of type G_1) fuse into a single conjugacy class within a group of type G_3^1 as well as within in a group of type DjM_4^2, but not within a group of type DjM_4^1.

Type: G_3^1, $G = \langle\, c, d, e, x, y \mid c^2, d^4, e^2, (cd)^2, [c, e], [d, e], x^3, (dx^{-1})^2, cxdx, [x, e],$ $y^3, (dy^{-1})^2, cd^{-1}ydy, [y, ed^2] \,\rangle$
$A = \langle c, d, x\rangle \cong S_4 \times C_2$ and $B = \langle c, d, y\rangle \cong S_4 \times C_2$, of order 48;
$C = \langle c, d, e\rangle \cong D_4 \times C_2$, $(s_u, s_v) = (5, 5)$

List of edge-transitive subgroups of G:

- Conjugacy class of sixteen index 16 non-normal subgroups isomorphic to G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to the wreath product $\mathrm{PSL}(2, 7) \wr C_2$.
- Index 2 normal subgroup H of type G_3, with standard generator inclusion given by $(c, d, x, y)_H \mapsto (cd^{-1}, d, y, x)_G$, and conjugation by c in G inducing an automorphism of H that takes $(c, d, x, y)_H$ to $(c, d, cdx, y)_H$.

Type: G_4, $G = \langle\, a, b, s, x, y \mid a^4, b^4, s^2, [a, b], sasb^{-1}, sbsa^{-1}, x^3, x^{-1}axb^{-1},$ $x^{-1}bxba, (xs)^2, y^3, y^{-1}absysa^2, y^{-1}sa^2yba^{-1}, a^{-1}sysay \,\rangle$
$A = \langle a, b, s, x\rangle$ and $B = \langle a, b, s, y\rangle$, of order 96;
$C = \langle a, b, s\rangle$, of order 32, $(s_u, s_v) = (5, 6)$

List of edge-transitive subgroups of G:

- Two conjugacy classes of thirty-two index 32 non-normal maximal subgroups of type G_1, with standard generator inclusions for a representative H of each class given by $(x, y)_H \mapsto (x, y)_G$ and $(x, y)_H \mapsto (x, ay)_G$, and quotient of G by the core of each subgroup being isomorphic to the alternating group A_{32}.

Note: These two conjugacy classes of subgroups (of type G_1) fuse into a single conjugacy within a group of type G_4^1.

Type: G_4^1, $G = \langle\, a,b,s,t,x,y \mid a^4, b^4, s^2, t^2, [a,b], sasb^{-1}, sbsa^{-1}, (ta)^2, (tb)^2, [s,t],$ $x^3, x^{-1}axb^{-1}, x^{-1}bxba, (xs)^2, [x,t],$ $y^3, y^{-1}absysa^2, y^{-1}sa^2yba^{-1}, a^{-1}sysay, tytsa^2y^{-1}a^2s \,\rangle$
$A = \langle a,b,s,t,x\rangle$ and $B = \langle a,b,s,t,y\rangle$, of order 192; $C = \langle a,b,s,t\rangle$, of order 64, $(s_u, s_v) = (7,7)$

List of edge-transitive subgroups of G:

- Conjugacy class of sixty-four index 64 non-normal subgroups isomorphic of type G_1, with standard generator inclusion for one of them (say H) given by $(x,y)_H \mapsto (x,y)_G$, and quotient of G by the core of each subgroup being isomorphic to the wreath product $A_{32} \wr C_2$.
- Index 2 normal subgroup H of type G_4, with standard generator inclusion given by $(a,b,s,x,y)_H \mapsto (a,b,s,x,y)_G$, and conjugation by t in G inducing an automorphism of H that takes $(a,b,s,x,y)_H$ to $(a^{-1},b^{-1},s,x,sa^2ya^2s)_H$.

Type: G_5, $G = \langle\, a,b,s,t,x,y \mid a^4, b^4, s^2, t^2, [a,b], sasb^{-1}, (ta)^2, (tb)^2, [s,t],$ $x^3, x^{-1}axb^{-1}, x^{-1}bxba, (xs)^2, [x,t], y^3,$ $y^{-1}absyba^{-1}, y^{-1}sa^2ysb^{-1}a^{-1}, y^{-1}tsa^2yb^{-1}a^{-1}, y^{-1}abyb^{-1}ast, tb^{-1}ybty \,\rangle$
$A = \langle a,b,s,t,x\rangle$ and $B = \langle a,b,s,t,y\rangle$, of order 192; $C = \langle a,b,s,t\rangle$, of order 64, $(s_u, s_v) = (6,5)$

List of edge-transitive subgroups of G:

- Three conjugacy classes of sixty-four index 64 non-normal maximal subgroups of type G_1, with standard generator inclusions for a representative H of each class given by $(x,y)_H \mapsto (x,y)_G$, $(x,y)_H \mapsto (b^2x,y)_G$ and $(x,y)_H \mapsto (b^{-1}x,y)_G$, and quotient of G by the core of each subgroup being isomorphic to the alternating group A_{64}.
- A fourth conjugacy class of sixty-four index 64 non-normal (and non-maximal) subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x,y)_H \mapsto (bx,y)_G$, and quotient of G by the core of each subgroup being isomorphic to a subgroup of index 3 in the wreath product $A_4 \wr A_{16}$.
- Conjugacy class of sixteen index 16 non-normal maximal subgroups of type G_2, with standard generator inclusion for one of them (say H) given by $(c,d,x,y)_H \mapsto (ab^{-1}s, y^{-1}ab^{-1}sy, bx, y)_G$, and quotient of G by the core of each subgroup being isomorphic to A_{16}.

Note: The first two conjugacy classes of subgroups of type G_1 (in the first bullet point above) fuse into a single conjugacy class within a group of type G_5^1.

Type: G_5^1,	$G = \langle\, a, b, s, t, v, x, y \mid a^4, b^4, s^2, t^2, v^2, [a, b], sasb^{-1}, (ta)^2, (tb)^2, [s, t],$ $vavb^{-2}a^{-1}, vbva^{-2}b, (vs)^2t, [v, t], x^3, x^{-1}axb^{-1}, x^{-1}bxba, (xs)^2, [x, t], [x, v],$ $y^3, y^{-1}absyba^{-1}, y^{-1}sa^2ysb^{-1}a^{-1}, y^{-1}tsa^2yb^{-1}a^{-1}, y^{-1}abyb^{-1}ast,$ $tb^{-1}ybty,\ b^{-1}tvbtvtb^3a^2tb^{-1}, [vtb, y]\,\rangle$
$A = \langle a, b, s, t, v, x\rangle$ and $B = \langle a, b, s, t, v, y\rangle$, of order 384; $C = \langle a, b, s, t, v\rangle$, of order 128, $(s_u, s_v) = (8, 7)$	

List of edge-transitive subgroups of G:

- Conjugacy class of 128 index 128 non-normal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (x, y)_G$, and quotient of G by the core of each subgroup being isomorphic to the wreath product $A_{64} \wr C_2$.
- Another conjugacy class of 64 index 128 non-normal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (bx, y)_G$, and quotient of G by the core of each subgroup being isomorphic to a subgroup of index $3 \cdot 2^{15}$ in the wreath product $S_4 \wr A_{16}$.
- A third conjugacy class of 64 index 128 non-normal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (b^{-1}x, y)_G$, and quotient of G_5^1 by the core of each subgroup being isomorphic to $A_{64} \times C_2$.
- Conjugacy class of 64 index 64 non-normal maximal subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (vt, ayb, b^2x)_G$, and quotient of G by the core of each subgroup being isomorphic to A_{64}. Also each subgroup in this class is the normaliser in G of a subgroup of index 128 described in the bullet point immediately above.
- Another conjugacy class of 64 index 64 non-normal subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (vt, yab, a^2x)_G$, and quotient of G by the core of each subgroup being isomorphic to a subgroup of index $3 \cdot 2^{15}$ in the wreath product $S_4 \wr A_{16}$, as in the second bullet point above. Also each subgroup in this class is the normaliser in G of a subgroup of index 128 described in the second bullet point above.
- Conjugacy class of 16 index 32 non-normal subgroups of type G_2, with standard generator inclusion for one of them (say H) given by $(c, d, x, y)_H \mapsto (st, a^2b^2t, ax, (bya)^{-1})_G$, and quotient of G by the core of each subgroup being isomorphic to $A_{16} \times C_2$.
- Conjugacy class of 16 index 16 non-normal maximal subgroups of type G_2^2, with standard generator inclusion for one of them (say H) given by $(c, d, x, y)_H \mapsto (v, a^2b^2stv, a^2x^{-1}, asy^{-1}a)_G$, and quotient of G by the core of each subgroup being isomorphic to A_{16}.
- Index 2 normal subgroup H of type G_5, with standard generator inclusion $(a, b, s, t, x, y)_H \mapsto (a, b, s, t, x, y)_G$, and conjugation by v in G inducing an automorphism of the subgroup H that takes (a, b, s, t, x, y) to $(ab^2, b^{-1}a^2, st, t, x, tbyb^{-1}t)$.

2.2. Inclusions within Djoković-Miller classes

In this subsection, we consider each of the seven Djoković-Miller classes in turn, describing each class by specifying a finite presentation for a representative G, which we again call the standard presentation of the class.

In each case, G is isomorphic to the amalgamated free product $A *_C B$, where $A = G_u$, $B = G_{\{u,v\}}$ and $C = A \cap B = G_{uv}$, for a fixed arc (u, v) of $\mathcal{T}_3$. We list generators for those subgroups A, B and C, and then list a representative of each conjugacy class of edge-transitive subgroups of G, and provide information about the sizes of those conjugacy classes and their Goldschmidt or Djoković-Miller types. Note that if a subgroup $H \le G$ has Goldschmidt or Djoković-Miller type $\mathcal{H}$, then it can be presented in terms of the standard presentation for the class $\mathcal{H}$, and then the inclusion of H in G is described by specifying the images of the standard generators for $\mathcal{H}$ as words in the standard generators for $\mathcal{G}$. The standard inclusions of these subgroups are given in the same manner as in the previous subsection.

To find out which of the Goldschmidt classes are included in which of the Djoković-Miller classes, again we made extensive use of Magma, in the obvious way, analogous to the approach taken in the previous subsection (but without the need for special treatment when the order of G_{uv} is large).

Type: DjM_1, $G = \langle h, a \mid h^3, a^2 \rangle$
$A = \langle h\rangle \cong C_3$ of order 3; $B = \langle a\rangle \cong C_2$ of order 2; $C = \{1\}$ (trivial)

List of edge-transitive subgroups of G:

- Index 2 semisymmetric normal subgroup H of type G_1, with standard generator inclusion given by $(x, y)_H \mapsto (h, aha)_G$, and conjugation by a in G inducing an automorphism of H that takes $(x, y)_H$ to $(y, x)_H$.

Type: DjM_2^1, $G = \langle h, p, a \mid h^3, p^2, a^2, (hp)^2, [a, p] \rangle$
$A = \langle h, p\rangle \cong S_3$ of order 6; $B = \langle p, a\rangle \cong C_2 \times C_2$ of order 4; $C = \langle p\rangle \cong C_2$

List of edge-transitive subgroups of G:

- Index 4 semisymmetric normal (and non-maximal) subgroup H of type G_1, with standard generator inclusion given by $(x, y)_H \mapsto (h, aha)_G$, and conjugation by a and p in G inducing automorphisms of H that take $(x, y)_H$ to $(y, x)_H$ and $(x^{-1}, y^{-1})_H$.
- Index 2 semisymmetric normal subgroup H of type G_1^1, with standard generator inclusion given by $(c, x, y)_H \mapsto (p, h, aha)_G$, and conjugation by a in G inducing an automorphism of H that takes $(c, x, y)_H$ to $(c, y, x)_H$.
- Two index 2 arc-transitive normal subgroups of type DjM_1, with standard generator inclusions given by $(h, a)_H \mapsto (h, a)_G$ and $(h, a)_H \mapsto (h, ap)_G$, and conjugation by p in G inducing an automorphism of H that takes $(h, a)_H$ to $(h^{-1}, a)_H$ in both cases.

Note: The last two subgroups (of type DjM_1) are conjugate within a group of type DjM_3.

Type: DjM_2^2, $G = \langle\, h, p, a \mid h^3, p^2, pa^2, (hp)^2 \,\rangle$
$A = \langle h, p\rangle \cong S_3$ of order 6; $B = \langle a\rangle \cong C_4$; $C = \langle p\rangle \cong C_2$

List of edge-transitive subgroups of G:

- Index 4 semisymmetric normal (and non-maximal) subgroup H of type G_1, with standard generator inclusion given by $(x, y)_H \mapsto (h, aha^{-1})_G$, and conjugation by a^{-1} and p in G inducing automorphisms of H that take $(x, y)_H$ to $(y, x^{-1})_H$ and $(x^{-1}, y^{-1})_H$ respectively.
- Index 2 semisymmetric normal subgroup H of type G_1^1, with standard generator inclusion given by $(c, x, y)_H \mapsto (p, h, aha^{-1})_G$, and conjugation by a^{-1} in G inducing an automorphism of H that takes $(c, x, y)_H$ to $(c, y, x^{-1})_H$.

Type: DjM_3, $G = \langle\, h, p, q, a \mid h^3, p^2, q^2, a^2, [p, q], apaq, [h, p], (hq)^2 \,\rangle$
$A = \langle h, p, q\rangle \cong S_3 \times C_2$ of order 12; $B = \langle a, p, q\rangle \cong D_4$ of order 8; $C = \langle p, q\rangle \cong C_2 \times C_2$ of order 4

List of edge-transitive subgroups of G:

- Index 8 semisymmetric normal (and non-maximal) subgroup H of type G_1, with standard generator inclusion given by $(x, y)_H \mapsto (h, aha)_G$, and conjugation by a, p and q in G inducing automorphisms of H that take $(x, y)_H$ to $(y, x)_H$, $(x, y^{-1})_H$ and $(x^{-1}, y)_H$, respectively.
- Index 4 semisymmetric normal (and non-maximal) subgroup H of type G_1^1, with standard generator inclusion given by $(c, x, y)_H \mapsto (pq, h, aha)_G$, and conjugation by a, p and q in G inducing automorphisms of H that take $(c, x, y)_H$ to $(c, y, x)_H$, $(c, x, y^{-1})_H$ and $(c, x^{-1}, y)_H$, respectively.
- Conjugacy class of two index 4 semisymmetric subgroups of type G_1^2, with standard generator inclusion for one of them (say H) given by $(c, x, y)_H \mapsto (p, aha, h)_G$ and the quotient of G by the core of each subgroup being isomorphic to D_4.
- Index 2 semisymmetric normal subgroup H of type G_1^3, with standard generator inclusion given by $(c, d, x, y)_H \mapsto (q, p, h, aha)_G$, and conjugation by a in G inducing an automorphism of H that takes $(c, d, x, y)_H$ to $(d, c, y, x)_H$.
- Conjugacy class of two index 4 arc-transitive non-normal subgroups of type DjM_1, with standard generator inclusion for one of them (say H) given by $(h, a)_H \mapsto (h, a)_G$, and quotient of G by the core of each subgroup being isomorphic to D_4.
- Index 2 arc-transitive normal subgroup H of type DjM_2^1, with standard generator inclusion given by $(h, p, a)_H \mapsto (h, pq, a)_G$, and conjugation by p in G inducing an automorphism of H that takes $(h, p, a)_H$ to $(h, p, ap)_H$.

- Index 2 arc-transitive normal subgroup H of type DjM_2^2, with standard generator inclusion given by $(h,p,a)_H \mapsto (h,pq,ap)_G$, and conjugation by p in G inducing an automorphism of H that takes $(h,p,a)_H$ to $(h,p,ap)_H$.

Type: DjM_4^1, $G = \langle\, h,p,q,r,a \mid h^3,p^2,q^2,r^2,a^2,[p,q],[p,r],p(qr)^2,[a,p],aqar, h^{-1}phq, h^{-1}qhpq,(hr)^2 \,\rangle$
$A = \langle h,p,q,r\rangle \cong S_4$ of order 24; $B = \langle a,p,q,r\rangle \cong D_4 \rtimes C_2$ of order 16; $C = \langle p,q,r\rangle \cong D_4$ of order 8

List of edge-transitive subgroups of G:

- Two conjugacy classes of eight index 16 semisymmetric non-normal subgroups of type G_1, with standard generator inclusion for a representative H of each class given by $(x,y)_H \mapsto (h,aha)_G$ and $(x,y)_H \mapsto (ph,aha)_G$, and the quotient of G by the core of each subgroup being isomorphic to $\mathrm{PGL}(2,7)$.
- Index 2 semisymmetric normal subgroup H of type G_3, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (pr,pqr,h^{-1},aha)_G$, and conjugation by a in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(cd,d^{-1},y^{-1},x^{-1})_H$.
- Two conjugacy classes of eight index 8 arc-transitive non-normal maximal subgroups of type DjM_1, with standard generator inclusion for one of them (say H) given by $(h,a)_H \mapsto (h,a)_G$ and $(h,a)_H \mapsto (h,ap)_G$, and quotient of G by the core of each subgroup (in each class) being isomorphic to $\mathrm{PGL}(2,7)$.

Note: The two conjugacy classes of subgroups of type G_1 fuse into a single conjugacy class within a group of type DjM_5, as do the two conjugacy classes of subgroups of type DjM_1.

Type: DjM_4^2, $G = \langle\, h,p,q,r,a \mid h^3,p^2,q^2,r^2,a^4,a^2p,[p,q],[p,r],p(qr)^2,a^{-1}qar, h^{-1}phq,\ h^{-1}qhpq,(hr)^2 \,\rangle$
$A = \langle h,p,q,r\rangle \cong S_4$ of order 24; $B = \langle a,p,q,r\rangle \cong C_8 \rtimes_3 C_2$ of order 16; $C = \langle p,q,r\rangle \cong D_4$ of order 8

List of edge-transitive subgroups of G:

- Conjugacy class of 16 index 16 semisymmetric subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x,y)_H \mapsto (aha^{-1},h)_G$, and quotient of G by the core of each subgroup being isomorphic to the wreath product $\mathrm{PSL}(2,7) \wr C_2$.
- Index 2 semisymmetric normal subgroup H of type G_3, with standard generator inclusion given by $(c,d,x,y)_H \mapsto (pr,pqr,h^{-1},aha^{-1})_G$, and conjugation by a^{-1} in G inducing an automorphism of H that takes $(c,d,x,y)_H$ to $(cd,d^{-1},y^{-1},x^{-1})_H$.

Type: DjM_5, $G = \langle\, h, p, q, r, s, a \mid h^3, p^2, q^2, r^2, s^2, a^2, [p, q], [p, r], [p, s], [q, r], [q, s], pq(rs)^2, apaq, aras, [h, p], h^{-1}qhr, h^{-1}rhpqr, (hs)^2 \,\rangle$
$A = \langle h, p, q, r, s\rangle \cong S_4 \times C_2$ of order 24; $B = \langle a, p, q, r, s\rangle \cong C_8 \rtimes V_4$ of order 32; $C = \langle p, q, r, s\rangle \cong D_4 \times C_2$ of order 16

List of edge-transitive subgroups of G:

- Conjugacy class of sixteen index 32 semisymmetric non-normal subgroups of type G_1, with standard generator inclusion for one of them (say H) given by $(x, y)_H \mapsto (h, aha)_G$. The quotient of G by the core of H is isomorphic to the automorphism group of the Biggs-Conway graph, namely a C_2-extension of the wreath product $\mathrm{PSL}(2,7) \wr C_2$, isomorphic to a semi-direct product of $\mathrm{PSL}(2,7) \times \mathrm{PSL}(2,7)$ by $C_2 \times C_2$.
- Index 4 semisymmetric normal (and non-maximal) subgroup H of type G_3, with standard generator inclusion given by $(c, d, x, y)_H \mapsto (pr, sr, aha, h^{-1})_G$, and conjugation by a and p in G inducing automorphisms of H that take $(c, d, x, y)_H$ to $(cd, d^{-1}, y^{-1}, x^{-1})_H$ and $(c, d, cdx, y)_H$, respectively.
- Index 2 semisymmetric normal subgroup H of type G_3^1, with standard generator inclusion given by $(c, d, e, x, y)_H \mapsto (qs, sr, p, h^{-1}, aha)_G$, and conjugation by a in G inducing an automorphism of H that takes $(c, d, e, x, y)_H$ to $(cd, d^{-1}, d^2e, y^{-1}, x^{-1})_H$.
- Conjugacy class of sixteen index 16 arc-transitive non-normal subgroups of type DjM_1, with standard generator inclusion for one of them (say H) given by $(h, a)_H \mapsto (h, a)_G$, and quotient of G by the core of each subgroup being isomorphic to a subgroup of index 2 in $\mathrm{PGL}(2,7) \wr C_2$.
- Index 2 arc-transitive normal subgroup H of type DjM_4^1, with standard generator for one of them (say H) given by $(h, p, q, r, a)_H \mapsto (h^{-1}, pq, qr, ps, a)_G$, and conjugation by p in G inducing an automorphism of H that takes $(h, p, q, r, a)_H$ to $(h, p, q, r, ap)_H$.
- Index 2 arc-transitive normal subgroup H of type DjM_4^2, with standard generator inclusion given by $(h, p, q, r, a)_H \mapsto (h^{-1}, pq, qr, ps, ap)_G$, and conjugation by p in G inducing an automorphism of H that takes $(h, p, q, r, a)_H$ to $(h, p, q, r, ap)_H$.

3. Finding all edge-transitive cubic graphs of order up to 10000

All connected arc-transitive cubic graphs on up to 10000 vertices were determined by the first author in 2011 (see [6]), but this list was not widely publicised before now.

The graphs were found with the help of an improved version of the `LowIndexNormal-Subgroups` command in Magma, using the same approach as was used to find all of order up to 2048 five years earlier. For example, those of type DjM_1 were determined by
(a) using `LowIndexNormalSubgroups` to find all normal subgroups N of index up to 30000 in the corresponding finitely-presented group G, such that the orders of the images in G/N of the stabilisers $A = G_u$, $B = G_{\{u,v\}}$ and $C = A \cap B = G_{uv}$ are preserved, and then for each such N,
(b) constructing the associated arc-transitive cubic graph Γ on which G/N has an arc-transitive action of type DjM_1, and

(c) checking that the (full) automorphism group of Γ has order $|G/N|$.

Step (c) can be performed either by using the `AutomorphismGroup` command for graphs in MAGMA, or by testing whether or not the subgroup N is normal in one of the finitely-presented groups associated with an amalgam that contains the given one.

The analogous process was used for each of the other six Djoković-Miller classes, applying the `LowIndexNormalSubgroups` command to find normal subgroups of index up to $10000c$, where $c = 6$ for DjM_2^1 and DjM_2^2, or $c = 12$ for DjM_3, or $c = 24$ for DjM_4^1 and DjM_4^2, or $c = 48$ for DjM_5.

The total number of connected arc-transitive cubic graphs on up to 10000 vertices is 3815, with the numbers of each Djoković-Miller type given in the next section.

We subsequently used an approach analogous to the one above, in order to find all connected semisymmetric cubic graphs on up to 10000 vertices. Here we applied `LowIndexNormalSubgroups` to find all normal subgroups of the appropriate index in each of the finitely-presented groups associated with the thirteen 'smallest' Goldschmidt amalgams, namely all except G_4^1, G_5 and G_5^1.

The graph construction in step (b) of the semisymmetric analogue of the above process differed a little from the arc-transitive version. First, let $\overline{G}$ denote the quotient G/N of the relevant finitely-presented group G by the normal subgroup N, and let $\overline{G_u} = G_uN/N$ and $\overline{G_v} = G_vN/N$ (which have the same orders as G_u and G_v and as each other). Then the edge-transitive graph Γ we construct is the bipartite graph with vertex set the union of the two right coset spaces $(\overline{G} : \overline{G_u})$ and $(\overline{G} : \overline{G_v})$, and the edges are all pairs $\{\overline{G_u}\,\overline{g}, \overline{G_v}\,\overline{g}\}$ obtained by right-multiplying the pair $\{\overline{G_u}, \overline{G_v}\}$ by elements of $\overline{G}$. Note that this graph Γ has order $2\,|\overline{G}:\overline{G_u}|$, and so $|G/N| = |\overline{G}| = |\overline{G_u}||V(\Gamma)|/2 = |G_u||V(\Gamma)|/2$.

Otherwise steps (a) and (c) were straightforward analogues of the corresponding steps in the arc-transitive case.

For the remaining three amalgams, namely G_4^1, G_5 and G_5^1, the above process was challenging because the upper bound on the index $|G:N|$ (of 960000, 960000 and 1920000 respectively) was too large. (The current implementation of `LowIndexNormalSubgroup` command in MAGMA computes normal subgroups of index up to at most 500000.) Instead we took a different approach, after observing that the finitely-presented groups associated with the G_4 and G_5 amalgams are perfect.

In the case of G_4^1, we first consider the finitely-presented group H associated with G_4. This group is perfect, and by the `LowIndexNormalSubgroups` computation undertaken for it, H has just two proper normal subgroups of index up to $960000/2 = 480000$, namely one of index 6048 and one of index 372000, with simple quotients $\mathrm{PSU}(3,3)$ and $\mathrm{PSL}(3,5)$. These give rise to semisymmetric graphs of type G_4^1, with orders $2 \cdot 6048/96 = 126$ and $2 \cdot 372000/96 = 7750$.

On the other hand, the `SimpleQuotientProcess` in MAGMA shows that the finitely-presented group G associated with G_4^1 has no non-abelian simple quotient of order up to 960000, and then since this group G has a unique subgroup of index 2, isomorphic to the group H considered above for G_4, it follows that G has at most two proper normal subgroups of index up to 960000, namely one with index 12096 and one with index

744000. Indeed the Homomorphisms command in Magma shows that there is a unique homomorphism from G onto $\mathrm{Aut(PSU}(3,3)) \cong \mathrm{P\Sigma U}(3,3)$, with kernel of index 12096, and similarly a unique homomorphism from G onto $\mathrm{Aut(PSL}(3,5)) \cong \mathrm{PSL}(3,5) \rtimes C_2$, with kernel of index 744000. Hence there are exactly two edge-transitive graphs of type G_4^1 with order at most 10000, namely the two semisymmetric graphs of orders 126 and 7750 mentioned in the paragraph above.

For G_5, every quotient of the associated group G up to the maximum conceivable order 960000 must have a non-abelian simple quotient, and in fact an application of the SimpleQuotientProcess in Magma [3] shows that this group G has only one non-abelian simple quotient of order up to 960000, namely the Mathieu group M_{12}, of order 95040. It follows that every non-trivial quotient Q of G of order at most 960000 has order $95040m$ for some $m \leq 10$, and hence is isomorphic to an extension by M_{12} of a soluble normal subgroup of order at most 10. But in that case, the normal subgroup must be central in the perfect quotient Q, and hence has order at most 2, because the Schur multiplier of M_{12} is cyclic of order 2. By a Magma computation using the Darstellungsgruppe function and the Homomorphisms command, however, it can be shown that the Schur cover of M_{12} (of order $2 \cdot 95040 = 190080$) is not a quotient of G, and hence the only possibility is that $Q \cong \mathrm{M}_{12}$.

Next, the Homomorphisms command in Magma shows there are just two homomorphisms from G onto M_{12}, up to equivalence under conjugation in $\mathrm{Aut}(\mathrm{M}_{12})$, but these have the same kernel, so both give rise to the same edge-transitive cubic graph of order $2 \cdot 95040/192 = 990$, namely a semisymmetric one of type G_5^1. Hence there is no edge-transitive graph of type G_5 with order at most 10000.

For G_5^1, the SimpleQuotientProcess in Magma shows that the associated finitely-presented group G has no non-abelian simple quotient of order up to 1920000, and then since G has a unique subgroup of index 2, isomorphic to the one considered above for G_5, it follows that there is just one edge-transitive graph of type G_5^1 with order at most 10000, namely the semisymmetric graph of order 990 mentioned in the paragraph above.

The total number of connected semisymmetric cubic graphs on up to 10000 vertices is 1043, with the numbers of each Goldschmidt type given in the next section.

4. Examples of edge-transitive cubic graphs of each type

Here we give some information about examples of connected finite edge-transitive cubic graphs of each type. For most of the types, we can even give the smallest possible example. When mentioned, the 'action type' indicates the type of action for each conjugacy class of arc-transitive subgroups of the automorphism group in the arc-transitive case, or for each conjugacy class of semisymmetric subgroups of the automorphism group in the semisymmetric case.

4.1. Examples with given arc-transitive type

(a) Type DjM_1:

The smallest example is the graph F026 in [5,9]. Other small examples have orders 38, 42, 56, 62, 74, 78, 86 and 98. There are 2522 examples of type DjM_1 with order up to 10000.

(b) Type DjM_2^1:

The smallest example is the complete graph K_4, and the next smallest is the 3-cube Q_3, known as F004 and F008 in [5,9], with both having action type $(\mathrm{DjM}_1, \mathrm{DjM}_2^1)$. Other small examples have orders 16, 24, 32, 48, 50, 54, 60, 64, 72, 84, 96 and 98. There are 1135 examples of type DjM_2^1 with order up to 10000.

(c) Type DjM_2^2:

The smallest example is the graph F448C which is found in [9] but not in [5], and has action type (DjM_2^2). The next smallest has order 896. There are 9 examples of type DjM_2^2 with order up to 10000.

(d) Type DjM_3:

The smallest example is the complete bipartite graph $K_{3,3}$, and the next smallest is the Petersen graph, known as F006 and F010 in [5,9], and these have action types $(\mathrm{DjM}_1, \mathrm{DjM}_2^1, \mathrm{DjM}_2^2, \mathrm{DjM}_3)$ and $(\mathrm{DjM}_2^1, \mathrm{DjM}_3)$ respectively. Other small examples have orders 18, 20, 28, 40, 56, 80 and 96. There are 129 examples of type DjM_3 with order up to 10000.

(e) Type DjM_4^1:

The smallest example is the Heawood graph F014 in [5,9], with action type $(\mathrm{DjM}_1, \mathrm{DjM}_4^1)$. The next smallest example is F102 in [5,9]. There are 13 examples of type DjM_4^1 with order up to 10000.

(f) Type DjM_4^2:

There are no examples of type DjM_4^2 with order up to 10000. The smallest example was shown in [8] to be a unique one of order 5314410 that happens to be a 3^{11}-fold cover of Tutte's 8-cage, with action type necessarily (DjM_4^2). A larger example (necessarily with the same action type) is a 3^{10}-fold cover of F468, with order 27634932, as mentioned in [13].

(g) Type DjM_5:

The smallest example is Tutte's 8-cage F030 in [5,9], and the next smallest is a triple cover of that, known as F090 in [5,9], with both having action type $(\mathrm{DjM}_4^1, \mathrm{DjM}_4^1, \mathrm{DjM}_5)$. There are 7 examples of type DjM_5 with order up to 10000.

4.2. Examples with given semisymmetric type

(a) Type G_1:

The smallest example is the Foster-Ljubljana graph of order 112, attributed to Foster and described in considerable detail in [12]. Other small examples were found in [11], with orders 336, 378, 400, and so on. There are 643 examples of type G_1 with order up to 10000.

(b) Type G_1^1:

The smallest example has order 144, with action type $(\mathrm{G}_1, \mathrm{G}_1^1)$. Other small examples found in [11] have orders 216, 336, 432, and so on. There are 222 examples of type G_1^1 with order up to 10000.

(c) Type G_1^2:

The smallest example has order 294, with action type $(\mathrm{G}_1, \mathrm{G}_1^2)$. Another small example found in [11] has order 504. There are 51 examples of type G_1^2 with order up to 10000.

(d) Type G_1^3:

The smallest example has order 120, with action type $(\mathrm{G}_1, \mathrm{G}_1^1, \mathrm{G}_1^2, \mathrm{G}_1^2, \mathrm{G}_1^3)$, indicating that the automorphism group has two different (normal) subgroups of index 2 acting with type G_1^2. Other small examples found in [11] have orders 220, 240, 336, and so on. There are 90 examples of type G_1^3 with order up to 10000.

(e) Type G_2:

There are no examples of type G_2 with order up to 10000. With the help of Magma [3], the smallest example can be shown to have order 25272, with automorphism group isomorphic to a split extension of C_3^3 by $\mathrm{PSL}(3,3)$, and with action type (G_2).

(f) Type G_2^1:

The smallest example has order 110, with action type $(\mathrm{G}_2, \mathrm{G}_2^1)$. Other small examples found in [11] have orders 182, 330, 506 and 546. There are 17 examples of type G_2^1 with order up to 10000.

(g) Type G_2^2:

There are no examples of type G_2^2 with order up to 10000. The smallest example we have found has order 2527220401920, with automorphism group isomorphic to the simple Mathieu group M_{24}, and with action type (G_2^2).

(h) Type G_2^3:

There are no examples of type G_2^3 with order up to 10000. We found an example with order 19!/12, coming from an action of Sym(19) with type G_2^3 that is not extendable to an action of a larger group with type G_2^4. The action type for this graph is $(\mathrm{G}_2, \mathrm{G}_2^3)$. The smallest example we have found has order 39366, with (soluble) automorphism group of order $472392 = 2^3 \cdot 3^{10}$, and action type $(\mathrm{G}_1, \mathrm{G}_1^2, \mathrm{G}_2, \mathrm{G}_2^3)$.

(i) Type G_2^4:

The smallest example of type G_2^4 is the well-known Gray graph, with order 54, with (soluble) automorphism group of order 1296, and action type $(\mathrm{G}_1, \mathrm{G}_1^1, \mathrm{G}_1^2, \mathrm{G}_1^2, \mathrm{G}_1^3, \mathrm{G}_2, \mathrm{G}_2^1,$

G_2^2, G_2^3, G_2^4), indicating that the automorphism group has many different subgroups that act semisymmetrically on the graph, including two different subgroups of index 8 acting with type G_1^2. There are 16 examples of type G_2^4 with order up to 10000.

(j) Type G_3:

There are no examples of type G_3 with order up to 10000. We found an easy example with order 15!/24, coming from an action of Alt(15) with type G_3 that is not extendable to an action of a larger group with type G_3^1, D_4^1 or D_4^2. The action type for this graph is (G_3). A smaller example is a graph of order 501645312, coming from the action of a group of order 6019743744 that is isomorphic to an extension of a group of order $2^{14}\,3^7$ by PSL(2, 7), having a transitive but imprimitive permutation representation on 28 points with 7 blocks of size 4 (making it a subgroup of the wreath product $\mathrm{Sym}(4) \wr \mathrm{PSL}(3,2)$, but it is not isomorphic to $\mathrm{Alt}(4) \wr \mathrm{PSL}(3,2)$). Again this action not extendable to one of a larger group of type G_3^1, D_4^1 or D_4^2. The action type for this graph is (G_1, G_3).

(k) Type G_3^1:

The smallest example of type G_3^1 is a graph of order 5760, with an (insoluble) automorphism group of order 138240, and with action type (G_3, G_3^1). This is the only example with order up to 10000,

(l) Type G_4:

There are no examples of type G_4 with order up to 10000. We easily found an example with order 28!/96, coming from an action of Alt(28) with type G_4 that is not extendable to an action of a larger group with type G_4^1. The action type for this graph is (G_4).

(m) Type G_4^1:

The smallest example of type G_4^1 is a graph of order 126, with automorphism group Aut(PSU(3, 3)) of order 12096, and action type (G_4, G_4^1). There is another of order 7750 with automorphism group PGL(3, 5) of order 372000, and the same action type. These are the only examples of type G_4^1 with order up to 10000.

(n) Type G_5:

There are no examples of type G_5 with order up to 10000. We found an easy example with order 62!/192, coming from an action of Alt(62) with type G_5 that is not extendable to an action of a larger group with type G_5^1. The action type for this graph is (G_5).

(o) Type G_5^1:

The smallest example of type G_5^1 is a graph of order 990 having automorphism group Aut(M_{12}) of order 190080, with action type (G_5, G_5^1). This is the only example of order up to 10000.

It remains an open question to find the smallest examples of connected finite semisymmetric cubic graphs of types G_2^2, G_2^3, G_3, G_4 and G_5. It would also be interesting to know precisely which action types are possible in the semisymmetric case, as was done in [13] for the arc-transitive case.

5. Existence of an infinite family for each amalgam type

Before continuing, we provide some theoretical background about graph coverings, and lifting groups of automorphisms along covering projections.

Let $\tilde{\Gamma}$ and Γ be two connected simple graphs, and let $\wp\colon \tilde{\Gamma} \to \Gamma$ be a graph morphism – that is, a function from $\mathrm{V}(\tilde{\Gamma})$ to $\mathrm{V}(\Gamma)$ such that $\wp(u) \sim_\Gamma \wp(v)$ whenever $u \sim_{\tilde{\Gamma}} w$.

If the restriction of $\wp$ to the neighbourhood $\tilde{\Gamma}(\tilde{v})$ is a bijection between $\tilde{\Gamma}(\tilde{v})$ and $\Gamma(\wp(\tilde{v}))$, for every $\tilde{v} \in \mathrm{V}(\tilde{\Gamma})$, then we say that $\wp$ is a *covering projection*, and in that case we call the pre-image $\wp^{-1}(v)$ of a vertex v of Γ a *vertex-fibre*. For any such $\wp$, the group of all automorphisms of $\tilde{\Gamma}$ that preserve every vertex-fibre set-wise is called the *group of covering transformations* of $\wp$, and is denoted by $\mathrm{CT}(\wp)$. It is easy to see that $\mathrm{CT}(\wp)$ acts faithfully and indeed semiregularly on each vertex-fibre. If it acts transitively (and hence regularly) on each fibre, then the covering projection $\wp$ is said to be *regular*.

Next, if $\tilde{g} \in \mathrm{Aut}(\tilde{\Gamma})$ and $g \in \mathrm{Aut}(\Gamma)$ satisfy $\wp(\tilde{v}^{\tilde{g}}) = \wp(\tilde{v})^g$ for every $\tilde{v} \in \mathrm{V}(\tilde{\Gamma})$, then we say that g *lifts* along $\wp$, and that $\tilde{g}$ is a *lift* of g, and also that $\tilde{g}$ *projects* along $\wp$ and that g is a *projection* of $\tilde{g}$. Moreover, if $G \leq \mathrm{Aut}(\Gamma)$ and every $g \in G$ lifts along $\wp$, then the set of all lifts of elements of G is called the *lift* of G, and similarly, if every element of $\tilde{G} \leq \mathrm{Aut}(\tilde{\Gamma})$ projects, then the set of all projections of elements of $\tilde{G}$ is called the *projection* of $\tilde{G}$.

With this terminology, the group $\mathrm{CT}(\wp)$ is precisely the lift of the trivial subgroup of $\mathrm{Aut}(\Gamma)$, and a lift of a projection of some group $\tilde{G} \leq \mathrm{Aut}(\tilde{\Gamma})$ is the group $\mathrm{CT}(\wp)\tilde{G}$. It is also easy to see that $\tilde{G} \leq \mathrm{Aut}(\tilde{\Gamma})$ projects along $\wp$ if and only if it normalises the covering transformation group $\mathrm{CT}(\wp)$.

Next let Γ be a graph, with arc-set $A = \mathrm{A}(\Gamma)$, and cycle-set $\mathcal{C}(\Gamma)$. Also let $\mathbb{Z}A$ be the free $\mathbb{Z}$-module over A, and let R be the sub-module $\langle\, x + x^{-1} : x \in A \,\rangle$ of $\mathbb{Z}A$, where x^{-1} denotes the reverse of an arc x. For a cycle C that traverses the arcs $x_0 = (v_0, v_1)$, $x_1 = (v_1, v_2), \ldots, x_{n-1} = (v_{n-1}, v_0)$ in that order, let a_C be the image of $x_0 + x_1 + \ldots + x_{n-1} \in \mathbb{Z}A$ in the quotient module $\mathbb{Z}A/R$.

Then the submodule $\langle\, a_C : C \in \mathcal{C}(\Gamma) \,\rangle$ of $\mathbb{Z}A$ is called the *first homology group* (or sometimes also *the integral cycle space*) of Γ, and is denoted by $\mathrm{H}_1(\Gamma;\mathbb{Z})$. Intuitively we may think of $\mathrm{H}_1(\Gamma;\mathbb{Z})$ as the submodule of $\mathbb{Z}A$ generated by all the cycles of Γ, where the reverse x^{-1} of each arc x is identified with the quantity $-x \in \mathbb{Z}A$.

Since every automorphism of Γ takes cycles to cycles, there exists a natural action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}_1(\Gamma;\mathbb{Z})$, preserving the structure of the $\mathbb{Z}$-module.

In very specific circumstances, this action could potentially be unfaithful, but in that case, a non-trivial automorphism of Γ would have to preserve each cycle of Γ while acting on it as a rotation (and not as a reflection). This happens, for example, when Γ itself is a cycle.

When this pathological situation does not arise, the following theorem can be applied to Γ.

Theorem 8. *[30, Theorem 6] Let p be an odd prime, let Γ be a finite connected graph such that the induced action of* $\mathrm{Aut}(\Gamma)$ *on* $\mathrm{H}_1(\Gamma;\mathbb{Z})$ *is faithful, and let* $G \leq \mathrm{Aut}(\Gamma)$*. Then there exists a regular covering projection* $\wp\colon \tilde{\Gamma} \to \Gamma$*, with* $\tilde{\Gamma}$ *finite, such that the maximal group that lifts along* $\wp$ *is* G*, and the group of covering transformations of* $\wp$ *is a* p*-group.*

Equipped with the above, we can now prove Theorem 6 from the Introduction. In fact we prove the following more detailed version of Theorem 6, in which 'a group of amalgam type' means a group acting edge-transitively with Goldschmidt type or Djoković-Miller type.

Theorem 9. *Let* Γ *be a finite cubic edge-transitive graph, and let* G *be an edge-transitive group of automorphisms of* Γ *of amalgam type* T*. Then for all but finitely many primes* p*, there exists a regular covering projection* $\wp\colon \tilde{\Gamma} \to \Gamma$ *such that* $\tilde{\Gamma}$ *is finite,* $\mathrm{Aut}(\tilde{\Gamma})$ *is the lift of* G *along* $\wp$*, and the group* $\mathrm{CT}(\wp)$ *of covering transformations of* $\wp$ *is a* p*-group. In particular, also* $\tilde{\Gamma}$ *has type* T*.*

Proof. We begin by proving that $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma;\mathbb{Z})$. This first part of the proof follows the proof of [30, Lemma 7] almost verbatim.

Assume to the contrary that the action of $\mathrm{Aut}(\Gamma)$ on $\mathrm{H}_1(\Gamma;\mathbb{Z})$ is not faithful. Then there exists an automorphism g fixing every element of $\mathrm{H}_1(\Gamma;\mathbb{Z})$, and a vertex v of Γ such that $v^g \neq v$. Now let u, w and z be the three neighbours of v. As Γ is finite and contains no vertices of valency 1, it contains a cycle, and then since Γ is edge-transitive, every one of its edges lies on a cycle. Moreover, the stabiliser $\mathrm{Aut}(\Gamma)_v$ acts transitively on the neighbourhood $\Gamma(v) = \{u, w, z\}$ of v, so it contains an element that cyclically permutes u, w and z, and hence each of the three 2-paths centred at v lies on a cycle.

Let C_1 be a cycle through the 2-arc (u, v, w), and C_2 be a cycle through the 2-arc (u, v, z), then fix an orientation of C_1 and C_2 in such a way that u is a predecessor of v in both C_1 and C_2, and consider C_1 and C_2 as elements of $\mathrm{H}_1(\Gamma;\mathbb{Z})$. By assumption, g preserves C_1 and C_2, as well as their orientations. In particular, the vertex v^g lies on C_1 and on C_2, and if we let P_i be the path from v^g to u following the cycle C_i in the positive direction with respect to the chosen orientation, for $i \in \{1, 2\}$, then since g preserves orientation, the vertex u^g lies on neither P_1 nor P_2. Now if we let C be the closed walk obtained by concatenating P_1 with the reverse of P_2, and consider it as an element of $\mathrm{H}_1(\Gamma;\mathbb{Z})$, then by assumption, C is fixed by g, but the vertex u belongs to C, while u^g does not, a contradiction. This proves that $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma;\mathbb{Z})$, as claimed.

The second part of the proof closely follows the proof of [30, Theorem 9].

Recall that the order of the edge-stabiliser in the automorphism group of a finite cubic edge-transitive graph is bounded above by the constant $c = 128$. Let n be the number of edges of Γ, and let p be any prime such that $p > nc$. Then since $\mathrm{Aut}(\Gamma)$ acts faithfully on $\mathrm{H}_1(\Gamma;\mathbb{Z})$, we may use Theorem 8 to obtain a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ with $\tilde{\Gamma}$ finite, such that the maximal group that lifts along $\wp$ is G, and the group $\tilde{K} = \mathrm{CT}(\wp)$ is a p-group.

Let $\tilde{G}$ be the lift of G along $\wp$. Then $\tilde{K}$ is a normal p-subgroup of $\tilde{G}$ such that $\tilde{G}/\tilde{K} \cong G$, and as G is the maximal group that lifts along $\wp$, the normaliser $N_{\tilde{A}}(\tilde{K})$ of $\tilde{K}$ in the automorphism group $\tilde{A} = \mathrm{Aut}(\tilde{\Gamma})$ is $\tilde{G}$. Let $\tilde{e}$ be an edge of $\tilde{\Gamma}$ and let $e = \wp(\tilde{e})$. Then by the definition of the constant c, it follows that $|\tilde{A}_{\tilde{e}}| \le c$. Next, because $\tilde{G}$ is transitive on the edges of $\tilde{\Gamma}$, we have $\tilde{A} = \tilde{A}_{\tilde{e}}\tilde{G}$, and hence $|\tilde{A} : \tilde{G}| = |\tilde{A}_{\tilde{e}} : \tilde{G}_{\tilde{e}}| \le |\tilde{A}_{\tilde{e}}| \le c < p$. Also $|\tilde{G} : \tilde{K}| = |G| = n|G_e| \le nc < p$, and it follows that $|\tilde{A} : \tilde{K}| = |\tilde{A} : \tilde{G}||\tilde{G} : \tilde{K}|$ is not divisible by p, and so $\tilde{K}$ is a Sylow p-subgroup of $\tilde{A}$. Then by Sylow theory, the number of Sylow p-subgroups of $\tilde{A}$ is $|\tilde{A} : N_{\tilde{A}}(\tilde{K})| = |\tilde{A} : \tilde{G}| < p$ and is congruent to 1 modulo p, so must be 1, and thus $\tilde{A} = N_{\tilde{A}}(\tilde{K}) = \tilde{G}$.

Finally, because $\tilde{G}$ is the lift of G along a regular covering projection, it has the same amalgam type as G, and this completes the proof. □

Note that it is easy to find a finite edge-transitive cubic graph admitting an edge-transitive group of type T, for every Goldschmidt or Djoković-Miller amalgam type T. A lot of them can be realised in $K_{3,3}$, while others are realised in larger graphs, and indeed we have shown in the previous section that for every such T there exists a finite edge-transitive cubic graph whose (full) automorphism group has type T. Hence Theorem 9 shows that there are infinitely many finite edge-transitive cubic graphs of each type, as required.

6. Asymptotic enumeration

In this section we will say that a function $f\colon \mathbb{N} \to \mathbb{R}$ is of type $n^{\log n}$ if there exist positive real constants a and b such that $n^{a \log n} \le f(n) \le n^{b \log n}$ holds for all sufficiently large integers n.

The following theorem essentially follows from [28, Theorem 1], which was stated in the more general setting of pro-p groups, and for the sake of completeness, we provide a sketch of its proof.

Theorem 10. *Let G be a group containing a free subgroup F of rank $r \ge 2$ as a normal subgroup of finite index, let p be a prime not dividing the index $|G{:}F|$, and let $f\colon \mathbb{N} \to \mathbb{R}$ be the function defined by $f(n) = |\mathcal{N}_n|$, where $\mathcal{N}_n = \{N \le F \mid N \trianglelefteq G \text{ and } |F{:}N| = p^\alpha \le n \text{ for some integer } \alpha\}$. Then f is of type $n^{\log n}$.*

Proof. First, $f(n)$ is clearly smaller than the number of all r-generated p-groups of order at most n, so an upper bound on $f(n)$ of the form $n^{b \log n}$ for some positive real constant b

follows from a theorem of Lubotzky [24, Theorem 1]. Hence we concentrate on obtaining a lower bound for $f(n)$ of the same form.

Let $F_1 \geq F_2 \geq F_3 \geq \dots$ be the lower p-central series for the group F, given by $F_1 = F$ and $F_{i+1} = [F, F_i]F_i^p$ for all $i \geq 1$. Then for each i the factor group F_i/F_{i+1} is an elementary abelian p-group, so can be viewed as a vector space V_i over $\mathrm{GF}(p)$. Next, define w_i and s_i by $|F_i : F_{i+1}| = p^{w_i}$ and $|F : F_i| = p^{s_i}$ (so that $s_1 = 0$ and $s_i = w_1 + \cdots + w_{i-1}$ for $i > 1$). Then $\dim(V_i) = w_i$ for all i. Also define $X_i = \{N \trianglelefteq G \mid F_{i+1} \leq N \leq F_i\}$, so that $f(n) \geq |X_i|$ for all i.

Since all the F_i are characteristic subgroups of F and hence normal in G, conjugation by G induces an action of G on each factor F_i/F_{i+1}, with kernel containing F, because F_i/F_{i+1} is central in F/F_{i+1}. Accordingly, this gives a linear representation $G/F \to \mathrm{GL}(V_i)$. Note also that there is a bijective correspondence between the set of invariant subspaces of this linear representation and the set X_i defined in the previous paragraph.

On the other hand, by [28, Lemma 1], there exists a positive constant c, depending solely on the group G/F_i and the prime p (which must be coprime to $|G/F|$), such that every linear representation of G/F on a w-dimensional vector-space over $\mathrm{GF}(p)$ has at least p^{cw^2} invariant subspaces. Together with the observations above, this implies that $|X_i| \geq p^{cw_i^2}$ and hence that $f(n) \geq p^{cw_i^2}$, for all i.

Now define $a = \frac{4c}{9r^2}$ (where r is the rank of the free subgroup F), and let n be any positive integer such that $n \geq |F : F_2| = p^{s_2}$, and for this n, let $j \geq 1$ be such that $p^{s_{j+1}} \leq n < p^{s_{j+2}}$.

We will prove that $3rw_j \geq 2s_{j+2}$, from which it follows that $9r^2w_j^2 \geq 4(s_{j+2})^2$, and hence that $cw_j^2 \geq \frac{4c}{9r^2}(s_{j+2})^2 = a(s_{j+2})^2 > a(\log_p n)^2$, and so $f(n) \geq p^{cw_j^2} \geq p^{a(\log_p n)^2} = n^{a \log_p n} = n^{(\frac{a}{\log p}) \log n}$, giving us a lower bound on $f(n)$ of the form we require.

To do this, we derive estimates of the values of the parameters w_j and s_{j+2}.

It follows from the work of Bryant and Kovacs [4] that the dimension w_j of the linear space F_j/F_{j+1} is $\sum_{i=1}^{j} r_i$, where r_i is the rank of the ith section $\gamma_i(F)/\gamma_{i+1}(F)$ of the lower central series of F, viewed as a $\mathbb{Z}$-module. The rank r_i was determined by Witt [33] and equals $\frac{1}{i}\sum_{d \mid i} \mu(d) r^{i/d}$, where μ denotes the arithmetic Möbius function (see [2, Lemma 20.6(iii)]). Accordingly

$$w_j = \sum_{i=1}^{j} \frac{1}{i} \sum_{d \mid i} \mu(d) r^{\frac{i}{d}} \qquad \text{and} \qquad s_{j+2} = \sum_{i=1}^{j+1} w_i.$$

As was proved in [2, Lemma 20.7] using a Stolz-Cesàr analogue of L'Hôspital's rule, the values of w_j and s_{j+2} can be asymptotically approximated as

$$w_j = \frac{r^{j+1}}{j(r+1)}(1 + o(1)), \quad \text{and} \quad s_{j+2} = \frac{r^{j+3}}{(j+2)(r+1)^2}(1 + o(1)), \quad \text{as } j \to \infty.$$

In particular, as $\frac{2}{3} \leq \frac{r}{r+1} < 1$ and $\frac{1}{j+2} < \frac{1}{j}$, it follows that for sufficiently large values of j, we have $w_j \geq \frac{2r^j}{3j}$ and $s_{j+2} \leq \frac{r^{j+1}}{j}$, which immediately gives us $3rw_j \geq 2s_{j+2}$, completing the proof. □

The above theorem has the following straightforward consequence.

Corollary 11. *Let G be a group containing a free subgroup F of rank $r \geq 2$ as a normal subgroup of finite index, and let p be a prime not dividing the index $|G : F|$. Also for every positive integer n, let $\mathcal{N}_n$ be the set defined in Theorem 10, and for subgroups K and N in $\mathcal{N}_n$, write $K \sim N$ whenever G/K is isomorphic to G/N, and let $\mathcal{N}_n^*$ denote the partition of $\mathcal{N}_n$ into the equivalence classes of the relation $\sim$. Then the function $g\colon \mathbb{N} \to \mathbb{R}$ given by $g(n) = |\mathcal{N}_n^*|$ is of type $n^{\log n}$.*

Proof. Since F has finite rank and finite index in G, the group G is generated by some finite set X, say of size d. Now for subgroups K and N in the same equivalence class of $\sim$, let $\pi\colon G \to G/K$ be the natural homomorphism, and let $\varphi\colon G/K \to G/N$ be an isomorphism. Then the (left-to-right) composite homomorphism $\pi\varphi\colon G \to G/N$ has kernel K. On the other hand, every homomorphism from $G = \langle X\rangle$ to G/N is determined by the images of the elements of X, and hence there are at most $|G/N|^d$ such homomorphisms, and hence at most $|G/N|^d$ possibilities for K. Finally, if $h = |G : F|$, then $|G/N| = |G/F||F/N| = hn$, and it follows that the equivalence class of N under $\sim$ has size at most $(hn)^d$, and therefore $g(n) \geq \frac{f(n)}{h^d n^d}$. Then since f is of type $n^{\log n}$, so is g. □

We can now strengthen Theorem 6 to the following one.

Theorem 12. *Let Γ be a connected finite cubic graph, and let G be an edge-transitive group of automorphisms of Γ. Then there exists a positive real constant a such that for all sufficiently large n, the number of (non-isomorphic) finite regular coverings of $\tilde{\Gamma}$ with $|V(\tilde{\Gamma})| \leq n$ whose automorphism group is precisely the lift of G is at least $n^{a \log n}$.*

Proof. By Theorem 9, there exists a regular covering projection $\wp\colon \tilde{\Gamma} \to \Gamma$ such that $\mathrm{Aut}(\tilde{\Gamma})$ is the lift of G. Now let $\wp_\circ\colon T_3 \to \tilde{\Gamma}$ be the universal covering projection from the cubic tree T_3 to $\tilde{\Gamma}$, and let A be the lift of $\mathrm{Aut}(\tilde{\Gamma})$ along $\wp_\circ$. Then $\mathrm{Aut}(\tilde{\Gamma}) \cong A/F$ for some free normal subgroup F of A, acting semiregularly on the vertices and on the edges of T_3. Note that the rank of F is equal to the rank of the group $\mathrm{H}_1(\tilde{\Gamma}, \mathbb{Z})$, which can be computed as $|V(\tilde{\Gamma})| - |E(\tilde{\Gamma})| + 1$. Moreover $\wp_\circ$ is equivalent to the quotient projection $T_3 \to T_3/F$. Hence we may assume that $\tilde{\Gamma} = T_3/F$ and $\mathrm{Aut}(\tilde{\Gamma}) = A/F$.

Next, let p be a prime not dividing $|A : F|$, and as before (but with A taking the role of G), for any given positive integer n, let $\mathcal{N}_n = \{N \leq F \mid N \trianglelefteq A$ and $|F : N| = p^\alpha \leq n$ for some $\alpha\}$. Then by Corollary 11, there exists a positive real constant a and a subset $\mathcal{N}_n^* \subseteq \mathcal{N}_n$ such that $|\mathcal{N}_n^*| \geq n^{a \log n}$ for every large enough n, with the property that for all K and N in $\mathcal{N}_n^*$, if $A/K \cong A/N$ then $K = N$.

Now take some $N \in \mathcal{N}_n^*$ and consider the quotient graph $\tilde{\Gamma}_N = T_3/N$. Note that since $N \leq F$, the quotient projection $\wp_N : T_3 \to \tilde{\Gamma}_N$ is a regular covering projection. Moreover, the group F/N acts semiregularly on the vertices and edges of $\tilde{\Gamma}$, and so the quotient projection $\wp_{F/N} : \tilde{\Gamma}_N \to \tilde{\Gamma}_N/(F/N)$ is also a covering projection. But $\tilde{\Gamma}_N/(F/N)$ is isomorphic to $(T_3/N)/(F/N) \cong T_3/F = \tilde{\Gamma}$ in a natural way, and so in this sense, we may conclude that $\wp_\circ = \wp_{F/N} \circ \wp_N$. Furthermore, the group $\mathrm{Aut}(\tilde{\Gamma}) = A/F$ lifts along $\wp_{F/N}$ to the group A/N. Then since the edge-stabiliser for a connected finite cubic edge-transitive graph has order at most 128, it follows that $|\mathrm{Aut}(\tilde{\Gamma}_N)| \leq 128|E(\tilde{\Gamma}_N)| \leq 128|A/N|$. In particular, if F/N is a p-group for some prime $p > 128|(A/N) : (F/N)| = 128|A : F| = 128|\mathrm{Aut}(\tilde{\Gamma})|$, then (just as in the proof of Theorem 9), F/N is a normal Sylow p-subgroup of $\mathrm{Aut}(\tilde{\Gamma}_N)$, and therefore $\mathrm{Aut}(\tilde{\Gamma}_N)$ projects via the quotient projection $\tilde{\Gamma}_N \to \tilde{\Gamma}$, so $\mathrm{Aut}(\tilde{\Gamma}_N)$ is equal to the lift of $\mathrm{Aut}(\tilde{\Gamma})$.

Thus we have shown that every $N \in \mathcal{N}_n^*$ yields a regular covering $\tilde{\Gamma}_N$ of $\tilde{\Gamma}$ (and hence also of Γ) such that $\mathrm{Aut}(\tilde{\Gamma}_N)$ is the lift of $\mathrm{Aut}(\tilde{\Gamma})$ along $\wp_{F/N} : \tilde{\Gamma}_N \to \tilde{\Gamma}$ (and hence also of G along $\wp \circ \wp_{F/N} : \tilde{\Gamma}_N \to \Gamma$). On the other hand, as $\mathrm{Aut}(\tilde{\Gamma}_K) = A/K \not\cong A/N = \mathrm{Aut}(\tilde{\Gamma}_N)$ for two distinct groups K and N in $\mathcal{N}_n^*$, it follows that $|\mathcal{N}_n^*|$ is bounded above by the number of regular coverings $\tilde{\Gamma}'$ of $\tilde{\Gamma}$ with $|V(\tilde{\Gamma}')| \leq n|V(\tilde{\Gamma})|$ whose automorphism group is the lift of G. Finally, because the function $n \mapsto |\mathcal{N}_n^*|$ is of type $n^{\log n}$, we find that for some positive real constant a and for all sufficiently large n, the number of these coverings is at most $n^{a \log n}$. □

Remark 13. As every Goldschmidt or Djoković-Miller amalgam T can be realised in some finite graph, and the amalgam type of a lift of a group G of automorphisms is the same as for G, a consequence of Theorem 12 is that there exists a positive real constant a such that the number of cubic edge-transitive graphs of order at most n with automorphism group of type T is at least $n^{a \log n}$. In fact, since every cubic edge-transitive graph Γ of order n is uniquely determined by its type T and the epimorphism from the corresponding universal group $A \leq \mathrm{Aut}(T_3)$ of type T to $\mathrm{Aut}(\Gamma)$, it follows that the number of cubic edge-transitive graphs of type T with order up to n is bounded above by the number of normal subgroups of index at most n in A times the maximum number of epimorphisms from A to a finite group of order at most n. By the same theorem of Lubotzky as used earlier, the first factor is at most $n^{b \log n}$ for a constant b depending only on the minimum number d of generators for A, while the second factor is at most n^d. Hence the growth rate of the number of connected edge-transitive cubic graphs of any given Goldschmidt or Djoković-Miller amalgam type T with order up to n is $n^{\log n}$.

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Data availability

Computational data is available at https://fostercensus.graphsym.net/, and further details can be made available upon request.

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