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On the interplay between productively Menger and productively Hurewicz spaces in models of $\mathfrak{b} = \mathfrak{d}$ [☆]

Dušan D. Repovš [a,*], Lyubomyr Zdomskyy [b]

[a] *Faculty of Education and Faculty of Mathematics and Physics, University of Ljubljana, & Institute of Mathematics, Physics and Mechanics, 1000 Ljubljana, Slovenia*
[b] *Institut für Diskrete Mathematik und Geometrie, Technische Universität Wien, Wiedner Hauptstrasse 8—10/104, 1040 Vienna, Austria*

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This article is devoted to the interplay between productively Menger and productively Hurewicz subspaces of the Cantor space. In particular, we show that in the Laver model for the consistency of the Borel's conjecture these two notions coincide and characterize Hurewicz spaces. On the other hand, it is consistent with CH that there are productively Hurewicz subspaces of the Cantor space which are not productively Menger.

1. Introduction

This work may be thought of as a continuation of our earlier paper [9], so we keep our introduction here short and refer the reader to that of [9]. Except for Theorem 1.1 and its proof at the end of Section 2, we consider only zero-dimensional metrizable separable spaces, i.e., subspaces of the Cantor space 2^ω up to a homeomorphism.

A topological space X has the *Menger* property (or, alternatively, is a Menger space) if for every sequence $\langle \mathcal{U}_n : n \in \omega\rangle$ of open covers of X there exists a sequence $\langle \mathcal{V}_n : n \in \omega\rangle$ such that each $\mathcal{V}_n$ is a finite subfamily of $\mathcal{U}_n$ and the collection $\{\cup\mathcal{V}_n : n \in \omega\}$ is a cover of X. We get an equivalent property if we demand

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* Corresponding author.
E-mail addresses: dusan.repovs@guest.arnes.si (D.D. Repovš), lzdomsky@gmail.com (L. Zdomskyy).
URLs: http://repovs.splet.arnes.si/ (D.D. Repovš), http://www.dmg.tuwien.ac.at/zdomskyy/ (L. Zdomskyy).

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that each $x \in X$ is covered by $\cup\mathcal{V}_n$ for infinitely many $n \in \omega$ (for this it suffices to split $\langle \mathcal{U}_n : n \in \omega\rangle$ into infinitely many mutually disjoint subsequences and apply the Menger property to each of them). If in the definition above we additionally require that $\{\cup\mathcal{V}_n : n \in \omega\}$ is a *γ-cover* of X (this means that the set $\{n \in \omega : x \notin \cup\mathcal{V}_n\}$ is finite for each $x \in X$), then we obtain the definition of the *Hurewicz property* introduced in [3]. However, we shall actually use the following characterizations of these properties established in [3] (see also [4, Theorems 4.3 and 4.4]): $X \subset 2^\omega$ is Menger (resp. Hurewicz) if and only if for every continuous $f : X \to \omega^\omega$, the range $f[X]$ is non-dominating (resp. bounded) with respect to the eventual dominance relation $\leq^*$.

One of the basic questions about a topological property is whether it is preserved by finite products, which led to the following definitions introduced in [7]: A topological space X is *productively Hurewicz* (resp. *productively Menger*), if $X \times Y$ is Hurewicz (resp. Menger) for all Hurewicz (resp. Menger) spaces Y. Since singletons are Hurewicz, productively Hurewicz (resp. productively Menger) spaces are Hurewicz (resp. Menger). If $\mathfrak{b} < \mathfrak{g}$, there are productively Menger spaces which are not even Hurewicz, see the discussion at the beginning of [11, p. 10]. However, if $\mathfrak{b} = \mathfrak{d}$, then productively Menger spaces are productively Hurewicz by [11, Theorem 4.8].

In this paper we show that the statement "$\mathfrak{b} = \mathfrak{d}$ and classes of productively Hurewicz and productively Menger spaces coincide" is independent from ZFC. More precisely, in one direction we use the key lemma of [5] in the style of [6] and prove the following

Theorem 1.1. *In the Laver model for the consistency of the Borel's conjecture, Hurewicz spaces are productively Menger (and hence also productively Hurewicz) in the realm of subspaces of 2^ω.*

Consequently, the product $X \times Y$ of a Hurewicz space X and a Menger space Y is Menger if it is Lindelöf.

The theorem above is an improvement of [13, Theorem 1.1], as follows from [11, Theorem 4.8].

The next result, whose proof relies on ideas from [11], shows that the conclusion of Theorem 1.1 is not a consequence of CH, so also not of $\mathfrak{b} = \mathfrak{d}$, the latter being the assumption in [11, Theorem 4.8].

Theorem 1.2. *The existence of a productively Hurewicz space which is not productively Menger is consistent with CH.*

We do not know whether the conclusion of Theorem 1.2 is actually a consequence of CH. The space we construct in the proof of Theorem 1.2 answers [11, Problems 7.6,7.8] in the negative.

We refer the reader to [1] for the definitions of cardinal characteristics we use, [2] for topological notions we use but not define, and to [9] for more motivation behind the research done in this paper.

2. Productively Hurewicz spaces in the Laver model

This section is mainly devoted to the proof of Theorem 1.1.

Definition 2.1. $X \subset 2^\omega$ is said to satisfy property (†), if for every function M assigning to each countable subset Q of X a Menger subset $M(Q) \cap Q = \emptyset$ of 2^ω, there exists a family $\mathsf{Q} \subset [X]^\omega$ of size $|\mathsf{Q}| = \omega_1$ such that $X \subset \bigcup_{Q\in\mathsf{Q}}(2^\omega \setminus M(Q))$. □

Let us note that under CH any $X \subset 2^\omega$ satisfies (†).

The following lemma is the key part of the proof of Theorem 1.1. Its proof is reminiscent of that of [6, Theorem 3.2]. We will use the notation from [5] with only differences being that smaller conditions in a forcing poset are stronger, i.e., carry more information about the generic filter, and the ground model is (nowadays standardly) denoted by V. We shall work in $V[G_{\omega_2}]$, where V satisfies GCH, G_{ω_2} is $\mathbb{P}_{\omega_2}$-generic and $\mathbb{P}_{\omega_2}$ is the iteration of length ω_2 with countable supports of the Laver forcing, see [5] for details. For

$\alpha \leq \omega_2$ we shall denote $G_{\omega_2} \cap \mathbb{P}_\alpha$ by G_α. For a Laver tree $T \subset \omega^{<\omega}$ we denote by $T\langle 0\rangle$ its root. If $s \in T$, $s \geq T\langle 0\rangle$, then we denote by $S_T(s)$ the family of all immediate successors of s in T.

As usually, $\forall^*$ means "for all but finitely many".

A subset C of ω_2 is called an ω_1-*club* if it is unbounded and for every $\alpha \in \omega_2$ of cofinality ω_1, if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.

Lemma 2.2. *In the Laver model every $X \subset 2^\omega$ with the Hurewicz property satisfies (†).*

Proof. First let us work in $V[G_{\omega_2}]$. Let M be such as in the definition of (†). By a standard closing-off argument there exists an ω_1-club $C \subset \omega_2$ such that for every $\alpha \in C$ the following conditions are satisfied:

- $X \cap V[G_\alpha] \in V[G_\alpha]$[1];
- For every $Q \in [X]^\omega \cap V[G_\alpha]$ and every continuous map $\phi : M(Q) \to \omega^\omega$ coded in $V[G_\alpha]$, there exists $h \in \omega^\omega \cap V[G_\alpha]$ such that $h \not\leq^* \phi(y)$ for any $y \in M(Q)$; and
- For every $E \in [2^\omega]^\omega \cap V[G_\alpha]$ disjoint from X there exists a G_δ set $O \supset E$ coded in $V[G_\alpha]$ such that $O \cap X = \emptyset$.

The existence of a G_δ-set required in the last item above is a well-known consequence of the Hurewicz property, see, e.g., [4, Theorem 5.7].

Let us fix $\alpha \in C$. We claim that $X \subset \bigcup_{Q\in\mathsf{Q}}(2^\omega \setminus M(Q))$, where $\mathsf{Q} = [X]^\omega \cap V[G_\alpha]$, which would complete our proof. By [5, Lemma 11], there is no loss of generality in assuming that $\alpha = 0$: We still have GCH in $V[G_\alpha]$, and the quotient forcing $\mathbb{P}_{[\alpha,\omega_2)}$ is again the iteration of length ω_2 with countable supports of the Laver forcing, defined in $V[G_\alpha]$. With this convention we have $V[G_\alpha] = V$, and hence the three items considered above hold for V instead of $V[G_\alpha]$.

Now we start working in V. Let $\dot X$ and $\dot M$ be $\mathbb{P}_{\omega_2}$-names for X and M, respectively. Suppose that, contrary to our claim, there exists $p \in G_{\omega_2}$ and a $\mathbb{P}_{\omega_2}$-name $\dot x$ such that p forces $\dot X$ to be Hurewicz and $\dot x \in \dot X \setminus \bigcup_{Q\in\mathsf{Q}}(2^\omega \setminus \dot M(Q))$. Applying [5, Lemma 14] to the sequence $\langle \dot a_i : i \in \omega\rangle$ such that $\dot a_i = \dot x$ for all $i \in \omega$, we get a condition $p' \leq p$ such that $p'(0) \leq_0 p(0)$, and a finite set U_s of reals for every $s \in p'(0)$ with $p'(0)\langle 0\rangle \leq s$, such that for each s as above the following property is satisfied:

$(1)_{p',s}$ $\forall n \in \omega\, \forall^* t \in S_{p'(0)}(s)\ \big(p'(0)_t\,\hat{}\,p' \restriction [1,\omega_2) \Vdash \exists u \in U_s\ (\dot x \restriction n = u \restriction n)\big)$.

Repeating the argument from the proof of [13, Lemma 2.3], namely the part before equation (5) there, we could, passing to a stronger condition, if necessary, assume that $U_s \subset X$ for all $s \in p'(0)$ such that $p'(0)\langle 0\rangle \leq s$. For this the third assumption on ordinals $\alpha \in C$ is crucial.

Claim 2.3. *If $s \in p'(0)$, $p'(0)\langle 0\rangle \leq s$, $u \in U_s$, and $n_0 \in \omega$ are such that there is no $r \leq p'(0)_s\,\hat{}\,p' \restriction [1,\omega_2)$ forcing $\dot x \restriction n_0 = u \restriction n_0$, then $(1)_{p',s}$ is still satisfied when U_s is replaced with $U_s \setminus \{u\}$.*

Proof. Suppose that $(1)_{p',s}$ with $U_s \setminus \{u\}$ instead of U_s fails for some $n \in \omega$. There is no loss of generality in assuming that $n \geq n_0$. This means that the set

$$T = \big\{t \in S_{p'(0)}(s)\ :\ p'(0)_t\,\hat{}\,p' \restriction [1,\omega_2) \not\Vdash \exists v \in U_s \setminus \{u\}\ (\dot x \restriction n = v \restriction n)\big\}$$

is infinite. For each $t \in T$ find $r_t \leq p'(0)_t\,\hat{}\,p' \restriction [1,\omega_2)$ such that

[1] Note that for a set $A \in V[G_{\omega_2}]$, the inclusion $A \subset V[G_\alpha]$ does not imply $A \in V[G_\alpha]$. For example, if $A \subset \omega$, then $A \subset V$ because $\omega \subset V$. Thus, in $V[G_{\omega_2}]$ there are ω_2-many subsets of ω, each of them is a subset of V, but only ω_1-many of them are *elements* of V.

$$r_t \Vdash \forall v \in U_s \setminus \{u\} \, (\dot{x} \upharpoonright n \neq v \upharpoonright n).$$

Let $t_1 \in T$ be such that

$$p'(0)_{t_1} {}^\frown p' \upharpoonright [1, \omega_2) \Vdash \exists v \in U_s \, (\dot{x} \upharpoonright n = v \upharpoonright n)\big).$$

From the two formulas displayed above it follows that $r_{t_1} \Vdash \dot{x} \upharpoonright n = u \upharpoonright n$, and hence also $r_{t_1} \Vdash \dot{x} \upharpoonright n_0 = u \upharpoonright n_0$ because $n \geq n_0$, which is impossible by our assumption. □

For a subset K of 2^ω we denote by $O_n(K)$ the set $\{z \in 2^\omega : z \upharpoonright n = y \upharpoonright n$ for some $y \in K\}$.

Claim 2.4. *For every $s \in p'(0)$, $p'(0)\langle 0 \rangle \leq s$, and every $t \in S_{p'(0)}(s)$ there exists $U'_t \subset U_t$ such that*

- *$(1)_{p',t}$ is still satisfied when U_t is replaced with U'_t; and*
- *for every $n \in \omega$ and all but finitely many $t \in S_{p'(0)}(s)$ we have $U'_t \subset O_n(U_s)$.*

Proof. Fix $n \in \omega$. Then by $(1)_{p',s}$ there exists $A \in [S_{p'(0)}(s)]^{<\omega}$ such that

$$p'(0)_t {}^\frown p' \upharpoonright [1, \omega_2) \Vdash \exists u \in U_s (u \upharpoonright n = \dot{x} \upharpoonright n) \tag{1}$$

for every $t \in S_{p'(0)}(s) \setminus A$. Note that (1) implies that if $t \in S_{p'(0)}(s) \setminus A$, $w \notin O_n(U_s)$, and $r \leq p'(0)_t {}^\frown p' \upharpoonright [1, \omega_2)$, then r forces $w \upharpoonright n \neq \dot{x} \upharpoonright n$. Applying Claim 2.3 we conclude that $(1)_{p',t}$ is satisfied for $t \in S_{p'(0)}(s) \setminus A$ if we replace U_t with $U_t \cap O_n(U_s)$.

Applying the same argument recursively for every $n \in \omega$ we can get an increasing sequence $\langle A_n : n \in \omega \rangle$ of finite subsets of $S_{p'(0)}(s)$ with $\bigcup_{n \in \omega} A_n = S_{p'(0)}(s)$ such that $(1)_{p',t}$ is satisfied for $t \in S_{p'(0)}(s) \setminus A_n$ if we replace U_t with $U_t \cap O_n(U_s)$. It remains to set $U'_t = U_t$ for all $t \in A_0$ and $U'_t = U_t \cap O_n(U_s)$ for all $t \in A_{n+1} \setminus A_n$ and note that these U'_t are as required. □

Claim 2.5. *Let $K \subset 2^\omega$ be compact, and for every $i \in \omega$ let $\langle U^i_m : m \in \omega \rangle$ be a sequence of finite subsets of 2^ω such that*

$$\forall i \in \omega \, \forall n \in \omega \, \forall^* m \in \omega \, (U^i_m \subset O_n(K)).$$

Then for every $i \in \omega$ there exists $m_i \in \omega$ such that $K \cup \bigcup_{i \in \omega} \bigcup_{m \geq m_i} U^i_m$ is compact.

Proof. It suffices to choose m_i such that $U^i_m \subset O_i(K)$ for all $m \geq m_i$, the standard details are left to the reader. □

After three auxiliary claims above, we are in a position to proceed with the proof of Lemma 2.2. Combining Claims 2.4 and 2.5, we can get a Laver condition $T \leq_0 p'(0)$ and $U'_t \subset U_t$ for every splitting node $t \in T$ such that

(*i*) letting $p'' = T {}^\frown p' \upharpoonright [1, \omega_2)$, we have $p'' \in G_{\omega_2}$ and $(1)_{p'',t}$ is satisfied for all splitting nodes $t \in T = p''(0)$;

(*ii*) For every $s \geq T\langle 0 \rangle$, $n \in \omega$, and all but finitely many $t \in S_T(s)$ we have $U'_t \subset O_n(U'_s)$; and

(*iii*) $K_m := \bigcup \big\{ U'_t \, : \, t \in T, T\langle 0 \rangle \leq t, |t| \leq m \big\}$ is a compact subset of 2^ω for all $m \in \omega$. (Note that $K_m = \emptyset$ for $m < m_0 := |T\langle 0 \rangle| = |p'(0)\langle 0 \rangle|$.)

Set $Q_* = \bigcup_{m\in\omega} K_m \in [X]^\omega$ and consider the map $\phi : 2^\omega \setminus Q_* \to \omega^\omega$ defined as follows:

$$\phi(z)(m) = \min\{n \in \omega : z \notin O_n(K_m)\}.$$

Since K_m is closed and $z \notin K_m$, ϕ is well-defined. The second item describing properties of ordinals in C yields $h \in \omega^\omega \cap V$ such that $h \not\leq^* \phi(y)$ for any $y \in \dot{M}(Q_*)$. It follows that

$$p'' \Vdash \dot{x} \in \dot{X} \setminus \bigcup_{Q\in\mathsf{Q}} (2^\omega \setminus \dot{M}(Q)) = \dot{X} \cap \bigcap_{Q\in\mathsf{Q}} \dot{M}(Q) \subset \dot{X} \cap \dot{M}(Q_*),$$

and hence

$$p'' \Vdash h \not\leq^* \phi(\dot{x}). \tag{2}$$

On the other hand, recursively removing finitely many immediate successors of every splitting node s of $T = p''(0)$, using (i) and (ii), we can get a Laver tree $T' \leq_0 T$ such that

$$T_t \,\hat{}\, p' \restriction [1, \omega_2) \Vdash \exists u \in U'_s \,(\dot{x} \restriction h(m) = u \restriction h(m)) \tag{3}$$

for all $m \geq m_0$, $s \in T' \cap \omega^m$, and $t \in S_{T'}(s)$. Equation (3) gives $T'_t \,\hat{}\, p' \restriction [1, \omega_2) \Vdash \dot{x} \in O_{h(m)}(K_m)$ for all m, s, t as above, and therefore

$$T' \,\hat{}\, p' \restriction [1, \omega_2) \Vdash \forall m \geq m_0 \,(\dot{x} \in O_{h(m)}(K_m)),$$

or, equivalently,

$$T' \,\hat{}\, p' \restriction [1, \omega_2) \Vdash \forall m \geq m_0 \,(\phi(\dot{x}) > h(m)),$$

which together with $T' \,\hat{}\, p' \restriction [1, \omega_2) \leq p''$ contradicts (2) and thus finishes our proof. □

The next lemma demonstrates the relation between (†) and products with Menger spaces.

Lemma 2.6. *Suppose that $\mathfrak{b} > \omega_1$. Let $Y \subset 2^\omega$ be a Menger space and $X \subset 2^\omega$ a Hurewicz space satisfying (†). Then $X \times Y$ is Menger.*

Proof. Fix a sequence $\langle \mathcal{U}_n : n \in \omega \rangle$ of countable covers of $X \times Y$ by clopen subsets of $2^\omega \times Y$. For every $Q \in [X]^\omega$ using that $Q \times Y$ is Menger, we can find a sequence $\langle \mathcal{V}^Q_n : n \in \omega \rangle$ such that $\mathcal{V}^Q_n \in [\mathcal{U}_n]^{<\omega}$ for all $n \in \omega$ and $Q \times Y \subset W^Q := \bigcap_{m\in\omega} \bigcup_{n\geq m} \cup\mathcal{V}^Q_n$. Then $(2^\omega \times Y) \setminus W^Q$ is Menger being an F_σ-subset of the Menger space $2^\omega \times Y$, and it is disjoint from $Q \times Y$, and hence its projection $M(Q)$ onto the first coordinate is a Menger subspace of 2^ω disjoint from Q.

Since X satisfies (†), there exists $\mathsf{Q} \subset [X]^\omega$ of size $|\mathsf{Q}| = \omega_1$ such that $X \subset \bigcup_{Q\in\mathsf{Q}} (2^\omega \setminus M(Q))$. Since $|\mathsf{Q}| < \mathfrak{b}$, there exists a sequence $\langle \mathcal{V}_n : n \in \omega \rangle$ such that $\mathcal{V}_n \in [\mathcal{U}_n]^{<\omega}$ for all $n \in \omega$, and for every $Q \in \mathsf{Q}$ there exists $m(Q) \in \omega$ such that $\mathcal{V}^Q_n \subset \mathcal{V}_n$ for all $n \geq m(Q)$. We claim that $X \times Y \subset \bigcup_{n\in\omega} \cup\mathcal{V}_n$. Indeed, given $\langle x, y \rangle \in X \times Y$, find $Q \in \mathsf{Q}$ such that $x \notin M(Q)$. This implies $\langle x, y \rangle \in W^Q$, and hence there exists $n \geq m(Q)$ with $\langle x, y \rangle \in \cup\mathcal{V}^Q_n$, consequently $\langle x, y \rangle \in \cup\mathcal{V}_n$ because $\mathcal{V}^Q_n \subset \mathcal{V}_n$, which completes our proof. □

Finally, we can prove the characterization of Hurewicz subspaces of 2^ω which holds in the Laver model and implies Theorem 1.1.

Proposition 2.7. *In the Laver model, for a subspace X of 2^ω the following conditions are equivalent:*

(1) *X is Hurewicz;*
(2) *X satisfies (†);*
(3) *X is productively Menger; and*
(4) *X is productively Hurewicz.*

Proof. The implication (1) → (2) is established in Lemma 2.2. The implication (2) → (3) is proved in Lemma 2.6 and thus requires only $\mathfrak{b} > \omega_1$. And finally, (3) → (4) follows from [11, Theorem 4.8(2)] because $\mathfrak{b} = \mathfrak{d}$ holds in the Laver model, while (4) → (1) is obvious. □

Finally, by nearly the same argument as at the end of [9] we can prove that Theorem 1.1 follows from Proposition 2.7. Again, we present its proof for the sake of completeness. A family $\mathcal{F} \subset [\omega]^\omega$ is called a *semifilter* if for every $F \in \mathcal{F}$ and $X \subset \omega$, if $|F \setminus X| < \omega$ then $X \in \mathcal{F}$. Each semifilter is considered with the topology inherited from the Cantor space 2^ω which we identify with $\mathcal{P}(\omega)$ via characteristic functions.

The proof of the second part of Theorem 1.1 uses characterizations of the Hurewicz and Menger properties obtained in [12]. Let $u = \langle U_n : n \in \omega\rangle$ be a sequence of subsets of a set X. For every $x \in X$ let $I_s(x, u, X) = \{n \in \omega : x \in U_n\}$. If every $I_s(x, u, X)$ is infinite (the collection of all such sequences u will be denoted by $\Lambda_s(X)$), then we shall denote by $\mathcal{U}_s(u, X)$ the smallest semifilter on ω containing all $I_s(x, u, X)$. By [12, Theorem 3], a Lindelöf topological space X is Hurewicz (Menger) if and only if for every $u \in \Lambda_s(X)$ consisting of open sets, the semifilter $\mathcal{U}_s(u, X)$ is Hurewicz (Menger). The proof given there also works if we consider only those $\langle U_n : n \in \omega\rangle \in \Lambda_s(X)$ such that all U_n's belong to a given base of X.

Proof of Theorem 1.1. Suppose that X is Hurewicz, Y is Menger, $X \times Y$ is Lindelöf, and fix $w = \langle U_n \times V_n : n \in \omega\rangle \in \Lambda_s(X \times Y)$ consisting of open sets. Set $u = \langle U_n : n \in \omega\rangle$, $v = \langle V_n : n \in \omega\rangle$, and note that $u \in \Lambda_s(X)$ and $v \in \Lambda_s(Y)$. It is easy to see that

$$\mathcal{U}_s(w, X \times Y) = \{A \cap B : A \in \mathcal{U}_s(u, X), B \in \mathcal{U}_s(v, Y)\},$$

and hence $\mathcal{U}_s(w, X \times Y)$ is a continuous image of $\mathcal{U}_s(u, X) \times \mathcal{U}_s(v, Y)$. By [12, Theorem 3] $\mathcal{U}_s(u, X)$ and $\mathcal{U}_s(v, Y)$ are Hurewicz and Menger, respectively, considered as subspaces of 2^ω, and hence their product is a Menger space by Proposition 2.7. Thus $\mathcal{U}_s(w, X \times Y)$ is Menger, being a continuous image of a Menger space. It now suffices to use [12, Theorem 3] again, in the other direction. □

3. Productively Hurewicz spaces in models of CH

In this section we prove Theorem 1.2.

Suppose that CH holds in the ground model V and fix $Y = \{y_\alpha : \alpha < \omega_1\} \subset [\omega]^\omega$ such that

- $y_\beta \subset^* y_\alpha$ for all $\beta > \alpha$;
- $y_{\alpha+1} \subset y_\alpha$ for all α;
- $y_\alpha \setminus y_{\alpha+1}$ is infinite for all α; and
- For every $y \in [\omega]^\omega$ there exists α with $y_\alpha \not\leq^* y$, where each element of $[\omega]^\omega$ is identified with its increasing enumerating function.

Thus Y is an unbounded tower in the terminology of [8].

In what follows, we shall work in $V[G]$, where G is $Fn(\omega_1, 2)$-generic over V. Here $Fn(\omega_1, 2)$ is the standard poset adding ω_1 many Cohen reals over V.

It is well-known that $Y = \{y_\alpha : \alpha < \omega_1\}$ is Menger in $V[G]$, see, e.g., [10, Theorem 11]. Moreover, Y is also unbounded in $V[G]$ since Cohen reals preserve the unboundedness of ground model unbounded sets. Fix an enumeration $[\omega]^\omega = \{z_\alpha : \alpha \in \omega_1\}$ and for every α pick $x_\alpha \in [y_\alpha]^\omega$ such that $x_\alpha \supset y_{\alpha+1}$, $|y_\alpha \setminus x_\alpha| = |x_\alpha \setminus y_{\alpha+1}| = \omega$, and $z_\alpha \leq^* (y_\alpha \setminus x_\alpha)$.

Since $x_\beta \subset y_\beta \subset^* y_{\alpha+1} \subset x_\alpha$ for any $\alpha < \beta$, we conclude that $\{x_\alpha : \alpha < \omega_1\}$ is an unbounded tower as well, and hence $X := \{x_\alpha : \alpha < \omega_1\} \cup [\omega]^{<\omega}$ is productively Hurewicz[2] by [7, Theorem 6.5(1)]. Now, Theorem 1.2 is a direct consequence of the following

Observation 3.1. *$X \times Y$ is not Menger, and therefore X is not productively Menger.*

Proof. Let $\oplus$ be the coordinate-wise addition modulo 2 in 2^ω, i.e., the standard operation turning 2^ω into a topological group. We shall show that $X \oplus Y$ is not a Menger subspace of 2^ω. Since no dominating subset of $[\omega]^\omega$ is Menger (see, e.g., [4, Theorem 4.4]) and $z_\alpha \leq^* y_\alpha \setminus x_\alpha = y_\alpha \oplus x_\alpha$, it remains to check that $X \oplus Y \subset [\omega]^\omega$. This is done through a routine consideration of $x_\alpha \oplus y_\beta$ for all possible α, β below.

1. $\alpha < \beta$. In this case $x_\alpha \setminus y_{\alpha+1} \subset^* x_\alpha \setminus y_\beta \subset x_\alpha \oplus y_\beta$, and $x_\alpha \setminus y_{\alpha+1}$ is infinite by the choice of x_α.
2. $\alpha = \beta$. Then $y_\alpha \setminus x_\alpha = x_\alpha \oplus y_\alpha$ is infinite by the choice of x_α.
3. $\alpha > \beta$. In this case $x_\alpha \subset y_\alpha \subset^* y_\beta$ and $y_\alpha \setminus x_\alpha$ is infinite. Thus, $y_\beta \setminus x_\alpha$ is infinite as well, and the latter difference is included into $x_\alpha \oplus y_\beta$. □

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[2] X is also a γ-set by the main result of [8].