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k-Domination invariants on Kneser graphs

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Abstract

In this follow-up to work of M.G. Cornet and P. Torres from 2023, where the k-tuple domination number and the 2-packing number in Kneser graphs $K(n,r)$ were studied, we are concerned with two variations, the k-domination number, $\gamma_k(K(n,r))$, and the k-tuple total domination number, $\gamma_{t\times k}(K(n,r))$, of $K(n,r)$. For both invariants we prove monotonicity results by showing that $\gamma_k(K(n,r)) \geq \gamma_k(K(n+1,r))$ holds for any $n \geq 2(k+r)$, and $\gamma_{t\times k}(K(n,r)) \geq \gamma_{t\times k}(K(n+1,r))$ holds for any $n \geq 2r+1$. We prove that $\gamma_k(K(n,r)) = \gamma_{t\times k}(K(n,r)) = k+r$ when $n \geq r(k+r)$, and that in this case every γ_k-set and $\gamma_{t\times k}$-set is a clique, while $\gamma_k(r(k+r)-1,r) = \gamma_{t\times k}(r(k+r)-1,r) = k+r+1$, for any $k \geq 2$. Concerning the 2-packing number, $\rho_2(K(n,r))$, of $K(n,r)$, we prove the exact values of $\rho_2(K(3r-3,r))$ when $r \geq 10$, and give sufficient conditions for $\rho_2(K(n,r))$ to be equal to some small values by imposing bounds on r with respect to n. We also prove a version of monotonicity for the 2-packing number of Kneser graphs.

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1 Introduction

Letting $n \geq 2r$ the *Kneser graph* $K(n, r)$ has the r-subsets of an n-set as its vertices and two vertices are adjacent in $K(n, r)$ if the corresponding sets are disjoint. The interest for Kneser graphs goes back to 1960s and 1970s when two classical theorems concerning their independence and chromatic number were proved [7, 20]. Many other graph invariants have been investigated in Kneser graphs, which makes them one of the most intensively studied classes of graphs. In particular, several authors have considered the domination number of Kneser graphs (see the most recent paper [21]), but only lower and upper bounds have been found in general, while the exact values of $\gamma(K(n, r))$ were found only when n is sufficiently large with respect to r. In three recent papers, the authors have considered Kneser graphs in relation with some variations of domination, namely, Grundy domination [2], Roman domination [24] and k-tuple domination [4]. This paper is a follow-up of the latter, where we extend our investigation from k-tuple domination to two similar variations, which we next present.

A *dominating set* in a graph G is a set S of vertices of G such that every vertex outside S is adjacent to at least one vertex in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set in G. A thorough treatise on dominating sets can be found in the so-called "domination books" [13, 14, 15, 18].

Let G be a graph and $k \in \mathbb{N}$. A set $D \subseteq V(G)$ is a *k-dominating set* of G if every vertex $u \in V(G) \setminus D$ has at least k neighbors in D (or, equivalently, if $|N_G(u) \cap D| \geq k$ for each $u \in V(G) \setminus D$ where $N_G(u)$ denotes the open neighborhood of the vertex u in G). The *k-domination number* of G is the minimum cardinality of a k-dominating set of G, and is denoted by $\gamma_k(G)$. A *γ_k-set* of G is a k-dominating set of cardinality $\gamma_k(G)$. See [3, 8, 9, 11] for a selection of papers considering k-domination.

A set $D \subseteq V(G)$ is a *k-tuple dominating set* of G if the closed neighborhood $N_G[u]$ of each vertex $u \in V(G)$ has at least k vertices in D (or, equivalently, if $|N_G[u] \cap D| \geq k$ for each $u \in V(G)$). The *k-tuple domination number* of G is the minimum cardinality of a k-tuple dominating set of G, and is denoted by $\gamma_{\times k}(G)$. A *$\gamma_{\times k}$-set* of G is a k-tuple dominating set of cardinality $\gamma_{\times k}(G)$. See [11, 12] for a selection of papers on k-tuple domination.

Finally, a set $D \subseteq V(G)$ is a *k-tuple total dominating set* of G if the open neighborhood of each vertex $u \in V(G)$ has at least k vertices in D (that is, if $|N_G(u) \cap D| \geq k$ for each $u \in V(G)$). The *k-tuple total domination number* of G is the minimum cardinality of a k-tuple total dominating set of G, and is denoted by $\gamma_{t\times k}(G)$. As usual, a *$\gamma_{t\times k}$-set* of G stands for a k-tuple total dominating set of G of cardinality $\gamma_{t\times k}(G)$. The k-tuple total domination number was studied, for example, in [16, 17], while all three invariants were studied for the case $k = 2$ in [1].

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Note that $\gamma_{\times k}(G)$ is defined only in graphs G with $\delta(G) \geq k-1$, while $\gamma_{t\times k}(G)$ is defined only if $\delta(G) \geq k$, where $\delta(G)$ denotes the minimum degree of vertices in G. One can derive immediately from the definitions that

$$\gamma_k(G) \leq \gamma_{\times k}(G) \leq \gamma_{t\times k}(G) \tag{1.1}$$

holds for any graph G in which the corresponding invariants are defined. Clearly, when $k = 1$, both $\gamma_{\times 1}(G)$ and $\gamma_1(G)$ correspond to the well-known domination number $\gamma(G)$, and $\gamma_{t\times 1}(G)$ corresponds to the total domination number $\gamma_t(G)$. In this paper, we denote by $\gamma_k(n,r)$, $\gamma_{\times k}(n,r)$ and $\gamma_{t\times k}(n,r)$ the k-domination number, the k-tuple domination number and the k-tuple total domination number, respectively, of the Kneser graph $K(n,r)$.

Concerning the domination number of Kneser graphs, it was shown to be monotonically decreasing when r is fixed and n grows [10] and a similar result was proved for the k-tuple domination number in [4]. In this vein we prove in Section 2 that $\gamma_{t\times k}(n,r) \geq \gamma_{t\times k}(n+1,r)$ and $\gamma_k(n,r) \geq \gamma_k(n+1,r)$ hold for any positive integers n and r, where $n \geq 2r+1$, respectively $n \geq 2(k+r)$. In Section 3, we obtain exact values for these two invariants in $K(n,r)$ when n is sufficiently large with respect to r. Notably, we prove that $\gamma_k(n,r) = \gamma_{t\times k}(n,r) = k+r$ as soon as $n \geq r(k+r)$, and, in addition, every γ_k-set and $\gamma_{t\times k}$-set is a clique in this case. When n is one less, we also get exact values, namely, if $k \geq 2$, we have $\gamma_k(r(k+r)-1,r) = \gamma_{t\times k}(r(k+r)-1,r) = k+r+1$. In Section 4, we study the 2-packing number, $\rho_2(K(n,r))$, of Kneser graphs, which is the largest cardinality of a set of vertices in the graph, which are pairwise at distance at least 3 apart. We continue the study from [4], where the 2-packing number was used for bounding the k-tuple domination number of $K(n,r)$, and $\rho_2(K(3r-2,r))$ was also determined for all values of r. Here we go a step further by considering $\rho_2(K(3r-3,r))$, where we obtain lower bounds when $r \leq 8$, prove that 4 is the exact value for $r = 9$, and that 3 is the exact value when $r \geq 10$. We also give sufficient conditions for $\rho_2(K(n,r))$ being equal to 3, resp. 4, by imposing bounds on r with respect to n. In addition, we prove a version of monotonicity, by showing that $\rho_2(K(n+1,r+1)) \geq \rho_2(K(n,r))$ as soon as $n \geq 2r+2$.

We conclude the introduction with several useful definitions. For $r,n \in \mathbb{N}$ with $r \leq n$, let $[r..n]$ and $[n]$ denote the sets $\{r,\dots,n\}$ and $\{1,\dots,n\}$ respectively. Given a set of vertices D in $K(n,r)$ and $x \in [n]$, the occurrences of the element x in D, denoted by $i_x(D)$, represent the number of vertices in D that contain the element x. This is, $i_x(D) = |\{u \in D : x \in u\}|$. For a positive integer a, we define $X_a(D)$ as the set of elements in $[n]$ such that their occurrences in D is equal to a, i.e., $X_a(D) = \{x \in [n] : i_x(D) = a\}$. Similarly, we define $X_a^{\geq}(D) = \{x \in [n] : i_x(D) \geq a\}$, and $X_a^{\leq}(D) = \{x \in [n] : i_x(D) \leq a\}$. When the set D is clear from the context, we shall omit it in the notation. It is important to note that the sum of the occurrences of all elements in D is equal to r times the cardinality of D, i.e., $\sum_{x\in[n]} i_x(D) = r|D|$.

2 Monotonicity

In [4] it is proved that the function $\gamma_{\times k}(n,r)$ is decreasing with respect to n. Using the same idea, we prove the analogous statement for the function $\gamma_{t\times k}(n,r)$. Regarding the k-domination number, we show that $\gamma_k(n,r)$ is decreasing with respect to n from $n_0 = 2(k+r)$.

Note that, using the standard notation for vertices of Kneser graphs, a vertex $u \in V(K(n,r))$ also belongs to $K(n+1,r)$, since it is represented as an r-subset of $[n] \subset [n+1]$. When r is fixed, we will simplify the notation by writing $N_n(u)$ for the neighborhood $N_{K(n,r)}(u)$, where $u \subset [n]$ with $|u| = r$. In this sense, the meaning of $N_{n+1}(u)$ should also be clear.

Theorem 2.1. *For any positive integers n and r, where $n \geq 2r+1$,*

$$\gamma_{t\times k}(n,r) \geq \gamma_{t\times k}(n+1,r).$$

That is, $\gamma_{t\times k}(n,r)$ is decreasing with respect to n.

Proof. Let D be a $\gamma_{t\times k}$-set of $K(n,r)$. Let us show that D is a k-tuple total dominating set of $K(n+1,r)$. Let $u \in V(K(n+1,r))$. If $u \in V(K(n+1,r)) \cap V(K(n,r))$, then $|N_{n+1}(u) \cap D| = |N_n(u) \cap D| \geq k$. If $u \in V(K(n+1,r)) \setminus V(K(n,r))$, then we have $u = \tilde{u} \cup \{n+1\}$ with $\tilde{u} \subseteq [n]$, $|\tilde{u}| = r-1$. Let $x \in [n] \setminus \tilde{u}$, and let $w = \tilde{u} \cup \{x\}$. Since $w \in V(K(n+1,r)) \cap V(K(n,r))$, then $|N_{n+1}(w) \cap D| \geq k$. On the other hand,

$$N_{n+1}(w) \cap D = \{v \in D : v \cap w = \emptyset\} \subseteq \{v \in D : v \cap \tilde{u} = \emptyset\} = \{v \in D : v \cap u = \emptyset\}.$$

Thus, we have $|N_{n+1}(u) \cap D| \geq |N_{n+1}(w) \cap D| \geq k$. Therefore, D is a k-tuple total dominating set of $K(n+1,r)$. Consequently, $\gamma_{t\times k}(n+1,r) \leq |D| = \gamma_{t\times k}(n,r)$. □

Lemma 2.2. *If $n \geq 2(k+r)$, D is a γ_k-set of $K(n,r)$ and $\tilde{u} \subseteq [n]$ with $|\tilde{u}| = r-1$, then there exists $x \in [n] \setminus \tilde{u}$ such that $\tilde{u} \cup \{x\} \notin D$.*

Proof. Let $[n] \setminus \tilde{u} = \{x_1, x_2, \ldots, x_{n-r+1}\}$. Suppose that $\tilde{u} \cup \{x\} \in D$ for each $x \in [n] \setminus \tilde{u}$, and let $u_i = \tilde{u} \cup \{x_i\}$ for each $i \in [n-r+1]$. Let $D' = (D \setminus \{u_1, \ldots, u_{k+1}\}) \cup \{v_1, \ldots, v_k\}$, where $v_i = \{x_{k+2}, \ldots, x_{k+r}, x_{k+r+i}\}$ for $i \in [k]$. It is possible since $n \geq 2(k+r)$.

Let us see that D' is a k-dominating set of $K(n,r)$. Let $w \in V(K(n,r)) \setminus D'$ and consider the following options:

- Let $w \cap \tilde{u} = \emptyset$ (and $w \neq u_i$ for each $i \in [k+1]$). Since $|w| = r$, $|w \cap \{x_{k+2}, \ldots, x_{n-r+1}\}| \leq r$. Hence there exist $|\{x_{k+2}, \ldots, x_{n-r+1}\}| - r = n-2r-k$ elements of $\{x_{k+2}, \ldots, x_{n-r+1}\}$ that are not contained in w. Denote those elements with $x_{i_1}, \ldots, x_{i_{n-2r-k}}$, i.e. $w \cap \{x_{i_1}, \ldots, x_{i_{n-2r-k}}\} = \emptyset$. Hence $w \cap u_{i_j} = \emptyset$ for any $j \in [n-2r-k]$ and thus $wu_{i_j} \in E(K(n,r))$. Since $\{x_{i_1}, \ldots, x_{i_{n-2r-k}}\} \subseteq \{x_{k+2}, \ldots, x_{n-r+1}\}$, vertices u_{i_j}, $j \in [n-2r-k]$, are contained in D', which implies that $|N(w) \cap D'| \geq n-2r-k \geq k$, where the last inequality holds as $n \geq 2(k+r)$.
- Let $w \cap \tilde{u} \neq \emptyset$ and $w \neq u_i$ for each $i \in [k+1]$. Then $|N(w) \cap D'| \geq |N(w) \cap D| \geq k$, since the vertices eliminated from D were not neighbors of w.
- Let $w = u_i$ for some $i \in [k+1]$. Then we have $|N(w) \cap D'| \geq |\{v_1, \ldots, v_k\}| = k$.

Therefore, D' is a k-dominating set of $K(n,r)$ of size $|D'| = |D| - 1 = \gamma_k(n,r) - 1$, and we arise to a contradiction. Thus, there is at least one element $x \in [n] \setminus \tilde{u}$ such that $\tilde{u} \cup \{x\} \notin D$. □

Theorem 2.3. *For any positive integers n and r, where $n \geq 2(k+r)$,*

$$\gamma_k(n,r) \geq \gamma_k(n+1,r).$$

Proof. Let $n \geq 2(k+r)$ and let D be a γ_k-set of $K(n,r)$. Let us show that D is a k-dominating set of $K(n+1,r)$. Let $u \in V(K(n+1,r)) \setminus D$. If $u \subseteq [n]$, then $|N_{n+1}(u) \cap D| = |N_n(u) \cap D| \geq k$. Otherwise, $u \in V(K(n+1,r)) \setminus V(K(n,r))$. It follows that $u = \tilde{u} \cup \{n+1\}$ with $\tilde{u} = \{a_1, \ldots, a_{r-1}\} \subseteq [n]$. By Lemma 2.2 there exists $x \in [n] \setminus \tilde{u}$ such that $w = \tilde{u} \cup \{x\} \notin D$. Thus, we have

$$\begin{aligned} N_n(w) \cap D = N_{n+1}(w) \cap D = \{v \in D : v \cap w = \emptyset\} = \\ = \{v \in D : v \cap \tilde{u} = \emptyset \wedge x \notin v\} \subseteq \{v \in D : v \cap \tilde{u} = \emptyset\} = N_{n+1}(u) \cap D. \end{aligned}$$

Consequently,

$$|N_{n+1}(u) \cap D| \geq |N_n(w) \cap D| \geq k.$$

Therefore, D is a k-dominating set of $K(n+1,r)$ and we have

$$\gamma_k(n+1,r) \leq |D| = \gamma_k(n,r).$$

□

Theorem 2.3 implies that $\gamma_k(n_1,r) \geq \gamma_k(n_2,r)$ holds for any $n_1 \leq n_2$, if n_1 is large enough. Anyway, monotonicity may not hold if n_1 is small. Consider the Kneser graphs $K(n,2)$, where $\gamma_2(5,2) < \gamma_2(6,2)$, $\gamma_2(6,2) = \gamma_2(7,2)$ and $\gamma_2(7,2) > \gamma_2(8,2)$, see Table 1.

3 Exact values for large n

In [4], the following result is stated.

Theorem 3.1 ([4])**.** *For any $n \geq 2r$, $\gamma_{\times k}(n,r) = k+r$ if and only if $n \geq r(k+r)$.*

Moreover, it is shown that except in the case $k=1$ and $r=2$, every $\gamma_{\times k}$-set of $K(n,r)$, where $n \geq r(k+r)$, is a clique. If $k=1$ and $r=2$, then in [19] it is proved that a $\gamma_{\times 1}$-set of $K(n,2)$ is either a clique or an independent set. In any case, if $\gamma_{\times k}(n,r) = k+r$, then it is possible to obtain a $\gamma_{\times k}$-set that is a clique. As a by-product, we get the following for the k-tuple total domination number.

Theorem 3.2. *For any $n \geq 2r$, $\gamma_{t\times k}(n,r) = k+r$ if and only if $n \geq r(k+r)$. Moreover, for any $n \geq r(k+r)$, every $\gamma_{t\times k}$-set is a clique.*

Proof. Let $n,r,k \in \mathbb{N}$. If $n \geq r(k+r)$, then $\gamma_{t\times k}(n,r) \geq \gamma_{\times k}(n,r) = k+r$. Since there exists a $\gamma_{\times k}$-set D which is a clique on $k+r$ vertices, it is also a k-tuple total dominating set of $K(n,r)$. Thus, $\gamma_{t\times k}(n,r) = k+r$. Conversely, if $\gamma_{t\times k}(n,r) = k+r$, then since any $\gamma_{t\times k}$-set D is a k-tuple dominating set, then $\gamma_{\times k}(n,r) \leq |D| = k+r$. Due to monotonicity of $\gamma_{\times k}(n,r)$ with respect to n, we have $\gamma_{\times k}(n,r) = k+r$, and by Theorem 3.1 it follows that $n \geq r(k+r)$. Hence D is a $\gamma_{\times k}$-set of $K(n,r)$ for $n \geq r(k+r)$ and thus D is a clique by the results described before the theorem. □

Remark 3.3. If D is a k-dominating set of the Kneser graph $K(n,r)$ and $w \in V(K(n,r)) \setminus D$, then w has nonempty intersection with at most $|D|-k$ vertices of D. Otherwise, $|N(w) \cap D| < k$ contradicting the fact that D is k-dominating.

Theorem 3.4. *If $k \geq 2$ and $n \geq k+2r$, then*

- $\gamma_k(n,r) = k+r$ *if and only if* $n \geq r(k+r)$*; moreover, every* γ_k*-set is a clique;*
- $\gamma_k(n,r) \geq k+r+1$ *if and only if* $n < r(k+r)$.

Proof. Let $n, r, k \in \mathbb{N}$ such that $n \geq k+2r$. We start the proof by showing that

$$\gamma_k(n,r) \leq k+r \Rightarrow n \geq r\cdot(k+r). \tag{3.1}$$

Thus, suppose that $\gamma_k(n,r) \leq k+r$ and let D be an arbitrary k-dominating set of $K(n,r)$ of cardinality $|D| = k+r$ (note that such a set exists, since $n \geq k+2r$, and thus we can obtain D by adding $k+r-\gamma_k(n,r)$ vertices to a $\gamma_k(n,r)$-set). We will show that D is a clique or, equivalently, that the vertices in D are pairwise disjoint. Assume D contains two non-adjacent vertices u and v. Hence there exists $j \in [n]$ such that $j \in u \cap v$ and consequently $i_j(D) \geq 2$. Let $a \in [n]$ such that $i_a = \max\{i_x(D) : x \in [n]\}$. Thus, $i_a \geq i_j \geq 2$. Let us see that $i_a = 2$. Otherwise, let $u_1, u_2, u_3 \in D$ such that $a \in u_1 \cap u_2 \cap u_3$, and $u_4, \ldots, u_{r+1} \in D \setminus \{u_1, u_2, u_3\}$. Let $b_1, \ldots, b_{r-1}$ be chosen in the following way:

$$\begin{aligned} &b_1 = a, \\ &b_i \in u_{i+2} \setminus \{b_1, \ldots, b_{i-1}\}, \text{ for } i \in [2..r-1], \end{aligned}$$

and let $b = \{b_1, \ldots, b_{r-1}\}$. Observe that there is at least one element $x \in [n] \setminus b$ such that $b \cup \{x\} \notin D$, since $|[n] \setminus b| - |D| = (n-r+1) - (k+r) = n - (k+2r) + 1 \geq 1$. Let $w = b \cup \{x\}$. We have $w \notin D$ and $w \cap u_i \neq \emptyset$ for every $i \in [r+1]$ which cannot be true because of Remark 3.3. Thus, $i_a = 2$. Let u_1 and u_2 be the two vertices of D that contain the element a.

If $r > 2$, let $u_3, \ldots, u_{r+1} \in D \setminus \{u_1, u_2\}$. Let $b_1, \ldots, b_r$ be chosen in the following way:

$$\begin{aligned} &b_1 = a, \\ &b_i \in u_{i+1} \setminus \{b_1, \ldots, b_{i-1}\}, \text{ for } i \in [2..r], \end{aligned}$$

and let $w = \{b_1, \ldots, b_r\}$. We have $w \cap u_i \neq \emptyset$ for every $i \in [r+1]$. By Remark 3.3, it follows that $w \in D$ and as $a \in w$, then w is either u_1 or u_2. Without loss of generality, $w = u_1$. Note that $b_i \in u_1 \cap u_{i+1}$ for every i, and since $i_a = \max\{i_x : x \in [n]\} = 2$, $b_i \notin u_j$ for every j different from 1 and $i+1$. Let us consider $x \in u_3 \setminus u_1$, and let $w' = w \setminus \{b_2\} \cup \{x\}$. We have $x \in w' \setminus u_1$ and $b_r \in w' \setminus u_2$. Thus, $a \in w'$ and $w' \notin \{u_1, u_2\}$ and therefore $w' \notin D$ but $w' \cap u_i \neq \emptyset$ for every $i \in [r+1]$, arising to a contradiction with Remark 3.3.

If $r = 2$, we have $u_1 = \{a, b\}$ and $u_2 = \{a, c\}$ for some $b, c \in [n]$. Since $|D| = k+r \geq 4$, then we can choose $u_3 \in D$ such that $u_3 \neq \{b, c\}$. Let $x \in u_3 \setminus \{b, c\}$. Since $i_a = 2$, $w = \{a, x\} \notin D$ and $w \cap u_i \neq \emptyset$ for $i \in [3]$, contradicting the fact that D is k-dominating.

Therefore, if $\gamma_k(n,r) \leq k+r$ and D is a k-dominating set of $K(n,r)$ of cardinality $k+r$, then the vertices in D are pairwise disjoint and as a consequence, $n \geq r|D| = r(k+r)$. In particular, if $\gamma_k(n,r) = k+r$, then $n \geq r(k+r)$.

Conversely, suppose that $n \geq r(k+r)$. We have $\gamma_k(n,r) \leq \gamma_{\times k}(n,r) = k+r$. Suppose that $\gamma_k(n,r) < k+r$ and let D be a k-dominating set of cardinality $|D| = k+r-1$. Let $u_1, \ldots, u_r \in D$. Consider

$$b_1 \in u_1,$$
$$b_i \in u_i \setminus \{b_1, \ldots, b_{i-1}\}, \text{ for } i \in [2..r],$$

and let $w = \{b_1, \ldots, b_r\}$. Let us note that $w \cap u_i \neq \emptyset$ for every $i \in [r]$. So, by Remark 3.3 it follows that $w \in D$. Let us note that we may assume that $w = u_j$ for some $j \in [r]$ by changing our initial choice of the vertices u_i. Notice that there exists $x \in [n] \setminus \left(\bigcup_{u\in D} u\right)$, since $\left|[n] \setminus \left(\bigcup_{u\in D} u\right)\right| \geq n - r|D| = n - r(k+r-1) \geq r$. Let $w' = w \setminus \{b_j\} \cup \{x\}$. We have $w' \notin D$ due to our choice of x, and $w' \cap u_i \neq \emptyset$ for each $i \in [r]$ (note that $u_j \cap w' = \{b_1, \ldots, b_r\} \setminus \{b_j\}$), arising to a contradiction with Remark 3.3.

Therefore, if $n \geq r(k+r)$, then $\gamma_k(n,r) = k+r$.

The proof of the second statement follows directly from the first statement of the theorem and from Equation 3.1. □

Corollary 3.5. *If $k \geq 2$ and $n \geq r(k+r)$, then*

$$\gamma_k(n,r) = \gamma_{\times k}(n,r) = \gamma_{t\times k}(n,r) = k+r.$$

Moreover, every γ_k-set, $\gamma_{\times k}$-set and $\gamma_{t\times k}$-set of $K(n,r)$, where $n \geq r(k+r)$, is a clique.

Proposition 3.6. *If $k \geq 2$, then*

$$\gamma_k(r(k+r)-1,r) = \gamma_{\times k}(r(k+r)-1,r) = \gamma_{t\times k}(r(k+r)-1,r) = k+r+1.$$

Proof. Let $k \geq 2$ and $n = r(k+r)-1$. Since $n > k+2r$, from Theorem 3.4 we have that $k+r+1 \leq \gamma_k(n,r) \leq \gamma_{\times k}(n,r) \leq \gamma_{t\times k}(n,r)$. We will give a k-tuple total dominating set D of cardinality exactly $k+r+1$.

It is enough to consider the set $D = A \cup B$, where

$$A = \big\{[r], [r-1] \cup \{r+1\}, [r..2r-1]\big\}$$
$$B = \big\{[2r..3r-1], [3r..4r-1], \ldots, [(k+r-1)r..(k+r)r-1]\big\}.$$

We have that A is an independent set of $|A| = 3$ vertices, B is a clique of $|B| = k+r-2$ vertices, and a is adjacent to b for each $a \in A$ and $b \in B$. Let $u \in V(K(n,r))$.

- If $u \in A$, then u is adjacent to every vertex in B, so $|N(u) \cap D| \geq k+r-2 \geq k$.
- If $u \in B$, then u is adjacent to every other vertex in D, so $|N(u) \cap D| \geq k+r > k$.
- If $u \notin D$ and $|u \cap [2r-1]| \geq 2$, then $|u \cap [2r..n]| \leq r-2$ and u has at least $|B| - (r-2) = k$ neighbors in B. So, $|N(u) \cap D| \geq k$.
- If $u \notin D$ and $|u \cap [2r-1]| = 1$, then $|u \cap [2r..n]| = r-1$ and u has at least $|B| - (r-1) = k-1$ neighbors in B. On the other hand, since there is no element contained in every vertex of A, we have that u has at least one neighbor in A. So, $|N(u) \cap D| = |N(u) \cap A| + |N(u) \cap B| \geq k$.

- Finally, if $u \notin D$ and $|u \cap [2r-1]| = 0$, then $|u \cap [2r..n]| = r$ and u has at least $|B| - r = k-2$ neighbors in B, and u is adjacent to every vertex in A. So, $|N(u) \cap D| = |N(u) \cap A| + |N(u) \cap B| \geq k+1 \geq k$. □

Combining the results of this section with results from [4], we obtain the table of values of the three types of k-domination invariants of $K(n,r)$ in the case $k=2$ and $r=2$; see Table 1.

n	$\gamma_2(n,2)$	$\gamma_{\times 2}(n,2)$	$\gamma_{t\times 2}(n,2)$
4	6	6	$\neg\exists$
5	4	6	8
6	5	6	6
7	5	5	5
≥ 8	4	4	4

Table 1: Domination invariants with $k=2$ for $K(n,2)$.

4 2-packing number

The 2-packing number of Kneser graphs was considered in [4]. Note that $\mathrm{diam}(K(n,r)) = 2$ as soon as $n \geq 3r-1$, and in such cases $\rho_2(K(n,r)) = 1$; see [23]. In [4], the authors studied the case, which is in a sense the closest to diameter 2 Kneser graphs, and this is when $n = 3r-2$. They obtained the exact values of the 2-packing number for all these Kneser graphs. In this paper, we consider the next case, which is when $n = 3r-3$. We again simplify the notation by writing $\rho_2(n,r)$ instead of $\rho_2(K(n,r))$.

We start by recalling a useful observation from [4], which follows from the fact that a set S is a 2-packing of a graph G if and only if no two vertices of S are adjacent nor have a common neighbor.

Observation 4.1 ([4, Remark 3])**.** *Let $2r+1 \leq n \leq 3r-2$. A set S is a 2-packing in $K(n,r)$ if and only if for every pair $u,v \in S$ we have $1 \leq |u \cap v| \leq (3r-1)-n$.*

Note that any two vertices of a 2-packing S are at distance at least 3, thus Observation 4.1 directly follows also from [23, Lemma 2].

From Observation 4.1 we derive that S is a 2-packing of the Kneser graph $K(n,r)$ if and only if S is an r-uniform family of subsets of $[n]$ whose pairwise intersections have cardinalities in the range $[1..3r-1-n]$. Intersecting families with such properties have been studied independently by several authors, and some of the known results yield upper bounds for the 2-packing number of $K(n,r)$. In particular, from [22, Theorem (Ray-Chaudhuri-Wilson, 1975)] we infer that

$$\rho_2(n,r) \leq \binom{n}{(3r-1)-n}. \tag{4.1}$$

In addition, from [5, Theorem 3.1] (see also [6]) we infer the bound

$$\rho_2(n,r) \leq \frac{(n-1)(n-2)\dots(2n-3r-1)}{(r-1)(r-2)\dots(n-2r+1)}.$$

In the case of $K(3r-3,r)$, Observation 4.1 yields that S is a 2-packing in $K(n,r)$ if and only if for every pair $u,v \in S$ we have

$$1 \leq |u \cap v| \leq 2.$$

Besides, by 4.1, $\rho_2(n,r) \leq \binom{n}{2} = \frac{1}{2}(n^2-n)$. We obtain lower bounds for $\rho_2(3r-3,r)$ when $r \leq 9$, and exact values for $\rho_2(3r-3,r)$ when $r > 9$. See Table 2. The lower bounds in Table 2 follow from constructions of the corresponding 2-packings, which we present in Table 3.

r	$\rho_2(3r-3,r)$
4	≥ 12
5	≥ 12
6	≥ 10
7	≥ 6
8	≥ 5
9	$= 4$
≥ 10	$= 3$

Table 2: 2-packing numbers of $K(3r-3,r)$.

Consider the Kneser graph $K(3r-3,r)$, where $r > 2$. Since a set S of three vertices $u,v,w \in V(K(3r-3,r))$ with $|u \cap v \cap w| = |u \cap v| = |u \cap w| = |v \cap w| = 2$ is a 2-packing (say $u = [1..r], v = \{1,2\} \cup [r+1..2r-2], w = \{1,2\} \cup [2r-1..3r-4]$), we infer $\rho_2(3r-3,r) \geq 3$. The next result proves the last line of Table 2.

Proposition 4.2. *If $r \geq 10$, then $\rho_2(3r-3,r) = 3$.*

Proof. Let P be a 2-packing of $K(3r-3,r)$, where $r \geq 10$, and let $\{u_1,u_2,u_3\} \subseteq P$. Since $|u_i \cap u_j| \leq 2$ for all $\{i,j\} \subset [3]$, we infer that $|[3r-3] \setminus (u_1 \cup u_2 \cup u_3)| \leq 3$. Suppose that $|P| \geq 4$, and let $z \in P$ be distinct from all u_i. Since $z \in P$, $|z \cap u_1| \leq 2$, $|z \cap u_2| \leq 2$, $|z \cap u_3| \leq 2$, and $|z \cap ([3r-3] \setminus (u_1 \cup u_2 \cup u_3))| \leq 3$. We derive that $|z| \leq 2+2+2+3 = 9$, which is a contradiction, since $r \geq 10$. Hence, $|P| = 3$. □

r	2-packing of $K(3r-3,r)$
4	{1, 2, 3, 5}, {1, 2, 6, 9}, {1, 2, 7, 8}, {1, 3, 4, 6}, {1, 4, 5, 8}, {1, 4, 7, 9}, {2, 3, 4, 7}, {2, 4, 5, 6}, {2, 4, 8, 9}, {3, 5, 7, 9}, {3, 6, 8, 9}, {5, 6, 7, 8}
5	{1, 2, 3, 4, 8}, {1, 2, 5, 10, 11}, {1, 2, 6, 9, 12}, {1, 3, 7, 9, 10}, {1, 4, 5, 7, 12}, {1, 6, 7, 8, 11}, {2, 3, 5, 6, 7}, {2, 4, 7, 9, 11}, {2, 7, 8, 10, 12}, {3, 4, 6, 10, 11}, {3, 5, 9, 11, 12}, {4, 5, 6, 8, 9}
6	{1, 2, 3, 4, 5, 11}, {1, 2, 7, 8, 9, 14}, {1, 3, 6, 9, 10, 15}, {1, 5, 6, 12, 13, 14}, {2, 4, 6, 7, 13, 15}, {2, 5, 8, 10, 12, 15}, {3, 4, 8, 10, 13, 14}, {3, 6, 7, 8, 11, 12}, {4, 9, 11, 12, 14, 15}, {5, 7, 9, 10, 11, 13}
7	{1, 2, 3, 4, 6, 11, 14}, {1, 3, 5, 8, 16, 17, 18}, {2, 4, 5, 7, 12, 15, 17}, {2, 8, 9, 12, 13, 14, 16}, {3, 7, 8, 9, 10, 11, 15}, {4, 5, 6, 9, 10, 13, 18},
8	{1, 2, 3, 4, 5, 6, 9, 18}, {1, 2, 7, 11, 12, 14, 20, 21}, {3, 7, 8, 9, 13, 15, 16, 20}, {4, 5, 10, 12, 13, 15, 17, 21}, {5, 6, 8, 10, 11, 14, 16, 19}

Table 3: 2-packings in $K(3r-3,r)$, for $r \leq 8$.

We suspect that the lower bounds in Table 2 are in fact exact values of $\rho_2(3r-3,r)$, and leave this as an open problem. The proof for the penultimate line in Table 2, as well as an alternative proof of Proposition 4.2, are given afterwards in Theorem 4.8.

We continue with several general results concerning the behavior of the 2-packing number in Kneser graphs.

Proposition 4.3. *If n, r and a are positive integers such that $2r+1 \leq n \leq 3r-2$ and $a \geq n-2r+1$, then*

$$\rho_2(n+2a, r+a) \geq 2\rho_2(n,r).$$

Proof. Let S be a 2-packing of $K(n,r)$ with $|S| = \rho_2(n,r)$. Consider the following set:

$$T = \Big\{v \cup [(n+1)..(n+a)], v \cup [(n+a+1)..(n+2a)] : v \in S\Big\}.$$

Clearly, $T \subset V(K(n+2a, r+a))$ and $|T| = 2\rho_2(n,r)$. Note that since $2r+1 \leq n \leq 3r-2$, it follows $2(r+a)+1 \leq n+2a \leq 3r+2a-2 < 3(r+a)-2$. To prove that T is a 2-packing in $K(n+2a, r+a)$, we need to prove, by Observation 4.1, that

$$1 \leq |u \cap v| \leq (3(r+a)-1) - (n+2a) = 3r-1-n+a$$

for any two vertices $u, v \in T$. Let $u, v \in T$, and let $\tilde{u} = u \cap [n]$, $\tilde{v} = v \cap [n]$. If $\tilde{u} \neq \tilde{v}$, then as $\tilde{u}, \tilde{v} \in S$, by Observation 4.1, we have $1 \leq |\tilde{u} \cap \tilde{v}| \leq 3r-1-n$, and since $|\tilde{u} \cap \tilde{v}| \leq |u \cap v| \leq |\tilde{u} \cap \tilde{v}| + a$, we have $1 \leq |u \cap v| \leq 3r-1-n+a$. On the other hand, if $\tilde{u} = \tilde{v}$, then $|u \cap v| = r$ and it follows $1 \leq |u \cap v| = r \leq 3r-1-n+a$ since $a \geq n-2r+1$. □

By applying Proposition 4.3 several times (note that if $K(n,r)$ is an odd graph, then $K(n+2a, r+a)$ is also an odd graph) we get the following.

Corollary 4.4. *If n, r and a are positive integers such that $n = 2r+1$ and $a \geq 2$, then*

$$\rho_2(n+2a, r+a) \geq 2^{\lfloor \frac{a}{2} \rfloor} \rho_2(n,r).$$

We follow with a version of monotonicity of the 2-packing number in Kneser graphs, which turns out to be non-decreasing when both n and r increase by the same value.

Theorem 4.5. *If n and r are positive integers such that $n \geq 2r+2$, then*

$$\rho_2(n+1, r+1) \geq \rho_2(n,r).$$

Proof. Note that if $n \geq 3r-1$, then $\rho_2(n,r) = 1$ and the result is straightforward. Thus, let us assume $2r+2 \leq n \leq 3r-2$. Let S be a 2-packing of $K(n,r)$ of size $\rho_2(n,r)$. Consider the set:

$$T = \Big\{v \cup \{n+1\} : v \in S\Big\}.$$

We have $T \subset V(K(n+1, r+1))$ and $|T| = \rho_2(n,r)$. Let us prove that T is a 2-packing of $K(n+1, r+1)$. In order to do so, since $2(r+1)+1 \leq n+1 \leq 3(r+1)-2$, by Observation 4.1 it is enough to see that $1 \leq |u \cap v| \leq (3(r+1)-1)-(n+1) = 3r+1-n$ for any pair of vertices in T. In fact, let $u, v \in T$, and let $\tilde{u} = u \cap [n]$, $\tilde{v} = v \cap [n]$. Since $\tilde{u}, \tilde{v} \in S$, we have $1 \leq |\tilde{u} \cap \tilde{v}| \leq 3r-1-n$. As $|u \cap v| = |\tilde{u} \cap \tilde{v}| + 1$, the result follows. □

Lemma 4.6. *Let r and t be positive integers such that $r \geq 3$ and $2 \leq t \leq \frac{r+3}{3}$. If $\rho_2(3r-t, r) \geq 4$, then there exists a 2-packing S of $K(3r-t, r)$ with $|S| = 4$ and $i_x(S) \leq 2$ for every $x \in [n]$.*

Proof. Let $n = 3r - t$ and let S be a 2-packing of $K(n, r)$ with cardinality $|S| = 4$. Note that since $r \geq 3$, we have $\frac{r+3}{3} \leq r - 1$. Thus $2 \leq t \leq r - 1$ and it follows that $2r + 1 \leq n \leq 3r - 2$. So, by Observation 4.1, we have that every pair of vertices in S intersect in at most $3r - 1 - n = t - 1$ elements. Let us assume $X_3^{\geq}(S) \neq \emptyset$ and let $x \in X_3^{\geq}(S)$.

If $x \in X_3(S)$, let u_1, u_2, u_3 be the vertices in S that contain the element x and u_4 the vertex in S which does not contain x. Note that

$$\left|u_1 \cap X_2^{\geq}(S)\right| = \left|\bigcup_{i=2}^{4}(u_1 \cap u_i)\right| \leq$$
$$\leq 1 + \underbrace{|(u_1 \cap u_2) \setminus \{x\}|}_{\leq t-2} + \underbrace{|(u_1 \cap u_3) \setminus \{x\}|}_{\leq t-2} + \underbrace{|u_1 \cap u_4|}_{\leq t-1} \leq 3(t-1) - 1.$$

Thus, since $t \leq \frac{r+3}{3}$, or equivalently $r \geq 3(t-1)$, there is at least one element $x_1 \in u_1 \cap X_1(S)$. Analogously, there exist $x_2 \in u_2 \cap X_1(S)$ and $x_3 \in u_3 \cap X_1(S)$. Let us consider $S' = \{v_1, v_2, v_3, v_4\}$, where

$$v_1 = (u_1 \setminus \{x\}) \cup \{x_2\}, \quad v_2 = (u_2 \setminus \{x\}) \cup \{x_3\}, \quad v_3 = (u_3 \setminus \{x\}) \cup \{x_1\}, \quad v_4 = u_4.$$

It is easy to see that for every $i \neq j$ we have $|u_i \cap u_j| = |v_i \cap v_j|$. Thus, S' is a 2-packing of $K(n, r)$ of size 4 and $|X_3^{\geq}(S')| = |X_3^{\geq}(S)| - 1$.

If $x \in X_4(S)$, let $S = \{u_1, u_2, u_3, u_4\}$. Now we have

$$\left|u_1 \cap X_2^{\geq}(S)\right| = \left|\bigcup_{i=2}^{4}(u_1 \cap u_i)\right| \leq 1 + \sum_{i=2}^{4} \underbrace{|(u_1 \cap u_i) \setminus \{x\}|}_{\leq t-2} \leq 3(t-1) - 2.$$

Thus, since $r \geq 3(t-1)$, there is at least two elements $x_1^1, x_1^2 \in u_1 \cap X_1(S)$. Analogously, there exist $x_i^1, x_i^2 \in u_i \cap X_1(S)$ for $i = 2, 3, 4$. Let us consider $S' = \{v_1, v_2, v_3, v_4\}$, where

$$\begin{aligned} v_1 &= (u_1 \setminus \{x\}) \cup \{x_4^1\}, \\ v_2 &= (u_2 \setminus \{x\}) \cup \{x_1^1\}, \\ v_3 &= (u_3 \setminus \{x, x_3^2\}) \cup \{x_1^2, x_2^1\}, \\ v_4 &= (u_4 \setminus \{x, x_4^2\}) \cup \{x_2^2, x_3^1\}. \end{aligned}$$

It is easy to see that for every $i \neq j$ we have $|u_i \cap u_j| = |v_i \cap v_j|$. Thus, S' is a 2-packing of $K(n, r)$ of size 4 and $|X_3^{\geq}(S')| = |X_3^{\geq}(S)| - 1$.

In any case, if $X_3^{\geq}(S) \neq \emptyset$, we build a 2-packing S' of size 4 with $|X_3^{\geq}(S')| < |X_3^{\geq}(S)|$. If $X_3^{\geq}(S') \neq \emptyset$, we repeat the procedure until we get a 2-packing of four vertices for which each element occurs at most twice. □

We can prove, analogously, the following lemma.

Lemma 4.7. *Let r and t be positive integers such that $r \geq 4$ and $2 \leq t \leq \frac{r+4}{4}$. If $\rho_2(3r-t,r) \geq 5$, then there exists a 2-packing S of $K(3r-t,r)$ with $|S| = 5$ and $i_x(S) \leq 2$ for every $x \in [n]$.*

Proof. Let $n = 3r - t$ and let S be a 2-packing of $K(n,r)$ with cardinality $|S| = 5$. Note that since $r \geq 3$, we have $\frac{r+4}{4} < r - 1$. Thus $2 \leq t < r - 1$ and it follows that $2r + 1 < n \leq 3r - 2$. Then, by Observation 4.1, every pair of vertices in S intersect in at most $3r - 1 - n = t - 1$ elements. Let us assume $X_3^{\geq}(S) \neq \emptyset$ and let $x \in X_3^{\geq}(S)$.

If $x \in X_3(S) \cup X_4(S)$, let u_1, u_2, u_3 be vertices in S that contain the element x and u_4, u_5 the remaining vertices in S such that $x \notin u_5$. Note that

$$\left|u_1 \cap X_2^{\geq}(S)\right| = \left|\bigcup_{i=2}^{5}(u_1 \cap u_i)\right| \leq$$

$$\leq 1 + \sum_{i=2,3} \underbrace{|(u_1 \cap u_i) \setminus \{x\}|}_{\leq t-2} + \sum_{i=4,5} \underbrace{|u_1 \cap u_i|}_{\leq t-1} \leq 4(t-1) - 1.$$

Thus, since $t \leq \frac{r+4}{4}$, or equivalently $r \geq 4(t-1)$, there is at least one element $x_1 \in u_1 \cap X_1(S)$. Analogously, there exist $x_2 \in u_2 \cap X_1(S)$ and $x_3 \in u_3 \cap X_1(S)$. Let us consider $S' = \{v_1, v_2, v_3, v_4, v_5\}$, where

$$v_1 = (u_1 \setminus \{x\}) \cup \{x_2\}, \quad v_2 = (u_2 \setminus \{x\}) \cup \{x_3\}, \quad v_3 = (u_3 \setminus \{x\}) \cup \{x_1\}, \quad v_4 = u_4, \quad v_5 = u_5.$$

Note that if $x \in X_4(S)$, then $x \in X_1(S')$. It is easy to see that for every $i \neq j$ we have $|u_i \cap u_j| = |v_i \cap v_j|$. Therefore, S' is a 2-packing of $K(n,r)$ of size 5 and $|X_3^{\geq}(S')| = |X_3^{\geq}(S)| - 1$.

If $x \in X_5(S)$, let $S = \{u_1, u_2, u_3, u_4, u_5\}$. We have

$$\left|u_1 \cap X_2^{\geq}(S)\right| = \left|\bigcup_{i=2}^{5}(u_1 \cap u_i)\right| \leq 1 + \sum_{i=2}^{5} \underbrace{|(u_1 \cap u_i) \setminus \{x\}|}_{\leq t-2} \leq 4(t-1) - 3.$$

Thus, since $r \geq 4(t-1)$, there are at least three elements $x_1^1, x_1^2, x_1^3 \in u_1 \cap X_1(S)$. Analogously, there exist $x_i^1, x_i^2, x_i^3 \in u_i \cap X_1(S)$ for $i = 2, 3, 4, 5$. Let us consider $S' = \{v_1, v_2, v_3, v_4, v_5\}$, where

$$\begin{aligned}
v_1 &= (u_1 \setminus \{x\}) \cup \{x_5^1\}, \\
v_2 &= (u_2 \setminus \{x\}) \cup \{x_1^1\}, \\
v_3 &= (u_3 \setminus \{x, x_3^3\}) \cup \{x_1^2, x_2^1\}, \\
v_4 &= (u_4 \setminus \{x, x_4^2, x_4^3\}) \cup \{x_1^3, x_2^2, x_3^1\}, \\
v_5 &= (u_5 \setminus \{x, x_5^2, x_5^3\}) \cup \{x_2^3, x_3^2, x_4^1\}.
\end{aligned}$$

It can be easily checked that for every $i \neq j$ we have $|u_i \cap u_j| = |v_i \cap v_j|$. It turns out that S' is a 2-packing of $K(n,r)$ of size 5 and $|X_3^{\geq}(S')| = |X_3^{\geq}(S)| - 1$.

In any case, if $X_3^{\geq}(S) \neq \emptyset$, we get a 2-packing S' of size 5 with $|X_3^{\geq}(S')| < |X_3^{\geq}(S)|$. If $X_3^{\geq}(S') \neq \emptyset$, we repeat the procedure until we get a 2-packing of five vertices for which each element occurs at most twice. □

From Proposition 4.2, and some other results of similar nature, one can suspect that $\rho_2(n,r)$ will be small if n is close to $3r$. The following result describes how close n has to be to $3r-3$ in order to have $\rho_2(n,r)$ equal to 3 or 4.

Theorem 4.8. *If r, n and t are positive integers such that $2 \le t \le r-1$ and $n = 3r-t$, then:*

(1) *if $t \le \frac{r+5}{5}$, then $\rho_2(n,r) = 3$;*

(2) *if $\frac{r+5}{5} < t \le \frac{2r+9}{9}$, then $\rho_2(n,r) = 4$.*

Proof. (1) Let us suppose $t \le \frac{r+5}{5}$, which is equivalent to $r \ge 5(t-1)$. Assume $\rho_2(n,r) \ge 4$ and let S be a packing of $K(n,r)$ of cardinality 4. Note that since $r \ge 5(t-1) > 3(t-1)$, we have $2 \le t \le \frac{r+3}{3}$ and $r \ge 3$. Thus, by Lemma 4.6, we may consider a 2-packing S for which $i_x(S) \le 2$ for every $x \in [n]$. Let us note that by Observation 4.1 we have $|u \cap v| \le t-1$ for each $u, v \in S$, and it follows

$$|X_2(S)| = \sum_{\substack{u,v\in S\\ u\neq v}} \underbrace{|u\cap v|}_{\le t-1} \le \binom{4}{2}(t-1) = 6(t-1).$$

On the other hand, since $|X_1(S)| \le n - |X_2(S)|$, we have

$$\begin{aligned} 4r = |S|r = \sum_{x\in[n]} i_x(S) &= 2|X_2(S)| + |X_1(S)| \\ &\le n + |X_2(S)| \\ &\le n + 6(t-1) \\ &\le 3r + 5t - 6. \end{aligned}$$

Thus, $r \le 5(t-1) - 1$ and we arise to a contradiction since $r \ge 5(t-1)$. Therefore $\rho_2(n,r) \le 3$. In order to see that it is equal to 3 it is enough to consider the set S given by $S = \{u_1, u_2, u_3\}$ where

$$u_1 = [r], \quad u_2 = [t-1] \cup [(r+1)..(2r-t+1)], \quad u_3 = \{1\} \cup [(2r-t+2)..(3r-t)].$$

We have $|u_1 \cap u_2| = t-1$ and $|u_1 \cap u_3| = |u_2 \cap u_3| = 1$. So, S is a 2-packing and $\rho_2(n,r) = 3$.

(2) Now, let us suppose $\frac{r+5}{5} < t \le \frac{2r+9}{9}$, which is equivalent to $\frac{9}{2}(t-1) \le r < 5(t-1)$. Assume $\rho_2(n,r) \ge 5$ and let S be a packing of $K(n,r)$ of size 5. Note that since $r \ge \frac{9}{2}(t-1) > 4(t-1)$, we have $2 \le t \le \frac{r+4}{4}$ and $r \ge 3$. Thus, by Lemma 4.7, we may consider a 2-packing S for which $i_x(S) \le 2$ for every $x \in [n]$. By Observation 4.1 we have $|u \cap v| \le t-1$ for each $u, v \in S$. Consequently,

$$|X_2(S)| = \sum_{\substack{u,v\in S\\ u\neq v}} \underbrace{|u\cap v|}_{\le t-1} \le \binom{5}{2}(t-1) = 10(t-1).$$

It follows that

$$\begin{aligned} 5r = |S|r = \sum_{x\in[n]} i_x(S) &= 2|X_2(S)| + |X_1(S)| \\ &\le n + |X_2(S)| \\ &\le n + 10(t-1) = 3r + 9t - 10. \end{aligned}$$

Thus, $r \leq \frac{9}{2}(t-1) - \frac{1}{2}$ and we arise to a contradiction since $r \geq \frac{9}{2}(t-1)$. Therefore $\rho_2(n,r) \leq 4$. In order to see that it is equal to 4 it is enough to consider the set S given by $S = \{u_1, u_2, u_3, u_4\}$ where

$$\begin{aligned} u_1 &= A_{12} \cup A_{13} \cup A_{14} \cup B_1 \\ u_2 &= A_{12} \cup A_{23} \cup A_{24} \cup B_2 \\ u_3 &= A_{13} \cup A_{23} \cup A_{34} \cup B_3 \\ u_4 &= A_{14} \cup A_{24} \cup A_{34} \cup B_4 \end{aligned}$$

where the sets in $\{\{A_{ij}\}_{i \neq j} \cup \{B_i\}_{i=1}^4\}$ are pairwise disjoint with $|A_{ij}| = t-1$, and $|B_i| = r - 3(t-1)$. Note that this is possible since

$$6(t-1) + 4(r - 3(t-1)) = 4r - 6(t-1) = \underbrace{(3r-t)}_{=n} + \underbrace{(r-(5t-6))}_{\leq 0} \leq n.$$

We have $|u_i \cap u_j| = t-1$ for every $i \neq j$. So, S is a packing and $\rho_2(n,r) = 4$. □

Corollary 4.9. *If* $\frac{14}{5}r - 1 \leq n \leq 3r - 2$, *then* $\rho_2(n,r) = 3$.

Corollary 4.10. *If* $\frac{25}{9}r - 1 \leq n < \frac{14}{5}r - 1$, *then* $\rho_2(n,r) = 4$.

5 Concluding remarks

In Section 2, we proved that γ_k is monotonically decreasing, where r is fixed and $n \geq 2(k+r)$ is growing. We wonder if the lower bound on n could be improved, and pose the following question.

Problem 5.1. Is there an integer n_0, where $n_0 < 2(k+r)$, such that for any $n \geq n_0$, we have $\gamma_k(n,r) \geq \gamma_k(n+1,r)$?

In Section 3, we established exact values of $\gamma_k(n,r)$ and $\gamma_{t\times k}(n,r)$ for all n, where $n \geq r(k+r) - 1$. It is thus natural to ask what are the values of these two invariants when n is smaller than $r(k+r) - 1$. In particular, the most interesting case seems to be that of odd graphs, that is, when $n = 2r+1$.

Problem 5.2. Determine or provide upper and lower bounds on $\gamma_k(2r+1,r)$, $\gamma_{\times k}(2r+1,r)$ and $\gamma_{t\times k}(2r+1,r)$, for $r > 2$.

We mentioned an open problem in Section 4, which is concerned with the exact values of $\rho_2(3r-3,r)$ where $r \leq 8$ (recall that for $r \geq 9$, we already established the values of $\rho_2(3r-3,r)$).

Problem 5.3. Is it true that the lower bounds in Table 2 are in fact exact values of $\rho_2(3r-3,r)$, where $r \leq 8$.

Regarding the 2-packing number of odd graphs $K(2r+1,r)$, we have proved in Corollary 4.4 that $\rho_2(2r'+1,r') \geq C \cdot \rho_2(2r+1,r)$ when $r' > r+1$, where $C = 2^{\left\lfloor \frac{r'-r}{2} \right\rfloor}$. In [4] it is shown that $\rho_2(7,3) = 7$ and $\rho_2(11,5) = 66$. So, we wonder whether this C could be improved.

Problem 5.4. Given an integer $r \geq 2$, is there some $C(a) > 2^{\lfloor \frac{a}{2} \rfloor}$ such that $\rho_2(2(r+a)+1, r+a) \geq C(a)\rho_2(2r+1,r)$?

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