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2-rainbow independent domination in complementary prisms

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Abstract
A function f that assigns values from the set $\{0, 1, 2\}$ to each vertex of a graph G is called a 2-rainbow independent dominating function, if the vertices assigned the value 1 form an independent set, the vertices assigned the value 2 form another independent set, and every vertex to which 0 is assigned has at least one neighbor in each of the mentioned independent sets. The weight of this function is the total number of vertices assigned nonzero values. The 2-rainbow independent domination number of G, $\gamma_{\mathrm{ri2}}(G)$, is the minimum weight of such a function. Motivated by a real-life application, we study the 2-rainbow independent domination number of the complementary prism $G\overline{G}$ of a graph G, which is constructed by taking G and its complement $\overline{G}$, and then adding edges between corresponding vertices. We provide tight bounds for $\gamma_{\mathrm{ri2}}(G\overline{G})$, and characterize graphs for which the lower bound, i.e. $\max\{\gamma_{\mathrm{ri2}}(G), \gamma_{\mathrm{ri2}}(\overline{G})\} + 1$, is attained. The obtained results can, in practice, enable the prediction of the cost estimate for a given communication or surveillance network.

Keywords Graph theory · Domination · 2-rainbow independent domination · Complementary prism

Mathematics Subject Classification 05C69

1 Introduction and preliminaries

The area of domination in graph theory is one of the fastest-growing and developing fields, which is not surprising given its versatile applications in real-life situations. Various types of domination have emerged from different needs. One such type is k-rainbow independent

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domination, introduced in Šumenjak et al. (2018), where it was indicated that this concept can be seen as a model for a problem in combinatorial optimization.

In this paper, we study 2-rainbow independent domination in the context of complementary prisms and demonstrate its applicability in the field of communication or surveillance network design. Complementary prisms model secure and efficient surveillance networks, especially in scenarios requiring redundancy and fault tolerance. By constructing a complementary prism of a graph, network designers can ensure that every node has alternative paths for data transmission, thereby enhancing the network's robustness against failures and attacks. More specifically, setting up a surveillance network in an area such as a house, school, hospital, or police station, the first step is determining the locations of surveillance devices like cameras and motion sensors to ensure full coverage. To avoid communication or surveillance blind spots, it is crucial to ensure network connectivity and redundancy. Connectivity means every device can communicate with all others, while redundancy provides alternative paths if direct communication fails.

In this context, 2-rainbow independent domination assigns values of 0, 1, or 2 to each device. Devices assigned values of 1 or 2 form independent sets, meaning no two devices within the same set are directly connected, thereby avoiding interference between devices operating on the same frequency. Devices with a value of 0 do not transmit signals directly but are crucial for network stability. They can communicate through all the frequencies used by the devices directly connected to them. These 0-valued devices function similarly to cost-effective repeaters, facilitating efficient communication at lower costs. To ensure effective communication, each 0-valued device must connect to devices with both values 1 and 2, preventing interference and enabling connectivity through intermediary devices. By applying this method to complementary prisms over a graph representing the original network, we can design networks with optional data transmission paths. This enhances robustness and reliability, optimizing device placement and operation, reducing energy consumption, and improving overall network stability.

In this sense the main objective of our paper is to explore the bounds of the 2-rainbow independent domination number of a graph and more specifically, to characterize extremal graphs attaining the lower bound, as it is our goal to have as many devices with value 0 assigned as possible, to reduce the energy consumption of a network.

To provide a comprehensive understanding of our model, we begin with an overview of the relevant graph theoretical background. In presenting our results, we adhere to the notation and terminology of graph theory as outlined in Hammack et al. (2011). Specifically, let G be a finite, simple graph with a vertex set $V(G)$ and an edge set $E(G)$. We use the following standard notation for graphs of order n: N_n represents an edgeless graph, K_n is a complete graph, C_n a cycle, P_n a path, and S_n denotes a star. The graph obtained from S_n by adding a single edge is denoted by S_n^+. A *broom* is a graph that consists of a path and a star attached at the one end of the path. For the brooms that are of our interest, the notation B_n is appropriate, since they all consist of a path P_3 to which $n-3$ leaves are attached, where $n \geq 4$.

For any vertex v in $V(G)$, the *open neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v. The *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. A vertex v is a *universal* vertex of G if $N[v] = V(G)$. A vertex v with $|N(v)| = 1$ is called a *leaf*.

A *dominating set* of a graph G is a subset $D \subseteq V(G)$ such that every vertex not in D is adjacent to at least one vertex in D. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

The concept of k-rainbow independent number was introduced in Šumenjak et al. (2018). For a function $f\colon V(G) \to \{0, 1, 2, \ldots, k\}$, let V_i be the set of vertices to which the value i is assigned by f, i.e. $V_i = \{x \in V(G) \mid f(x) = i\}$. A function $f\colon V(G) \to \{0, 1, \ldots, k\}$ is called a *k-rainbow independent dominating function* (kRiDF) of G if the following conditions are met: V_i is independent for all $i \in [k]$, and for every $x \in V_0$, there is at least one neighbor in each V_i. The definition implies that a k-rainbow independent dominating function f can be represented by the ordered k-tuple $(V_0, V_1, \ldots, V_k)$ determined by f, which is convenient to work with. Note that if $V_0 \neq \emptyset$, then the corresponding k-tuple $(V_0, V_1, \ldots, V_k)$ forms a partition of $V(G)$. Vertices from V_0 are called *empty*, and all vertices in $V(G) \setminus V_0$ are *nonempty*. If $f(v) = i$, we say that v is *labelled* by i, and if every vertex in the subset $A \subseteq V(G)$ is labelled by i, we simply write $f(A) = i$. The weight of a kRiDF f is defined as $w(f) = \sum_{i=1}^{k} |V_i|$, or equivalently $w(f) = n - |V_0|$, where n is the order of the graph. The *k-rainbow independent domination number* of a graph, denoted by $\gamma_{\mathrm{ri}k}(G)$, is the minimum weight of a kRiDF of G.

In what follows, we set the parameter k to 2, and since we will explore only 2-rainbow independent domination (not ordinary domination), we will say that a vertex is *dominated* under a 2RiDF if it either belongs to $V_1 \cup V_2$ or has at least one neighbor in V_1 and at least one neighbor in V_2. Similarly, a vertex in V_0 is *undominated* if, in its neighborhood, the label 1 or 2 (or both) is missing. We will use the term *$\gamma_{\mathrm{ri2}}(G)$-function* (or simply γ_{ri2}-function when no confusion is likely) for a 2RiDF of G with weight $\gamma_{\mathrm{ri2}}(G)$.

In the seminal paper (Šumenjak et al. 2018) on the topic several basic properties of k-rainbow independent domination were derived. For us the following will be of use.

Lemma 1 [Šumenjak et al. (2018)] *For any graph G of order n, $\gamma_{\mathrm{ri2}}(G) = n$ if and only if every connected component of G is isomorphic either to K_1 or K_2. In addition, if $\gamma_{\mathrm{ri2}}(G) = n$, then $\gamma_{\mathrm{ri2}}(\overline{G}) = 2$.*

The main contribution in Šumenjak et al. (2018) is the following Nordhaus-Gaddum type result.

Theorem 2 [Šumenjak et al. (2018)] *If G is a graph of order n where $n \geq 3$, then $5 \leq \gamma_{\mathrm{ri2}}(G) + \gamma_{\mathrm{ri2}}(\overline{G}) \leq n + 3$, and the bounds are sharp.*

The authors posed a problem whether C_5 is the only graph for which the upper bound is attained in the Nordhaus-Gaddum type inequality. Zhu (2021) proved this to be true by using the following observation.

Theorem 3 [Zhu (2021)] *Let G be a connected graph of order $n \geq 3$. Then, $\gamma_{\mathrm{ri2}}(G) = n-1$ if and only if G is isomorphic to one among S_n, S_n^+, B_n (for $n \geq 4$) and C_5.*

Suppose G is a graph with connected components $G_1, G_2, \ldots, G_k$. Since $\gamma_{\mathrm{ri2}}(G) = \sum_{i=1}^{k} \gamma_{\mathrm{ri2}}(G_i)$, the following immediately follows.

Theorem 4 [Zhu (2021)] *Let G be a graph of order $n \geq 3$. Then, $\gamma_{\mathrm{ri2}}(G) = n - 1$ if and only if G has one connected component G_1 isomorphic to a graph from $\{S_{n_1}, S_{n_1}^+, B_{n_1}, C_5\}$, where $n_1 = |V(G_1)|$, and all the other components of G are isomorphic to K_1 or K_2.*

For the sake of completeness, we also provide a brief overview of the remaining literature on the topic. In Brezovnik et al. (2019) Kraner Šumenjak and Brezovnik showed that the problem of determining whether a graph has a k-rainbow independent dominating function of a given weight is NP-complete for bipartite graphs and that there exists a linear-time algorithm to compute $\gamma_{\mathrm{ri2}}(G)$ of trees. They have also considered this type of domination in the lexicographic products. Gabrovšek et al. (2023) studied several infinite families of graphs and provided exact values of some Cartesian products of paths and cycles as well as of generalized Petersen graphs. If the 2-rainbow independent domination number of G remains unchanged under removal of any vertex, G is 2-*rainbow independent domination stable*. Shi et al. (2019) characterized 2-rainbow independent domination stable trees and investigated the effect of edge removal on 2-rainbow independent domination number in trees.

For an interested reader, let us remark that in the literature the concept of *independent* k-*rainbow domination* appears which should not be confused with k-rainbow independent domination, see Šumenjak et al. (2018) for further clarification.

Before introducing the structure of our interest, complementary prisms, introduced by Haynes et al. in Haynes et al. (2007), recall that the *complement* of a graph G, denoted by $\overline{G}$, is a graph with the same vertex set as G, but two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. From the definition, it is clear that $\overline{\overline{G}} = G$. The *complementary prism* of a graph G, denoted by $G\overline{G}$, is a graph constructed by taking the disjoint union of G and its complement $\overline{G}$, and then adding edges between corresponding vertices in G and $\overline{G}$. Note that the later form a perfect matching, thus we call them *perfect edges*. If for a perfect edge $v\overline{v}$ and a 2-rainbow independent dominating function f we have $f(v) = f(\overline{v}) = 0$, then $v\overline{v}$ is called an *empty perfect edge*.

To enhance the clarity of our presentation, we will call the subgraph of $G\overline{G}$, induced by $V(G)$, the G-*layer*, and the subgraph of $G\overline{G}$, induced by $V(\overline{G})$, the $\overline{G}$-*layer*. Also, we will use $\overline{v}$ to denote the vertex in the $\overline{G}$-layer that corresponds to $v \in V(G)$. Similarly, if A is a subset of $V(G)$ in a complementary prism, then $\overline{A}$ denotes the set of vertices in the $\overline{G}$-layer that correspond to vertices in A. When no confusion is likely, we will denote vertices in the G-layer with the same notation as in the base graph G.

We will also use the notion of a *clique* in a graph G, which is a subset of vertices such that every two distinct vertices in the clique are adjacent. It is clear that in $G\overline{G}$, each of the sets $\overline{V_1}$ and $\overline{V_2}$ induces a clique provided that $f = (V_0, V_1, V_2)$ is a 2RiDF on $G\overline{G}$.

In the next section, we first provide a tight upper bound on the 2-rainbow independent domination number of a complementary prism and present exact formulas for certain classes of graphs needed in the sequel. In Sect. 3, we introduce the main nontrivial tool needed for exploring additional graph classes to establish and achieve the lower bound. By combining all the results, we ultimately prove the following main theorem.

Theorem 5 *Let G be a graph of order $n \in \mathbb{N}$. Then $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq \max\{\gamma_{\mathrm{ri2}}(G), \gamma_{\mathrm{ri2}}(\overline{G})\} + 1$ with equality if and only if G or $\overline{G}$ is a disjoint union of one connected component G_1 isomorphic to a graph from $\{K_1, K_2, B_{n_1}\}$, where $n_1 \geq 4$, and all other (possible) connected components isomorphic to K_1.*

2 Upper bound and special cases

We start with a straightforward observation regarding the upper bound of the 2-rainbow independent domination number of a complementary prism. As indicated by subsequent results (e.g., Lemma 9), this bound is tight.

Proposition 6 *Let G be a graph of order $n \in \mathbb{N}$. Then $\gamma_{\mathrm{ri2}}(G\overline{G}) \leq n + 2$.*

Proof Let $f : V(G\overline{G}) \to \{0, 1, 2\}$ be a γ_{ri2}-function, and $A = \{v \in V(G) | \, f(v) = 1\}$, $B = \{v \in V(G) | \, f(v) = 2\}$ and $C = \{v \in V(G) | \, f(v) = 0\}$. Since vertices in A and B, respectively, are independent, vertices in $\overline{A}$ and $\overline{B}$, respectively, induce a clique. Therefore in $\overline{A}$ at most one vertex is labeled by 2 and none of them is labeled by 1 under

f. Similarly, in $\overline{B}$ at most one vertex is labeled by 1 and none of them is labeled by 2. Also note that vertices in $\overline{C}$ contribute at most $|\overline{C}|$ to the weight of f. This readily implies that $\gamma_{\rm ri2}(G\overline{G}) \le |A| + |B| + 2 + |\overline{C}| = n + 2$. □

Next, we present several properties of complementary prisms over graphs whose 2-rainbow independent domination number of the base graph equals the order of the graph. These observations will be instrumental in the proofs of the main result.

Lemma 7 *Let $n \in \mathbb{N}$. If $G \in \{K_n, N_n\}$, then $\gamma_{\rm ri2}(G\overline{G}) = n + 1$.*

Proof Since $\overline{N_n} = K_n$, we may assume G to be a complete graph. Then vertices in the $\overline{G}$-layer of $G\overline{G}$ are all leaves and thus nonempty. Hence $\gamma_{\rm ri2}(G\overline{G}) \ge n$, where the equality immediately leads to a contradiction, since then all vertices in the G-layer must be empty and so they are not dominated. Thus $\gamma_{\rm ri2}(G\overline{G}) \ge n+1$. On the other hand, $f : V(G\overline{G}) \to \{0, 1, 2\}$ defined by $f(\overline{v}) = 2$ for every $\overline{v} \in V(\overline{G})$, $f(x) = 1$ for a vertex x in the G-layer, and $f(v) = 0$ for every $v \in V(G) \setminus \{x\}$, is a 2RiDF of $G\overline{G}$, thus the equality holds. □

Proposition 8 *Let G be a graph of order $n \in \mathbb{N}$. Then*

(i) $\gamma_{\rm ri2}(G\overline{G}) = 2$ if and only if $G = K_1$,
(ii) $\gamma_{\rm ri2}(G\overline{G}) = 3$ if and only if $G \in \{K_2, N_2\}$.

Proof Let G be a graph on n vertices, $\gamma_{\rm ri2}(G\overline{G}) \in \{2, 3\}$ and $f : V(G\overline{G}) \to \{0, 1, 2\}$ a $\gamma_{\rm ri2}$-function. If there is an empty perfect edge $x\overline{x}$, then x is dominated within the G-layer, and $\overline{x}$ is dominated within the $\overline{G}$-layer. In this case, the weight of f is at least 4, a contradiction. Thus, every perfect edge has a nonempty endvertex, and consequently $\gamma_{\rm ri2}(G\overline{G}) \ge n$.

If $\gamma_{\rm ri2}(G\overline{G}) = 2$, then $n \in \{1, 2\}$. However, $n = 2$ immediately yields a contradiction with Lemma 7. For $\gamma_{\rm ri2}(G\overline{G}) = 3$, we infer that $n \in \{1, 2, 3\}$. By Lemma 7, $n \ne 1$ and $G \notin \{K_3, N_3\}$. Thus, either $G \in \{K_2, N_2\}$ or G is a disjoint union of K_1 and K_2 or $G = P_3$. Note that the last two cases are not possible since $G\overline{G}$ is a 5-cycle with one vertex attached to an arbitrary vertex of the cycle; one can quickly verify that the 2-rainbow independent number of this graph is 4, a contradiction. Hence, $G \in \{K_2, N_2\}$.

By Lemma 7, the opposite implications immediately follow. □

We remark that the result (i) in the proposition above can be derived from Proposition 2.2 from Šumenjak et al. (2018), however, our reasoning is more direct.

By Lemma 1 we already know that if every connected component of a graph is isomorphic either to K_1 or K_2, then its 2-rainbow independent domination number equals the order of the graph. In the next result, we see that for the complementary prism of such a graph, this number increases by 1 or 2, depending on the number of connected components isomorphic to K_2.

Lemma 9 *Let G be a graph of order $n \ge 3$, and let m and k be integers such that $m \ge 1$, $k \ge 0$ and $n = 2m + k$. If G is a disjoint union of m copies of K_2 and k copies of K_1, then*

$$\gamma_{\rm ri2}(G\overline{G}) = \begin{cases} n+1 \ ; \ m = 1 \\ n+2 \ ; \ m \ge 2 \end{cases} .$$

Proof Let S be a set of isolated vertices in G (if $k > 0$), and $T = \{x_1, \dots, x_m, y_1, \dots, y_m\}$ the set of all other vertices in G, where $x_i y_i \in E(G)$ for every $i \in \{1, 2, \dots, m\}$. By Proposition 6 we already know that $\gamma_{\rm ri2}(G\overline{G}) \le n+2$. However, this bound can be improved if $m = 1$. Namely, by defining $g : V(G\overline{G}) \to \{0, 1, 2\}$ with $g(S) = 1$, $g(\overline{S}) = 0$, $g(x_1) = 0$, $g(\overline{x_1}) = 2$, $g(y_1) = 1$ and $g(\overline{y_1}) = 2$, we clearly have a 2RiDF of weight $n + 1$.

Now, let $f : V(G\overline{G}) \to \{0, 1, 2\}$ be a γ_{ri2}-function. Observe that since all vertices in the G-layer are of degree at most 2, there is no empty perfect edge in $G\overline{G}$. Hence $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq n$.

Suppose $\gamma_{\mathrm{ri2}}(G\overline{G}) = n$. Then exactly one endvertex of every perfect edge is nonempty. As every vertex in S is a leaf in $G\overline{G}$ and is thus nonempty, we have $f(\overline{S}) = 0$. To ensure all vertices in $\overline{S}$ are dominated, there must be $\overline{x} \in V(\overline{G})\setminus\overline{S}$ such that $f(\overline{x}) \neq 0$, say $f(\overline{x}) = 1$. Thus $f(x) = 0$ and y, the neighbor of x in the G-layer has label 2. Consequently, $f(\overline{y}) = 0$. Now, if $m = 1$, this is a contradiction since $\overline{y}$ is not dominated (it does not have a vertex with label 1 in its neighborhood). However, if $m \geq 2$, then there is $\overline{z} \in V(\overline{G})$, a neighbor of $\overline{y}$, such that $f(\overline{z}) = 1$, a contradiction as well, since $\overline{x}$ and $\overline{z}$ are adjacent. So, $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq n+1$. In fact, when $m = 1$, we have shown that $\gamma_{\mathrm{ri2}}(G\overline{G}) = n + 1$.

Furthermore, if $m \geq 2$, then $\gamma_{\mathrm{ri2}}(G\overline{G}) \in \{n + 1, n + 2\}$. Suppose (to the contrary) that $\gamma_{\mathrm{ri2}}(G\overline{G}) = n+1$. Since no empty perfect edge exists, there is exactly one perfect edge with both endvertices being nonempty, and all other perfect edges have precisely one nonempty endvertex. Now we consider two cases.

In the first case we assume that the perfect edge with both endvertices being nonempty has one endvertex, say v, in S. Without loss of generality let $f(v) = 1$ and $f(\overline{v}) = 2$. Then $f(\overline{S} \setminus \{\overline{v}\}) = 0$. Note also that in $\overline{T}$ there is no vertex labelled by 2 under f. Moreover, $\overline{T}$ contains at most two vertices with label 1, otherwise the independence condition is violated. The case when $f(\overline{T}) = 0$ is immediately excluded since all vertices in T are then nonempty, i.e. for every pair of vertices x_i, y_i, one of them has label 1 and the other is labelled by 2, say $f(x_1) = 1$ and $f(y_1) = 2$, but this readily means that $\overline{y_1}$ is not dominated (as there is no vertex with label 1 in the $\overline{G}$-layer). Next, if $\overline{T}$ contains one vertex with label 1, say $\overline{x_1}$, then $f(x_1) = 0$ and therefore $f(y_1) = 2$ and $f(\overline{y_1}) = 0$. Once again, $\overline{y_1}$ is undominated since it is not adjacent to $\overline{x_1}$. In the last subcase we may without loss of generality assume that $f(\overline{x_1}) = f(\overline{y_1}) = 1$. But then x_1 and y_1 are both empty and thus undominated, a contradiction again.

Finally, assume that the perfect edge with both endvertices being nonempty has one endvertex, say x_1, in T, and let $f(x_1) = 1$ and $f(\overline{x_1}) = 2$. Then $f(\overline{S}) = 0$, and $f(\overline{y_1}) \neq 1$ (otherwise y_1 is not dominated). If $f(\overline{y_1}) = 0$, then $f(y_1) = 2$, and there has to exist a vertex in $V(\overline{T})\setminus\{\overline{x_1}, \overline{y_1}\}$, say $\overline{x_2}$, with label 1. This implies $f(x_2) = 0$, $f(y_2) = 2$ and $f(\overline{y_2}) = 0$. Since $\overline{y_2}$ is not adjacent to $\overline{x_2}$ and all vertices (if they exist) in $V(\overline{T})\setminus\{\overline{x_1}, \overline{y_1}, \overline{x_2}, \overline{y_2}\}$ are clearly empty, $\overline{y_2}$ is not dominated. So it must hold $f(\overline{y_1}) = 2$. Then $f(y_1) = 0$, and no vertex in $V(\overline{T}) \setminus \{\overline{x_1}, \overline{y_1}\}$ has label 2. Moreover, this set contains a vertex with label 1, say $\overline{x_2}$, otherwise all vertices in $T \setminus \{x_1, y_1\}$ are nonempty, which leads to a contradiction as we have seen before. Thus $f(x_2) = 0$, $f(y_2) = 2$ and $f(\overline{y_2}) = 0$. But this is impossible since $\overline{y_2}$ is not dominated. With this final contradiction we have arrived to the conclusion that if $m \geq 2$, then $\gamma_{\mathrm{ri2}}(G\overline{G}) = n + 2$. □

3 Lower bound and characterization of extremal graphs

First, we introduce a crucial property that is fundamental for deriving subsequent results.

Proposition 10 *Let G be a graph of order $n \in \mathbb{N}$ and f a γ_{ri2}-function on $G\overline{G}$. If there exists an empty perfect edge under f, then* $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq \max\{\gamma_{\mathrm{ri2}}(G), \gamma_{\mathrm{ri2}}(\overline{G})\} + 2$.

Proof Let G be a graph and assume without loss of generality that $\gamma_{\mathrm{ri2}}(G) \geq \gamma_{\mathrm{ri2}}(\overline{G})$. Suppose to the contrary that

$$\gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(G) + 1. \tag{1}$$

Let $f : V(G\overline{G}) \to \{0, 1, 2\}$ be a γ_{ri2}-function such that there exists an empty perfect edge. Thus in the G-layer as well as in the $\overline{G}$-layer we have vertices with both kind of labels, 1 and 2. Let $A = \{v \in V(G) | f(v) = 1\}$, $B = \{v \in V(G) | f(v) = 2\}$, and note that $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq |A| + |B| + 2$.

Let $\{C_0, C_1, C_2\}$ be the partition of $C = V(G) \backslash (A \cup B)$ such that $f(\overline{C_0}) = 0$, $f(\overline{C_1}) = 1$ and $f(\overline{C_2}) = 2$. Note that $C_0 \neq \emptyset$, however, C_1 and C_2 might be empty sets. If C_1 is nonempty, let $C_1 = D_1 \cup D_2$ such that D_1 contains vertices in C_1 that have no neighbor in A, and D_2 contains vertices in C_1 that have a neighbor in A. Similarly, let $C_2 = E_1 \cup E_2$ such that E_1 contains vertices in C_2 having no neighbor in B, and E_2 consists of vertices in C_2 having a neighbor in B. Further, since vertices in $\overline{A}$ induce a clique, $\overline{A}$ contains at most one vertex with label 2, and since vertices in $\overline{B}$ are pairwise adjacent, at most one of them is labelled by 1. Thus the weight of f equals $|A| + |B| + |\overline{C_1}| + |\overline{C_2}| + k$, where $k = 0$ if all vertices in $\overline{A}$ and $\overline{B}$ are empty, $k = 1$ if there exists one nonempty vertex in $\overline{A} \cup \overline{B}$, and $k = 2$ if we have two nonempty vertices in $\overline{A} \cup \overline{B}$. So, we have

$$|A| + |B| + 2 \leq \gamma_{\mathrm{ri2}}(G\overline{G}) = |A| + |B| + |\overline{C_1}| + |\overline{C_2}| + k, \tag{2}$$

where $k \in \{0, 1, 2\}$.

Now consider the graph G, having in mind that it is isomorphic to the G-layer of $G\overline{G}$, and let a function $g : V(G) \to \{0, 1, 2\}$ be defined as follows:

- $g(A) = 1$, $g(B) = 2$, $g(C_0) = 0$,
- if there exists $d \in D_1$, then $g(d) = 1$ and $g(v) = 0$ for every $v \in C_1 \backslash \{d\}$; and if $D_1 = \emptyset$ then $g(C_1) = 0$,
- if there exists $e \in E_1$, then $g(e) = 2$ and $g(v) = 0$ for every $v \in C_2 \backslash \{e\}$; and if $E_1 = \emptyset$ then $g(C_2) = 0$.

Let us verify that g is a 2RiDF on G. The independence condition is clearly satisfied: all vertices in A (B, respectively) are independent, and, if it exists, d (e, respectively) with the label 1 (2, respectively) has no neighbors in A (B, respectively). Now let us consider vertices in C. The construction of g implies that vertices in C_0 are clearly dominated (they must have a neighbor in A and a neighbor in B, otherwise f is not a 2RiDF on $G\overline{G}$, a contradiction). Next, note that every vertex in C_1 is adjacent to a vertex in B, and that every vertex in C_2 is adjacent to a vertex in A. Thus vertices in $D_2 \cup E_2$ are clearly dominated as well. Finally, if $D_1 \neq \emptyset$, then every vertex in $D_1 \backslash \{d\}$ (if it exists) has a neighbor in B and is adjacent to d with the label 1, since vertices in C_1 induce a clique. Similarly, if $E_1 \neq \emptyset$, then every vertex in $E_1 \setminus \{e\}$ (if it exists) has a neighbor in A and is adjacent to e with the label 2, since C_2 induces a clique as well. Let t be the contribution of vertices d and e to the weight of g; i.e. $t = 0$ if d and e do not exist, $t = 1$ if one of them exists, and $t = 2$ if they both exist. So

$$\gamma_{\mathrm{ri2}}(G) \leq |A| + |B| + t \tag{3}$$

for $t \in \{0, 1, 2\}$. Combining (1), (2) and (3) we derive

$$|A| + |B| + 2 \leq |A| + |B| + |\overline{C_1}| + |\overline{C_2}| + k \leq |A| + |B| + t + 1 \leq |A| + |B| + 3$$

which simplifies to

$$2 \leq |\overline{C_1}| + |\overline{C_2}| + k \leq t + 1 \leq 3. \tag{4}$$

It is clear that $t \neq 0$. Before we consider the cases $t = 1$ and $t = 2$ let us verify the following claim: if $f(\overline{A}) = 0$ and $|C_2| = 1$ we obtain a contradiction. Indeed, if $f(\overline{A}) = 0$ and $C_2 = \{c\}$, then each vertex in $\overline{A}$ has to be adjacent to $\overline{c}$, meaning that no vertex in A is adjacent to c, which is not possible (since in that case c is not dominated under f). Consequently, from $f(\overline{A}) = 0$ it follows $|C_2| \geq 2$, otherwise $|C_2| = 0$ and vertices from $\overline{A}$ are undominated under f. Similarly, $f(\overline{B}) = 0$ yields $|C_1| \geq 2$. Since $|C_1| + |C_2| \leq 3$ by (4) we immediately have a contradiction if $f(\overline{A}) = 0 = f(\overline{B})$. Hence, $k \neq 0$.

First, consider the case when $t = 1$, i.e. exactly one of d and e exists. Since $k \neq 0$, and from $|C_1| + |C_2| + k = 2$ by (4), we may conclude that $|C_1| + |C_2| = 1$ and $k = 1$. However, that means $f(\overline{A}) = 0$ or $f(\overline{B}) = 0$ and consequently $|C_1| + |C_2| \geq 2$, a contradiction. Thus, $t = 2$, i.e. $|C_1| \geq 1$ and $|C_2| \geq 1$. As $|C_1| + |C_2| + k \leq 3$ by (4) and $k \neq 0$, we have $|C_1| = |C_2| = k = 1$, which leads to a final contradiction. □

By Proposition 8 it is clear that $\gamma_{ri2}(G\overline{G}) \geq \max\{\gamma_{ri2}(G), \gamma_{ri2}(\overline{G})\} + 1$ as soon as $\gamma_{ri2}(G\overline{G}) \in \{2, 3\}$. With the above results, we have established the main tools to demonstrate that this inequality holds in general.

Lemma 11 *For any graph G we have $\gamma_{ri2}(G\overline{G}) \geq \max\{\gamma_{ri2}(G), \gamma_{ri2}(\overline{G})\} + 1$.*

Proof Without loss of generality suppose that $\gamma_{ri2}(G) \geq \gamma_{ri2}(\overline{G})$. Let $f : V(G\overline{G}) \to \{0, 1, 2\}$ be a $\gamma_{ri2}(G\overline{G})$-function. If there is an empty perfect edge under f, we are done by Proposition 10. Thus, assume no empty perfect edge exists, and suppose to the contrary that $\gamma_{ri2}(G\overline{G}) \leq \gamma_{ri2}(G)$. Then for each pair of vertices $v, \overline{v}$ at least one of them is nonempty, which further implies that $\gamma_{ri2}(G\overline{G}) \geq n$. Consequently $n \leq \gamma_{ri2}(G\overline{G}) \leq \gamma_{ri2}(G) \leq n$, i.e. $\gamma_{ri2}(G\overline{G}) = \gamma_{ri2}(G) = n$. Hence, Lemma 1 provides that G is a disjoint union of graphs isomorphic to K_1 or K_2, and by Lemma 9 we have $\gamma_{ri2}(G\overline{G}) > n$, a contradiction. □

It turns out that the bound in Lemma 11 is tight. The remainder of this paper is dedicated to characterizing the graphs that achieve this bound. To do so, we first examine special complementary prisms. Notably, $C_5\overline{C_5}$ is the well-known Petersen graph, which we will denote by P.

Lemma 12 *Let P be the Petersen graph. Then $\gamma_{ri2}(P) = 6$.*

Proof The function presented in Fig. 1 is clearly a 2RiDF of P. Thus $\gamma_{ri2}(P) \leq 6$. Further, by Lemma 11 and the fact that $\gamma_{ri2}(C_5) = \gamma_{ri2}(\overline{C_5}) = 4$, we obtain that $\gamma_{ri2}(P) \in \{5, 6\}$. We will show that the assumption that there is a 2RiDF of weight 5 leads to a contradiction. To do so, we use the notation of vertices in $V(P)$ as depicted on the left-hand side of Fig. 1, and we assume that there is no empty perfect edge in P, as the opposite would contradict Proposition 10.

Recall that the Petersen graph is vertex-transitive. Therefore, we may without loss of generality suppose that $f(a) = 0$, $f(\overline{a}) = 1$ and $f(b) = 2$. Hence, $f(\overline{b}) = 0$, otherwise at least one of the perfect edges $c\overline{c}$, $d\overline{d}$ and $e\overline{e}$ is empty, a contradiction. As $\overline{d}$ is adjacent to $\overline{a}$, it must be $f(\overline{e}) = 1$ and in turn, $f(e) = 0$, otherwise, if $f(e) = 2$, a contradiction is obtained due to the existence of an empty perfect edge. Now we derive that $f(d) = 2$ and $f(\overline{d}) = 0$ thus exactly one of vertices c and $\overline{c}$ is nonempty. However, in either case the empty one is not dominated, a contradiction. □

Before proceeding to the next specific complementary prism, consider the following. Let G be a graph of order $n \geq 4$ with $\gamma_{ri2}(G) = n - 1$. Suppose that $\gamma_{ri2}(\overline{G}) > \gamma_{ri2}(G)$, i.e. $\gamma_{ri2}(\overline{G}) = n$. Then, by Lemma 1, we have $\gamma_{ri2}(G) = 2$, a contradiction. To facilitate future references, we state this observation as follows.

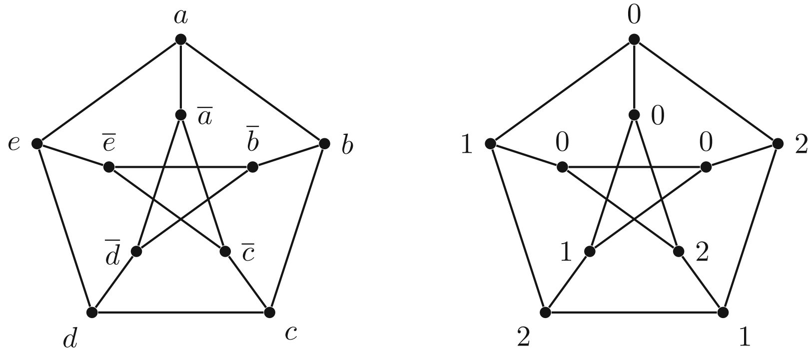

Fig. 1 Petersen graph (left) and one of its 2RiDFs (right)

Observation 13 *If G is a graph of order $n \geq 4$ with $\gamma_{\rm ri2}(G) = n-1$, then $\gamma_{\rm ri2}(G) \geq \gamma_{\rm ri2}(\overline{G})$.*

Proposition 14 *Let $n \geq 3$. If $G \in \{S_n, S_n^+\}$, then $\gamma_{\rm ri2}(G\overline{G}) = n+1$.*

Proof Let $G \in \{S_n, S_n^+\}$. If $G = S_3$, then by Lemma 9 we have $\gamma_{\rm ri2}(G\overline{G}) = 4$, and if $G = S_3^+$, the same holds by Lemma 7. So we may assume $n \geq 4$. By Observation 13, Lemma 11 and Theorem 3 we have $\gamma_{\rm ri2}(G\overline{G}) \geq n$. Let $c, x_1, x_2, \ldots, x_{n-1}$ denote vertices of S_n where c is the central vertex of the star, and let S_n^+ be obtained from S_n by joining x_1 and x_2. Now let g be a function that assigns values from $\{0, 1, 2\}$ to vertices of $G\overline{G}$ as follows:

- $g(c) = 0$ and $g(\overline{c}) = 1$,
- $g(x_1) = 2$ and $g(\overline{x_1}) = 1$,
- $g(x_i) = 2$ and $g(\overline{x_i}) = 0$ for $i \in \{3, 4, \ldots, n-1\}$,
- $g(x_2) = 2$ and $g(\overline{x_2}) = 0$, if $G = S_n$, and
- $g(x_2) = 0$ and $g(\overline{x_2}) = 1$, if $G = S_n^+$.

As g is clearly a 2RiDF on $G\overline{G}$, we have $\gamma_{\rm ri2}(G\overline{G}) \leq n+1$.

Let f be a $\gamma_{\rm ri2}(G\overline{G})$-function. If there is an empty perfect edge under f, then by Proposition 10 and Observation 13 we have $\gamma_{\rm ri2}(G\overline{G}) \geq \gamma_{\rm ri2}(G) + 2 = n+1$, so we are done in this case. Thus assume there is no empty perfect edge, and recall that $\gamma_{\rm ri2}(G\overline{G}) \in \{n, n+1\}$. Suppose (to the contrary) that $\gamma_{\rm ri2}(G\overline{G}) = n$. This means that exactly one endvertex of every perfect edge is nonempty. Since $\overline{c}$ is a leaf in $G\overline{G}$, $f(\overline{c}) \neq 0$, say $f(\overline{c}) = 1$, thus $f(c) = 0$.

If $G = S_n$, then every vertex in the G-layer except c is of degree 2 and thus nonempty, so the corresponding vertices in the $\overline{G}$-layer are empty, which is a contradiction since they are not dominated (they only have one nonempty neighbor $\overline{c}$).

Now consider the case when $G = S_n^+$. By the same reasoning as above we conclude that $x_3, x_4, \ldots, x_{n-1}$ are all nonempty, so $\overline{x_3}, \overline{x_4}, \ldots, \overline{x_{n-1}}$ are all empty. Thus at least one of $\overline{x_1}, \overline{x_2}$ must be nonempty, say $f(\overline{x_1}) \neq 0$, so $f(x_1) = 0$. To guarantee x_1 is dominated, x_2 has to be nonempty, so $f(\overline{x_2}) = 0$. But this is a final contradiction since x_2 has only one nonempty neighbor. □

Recall that for $n \geq 4$, we have $\gamma_{\rm ri2}(B_n) = n-1$. We next show that for $B_n\overline{B_n}$ any $\gamma_{\rm ri2}$-function has the weight increased by one with respect to a $\gamma_{\rm ri2}$-function on the base graph.

Lemma 15 *Let $n \geq 4$. Then $\gamma_{\rm ri2}(B_n\overline{B_n}) = n$ and for every $\gamma_{\rm ri2}(B_n\overline{B_n})$-function f exactly two vertices in the $\overline{B_n}$-layer are nonempty. Moreover, to these two nonempty vertices the same color is assigned under f.*

Proof By Observation 13, $\max\{\gamma_{\mathrm{ri2}}(B_n), \gamma_{\mathrm{ri2}}(\overline{B_n})\} = n-1$, therefore $\gamma_{\mathrm{ri2}}(G\overline{G}) \geq n$ by Lemma 11. As described in the introduction, B_n is obtained by attaching $n-3$ leaves $x_1, x_2, \ldots, x_{n-3}$ to one endvertex of a path abc, say c. Now define $g : V(B_n\overline{B_n}) \to \{0, 1, 2\}$ by $g(x_i) = 1$ for $i \in \{1, 2, \ldots, n-3\}$, $g(a) = 1$, $g(\overline{b}) = g(\overline{c}) = 2$ and all other vertices in $V(B_n\overline{B_n})$ are empty under g. It is easy to see that g is a 2RiDF on $B_n\overline{B_n}$. Thus, $\gamma_{\mathrm{ri2}}(B_n\overline{B_n}) \leq n$, and hence $\gamma_{\mathrm{ri2}}(B_n\overline{B_n}) = n$. In what follows, we show that g is a unique γ_{ri2}-function on $B_n\overline{B_n}$, aside from the potential interchange of 1s and 2s.

In order to do this, let $f : V(B_n\overline{B_n}) \to \{0, 1, 2\}$ be an arbitrary γ_{ri2}-function. Then its weight is n. Note that there is no empty perfect edge under f, otherwise by Observation 13, Lemma 10 and Theorem 3 we derive that $\gamma_{\mathrm{ri2}}(B_n\overline{B_n}) \geq n+1$, a contradiction. Thus, every perfect edge has exactly one nonempty endvertex.

First, suppose c is nonempty, say $f(c) = 1$. Then $f(\overline{c}) = 0$ and hence, $f(\overline{a}) = 2$ and $f(a) = 0$. By the independence condition $f(b) \neq 1$, which further implies that a is not dominated under f, a contradiction. Therefore, $f(c) = 0$ and $f(\overline{c}) \neq 0$. Suppose without loss of generality that $f(\overline{c}) = 2$. Observe that $\overline{x_1}, \overline{x_2}, \ldots, \overline{x_{n-3}}$ must be empty. Indeed, if one of them is nonempty, say $f(\overline{x_1}) \neq 0$, we have $f(x_1) = 0$, meaning that x_1 is not dominated. The case $f(b) \neq 0$ and $f(\overline{b}) = 0$ leads to a contradiction since $\overline{b}$ is adjacent to only one nonempty vertex b. Now, from $f(b) = 0$ it follows $f(\overline{b}) \neq 0$ and $f(a) \neq 0$. Consequently, $f(\overline{a}) = 0$ implies that $f(a) = 1$, since $\overline{a}$ is adjacent to only one nonempty vertex $\overline{c}$ in the $\overline{B_n}$-layer. Finally, $f(\overline{b}) = 2$ (otherwise b is not dominated), and $f(x_i) = 1$ for every $i \in \{1, 2, \ldots, n-3\}$. To sum up, in the $\overline{B_n}$-layer only $\overline{b}$ and $\overline{c}$ are nonempty and they are of the same color. □

To characterize the graphs that achieve equality in Lemma 11, we must examine the impact on the 2-rainbow independent domination number of a complementary prism when isolated vertices are added to the base graph.

Lemma 16 *Let $H = C_5$, or $H = B_n$ for $n \geq 4$, or $H \in \{S_n, S_n^+\}$ for $n \geq 3$. If G is a disjoint union of H and $k \geq 1$ copies of K_1, then $\gamma_{\mathrm{ri2}}(G\overline{G}) = \gamma_{\mathrm{ri2}}(H\overline{H}) + k$.*

Proof Let $S = \{x_1, x_2, \ldots, x_k\}$ be the set of all isolated vertices in G and let $f_H : V(H\overline{H}) \to \{0, 1, 2\}$ be a γ_{ri2}-function. Clearly there is a nonempty vertex in the $\overline{H}$-layer (otherwise vertices in $V(\overline{H})$ are not dominated as they have at most one nonempty neighbor), say $f_H(\overline{a}) = 1$. Then $f : V(G\overline{G}) \to \{0, 1, 2\}$ defined with $f(v) = f_H(v)$ for every $v \in V(H\overline{H})$, $f(x_i) = 2$ and $f(\overline{x_i}) = 0$ for every $i \in \{1, 2, \ldots, k\}$, is clearly a 2RiDF on $G\overline{G}$, thus $\gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(H\overline{H}) + k$. In order to see that in fact the equality holds, let us assume to the contrary that $\gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(H\overline{H}) + k - 1$. By Lemma 11 and Observation 13 we further infer

$$\gamma_{\mathrm{ri2}}(G) + 1 \leq \gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(H\overline{H}) + k - 1. \tag{5}$$

First we consider the case when $H = B_n$. Recall that $\gamma_{\mathrm{ri2}}(B_n) = n-1$ and $\gamma_{\mathrm{ri2}}(B_n\overline{B_n}) = n$ by Lemma 15. Applying (5) we derive

$$n + k = \gamma_{\mathrm{ri2}}(G) + 1 \leq \gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(B_n\overline{B_n}) + k - 1 = n + k - 1,$$

a contradiction.

Thus, $H \in \{C_5, S_n, S_n^+\}$. In this case (5) simplifies to

$$n + k \leq \gamma_{\mathrm{ri2}}(G\overline{G}) \leq \gamma_{\mathrm{ri2}}(H\overline{H}) + k - 1 = n + k$$

by Lemmas 12 and 14. Let $g : V(G\overline{G}) \to \{0, 1, 2\}$ be a γ_{ri2}-function, i.e. $w(g) = n + k$. Clearly, all leaves that in $G\overline{G}$ correspond to vertices in S are nonempty with respect to g.

If, in addition, all corresponding vertices in $\overline{S}$ are empty, then all vertices from $V(H\overline{H})$ are dominated within $H\overline{H}$. This implies $\gamma_{ri2}(H\overline{H}) = n$. However, as observed above we have $\gamma_{ri2}(H\overline{H}) = n + 1$, a contradiction.

Therefore, there is a nonempty vertex in $\overline{S}$. Without loss of generality let us assume $g(x_1) = 1$ and $g(\overline{x_1}) = 2$. Since in the $\overline{G}$-layer, $\overline{x_1}$ is a universal vertex, $g(\overline{v}) \neq 2$ for every $\overline{v} \in V(\overline{G})\backslash\{x_1\}$. Note that in $V(G\overline{G})\backslash(V(S) \cup \{\overline{x_1}\})$ there are exactly $n-1$ nonempty vertices, and since in $G\overline{G}$ there are exactly n perfect edges between vertices in $V(H)$ and $V(\overline{H})$ there exists an empty perfect edge. Applying Proposition 10 and Observation 13 we infer that

$$\gamma_{ri2}(G\overline{G}) \geq \max\{\gamma_{ri2}(G), \gamma_{ri2}(\overline{G})\} + 2 = \gamma_{ri2}(G) + 2 = \gamma_{ri2}(H) + k + 2 = n + k + 1,$$

a final contradiction. □

With all the above results we are now ready to prove the main theorem in which we give the lower bound on the 2-rainbow independent domination number of a complementary prism and characterize extremal graphs that attain this bound.

Theorem 5 *Let G be a graph of order $n \in \mathbb{N}$. Then $\gamma_{ri2}(G\overline{G}) \geq \max\{\gamma_{ri2}(G), \gamma_{ri2}(\overline{G})\} + 1$ with equality if and only if G or $\overline{G}$ is a disjoint union of one connected component G_1 isomorphic to a graph from $\{K_1, K_2, B_{n_1}\}$, where $n_1 \geq 4$, and all other (possible) connected components isomorphic to K_1.*

Proof For $n \leq 2$ the claim is a straightforward consequence of Proposition 8. In what follows, let $n \geq 3$, and without loss of generality let G be a graph of order n such that $\gamma_{ri2}(G) \geq \gamma_{ri2}(\overline{G})$.

The desired inequality holds by Lemma 11. Also, if G or its complement is one of the graphs from the theorem, then Lemmas 7, 9, 15 and 16, respectively, assure that the equality is attained for these graphs. Thus it only remains to consider the structure of a graph G if

$$\gamma_{ri2}(G\overline{G}) = \max\{\gamma_{ri2}(G), \gamma_{ri2}(\overline{G})\} + 1 = \gamma_{ri2}(G) + 1. \tag{6}$$

Let $f : V(G\overline{G}) \to \{0, 1, 2\}$ be a γ_{ri2}-function. Applying Lemma 7 we can further assume that $G \neq N_n$. If $\gamma_{ri2}(G) \leq n-2$, then $\gamma_{ri2}(G\overline{G}) \leq n-1$ and there is an empty perfect edge under f. Thus, by Proposition 10 it holds that $\gamma_{ri2}(G\overline{G}) \geq \gamma_{ri2}(G) + 2$, a contradiction. For $\gamma_{ri2}(G) = n$, Lemma 1 provides that every connected component of G is isomorphic either to K_1 or K_2. Lemma 9 further implies that G is a disjoint union of K_2 and N_{n-2}.

In the last case, when $\gamma_{ri2}(G) = n-1$ and consequently $\gamma_{ri2}(G\overline{G}) = n$, we have by Theorem 4 that G has one connected component G_1 isomorphic to a graph from $\{S_{n_1}, S^+_{n_1}, B_{n_1}, C_5\}$ where $n_1 = |V(G_1)|$, and all the other connected components of G (if they exist) are isomorphic to K_1 or K_2. Note that $\gamma_{ri2}(G_1) = n_1 - 1$.

Denote by S the set of all isolated vertices in G, and by T the set of vertices that belong to connected components isomorphic to K_2 and note that S or T or both can be empty sets. Also observe that there is no empty perfect edge under f, otherwise we have a contradiction by Proposition 10. Therefore for every perfect edge exactly one of its endvertices is nonempty.

If T is empty, then G is a disjoint union of G_1 and N_k, where $k \geq 0$, so by Lemma 16 it holds $\gamma_{ri2}(G\overline{G}) = \gamma_{ri2}(G_1\overline{G_1}) + k$, and by (6) we derive

$$\gamma_{ri2}(G\overline{G}) = \gamma_{ri2}(G) + 1 = \gamma_{ri2}(G_1) + k + 1 = n_1 + k.$$

Combining both equalities, we infer $\gamma_{ri2}(G_1\overline{G_1}) = n_1$, which is not possible unless G_1 is a broom. Namely, if G is either S_{n_1} or $S^+_{n_1}$, then by Lemma 14, $\gamma_{ri2}(G_1\overline{G_1}) = n_1 + 1$, and if $G = C_5$, then by Lemma 12, $\gamma_{\mathrm{ri2}}(G\overline{G}) = 6$, but $\gamma_{\mathrm{ri2}}(C_5) = 4$, clearly a contradiction in each case. Thus $G_1 = B_{n_1}$.

The remaining case to consider is the case when $T \neq \emptyset$. Take any vertices $x, y \in T$ that induce a connected subgraph of G, isomorphic to K_2. Since they are of degree two in $G\overline{G}$, they cannot both be empty. So without loss of generality assume $f(x) = 1$. Thus $f(\overline{x}) = 0$. If $f(y) = 0$, then $f(\overline{y}) = 2$. Note that every vertex in the $\overline{G}$-layer except $\overline{x}$ is adjacent to $\overline{y}$, so there is no other vertex in $V(\overline{G}) \setminus \{\overline{x}, \overline{y}\}$ with label 2, meaning that $\overline{x}$ is not dominated in $G\overline{G}$ as it is not adjacent to $\overline{y}$, a contradiction. Therefore $f(y) = 2$ and thus $f(\overline{y}) = 0$. Since x and y were arbitrarily chosen adjacent vertices from T, we derive that $f(\overline{T}) = 0$. Recall that, if S is nonempty, $f(\overline{S}) = 0$ as well. So all vertices from $S \cup T$ are nonempty, and since

$$\gamma_{\mathrm{ri2}}(G\overline{G}) = \gamma_{\mathrm{ri2}}(G) + 1 = \gamma_{\mathrm{ri2}}(G_1) + \gamma_{\mathrm{ri2}}(S) + \gamma_{\mathrm{ri2}}(T) + 1 = n_1 + \gamma_{\mathrm{ri2}}(S) + \gamma_{\mathrm{ri2}}(T),$$

in $G_1\overline{G_1}$ we have exactly n_1 nonempty vertices. In addition, vertices in $G\overline{G}$ that belong to $V(G_1) \cup V(\overline{G_1})$ are dominated within the subgraph induced by this set, so by Lemmas 12 and 14 we conclude $G_1 = B_{n_1}$. In other words, f restricted to $V(G_1\overline{G_1})$ is a γ_{ri2}-function on $G_1\overline{G_1}$. However, to ensure that $\overline{x}$ and $\overline{y}$ are dominated, there exist $u, v \in V(G)$ such that $f(\overline{u}) = 1$ and $f(\overline{v}) = 2$. But this contradicts Lemma 15 by which vertices from $\overline{G_1}$ cannot have different labels. □

4 Conclusion

The main goal of our research in this paper was achieved by establishing the tight lower bound on the 2-rainbow independent domination number of a complementary prism over a given graph in terms of this parameter on the base graph and of its complement, respectively. Furthermore, we have identified all the graphs attaining this bound, revealing all possible configurations of the most cost-effective networks that meet the conditions outlined in the introductory section. While the most expensive configurations may be less interesting from the practical point of view, the natural question of characterizing graphs attaining the upper bound remains open. Additionally, we believe that the tight upper bound in terms of $\gamma_{\mathrm{ri2}}(G)$ and $\gamma_{\mathrm{ri2}}(\overline{G})$ can be determined.

During our studies, we have observed that if $H = C_5$, or $H = B_n$ for $n \geq 4$, or $H \in \{S_n, S_n^+\}$ for $n \geq 3$, the 2-rainbow independent domination number of the corresponding complementary prism increases by 1 for each added isolated vertex (see Lemma 16). However, this is not the case for all graphs. For instance, when C_4 gradually has isolated vertices added one by one, the 2-rainbow independent domination number of the corresponding complementary prism in some cases remains unchanged, i.e. $\gamma_{\mathrm{ri2}}(G\overline{G}) = 6$ for $G \in \{C_4, C_4 \cup K_1, C_4 \cup K_1 \cup K_1\}$. However, in general we believe the following holds.

Conjecture 6 *Let H be a disjoint union of a graph G and K_1. Then $\gamma_{\mathrm{ri2}}(H\overline{H}) = \gamma_{\mathrm{ri2}}(G\overline{G})$ or $\gamma_{\mathrm{ri2}}(H\overline{H}) = \gamma_{\mathrm{ri2}}(G\overline{G}) + 1$.*

If the conjecture holds, another interesting question is a characterization of graphs for which adding of an isolated vertex to a base graph does not increase the 2-rainbow independent domination number.

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