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The exact region determined by Blomqvist's beta, Spearman's footrule and Gini's gamma

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ABSTRACT

To quantify the degree of association between random variables, concordance measures are employed. To express such a degree, a single measure might give too much space, so several are used for comparison. In this paper we study the ternary relation between three well-known (weak) concordance measures, namely Blomqvist's beta, Spearman's footrule and Gini's gamma. In other words, given the values of Blomqvist's beta and Spearman's footrule, we determine the degree of freedom a copula has at taking the value of Gini's gamma. We explicitly determine the 3-dimensional region representing the relation. We also provide copulas where bounds of the region are attained.

1. Introduction

Two continuous random variables are concordant when large values of the first are associated with large values of the second. It is often more appropriate to study the association of random variables instead of their linear correlation due to the invariance of measures of concordance to monotonously increasing transformations. This directly implies that measures of concordance are independent of marginal distributions of continuous random variables. Rather, they rely solely on the copula modeling the dependence of the two random variables.

The most commonly used concordance measures are Spearman's rho, Kendall's tau, Gini's gamma and Blomqvist's beta, and a weak concordance measure Spearman's footrule. These measures have been studied extensively since their introduction. Recent references for bivariate concordance measures include [1–7] and their multivariate generalizations were studied in [8–11], to name just a few.

Recently, studying binary relations between five most common (weak) measures of concordance on copulas has become increasingly popular. Since Spearman's rho and Kendall's tau are the most common and well known concordance measures, it is only natural to try and determine their relation. The problem of the exact region determined by them was open from 1960's and was solved only recently by Schreyer, Paulin and Trutschnig in [12]. The relation between Blomqvist's beta and other (possibly weak) measures of concordance is relatively easy to tackle due to the nature of beta, while relations between others might and do prove to be more challenging. The region determined by Gini's gamma and Spearman's footrule is characterized in [13]. The problem of the region determined by Spearman's rho and Spearman's footrule is partially solved in [14], see also [15]. Characterizing the upper bound of the region appears to be a very hard problem that has so far evaded the solution. The region determined by Spearman's rho and Gini's gamma is still open.

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In this paper we take a step further and characterize the exact region representing the ternary relation between Blomqvist's beta, Spearman's footrule and Gini's gamma. Given the values of Blomqvist's beta and Spearman's footrule for some copula, we give the lower and the upper bound that Gini's gamma can take. We also provide copulas where these bounds are attained.

2. Preliminaries

Let $\mathbb{I}$ be the unit interval $[0,1] \subseteq \mathbb{R}$ and $x_1, x_2, y_1, y_2 \in \mathbb{I}$ be such that $x_1 \leq x_2$ and $y_1 \leq y_2$. The Cartesian product of intervals $B = [x_1, x_2] \times [y_1, y_2]$ is called a *rectangle* in $\mathbb{I}^2$. Let $H : \mathbb{I}^2 \to \mathbb{R}$ be a real function. We define the H-volume of B as $\mathrm{Vol}_H(B) = H(x_2, y_2) - H(x_2, y_1) - H(x_1, y_2) + H(x_1, y_1)$. A *bivariate copula* is a function $C : \mathbb{I}^2 \to \mathbb{I}$ with the following properties:

- $C(0, y) = C(x, 0) = 0$ for all $x, y \in \mathbb{I}$ (C is *grounded*),
- $C(x, 1) = x$ and $C(1, y) = y$ for all $x, y \in \mathbb{I}$ (C has *uniform marginals*), and
- $\mathrm{Vol}_C(B) \geq 0$ for every rectangle $B \subseteq \mathbb{I}^2$ (C is 2*–increasing*).

Bivariate copulas are therefore functions of two variables which couple bivariate distribution functions with their one-dimensional marginal distribution functions, a famous Theorem by Sklar [16].

Let $\mathcal{C}$ be the set of all bivariate copulas. We introduce some transformations that are naturally defined on $\mathcal{C}$: We denote by C^t the transpose of the copula C, i.e., $C^t(x, y) = C(y, x)$. By C^{σ_1} and C^{σ_2} we denote the two reflections of a copula C defined by $C^{\sigma_1}(x, y) = y - C(1 - x, y)$ and $C^{\sigma_2}(x, y) = x - C(x, 1 - y)$ (see [17, §1.7.3]), and by $\widehat{C} = (C^{\sigma_1})^{\sigma_2}$ the survival copula of C. We write $C \leq D$ if $C(x, y) \leqslant D(x, y)$ for all $(x, y) \in \mathbb{I}^2$. This is the so-called *pointwise order* of copulas. It is well known that $\mathcal{C}$ is a partially ordered set with respect to the order, but not a lattice [18, Theorem 2.1], and that $W(x, y) = \max\{0, x+y-1\}$ and $M(x, y) = \min\{x, y\}$ are the lower and upper bounds of all copulas, respectively. Copulas W and M are called *Fréchet-Hoeffding lower and upper bounds*. We denote the main diagonal of copula C by δ_C and its opposite diagonal by ω_C, i.e.,

$$\delta_C(x) = C(x, x), \quad \text{and} \quad \omega_C(x) = C(x, 1 - x).$$

The function δ_C satisfies $0 \leq \delta_C(x) \leq x$, $\delta_C(1) = 1$ and it is increasing and 2-Lipschitz. The function ω_C satisfies $\omega_C(0) = \omega_C(1) = 0$ and it is nonnegative and 1-Lipschitz (see [19]).

A mapping $\kappa : \mathcal{C} \to [-1, 1]$ is called a *concordance measure* if it satisfies the following properties (see [17, Definition 2.4.7]):

(C1) $\kappa(C) = \kappa(C^t)$ for every $C \in \mathcal{C}$.
(C2) $\kappa(C) \leqslant \kappa(D)$ when $C \leqslant D$.
(C3) $\kappa(M) = 1$.
(C4) $\kappa(C^{\sigma_1}) = \kappa(C^{\sigma_2}) = -\kappa(C)$.
(C5) If a sequence of copulas C_n, $n \in \mathbb{N}$, converges pointwise to $C \in \mathcal{C}$, then $\lim_{n\to\infty} \kappa(C_n) = \kappa(C)$.

Certain properties that are sometimes stated in definitions of a concordance measure follow from the properties listed above. Indeed, a concordance measure also satisfies the following properties (see [4, §3] for more details):

(C6) $\kappa(\Pi) = 0$, where Π is the independence copula $\Pi(u, v) = uv$.
(C7) $\kappa(W) = -1$.
(C8) $\kappa(C) = \kappa(\widehat{C})$ for every $C \in \mathcal{C}$.

The four most commonly used concordance measures of a copula C are Spearman's rho, Kendall's tau, Gini's gamma, and Blomqvist's beta. If we replace property (C4) with property (C6) in the definition of a concordance measure, we get a *weak concordance measure* (see [5]). Spearman's footrule is an example of a weak concordance measure. The range of a concordance measure is the interval $[-1, 1]$, while the range of Spearman's footrule is equal to $\left[-\frac{1}{2}, 1\right]$ (see [11, §4]). In this paper we are going to consider the relation between Blomqvist's beta (β), Spearman's footrule (ϕ) and Gini's gamma (γ). They are defined as follows:

$$\beta(C) = 4\,C\left(\tfrac{1}{2}, \tfrac{1}{2}\right) - 1, \tag{1}$$

$$\phi(C) = 6\int_0^1 \delta_C(x)dx - 2, \tag{2}$$

$$\gamma(C) = 4\int_0^1 \delta_C(x)dx + 4\int_0^1 \omega_C(x)dx - 2. \tag{3}$$

In recent years the relations between pairs of (weak) concordance measures got lots of attention. The exact region determined by Kendall's tau and Spearman's rho was determined in [12]. The regions determined by Blomqvist's beta and Spearman's rho, Kendall's tau, and Gini's gamma are given in [20] as an exercise for the reader. The region determined by Blomqvist's beta and Spearman's footrule was given in [21]. The region determined by Spearman's footrule and Gini's gamma was given in [13], the regions determined by Spearman's footrule and Kendall's tau and determined by Gini's gamma and Kendall's tau in [22]. The region between Spearman's footrule and Spearman's rho was partially determined in [14] (see also [15]). The region determined by Gini's gamma and Spearman's rho is still open. Here we give the results we are going to need in the sequel.

Proposition 2.1 ([13]). *For any copula $C \in \mathcal{C}$ we have*

$$\tfrac{4}{3}\phi(C) - \tfrac{1}{3} \le \gamma(C) \le \min\{\tfrac{4}{3}\phi(C) + \tfrac{1}{6}, \tfrac{2}{3}\phi(C) + \tfrac{1}{3}\}.$$

The bounds are attained by shuffles of M.

The exact region determined by Spearman's footrule and Gini's gamma

$$\Omega_{\phi,\gamma} = \{(\phi(C), \gamma(C)) \in [-\tfrac{1}{2}, 1] \times [-1, 1] : C \in \mathcal{C}\}$$

is quadrilateral $A_1B_1C_1D_1$ with vertices $A_1(-\frac{1}{2}, -1), B_1(1, 1), C_1(\frac{1}{4}, \frac{1}{2}), D_1(-\frac{1}{2}, -\frac{1}{2})$ depicted in Fig. 8.

Proposition 2.2. *For any copula $C \in \mathcal{C}$ we have*

$$\tfrac{3}{16}(1 + \beta(C))^2 - \tfrac{1}{2} \le \phi(C) \le 1 - \tfrac{3}{8}(1 - \beta(C))^2, \tag{4}$$

$$\tfrac{3}{8}(1 + \beta(C))^2 - 1 \le \gamma(C) \le 1 - \tfrac{3}{8}(1 - \beta(C))^2. \tag{5}$$

The bounds are attained by shuffles of M.

3. Main result

In this section we locate the exact three-dimensional region determined by Blomqvist's beta, Spearman's footrule and Gini's gamma. In other words, we determine all possible triples $(\beta(C), \phi(C), \gamma(C))$, where C runs over the set of all copulas $\mathcal{C}$. We will find the upper and lower bound for $\gamma(C)$, if $\beta(C)$ and $\phi(C)$ are given and satisfy Proposition 2.2, and show that the bounds can be attained. Proposition 2.1 already gives two bounds which will be part of our final result, namely,

$$\tfrac{4}{3}\phi(C) - \tfrac{1}{3} \le \gamma(C) \le \tfrac{4}{3}\phi(C) + \tfrac{1}{6}. \tag{6}$$

In next two propositions we prove two more bounds.

Proposition 3.1. *For any copula $C \in \mathcal{C}$ we have*

$$\gamma(C) \le \tfrac{2}{3}\phi(C) + \tfrac{1}{4}\beta(C) - \tfrac{1}{8}\beta(C)^2 + \tfrac{5}{24}.$$

Proof. Let C be any copula and let $b = \omega_C(\frac{1}{2}) = C(\frac{1}{2}, \frac{1}{2}) = \frac{\beta(C)+1}{4}$. Since ω_C is 1-Lipschitz, we have for any $x \in [0, \frac{1}{2}]$ that $\omega_C(\frac{1}{2}) - \omega_C(x) \ge -(\frac{1}{2} - x)$, so that $\omega_C(x) \le -x + b + \frac{1}{2}$. Furthermore, for any $x \in [\frac{1}{2}, 1]$ 1-Lipschitz property implies that $\omega_C(x) - \omega_C(\frac{1}{2}) \le x - \frac{1}{2}$, and thus $\omega_C(x) \le x + b - \frac{1}{2}$. Since $\omega_C(0) = \omega_C(1) = 0$, we have also $\omega(x) \le \min\{x, 1 - x\}$ for any $x \in \mathbb{I}$. It follows that

$$\omega_C(x) \le \omega_0(x) = \begin{cases} x; & \text{if } 0 \le x \le \frac{1}{4} + \frac{b}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{4} + \frac{b}{2} < x \le \frac{1}{2}, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{3}{4} - \frac{b}{2}, \\ 1 - x; & \text{if } \frac{3}{4} - \frac{b}{2} < x \le 1. \end{cases}$$

From equality (2) it follows that $\int_0^1 \delta_C(x)dx = \frac{\phi(C)+2}{6}$. Now,

$$\begin{aligned} \gamma(C) &= 4\int_0^1 \delta_C(x)dx + 4\int_0^1 \omega_C(x)dx - 2 \\ &\le 4\int_0^1 \delta_C(x)dx + 4\int_0^1 \omega_0(x)dx - 2 \\ &= 4 \cdot \frac{\phi(C) + 2}{6} + 4 \cdot \left(\tfrac{1}{2}b - \tfrac{1}{2}b^2 + \tfrac{1}{8}\right) - 2 \\ &= \tfrac{2}{3}\phi(C) + 2b - 2b^2 - \tfrac{1}{6} \\ &= \tfrac{2}{3}\phi(C) + \tfrac{1}{4}\beta(C) - \tfrac{1}{8}\beta(C)^2 + \tfrac{5}{24}, \end{aligned}$$

which finishes the proof. □

Proposition 3.2. *For any copula $C \in \mathcal{C}$ we have*

$$\gamma(C) \ge \tfrac{2}{3}\phi(C) + \tfrac{1}{2}\beta(C) + \tfrac{1}{4}\beta(C)^2 - \tfrac{5}{12}.$$

Proof. As above, for any copula C let $b = \omega_C(\frac{1}{2}) = C(\frac{1}{2}, \frac{1}{2}) = \frac{\beta(C)+1}{4}$. Since ω_C is 1-Lipschitz, we have for any $x \in [0, \frac{1}{2}]$ that $\omega_C(\frac{1}{2}) - \omega_C(x) \le \frac{1}{2} - x$, so that $\omega_C(x) \ge x + b - \frac{1}{2}$. Furthermore, for any $x \in [\frac{1}{2}, 1]$ 1-Lipschitz property implies that $\omega_C(x) - \omega_C(\frac{1}{2}) \ge -(x - \frac{1}{2})$, and thus $\omega_C(x) \le -x + b + \frac{1}{2}$. Since ω_C is nonnegative, it follows that

$$\omega_C(x) \ge \omega_1(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - b, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} - b < x \le \frac{1}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + b, \\ 0; & \text{if } \frac{1}{2} + b < x \le 1. \end{cases}$$

Now,

$$\begin{aligned} \gamma(C) &\ge 4\int_0^1 \delta_C(x)dx + 4\int_0^1 \omega_1(x)dx - 2 \\ &= 4 \cdot \frac{\phi(C)+2}{6} + 4 \cdot b^2 - 2 \\ &= \tfrac{2}{3}\phi(C) + 4b^2 - \tfrac{2}{3} \\ &= \tfrac{2}{3}\phi(C) + \tfrac{1}{2}\beta(C) + \tfrac{1}{4}\beta(C)^2 - \tfrac{5}{12}, \end{aligned}$$

which finishes the proof. □

In next examples we give six families of shuffles of M, which attain the proved bounds. A shuffle of M

$$M(n, J, \pi, \varepsilon)$$

is determined by a positive integer n, a partition of interval $\mathbb{I}$ into n pieces

$$J = \{[0, x_1], [x_1, x_2], \dots, [x_{n-2}, x_{n-1}], [x_{n-1}, 1]\},$$

shortly written as an $(n-1)$-tuple $J = (x_1, x_2, \dots, x_{n-1})$ where $0 \le x_1 \le x_2 \le \dots \le x_{n-1} \le 1$, a permutation $\pi \in S_n$, written as an n-tuple of images $\pi = (\pi(1), \pi(2), \dots, \pi(n))$, and a mapping $\varepsilon : \{1, 2, \dots, n\} \to \{-1, 1\}$, written as an n-tuple of images $\varepsilon = (\varepsilon(1), \varepsilon(2), \dots, \varepsilon(n))$. For more details see [20, §3.2.3]. Notice that we allow some of the intervals in the partition J to be singletons.

Example 3.3. Let $b \in [0, \frac{1}{2}]$ and let C_b be a shuffle of M

$$C_b = M(4, (\tfrac{1}{2} - b, \tfrac{1}{2}, \tfrac{1}{2} + b), (4, 2, 3, 1), (-1, -1, -1, -1)).$$

Notice that $C_0 = W$ and $C_{\frac{1}{2}} = M(2, \frac{1}{2}, (1, 2), (-1, -1))$. We have $C_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{C_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - \frac{b}{2}, \\ 2x + b - 1; & \text{if } \frac{1}{2} - \frac{b}{2} < x \le \frac{1}{2}, \\ b; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + \frac{b}{2}, \\ 2x - 1; & \text{if } \frac{1}{2} + \frac{b}{2} < x \le 1, \end{cases}$$

and

$$\omega_{C_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - b, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} - b < x \le \frac{1}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + b, \\ 0; & \text{if } \frac{1}{2} + b < x \le 1. \end{cases}$$

The mass distribution of copula C_b, the graphs of functions δ_{C_b} and ω_{C_b}, and 3D plot of copula C_b are depicted in Fig. 1. It follows that

$$\beta(C_b) = 4b - 1, \phi(C_b) = 3b^2 - \tfrac{1}{2} \text{ and } \gamma(C_b) = 6b^2 - 1,$$

so that

$$\phi(C_b) = \tfrac{3}{16}(1 + \beta(C_b))^2 - \tfrac{1}{2} \quad \text{and} \quad \gamma(C_b) = \tfrac{2}{3}\phi(C_b) + \tfrac{1}{2}\beta(C_b) + \tfrac{1}{4}\beta(C_b)^2 - \tfrac{5}{12},$$

and both the lower bound from inequality (4) and Proposition 3.2 are attained.

Example 3.4. Let $b \in [0, \frac{1}{2}]$ and let C_b be a shuffle of M

$$D_b = M(2, (\tfrac{1}{2} - b, \tfrac{1}{2} + b), (3, 2, 1), (-1, 1, -1)).$$

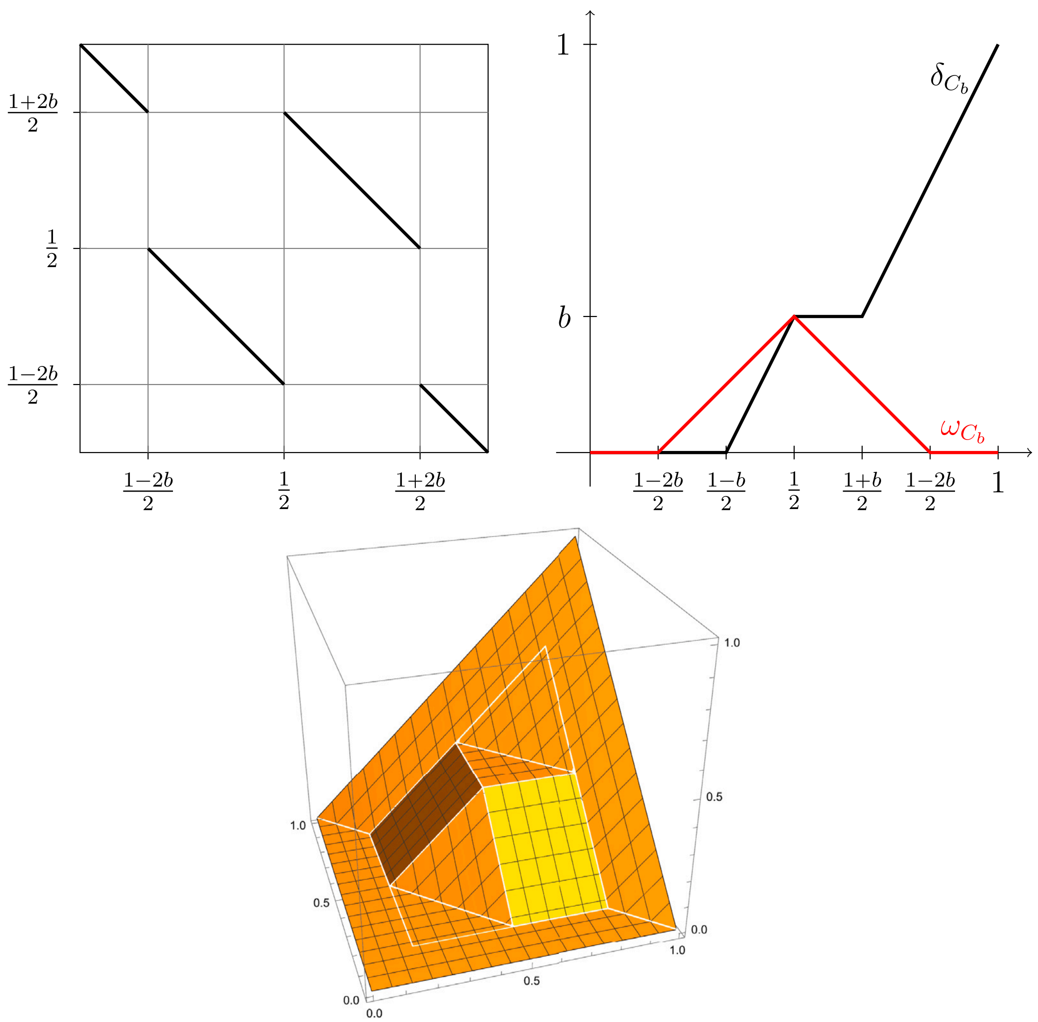

Fig. 1. The mass distribution of copula C_b (top left), the graphs of functions δ_{C_b} and ω_{C_b} (top right), and a 3D plot of copula C_b (bottom) from Example 3.3.

As in the previous example, $D_0 = W$, but $D_{\frac{1}{2}} = M$. We have $D_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{D_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - b, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} - b < x \le \frac{1}{2} + b, \\ 2x - 1; & \text{if } \frac{1}{2} + b < x \le 1, \end{cases}$$

and

$$\omega_{D_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - b, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} - b < x \le \frac{1}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + b, \\ 0; & \text{if } \frac{1}{2} + b < x \le 1. \end{cases}$$

Fig. 2 depicts the mass distribution of copula D_b, the graphs of functions δ_{D_b} and ω_{D_b}, and 3D plot of copula D_b. It follows that

$$\beta(D_b) = 4b - 1, \phi(D_b) = 6b^2 - \tfrac{1}{2} \text{ and } \gamma(D_b) = 8b^2 - 1,$$

so that

$$\gamma(D_b) = \tfrac{4}{3}\phi(D_b) - \tfrac{1}{3} = \tfrac{2}{3}\phi(D_b) + \tfrac{1}{2}\beta(D_b) + \tfrac{1}{4}\beta(D_b)^2 - \tfrac{5}{12},$$

and both the lower bound from inequality (6) and Proposition 3.2 are attained.

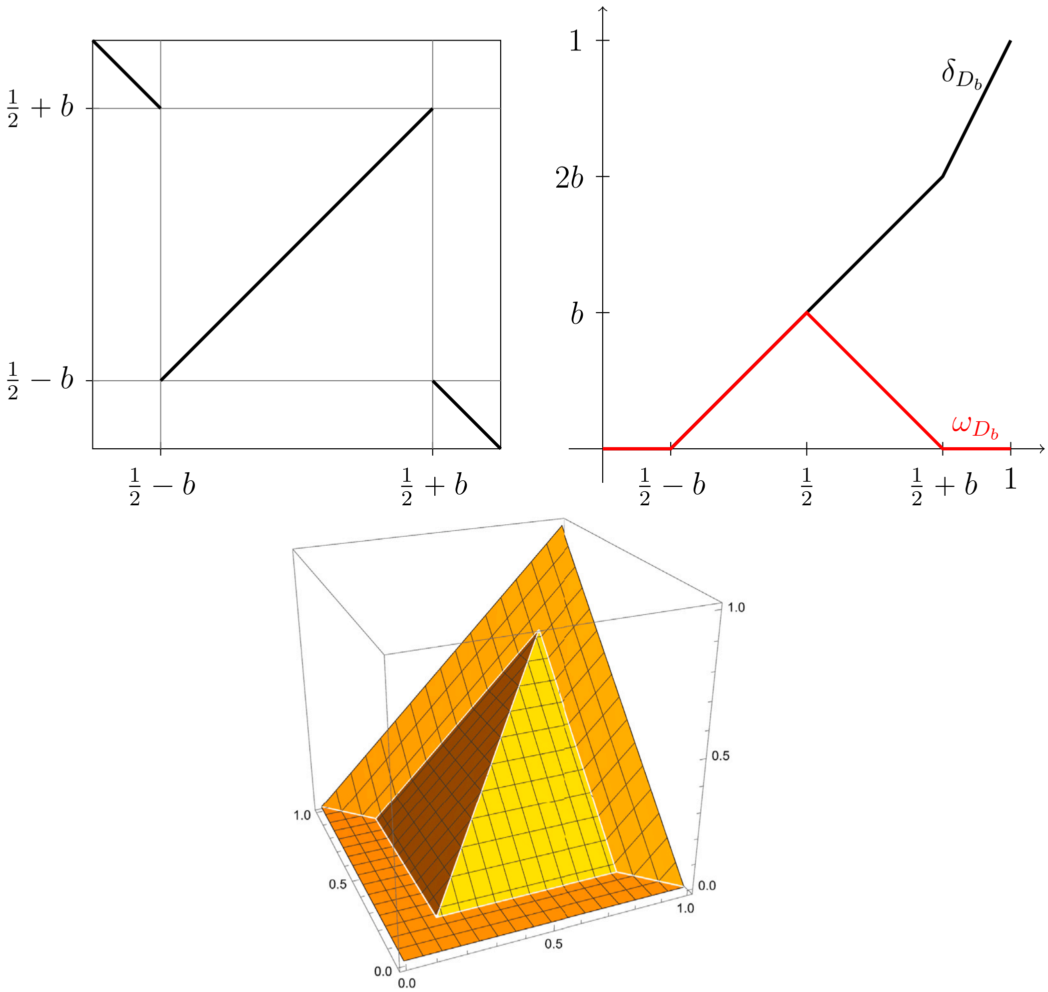

Fig. 2. The mass distribution of copula D_b (top left), the graphs of functions δ_{D_b} and ω_{D_b} (top right), and a 3D plot of copula D_b (bottom) from Example 3.4.

Example 3.5. Let $b \in [0, \frac{1}{2}]$ and let E_b be a shuffle of M

$$E_b = M(3, (b, 1-b), (1, 2, 3), (1, -1, 1)).$$

Again $E_0 = W$ and $E_{\frac{1}{2}} = M$. We have $E_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{E_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le b, \\ b; & \text{if } b < x \le \frac{1}{2}, \\ 2x + b - 1; & \text{if } \frac{1}{2} < x \le 1 - b, \\ x; & \text{if } 1 - b < x \le 1, \end{cases}$$

and

$$\omega_{E_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le b, \\ b; & \text{if } b < x \le 1 - b, \\ 1 - x; & \text{if } 1 - b < x \le 1. \end{cases}$$

The mass distribution of copula E_b, the graphs of functions δ_{E_b} and ω_{E_b}, and 3D plot of copula E_b can be found in Fig. 3. It follows that

$$\beta(E_b) = 4b - 1, \phi(E_b) = -6b^2 + 6b - \tfrac{1}{2} \quad \text{and} \quad \gamma(E_b) = -8b^2 + 8b - 1,$$

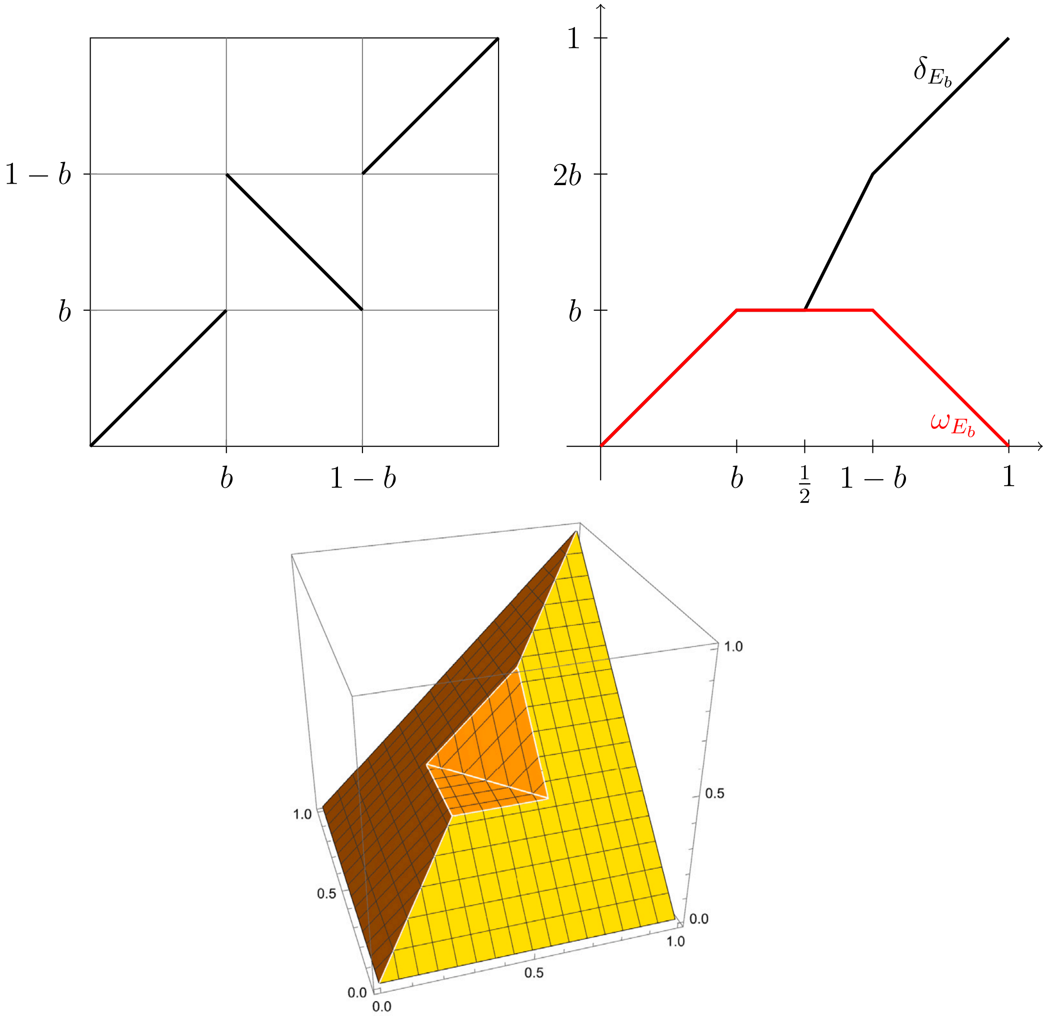

Fig. 3. The mass distribution of copula E_b (top left), the graphs of functions δ_{E_b} and ω_{E_b} (top right), and a 3D plot of copula E_b (bottom) from Example 3.5.

so that

$$\phi(E_b) = 1 - \tfrac{3}{8}(1 - \beta(E_b))^2 \quad \text{and} \quad \gamma(E_b) = \tfrac{4}{3}\phi(E_b) - \tfrac{1}{3},$$

and the upper bound from inequality (4) and the lower bound from inequality (6) are attained.

Example 3.6. Let $b \in [0, \frac{1}{2}]$ and let F_b be a shuffle of M

$$F_b = M(4, (b, \tfrac{1}{2}, 1 - b), (1, 3, 2, 4), (1, 1, 1, 1)).$$

Note that $F_0 = M(2, \frac{1}{2}, (2, 1), (1, 1))$ and $F_{\frac{1}{2}} = M$. We have $F_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{F_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le b, \\ b; & \text{if } b < x \le \frac{1}{2}, \\ 2x + b - 1; & \text{if } \frac{1}{2} < x \le 1 - b, \\ x; & \text{if } 1 - b < x \le 1, \end{cases}$$

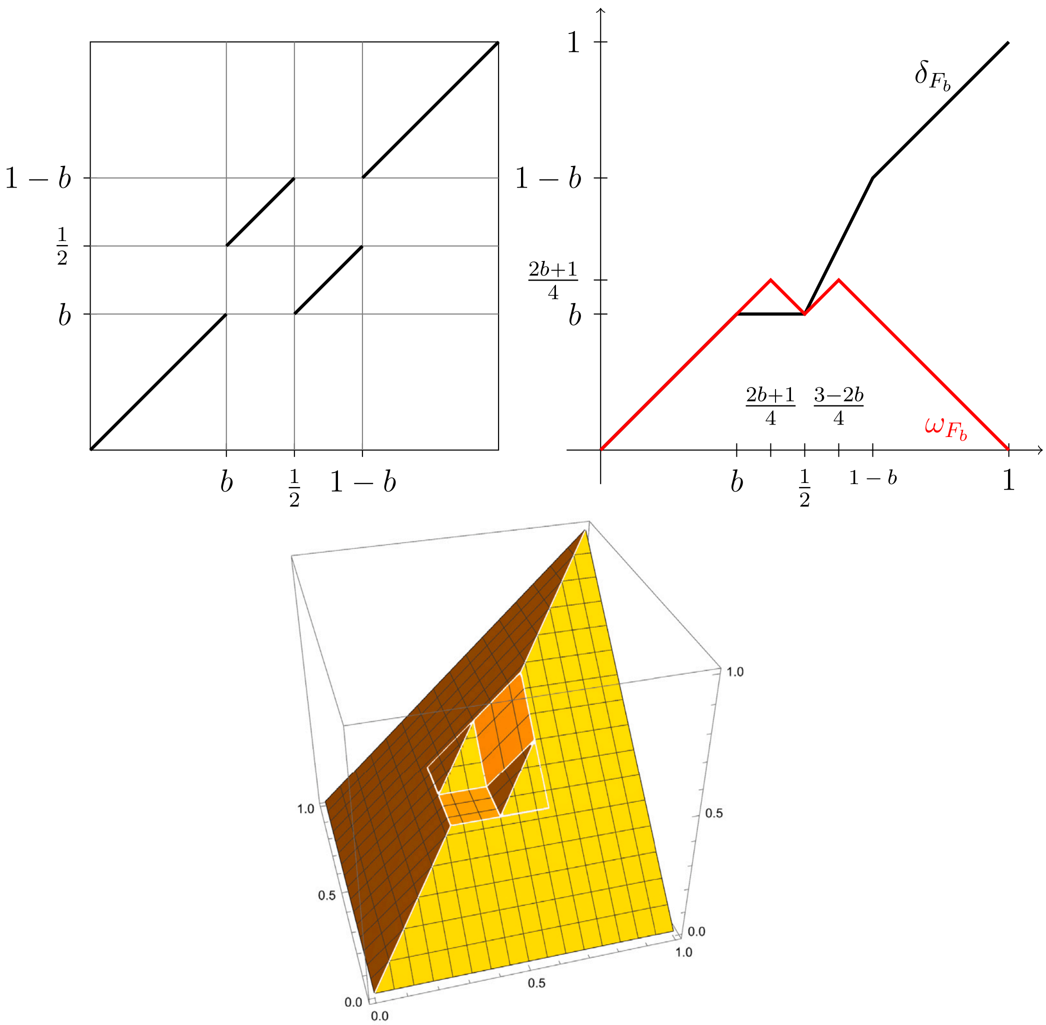

Fig. 4. The mass distribution of copula F_b (top left), the graphs of functions δ_{F_b} and ω_{F_b} (top right), and a 3D plot of copula F_b (bottom) from Example 3.6.

and

$$\omega_{F_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le \frac{1}{4} + \frac{b}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{4} + \frac{b}{2} < x \le \frac{1}{2}, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{3}{4} - \frac{b}{2}, \\ 1 - x; & \text{if } \frac{3}{4} - \frac{b}{2} < x \le 1. \end{cases}$$

Fig. 4 shows the mass distribution of copula F_b, the graphs of functions δ_{F_b} and ω_{F_b}, and 3D plot of copula F_b. It follows that

$$\beta(F_b) = 4b - 1, \phi(F_b) = -6b^2 + 6b - \tfrac{1}{2} \text{ and } \gamma(F_b) = -6b^2 + 6b - \tfrac{1}{2},$$

so that

$$\phi(F_b) = 1 - \tfrac{3}{8}(1 - \beta(F_b))^2 \quad \text{and} \quad \gamma(F_b) = \tfrac{2}{3}\phi(F_b) + \tfrac{1}{4}\beta(F_b) - \tfrac{1}{8}\beta(F_b)^2 + \tfrac{5}{24},$$

and both the upper bound from inequality (4) and Proposition 3.1 are attained.

Example 3.7. Let $b \in [0, \frac{1}{2}]$ and let G_b be a shuffle of M

$$G_b = M(6, (\tfrac{1}{4} - \tfrac{b}{2}, \tfrac{1}{4} + \tfrac{b}{2}, \tfrac{1}{2}, \tfrac{3}{4} - \tfrac{b}{2}, \tfrac{3}{4} + \tfrac{b}{2}), (4, 2, 6, 1, 5, 3), (1, -1, 1, 1, -1, 1)).$$

Notice that $G_0 = M(2, \frac{1}{2}, (2,1), (1,1))$ and $G_{\frac{1}{2}} = M(2, \frac{1}{2}, (1,2), (-1,-1))$. We have $G_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{G_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{4}, \\ 2x - \frac{1}{2}; & \text{if } \frac{1}{4} < x \le \frac{1}{4} + \frac{b}{2}, \\ b; & \text{if } \frac{1}{4} + \frac{b}{2} < x \le \frac{1}{2}, \\ 2x + b - 1; & \text{if } \frac{1}{2} < x \le \frac{3}{4} - \frac{b}{2}, \\ \frac{1}{2}; & \text{if } \frac{3}{4} - \frac{b}{2} < x \le \frac{3}{4}, \\ 2x - 1; & \text{if } \frac{3}{4} < x \le 1, \end{cases}$$

and

$$\omega_{G_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le \frac{1}{4} + \frac{b}{2}, \\ b - x + \frac{1}{2}; & \text{if } \frac{1}{4} + \frac{b}{2} < x \le \frac{1}{2}, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{3}{4} - \frac{b}{2}, \\ 1 - x; & \text{if } \frac{3}{4} - \frac{b}{2} < x \le 1. \end{cases}$$

The mass distribution of copula G_b, the graphs of functions δ_{G_b} and ω_{G_b}, and 3D plot of copula G_b are drawn in Fig. 5. It follows that

$$\beta(G_b) = 4b - 1, \phi(G_b) = -3b^2 + 3b - \tfrac{1}{2} \text{ and } \gamma(G_b) = -4b^2 + 4b - \tfrac{1}{2},$$

so that

$$\gamma(G_b) = \tfrac{4}{3}\phi(G_b) + \tfrac{1}{6} = \tfrac{2}{3}\phi(G_b) + \tfrac{1}{4}\beta(G_b) - \tfrac{1}{8}\beta(G_b)^2 + \tfrac{5}{24},$$

and both the upper bound from inequality (6) and Proposition 3.1 are attained.

Example 3.8. Let $b \in [0, \frac{1}{2}]$ and let H_b be a shuffle of M

$$H_b = M(6, (\tfrac{b}{2}, \tfrac{1}{2} - \tfrac{b}{2}, \tfrac{1}{2}, \tfrac{1}{2} + \tfrac{b}{2}, 1 - \tfrac{b}{2}), (3,5,1,6,2,4), (1,1,1,1,1,1)).$$

Observe that $H_0 = M(2, \frac{1}{2}, (2,1), (1,1))$ and $H_{\frac{1}{2}} = M(4, (\frac{1}{4}, \frac{1}{2}, \frac{3}{4}), (2,1,4,3), (1,1,1,1))$. We have $H_b(\frac{1}{2}, \frac{1}{2}) = b$,

$$\delta_{H_b}(x) = \begin{cases} 0; & \text{if } 0 \le x \le \frac{1}{2} - \frac{b}{2}, \\ 2x + b - 1; & \text{if } \frac{1}{2} - \frac{b}{2} < x \le \frac{1}{2}, \\ b; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + \frac{b}{2}, \\ 2x - 1; & \text{if } \frac{1}{2} + \frac{b}{2} < x \le 1, \end{cases}$$

and

$$\omega_{H_b}(x) = \begin{cases} x; & \text{if } 0 \le x \le \frac{1}{4}, \\ -x + \frac{1}{2}; & \text{if } \frac{1}{4} < x \le \frac{1}{2} - \frac{b}{2}, \\ x + b - \frac{1}{2}; & \text{if } \frac{1}{2} - \frac{b}{2} < x \le \frac{1}{2}, \\ -x + b + \frac{1}{2}; & \text{if } \frac{1}{2} < x \le \frac{1}{2} + \frac{b}{2}, \\ x - \frac{1}{2}; & \text{if } \frac{1}{2} + \frac{b}{2} < x \le \frac{3}{4}, \\ 1 - x; & \text{if } \frac{3}{4} < x \le 1. \end{cases}$$

The mass distribution of copula H_b, the graphs of functions δ_{H_b} and ω_{H_b}, and 3D plot of copula H_b are shown in Fig. 6. It follows that

$$\beta(H_b) = 4b - 1, \phi(H_b) = 3b^2 - \tfrac{1}{2} \text{ and } \gamma(H_b) = 4b^2 - \tfrac{1}{2},$$

so that

$$\phi(H_b) = \tfrac{3}{16}(1 + \beta(H_b))^2 - \tfrac{1}{2} \quad \text{and} \quad \gamma(H_b) = \tfrac{4}{3}\phi(H_b) + \tfrac{1}{6},$$

and the lower bound from inequality (4) and the upper bound from inequality (6) are attained.

We are now able to prove our main theorem.

Theorem 3.9. *Let*

$$\Omega_{\beta,\phi,\gamma} = \{(\beta(C), \phi(C), \gamma(C)) \in [-1,1] \times [-\tfrac{1}{2}, 1] \times [-1,1] : C \in \mathcal{C}\}$$

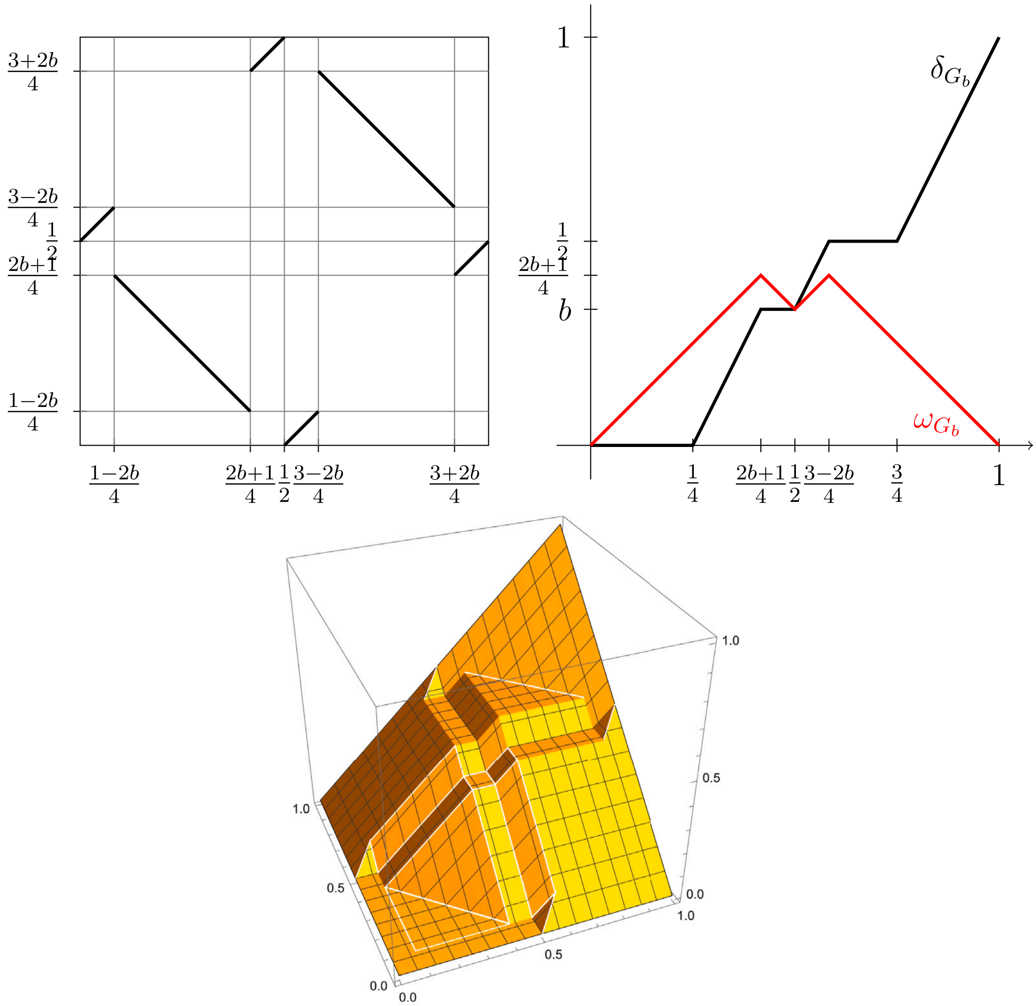

Fig. 5. The mass distribution of copula G_b (top left), the graphs of functions δ_{G_b} and ω_{G_b} (top right), and a 3D plot of copula G_b (bottom) from Example 3.7.

be the exact region determined by Blomqvist's beta, Spearman's footrule and Gini's gamma. Then $\Omega_{\beta,\phi,\gamma}$ *equals the region*

$$\Omega = \{ (\beta,\phi,\gamma) \in [-1,1]\times[-\tfrac{1}{2},1]\times[-1,1] : \tfrac{3}{16}(1+\beta)^2 - \tfrac{1}{2} \le \phi \le 1 - \tfrac{3}{8}(1-\beta)^2,$$
$$\max\{\tfrac{4}{3}\phi - \tfrac{1}{3}, \tfrac{2}{3}\phi + \tfrac{1}{2}\beta + \tfrac{1}{4}\beta^2 - \tfrac{5}{12}\} \le \gamma \le \min\{\tfrac{4}{3}\phi + \tfrac{1}{6}, \tfrac{2}{3}\phi + \tfrac{1}{4}\beta - \tfrac{1}{8}\beta^2 + \tfrac{5}{24}\} \} .$$

Proof. The inclusion $\Omega_{\beta,\phi,\gamma} \subseteq \Omega$ follows from Propositions 2.1, 2.2, 3.1, and 3.2. The region Ω is bounded by six surfaces

$$S_1: \quad \phi = \tfrac{3}{16}(1+\beta)^2 - \tfrac{1}{2},$$
$$S_2: \quad \gamma = \tfrac{2}{3}\phi + \tfrac{1}{2}\beta + \tfrac{1}{4}\beta^2 - \tfrac{5}{12},$$
$$S_3: \quad \gamma = \tfrac{4}{3}\phi - \tfrac{1}{3},$$
$$S_4: \quad \phi = 1 - \tfrac{3}{8}(1-\beta)^2,$$
$$S_5: \quad \gamma = \tfrac{2}{3}\phi + \tfrac{1}{4}\beta - \tfrac{1}{8}\beta^2 + \tfrac{5}{24},$$
$$S_6: \quad \gamma = \tfrac{4}{3}\phi + \tfrac{1}{6}.$$

For any fixed β the intersection of any of these surfaces with the plane $\beta = \beta_0$ is a line. Let $b = \frac{\beta_0+1}{4}$ and denote by Σ the plane $\beta = 4b-1$. We have the following intersections:

$$S_1 \cap S_2 \cap \Sigma: \quad C(4b-1, 3b^2 - \tfrac{1}{2}, 6b^2 - 1),$$
$$S_2 \cap S_3 \cap \Sigma: \quad D(4b-1, 6b^2 - \tfrac{1}{2}, 8b^2 - 1),$$

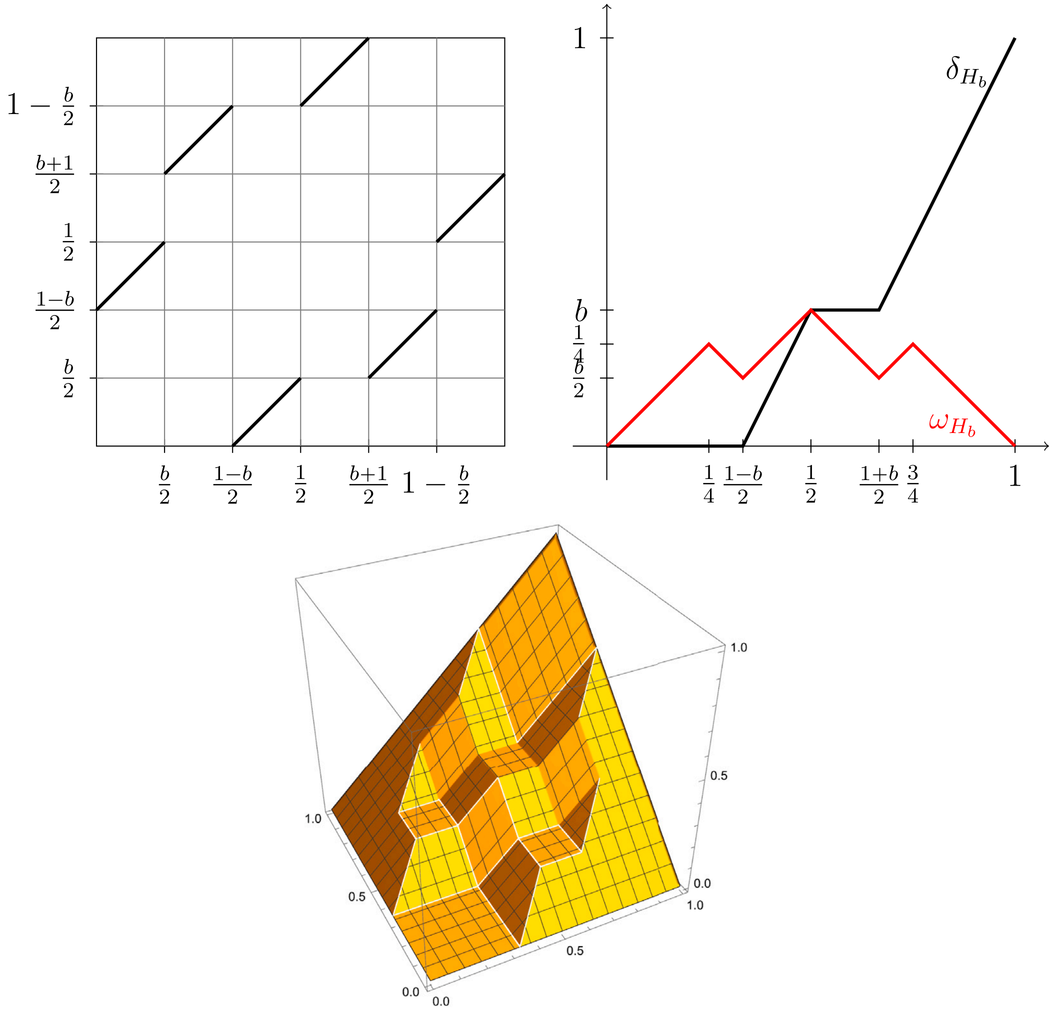

Fig. 6. The mass distribution of copula H_b (top left), the graphs of functions δ_{H_b} and ω_{H_b} (top right), and a 3D plot of copula H_b (bottom) from Example 3.8.

$$
\begin{aligned}
S_3 \cap S_4 \cap \Sigma: &\quad E(4b-1, -6b^2+6b-\tfrac{1}{2}, -8b^2+8b-1),\\
S_4 \cap S_5 \cap \Sigma: &\quad F(4b-1, -6b^2+6b-\tfrac{1}{2}, -6b^2+6b-\tfrac{1}{2}),\\
S_5 \cap S_6 \cap \Sigma: &\quad G(4b-1, -3b^2+3b-\tfrac{1}{2}, -4b^2+4b-\tfrac{1}{2}),\\
S_6 \cap S_1 \cap \Sigma: &\quad H(4b-1, 3b^2-\tfrac{1}{2}, 4b^2-\tfrac{1}{2}).
\end{aligned}
$$

So the intersection of the region Ω with the plane Σ is the hexagon $CDEFGH$. This hexagon has three pairs of parallel sides: the projections of the sides EF and HC to the ϕ-γ plane are vertical, the projections of the sides CD and FG have slope $\frac{2}{3}$, and the projections of the sides DE and GH have slope $\frac{4}{3}$. Each of the vertices C, D, E, F, G, and H is attained by copula C_b, D_b, E_b, F_b, G_b, and H_b from Examples 3.3–3.8, respectively. In the case $b = 0$ the hexagon is reduced to the line segment from $C(-1, -\frac{1}{2}, -1)$ to $F(-1, -\frac{1}{2}, -\frac{1}{2})$. In the case $b = 0$ the hexagon is reduced to the line segment from $C(1, \frac{1}{4}, \frac{1}{2})$ to $D(1, 1, 1)$. Since Blomqvist's beta, Spearman's footrule and Gini's gamma are all linear functions of copulas, a convex combination of two copulas C_1 and C_2, which is a copula, is mapped by β, ϕ, and γ to the convex combination of their images. This means that any point (β_0, ϕ, γ) in the hexagon $CDEFGH$ is attained by some copula with $\beta(C) = \beta_0$, and thus any point $(\beta, \phi, \gamma) \in \Omega$ is attained by some copula $C \in \mathcal{C}$. □

In Fig. 7 the region Ω is shown. The curves $S_1 \cap S_2$, $S_2 \cap S_3$, $S_3 \cap S_4$, $S_4 \cap S_5$, $S_5 \cap S_6$, and $S_6 \cap S_1$, are drawn blue, black, orange, red, magenta, and green, respectively. In Fig. 8 the projection of the hexagon $CDEFGH$ to the ϕ-γ plane is drawn in red for the case $\beta_0 = \frac{1}{3}$. The whole region $\Omega_{\phi,\gamma} = \{(\phi(C), \gamma(C)) \in [-\frac{1}{2}, 1] \times [-1, 1] : C \in \mathcal{C}\}$ is drawn in black. In the case $\beta_0 = \frac{1}{3}$ we have $C(-\frac{1}{6}, -\frac{1}{3})$, $D(\frac{1}{6}, -\frac{1}{9})$, $E(\frac{5}{6}, \frac{7}{9})$, $F(\frac{5}{6}, \frac{5}{6})$, $G(\frac{1}{6}, \frac{7}{18})$, $H(-\frac{1}{6}, -\frac{1}{18})$.

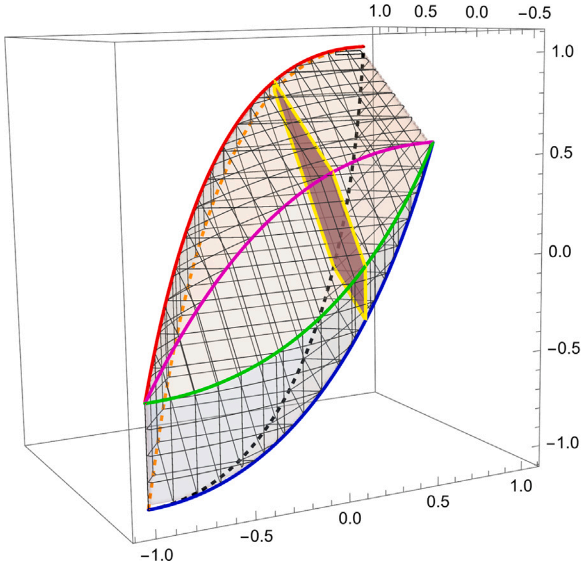

Fig. 7. The region Ω from Theorem 3.9. The hexagon $CDEFGH$ for the case $\beta_0 = \frac{1}{3}$ has edges depicted in yellow.

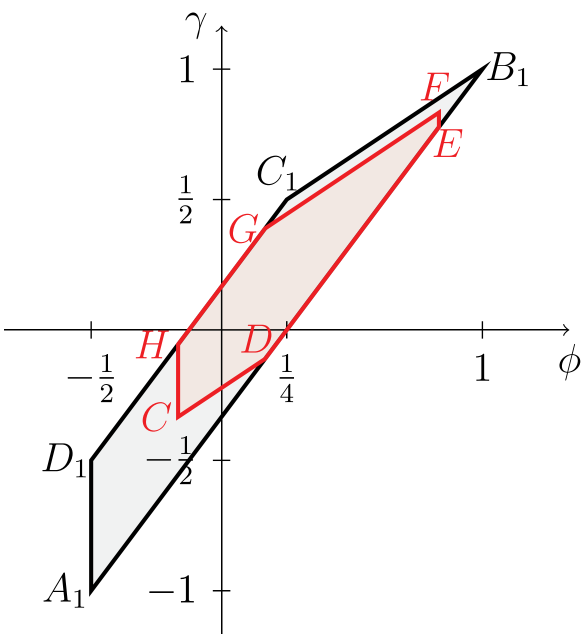

Fig. 8. The hexagon $CDEFGH$ for the case $\beta_0 = \frac{1}{3}$ (red) and the region $\Omega_{\phi,\gamma}$ (black).

It is obvious that the projection of the region Ω to the β-ϕ plane is the region determined by Blomqvist's beta and Spearman's footrule

$$\begin{aligned}\Omega_{\beta,\phi} &= \{(\beta(C),\phi(C)) \in [-1,1]\times[-\tfrac{1}{2},1] : C \in \mathcal{C}\}\\ &= \{(\beta,\phi) \in [-1,1]\times[-\tfrac{1}{2},1] : \tfrac{3}{16}(1+\beta)^2 - \tfrac{1}{2} \le \phi \le 1 - \tfrac{3}{8}(1-\beta)^2\}.\end{aligned}$$

It is also clear that the projection of the region Ω to the ϕ-γ plane is the region $\Omega_{\phi,\gamma}$, i.e. the quadrilateral $A_1B_1C_1D_1$ in Fig. 8. A short calculation shows that the projection of the region Ω to the β-γ plane is the region determined by Blomqvist's beta and Gini's gamma

$$\Omega_{\beta,\gamma} = \{(\beta(C),\gamma(C)) \in [-1,1]^2 : C \in \mathcal{C}\} = \{(\beta,\gamma) \in [-1,1]^2 : \tfrac{3}{8}(1+\beta)^2 - 1 \le \gamma \le 1 - \tfrac{3}{8}(1-\beta)^2\}.$$

Fig. 8 shows that for any copula C with given value of $\phi(C)$, the spread of $\gamma(C)$, i.e., the difference between the maximal possible value and the minimal possible value that $\gamma(C)$ can attain, is at most $\frac{1}{2}$. If $\phi(C)$ is close to 1, this spread is even smaller. On average the spread of $\gamma(C)$ given $\phi(C)$ is

$$\frac{\text{area}(\Omega_{\phi,\gamma})}{\text{length}([-\frac{1}{2},1])} = \frac{3}{8} = 0.375.$$

If we know both, $\beta(C)$ and $\phi(C)$, the spread of $\gamma(C)$ can still be equal to $\frac{1}{2}$, for example in the case $\beta(C) = -\frac{1}{2}, \phi(C) = -\frac{1}{4}$, when $\gamma(C) \in [-\frac{2}{3}, -\frac{1}{6}]$. But the average spread of $\gamma(C)$ given $\beta(C)$ and $\phi(C)$ is

$$\frac{\text{volume}(\Omega_{\beta,\phi,\gamma})}{\text{area}(\Omega_{\beta,\phi})} = \frac{19}{60} \approx 0.3167.$$

So, if we know $\beta(C)$ and $\phi(C)$ on average the spread of $\gamma(C)$ is reduced by 15.6% with respect to the average spread of $\gamma(C)$ given $\phi(C)$ only.

4. Concluding remarks

In this paper we characterize the exact region representing the ternary relation between Blomqvist's beta, Spearman's footrule and Gini's gamma. Given the values of Blomqvist's beta and Spearman's footrule for some copula, we give the lower and the upper bound that Gini's gamma can take. We also provide copulas where these bounds are attained.

Future work might include studying other combinations of triplets of classical (weak) concordance measures, and relations between other (weak) concordance measures like convex (weak) concordance measures introduced in [23,24].

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Data availability

No data was used for the research described in the article.

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